A proposal for a proper definition of higher-order rigidity

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Let $\mathcal{F}$ be a bar-and-joint-framework in the d-dimensional Euclidean space $\mathbb{E}^d$ with vertex set

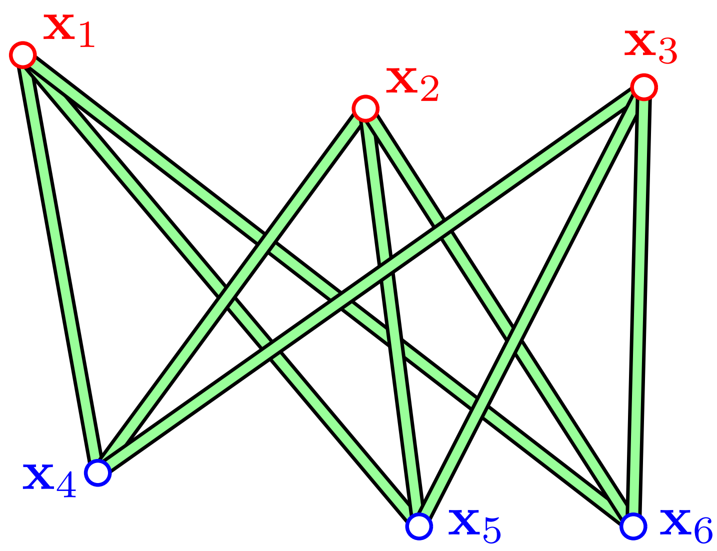
$$V = \{\mathbf{x}_1, \ldots, \mathbf{x}_v\},$$
$$\mathbf{x}_i \in \mathbb{R}^d \text{ for all } i \in I := \{1, \ldots, v\}$$

and edge set

$$E \subset \left\{(i,j) \mid i < j,\ (i,j) \in I^2\right\}.$$

We denote the edge lengths by

$$l_{ij} := \|\mathbf{x}_i - \mathbf{x}_j\| \text{ for all } (i,j) \in E.$$



e.g., bipartite framework:

$$V = \{\mathbf{x}_1, \ldots, \mathbf{x}_6\},$$
$$E = \{(1,4),\ (1,5), \ldots, (3,6)\}$$

1. The definition of infinitesimal flexibility

Definition: *'classical'*
[Rembs 1932, Sabitov 1989, Tarnai 1989, Connelly 1994, ...]

$\mathcal{F}$ is called *infinitesimally flexible of order* n if for each vertex, i.e., for each $i \in I$, there is a polynomial function

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1}\, t + \ \dots\ + \mathbf{x}_{i,n}\, t^n, \quad n \geq 1,$$

such that

1. the replacement of $\mathbf{x}_i$ by $\mathbf{x}_i' \in \mathbb{R}[t]^d$ in the equations for the edge lengths gives stationary values of multiplicity $\geq n$ at $t = 0$, i.e.,
$$\|\mathbf{x}_i' - \mathbf{x}_j'\| - l_{ij} = o(t^n) \quad \forall (i,j) \in E, \text{ and}$$

2. in order to exclude *trivial* flexes, the *velocity vectors* $\mathbf{x}_{1,1}, \dots, \mathbf{x}_{n,1}$ do not originate from any motion of $\mathcal{F}$ as a rigid body.

Definition of infinitesimal rigidity

For the sake of brevity we write

$$X_0 := \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_v \end{pmatrix}, \quad X_1 := \begin{pmatrix} \mathbf{x}_{1,1} \\ \vdots \\ \mathbf{x}_{v,1} \end{pmatrix}, \quad \ldots, \; X_n := \begin{pmatrix} \mathbf{x}_{1,n} \\ \vdots \\ \mathbf{x}_{v,n} \end{pmatrix}$$

and we call

$$X(t) := X_0 + X_1\, t + \ldots + X_n\, t^n$$

a flex of order n. We say that this is *a flex of $\mathcal{F}$* (or: $\mathcal{F}$ *admits* this flex) if property 1. holds.

Definition:

The framework $\mathcal{F}$ is called *infinitesimally rigid of order n*, if any n-th order flex of $\mathcal{F}$ is trivial.

Conditions for infinitesimal flexibility

Comparing the coefficients of $t, t^2, \ldots, t^n$ in

$$(\mathbf{x}_i' - \mathbf{x}_j')^2 - l_{ij}^2 = o(t^n) \quad \text{for} \quad \mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1}\, t + \ \ldots\ + \mathbf{x}_{i,n}\, t^n$$

results in a sequence of systems of linear equations for the unknowns $\mathbf{x}_{i,k}$

$$\begin{aligned}
(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}_{i,1} - \mathbf{x}_{j,1}) &= 0\,, \\
(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}_{i,2} - \mathbf{x}_{j,2}) &= -\tfrac{1}{2}\,(\mathbf{x}_{i,1} - \mathbf{x}_{j,1})\cdot(\mathbf{x}_{i,1} - \mathbf{x}_{j,1}), \\
(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}_{i,3} - \mathbf{x}_{j,3}) &= -(\mathbf{x}_{i,1} - \mathbf{x}_{j,1})\cdot(\mathbf{x}_{i,2} - \mathbf{x}_{j,2}), \\
(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}_{i,4} - \mathbf{x}_{j,4}) &= -(\mathbf{x}_{i,1} - \mathbf{x}_{j,1})\cdot(\mathbf{x}_{i,3} - \mathbf{x}_{j,3}) - \\
&\quad -\tfrac{1}{2}\,(\mathbf{x}_{i,2} - \mathbf{x}_{j,2})\cdot(\mathbf{x}_{i,2} - \mathbf{x}_{j,2}), \\
(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}_{i,5} - \mathbf{x}_{j,5}) &= -(\mathbf{x}_{i,1} - \mathbf{x}_{j,1})\cdot(\mathbf{x}_{i,4} - \mathbf{x}_{j,4}) - \\
&\quad -(\mathbf{x}_{i,2} - \mathbf{x}_{j,2})\cdot(\mathbf{x}_{i,3} - \mathbf{x}_{j,3}), \\
\ldots \quad &\quad \ldots
\end{aligned}$$

$$\begin{aligned}
1 &= 0 \\
2 &= 1.1 \\
3 &= 1.2 \\
4 &= 1.3 + 2.2 \\
5 &= 1.4 + 2.3 \\
&\ldots
\end{aligned}$$

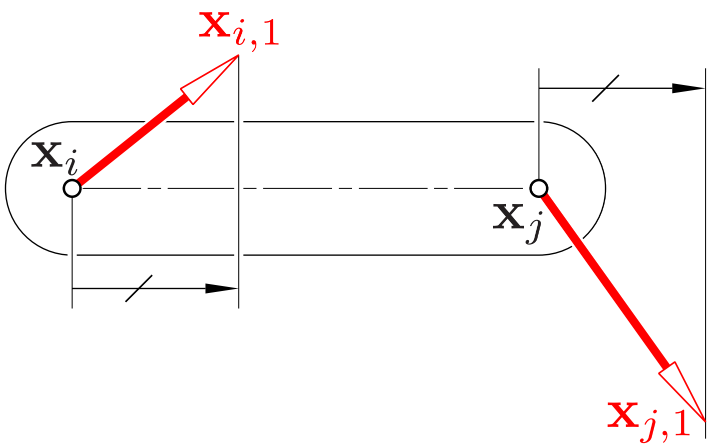
Conditions for infinitesimal flexibility

The number e of equations in each linear system equals the number of edges of $\mathcal{F}$. The unknowns vectors $\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{v,k}$ contain vd unknown coordinates.

The $(e \times vd)$-matrix of coefficients on the left side is always the same and called *rigidity matrix* $R_{\mathcal{F}}$ of $\mathcal{F}$, e.g., for K_{33} the 9×12-matrix reads:

$$R_{K_{33}} = \begin{pmatrix} (\mathbf{x}_1 - \mathbf{x}_4) & \mathbf{o} & \mathbf{o} & (\mathbf{x}_4 - \mathbf{x}_1) & \mathbf{o} & \mathbf{o} \\ (\mathbf{x}_1 - \mathbf{x}_5) & \mathbf{o} & \mathbf{o} & \mathbf{o} & (\mathbf{x}_5 - \mathbf{x}_1) & \mathbf{o} \\ (\mathbf{x}_1 - \mathbf{x}_6) & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & (\mathbf{x}_6 - \mathbf{x}_1) \\ \mathbf{o} & (\mathbf{x}_2 - \mathbf{x}_4) & \mathbf{o} & (\mathbf{x}_4 - \mathbf{x}_2) & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & (\mathbf{x}_2 - \mathbf{x}_5) & \mathbf{o} & \mathbf{o} & (\mathbf{x}_5 - \mathbf{x}_2) & \mathbf{o} \\ \mathbf{o} & (\mathbf{x}_2 - \mathbf{x}_6) & \mathbf{o} & \mathbf{o} & \mathbf{o} & (\mathbf{x}_6 - \mathbf{x}_2) \\ \mathbf{o} & \mathbf{o} & (\mathbf{x}_3 - \mathbf{x}_4) & (\mathbf{x}_4 - \mathbf{x}_3) & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & (\mathbf{x}_3 - \mathbf{x}_5) & \mathbf{o} & (\mathbf{x}_5 - \mathbf{x}_3) & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & (\mathbf{x}_3 - \mathbf{x}_6) & \mathbf{o} & \mathbf{o} & (\mathbf{x}_6 - \mathbf{x}_3) \end{pmatrix}$$

Geometric meaning of the first two systems

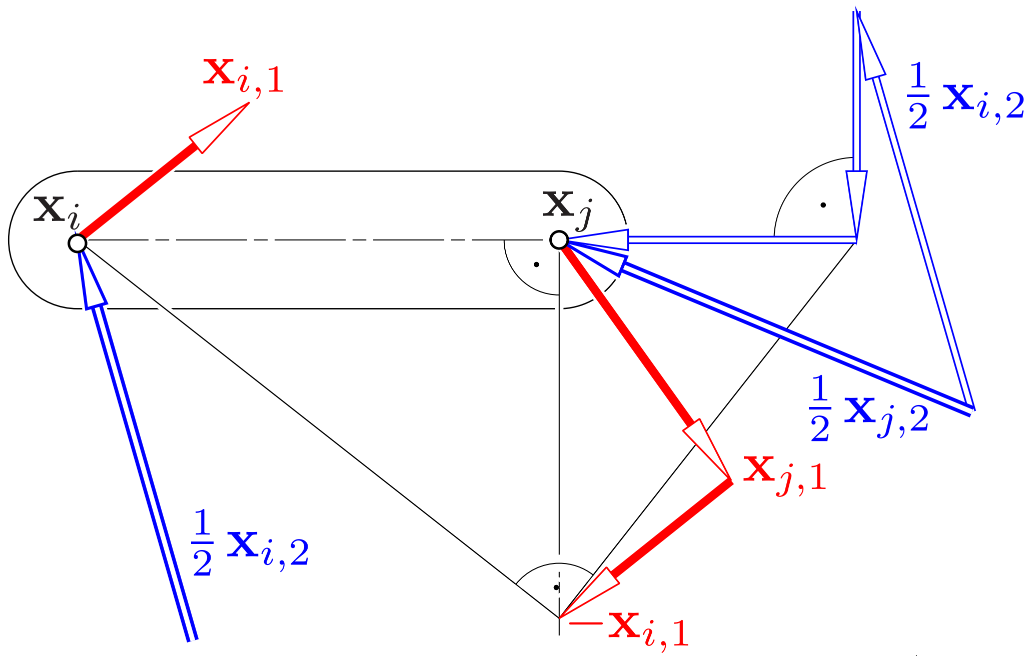


$$(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}_{i,1} - \mathbf{x}_{j,1}) = 0 \iff$$

Projection Theorem — the condition for velocity vectors $\mathbf{x}_{i,1}$, $\mathbf{x}_{j,1}$ at the endpoints $\mathbf{x}_i, \mathbf{x}_j$ of any bar of $\mathcal{F}$.

$$(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}_{i,2} - \mathbf{x}_{j,2}) = -\tfrac{1}{2}(\mathbf{x}_{i,1} - \mathbf{x}_{j,1})\cdot(\mathbf{x}_{i,1} - \mathbf{x}_{j,1}) \iff$$

condition for the acceleration vectors $\frac{1}{2}\mathbf{x}_{i,2}$, $\frac{1}{2}\mathbf{x}_{j,2}$ of $\mathbf{x}_i$, $\mathbf{x}_j$ for $(i,j) \in E$.

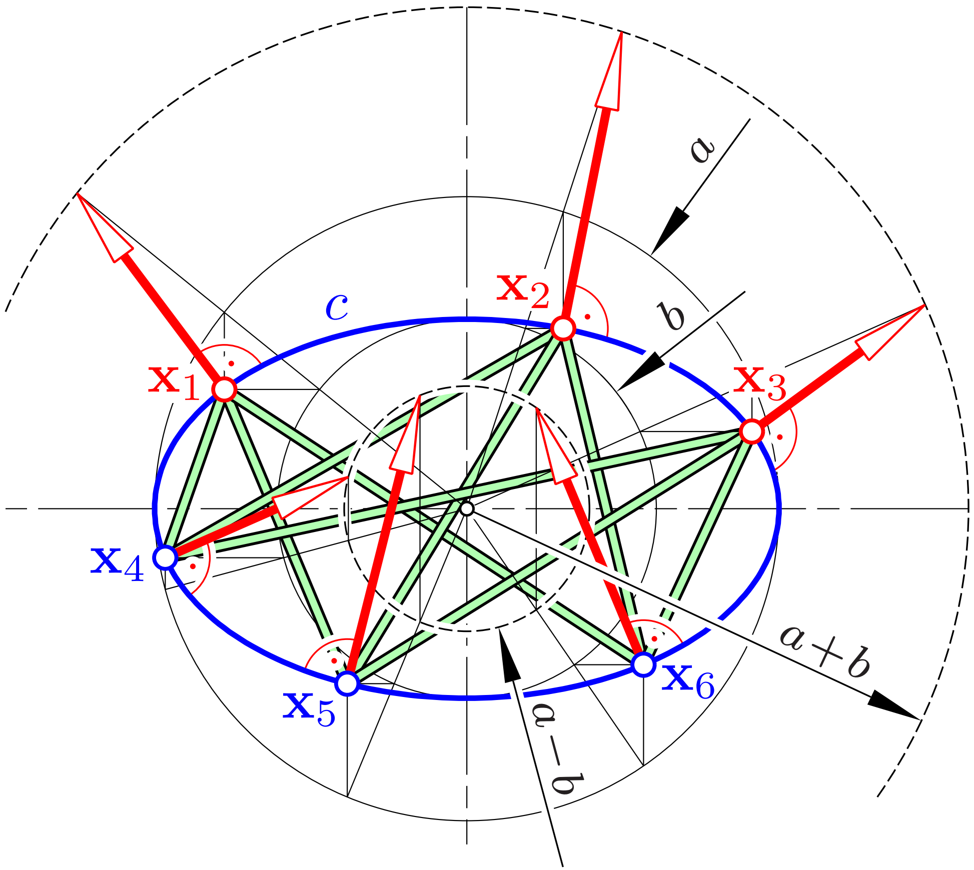


1st order infinitesimal flexibility

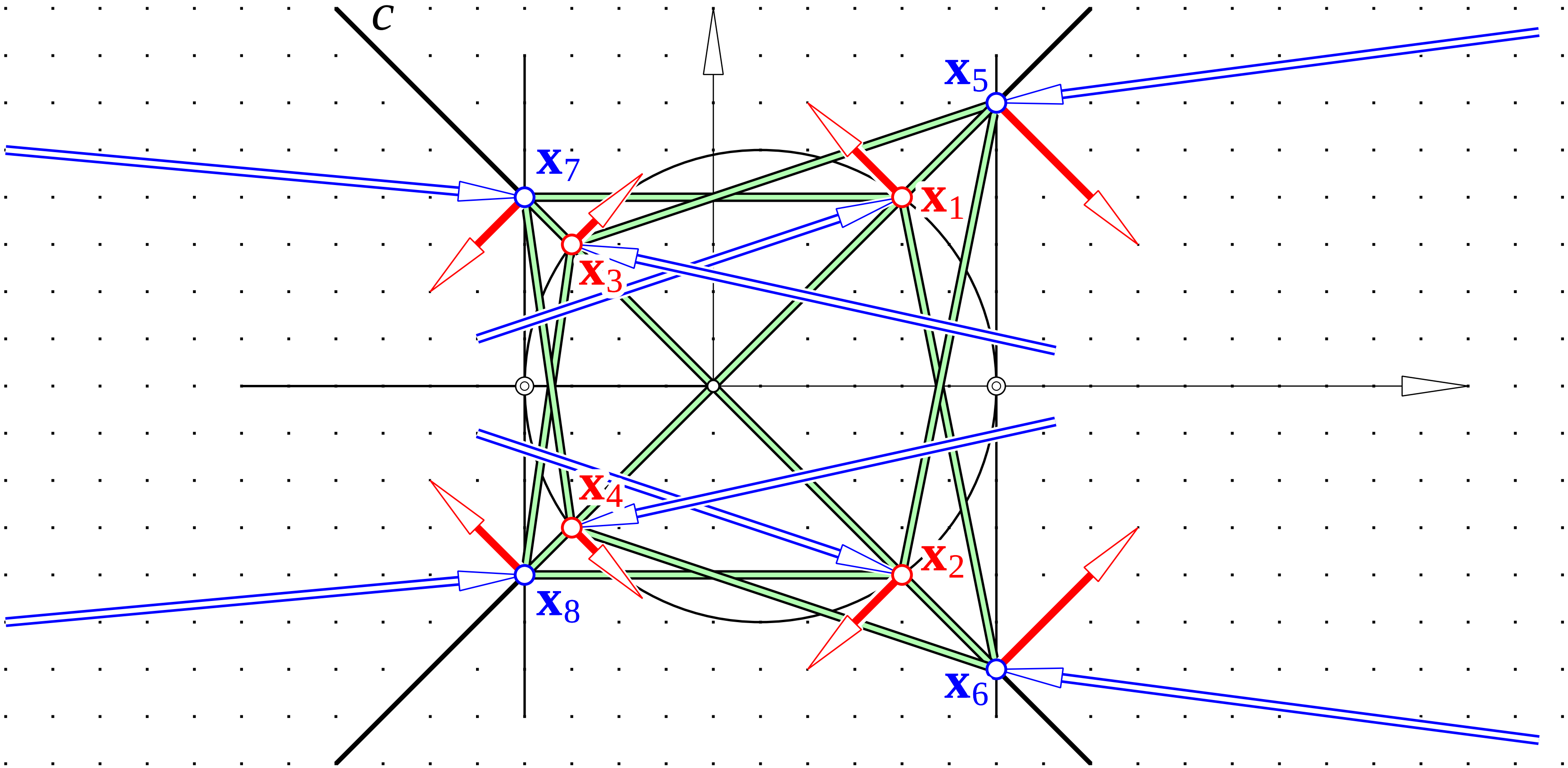
First order flexes $X(t) = X_0 + X_1\, t$ of $\mathcal{F}$ result from the solution of the homogeneous system $R_{\mathcal{F}} \cdot X_1 = 0\,.$

The existence of nontrivial first order flexes is equivalent to

$$\mathrm{rk}(R_{\mathcal{F}}) < vd - \frac{d(d+1)}{2}\,.$$



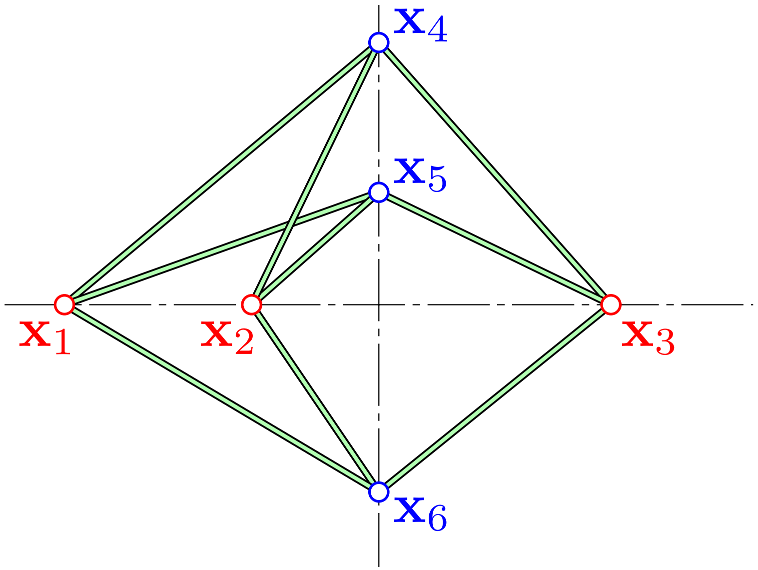
2nd order infinitesimal flexibility



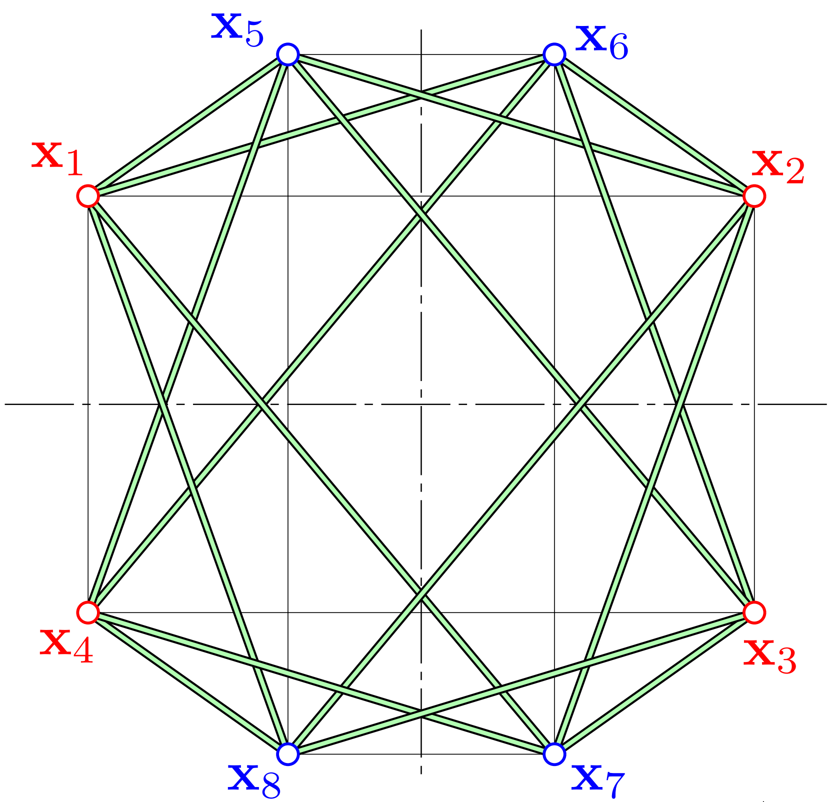
In special cases the bipartite framework is even flexible of second order.

Continuous flexibility

V. Alexandrov (1998): For each framework there is a sufficiently large n such that any nontrivial n-th-order flex can be extended to an analytical flex.

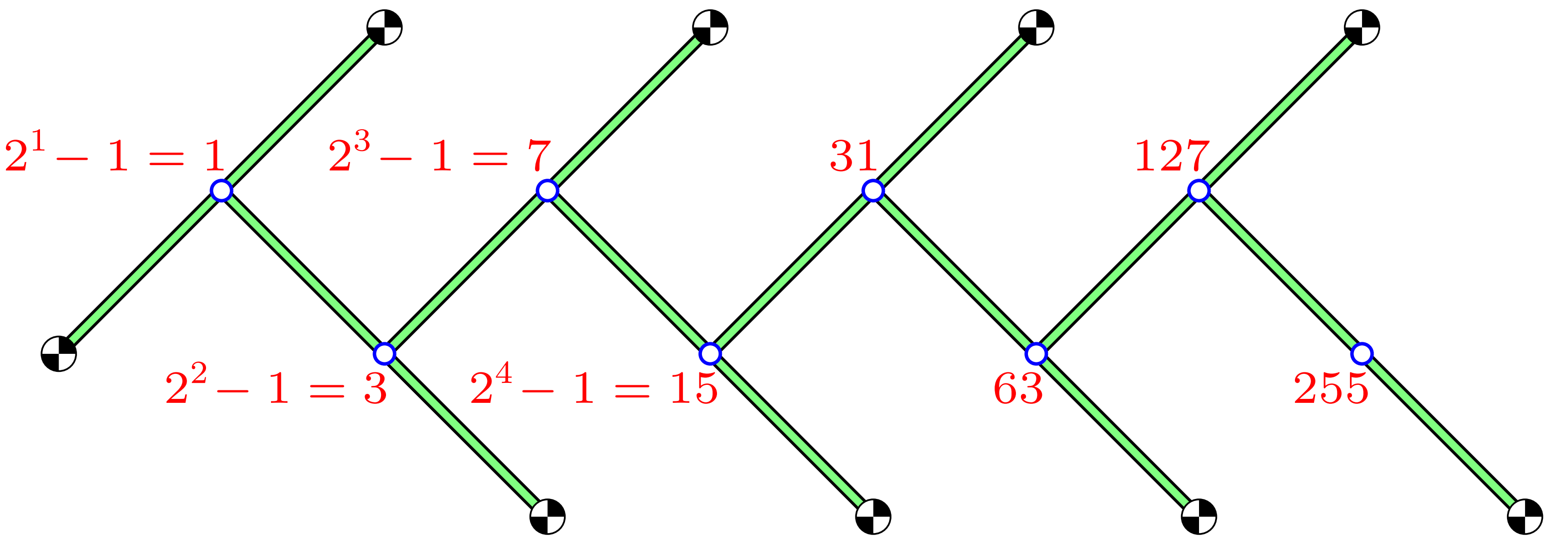


two bipartite frameworks with continuous flexibility — according to Dixon (1899)



Arbitrarily high infinitesimal flexibility

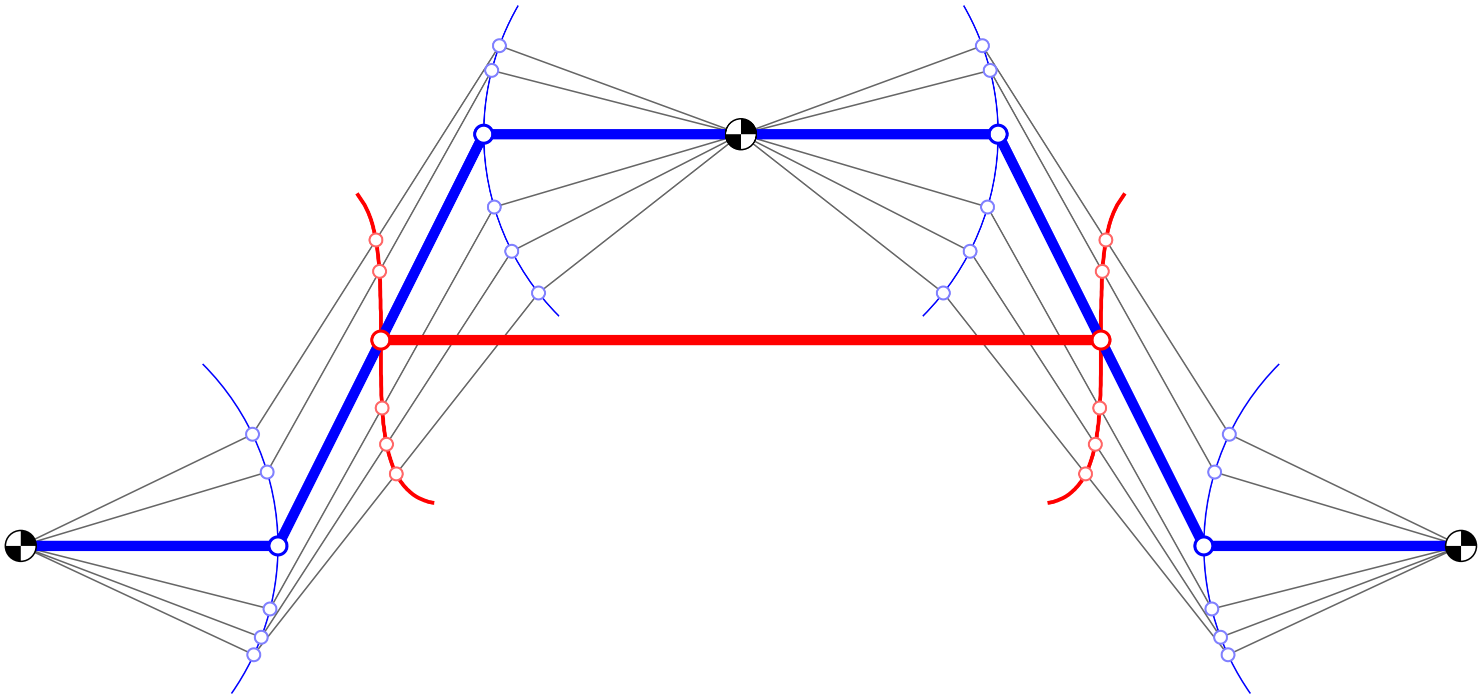
T. TARNAI (1989): There are frameworks with infinitesimal flexibility of arbitrarily high order $2^m - 1$, e.g., the pinned framework (Leonardo DA VINCI ?)



A cusp in the configuration space

R. Connelly, H. Servatius (1994):

There is an example of a continuously flexible pinned framework with a *standstill* in its initial position:



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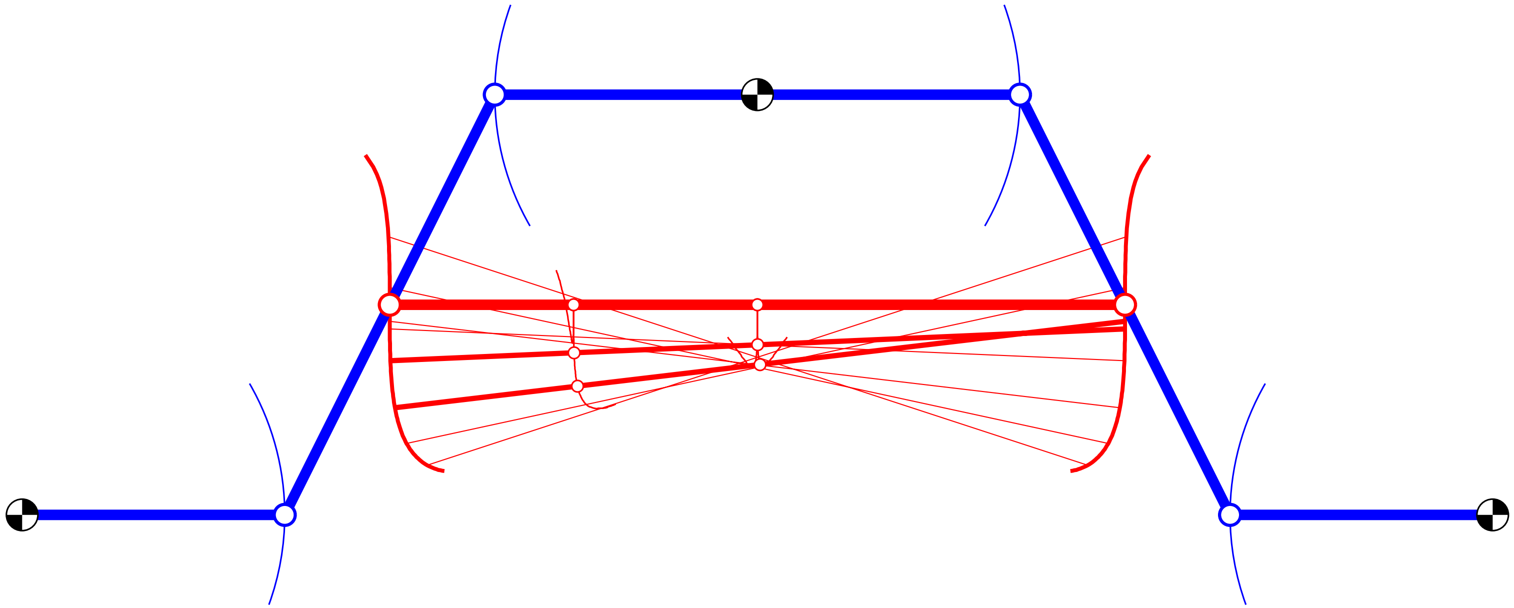
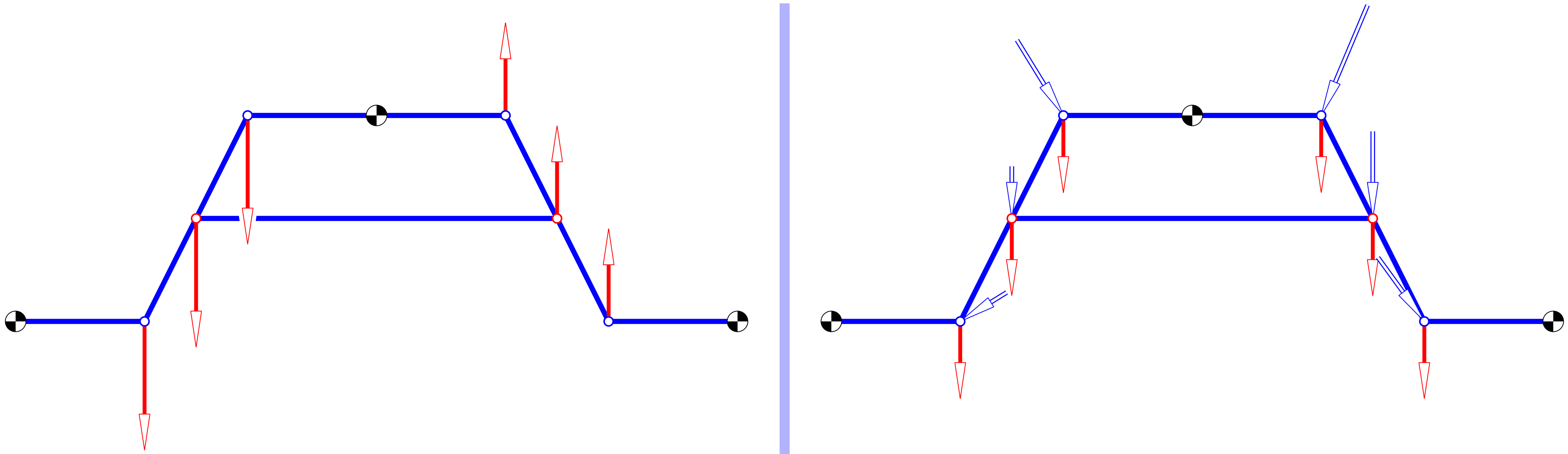


A cusp in the configuration space



nontrivial first order flex | nontrivial second order flex

The admitted third order flexes are of type

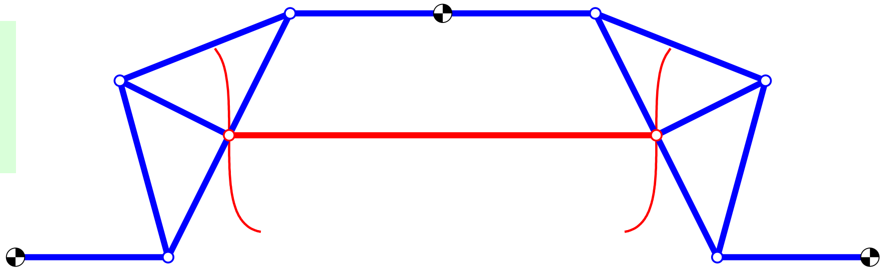
$$X_0 + \mathbf{0}\,.\,t + X_2\,t^2 \;\; + X_3\,t^3$$

and therefore trivial.

$$\begin{aligned} 1 &= 0 \\ 2 &= 1.1 \\ 3 &= 1.2 \\ 4 &= 1.3 + 2.2 \end{aligned}$$

The dilemma

continuously flexible $\overset{?}{\Longrightarrow}$ *3rd-order rigid*



The way out? *"Flexes with* $X_1 = \mathbf{0}$ *can be nontrivial."*

But then from any nontrivial first-order flex $X_0 + X_1\,t$ we obtain a nontrivial second-order flex $X_0 + X_1\,t^2$, i.e.

first-order flexible $\overset{?}{\Longrightarrow}$ *second-order flexible*

$1 = 0$
$2 = 1.1$
$3 = 1.2$
$4 = 1.3 + 2.2$
$5 = 1.4 + 2.3$

Which flex is trivial?

Definition:
The flex $X(t) := X_0 + X_1\, t + \ldots + X_n\, t^n$ of framework $\mathcal{F}$ is trivial if it originates from a motion of $\mathcal{F}$ as a rigid body.

1st order: At each instant of a rigid body motion in $\mathbb{E}^d$ there is a constant vector $\mathbf{c}_1 \in \mathbb{R}^d$ and a skew-symmetric matrix $C_1 \in \mathbb{R}^{d\times d}$ such that

$$\mathbf{x}_{i,1} = \mathbf{c}_1 + C_1\,\mathbf{x}_i \quad \text{for all } i \in I.$$

We say briefly: The component $X_1 = S_1 := (\mathbf{x}_{i,1}, \ldots, \mathbf{x}_{v,1})^T$ is of *S-type*.

These trivial solutions S_1 of $R_{\mathcal{F}} \cdot X_1 = 0$ constitute a subspace of dimension

$$d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$$

in the nullspace of $R_{\mathcal{F}}$.

Which flex is trivial?

Any other solution X_1 gives a nontrivial flex of $\mathcal{F}$, hence

$$\mathcal{F} \text{ is infinitesimally flexible of order 1} \iff \operatorname{rk}(R_{\mathcal{F}}) < vd - \frac{d(d+1)}{2}.$$

2nd order: $(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}_{i,2} - \mathbf{x}_{j,2}) = -\frac{1}{2}(\mathbf{x}_{i,1} - \mathbf{x}_{j,1})\cdot(\mathbf{x}_{i,1} - \mathbf{x}_{j,1}) \implies$
$\exists\ \mathbf{s}_2 \in \mathbb{R}^d$ and $C_2 \in \mathbb{R}^{d\times d}$ with $C_2^T = -C_2$ and

$$\mathbf{x}_{i,2} = \mathbf{c}_2 + \left(C_2 - \tfrac{1}{2}\,C_1^T C_1\right)\mathbf{x}_i \quad \text{for all } i \in I.$$

The higher derivatives for trivial flexes are as follows:

$$\begin{aligned}
\mathbf{x}_{i,3} &= \mathbf{c}_3 + \left(C_3 - C_1^T C_2\right)\mathbf{x}_i \\
\mathbf{x}_{i,4} &= \mathbf{c}_4 + \left(C_4 - C_1^T C_3 - \tfrac{1}{2}\,C_2^T C_2\right)\mathbf{x}_i \\
\mathbf{x}_{i,5} &= \mathbf{c}_5 + \left(C_5 - C_1^T C_4 - C_2^T C_3\right)\mathbf{x}_i \\
&\dots \quad \dots \quad \dots
\end{aligned}$$

$\mathbf{c}_j \in \mathbb{R}^d, \quad C_j \in \mathbb{R}^{d\times d}$
with $C_j^T = -C_j$.

Which flex is trivial?

Lemma:
Let $X(t) := X_0 + X_k\,t^k + \ldots + X_n\,t^n$ with $k \geq 1$ be an n-th order flex.

$$X_k \text{ is not of S-type} \implies X(t) \text{ is nontrivial.}$$

If $X(t) := X_0 + X_1\,t + \ldots + X_n\,t^n$ is a flex of $\mathcal{F}$, then also

$$X(t) := X_0 + (X_1 + S_1)\,t + (X_2 + S_2)\,t^2 + \ldots + (X_n + S_n)\,t^n$$

is a flex of $\mathcal{F}$, but also

$$X(t) := X_0 + (X_1 + S_1)\,t + (X_2 + S_2 + X_1')\,t^2 + \ldots + (X_n + S_n + X_1^{(n-1)})\,t^n$$

with $X_1, X_1', X_1'', \ldots, X_1^{(n-1)}$ not of S-type, but in the nullspace von $R_{\mathcal{F}}$.

But this is not the only possibility to modify flexes of $\mathcal{F}$.

Parameter substitutions of flexes

Let $X(t)$ be an n-th order flex of $\mathcal{F}$ and

$$t := a_1\bar{t} + a_2\bar{t}^2 + \ldots + a_n\bar{t}^n + \ldots \text{ with } a_1 \neq 0$$

be a regular polynomial parameter substitution. Then by replacing t in $X(t)$ we obtain the flex $\overline{X}(\bar{t})$ of $\mathcal{F}$.

Lemma:
If $X(t)$ keeps all edge lengths l_{ij} of $\mathcal{F}$ stationary of multiplicity $\geq n$ at $t = 0$, then the same holds for $\overline{X}(\bar{t})$ at $\bar{t} = 0$, and vice versa.

However, any substition of order $p > 1$, i.e.,

$$t = \bar{t}^p(a_0 + a_1\bar{t} + \ldots) \text{ with } a_0 \neq 0$$

will give a flex which keeps the lengths stationary of multiplicity $\geq pn$. Such flex will be called reducible.

Reducible and irreducible flexes

Definition:
Two flexes $X(t)$ and $\overline{X}(\bar{t})$ are called *equivalent* if they are of the same order and $\overline{X}(\bar{t})$ results from $X(t)$ by imposing trivial flexes and regular parameter substitutions.

E.g., $X(t) = X_0 + X_1\,t + X_3\,t^3$ and $\overline{X}(\bar{t}) = X_0 + X_1\,\bar{t} + (X_1 + S_2)\,\bar{t}^2 + X_3\,\bar{t}^3$ are equivalent (hint: $t := \bar{t} + \bar{t}^2$).

Definition:
An n-th order flex $X(t)$ is called *reducible* if there is an equivalent $\overline{X}(\bar{t})$ in which all exponents of $\bar{t}$ have a common divisor $p > 1$.

Flexes which are not reducible are called *irreducible*.

Modified definition of infinitesimal flexibility

Definition: *('modified')*
$\mathcal{F}$ is called *infinitesimally flexible of order n* if there is an *irreducible* flex

$$X(t) := X_0 + X_k t^k + \ldots + X_n t^n, \quad 1 \leq k \leq n,$$

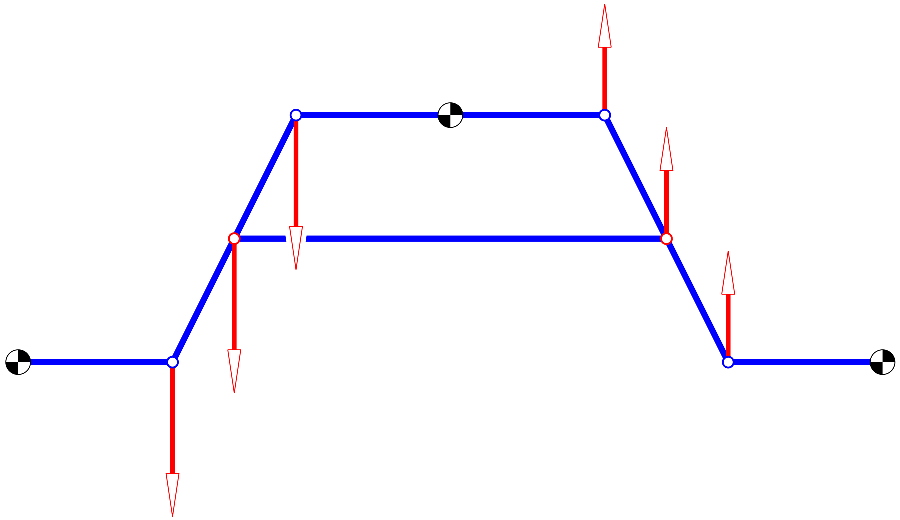
with $X_k \neq \mathbf{0}$ and not of S-type, which keeps the lengths of all edges stationary of multiplicity $\geq n$ at $t = 0$.

Revisiting the previous dilemma: The transition from $X_0 + X_1 t$ to $X_0 + X_1 t^2$ gives a reducible flex; so

$$\text{first-order flexible} \not\Rightarrow \text{second-order flexible.}$$

Revisiting the dilemma above

The flex $X_0 + X_2\,t^2 + X_3\,t^3$ of the CONNELLY-SERVATIUS framework is nontrivial as X_2 is not of S-type.



However, if X_3 is a multiple of X_2, then this 3rd-order flex is reducible, as

$$X_0 + X_2\,t^2 + aX_2\,t^3 = X_0 + X_2(t^2 + at^3) \text{ and } \bar{t} = t + \tfrac{a}{2}\,t^2 \implies$$

$$\bar{t}^2 = t^2 + at^3 \text{ and } X_0 + X_2\,\bar{t}^2 + 0\,\bar{t}^3.$$

Hence it is necessary that $\{X_2, X_3\}$ are linearly independent in the nullspace of $R_{\mathcal{F}}$.

Finally, a necessary condition

Theorem:
If $\mathcal{F}$ is n-th order flexible due to the flex $X_0 + X_k t^k + \ldots + X_n t^n$, $k > 1$, then $\mathrm{rk}(R_{\mathcal{F}}) \leq vd - \frac{d(d+1)}{2} - 2$.

Proof: Let $X_{i.k+j}\, t^{\,i.k+j}$ with $1 \leq j < k$ be the first term with an exponent which is not an integer multiple of k. Then $X_{i.k+j}$ is included in the nullspace of $R_{\mathcal{F}}$.

Suppose $\mathrm{rk}(R_{\mathcal{F}}) = vd - \frac{d(d+1)}{2} - 1$: Then (after imposing a suitable trivial flex $S\, t^{i.k+j}$) we obtain $X_{i.k+j} = a X_k$. We eliminate this term by substituting

$$\bar{t} = t + \frac{a}{k} t^{(i-1)k+1+j} \implies \bar{t}^k = t^k + a\, t^{\,i.k+j} + \ldots$$

Iteration leads to a reducible flex. $\square$

Literatur

- V. Alexandrov: *Sufficient Conditions for the Extendibility of an n-th Order Flex of Polyhedra*. Beitr. Algebra Geom. **39**, no. 2, 367–378 (1998).
- R. Connelly, H. Servatius: *Higher-order rigidity – What is the proper definition?* Discrete Comput. Geom. **11**, no. 2, 193–200 (1994).
- A.C. Dixon: *On certain deformable frameworks*. Mess. Math. **29**, 1–21 (1899/1900).
- I. Sabitov: *Local Theory of Bendings of Surfaces*. In Yu.D. Burago, V.A. Zalgaller (eds.): Geometry III, Theory of Surfaces. Encycl. of Math. Sciences, vol. 48, Springer-Verlag 1992, 179–250.

- I. Sabitov: *On the problem of invariance of the volume of a flexible polyhedron*. Russian Math. Surveys **50**, no. 2, 451–452 (1995).

- I. Sabitov: *The Volume as a Metric Invariant of Polyhedra*. Discrete Comput. Geom. **20**, 405–425 (1998).

- H. Stachel: *Infinitesimal Flexibility of Higher Order for a Planar Parallel Manipulator*. In G. Karáné, H. Sachs, F. Schipp (eds.): Topics in Algebra, Analysis and Geometry. BPR Kiadó 1999, 343–353.

- H. Stachel: *Higher-Order Flexibility for a Bipartite Planar Framework*. In A. Kecskeméthy, M. Schneider, C. Woernle (eds.): Advances in Multibody Systems and Mechatronics. Inst. f. Mechanik und Getriebelehre, TU Graz, Duisburg 1999, 345–357.

- H. Stachel: *Higher Order Flexibility of Octahedra*. Period. Math. Hung. **39**, 225–240 (1999).

- T. Tarnai: *Higher order infinitesimal mechanisms*. Acta Technica Acad. Sci. Hung. **102** (3–4), 363–378 (1989).

- R. Walker: *Algebraic Curves*. Springer Verlag 1978.