Black Hole Entropy, Finite Geometry and Mermin Squares

Péter Lévay

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The Black Hole Analogy

QUANTUM INFORMATION THEORY ↔ STRING THEORY

Multipartite Entanglement ↔ Black Hole solutions

The main correspondence is between certain multipartite entanglement measures and the black hole entropy.

The talk is based on a recent paper

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Plan of the talk

1. Black hole entropy in $D = 5$ and $D = 4$.


3. The finite geometry of the $D = 5$ black hole entropy.


5. The finite geometry of the $D - 5$ lift.

6. Finite subgroups of the $U$-duality group.

7. Conclusions.
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Black Hole Entropy in $D = 4$ and $D = 5$

The Bekenstein-Hawking entropy formula

$$S = k \frac{A}{4l_D^2}, \quad l_D^2 = \frac{\hbar G_D}{c^3}$$

for Reissner-Nordström type solutions arising from M-theory/String theory compactifications are described by **cubic** ($D = 5$) and **quartic** ($D = 4$) invariants as

$$S = \pi \sqrt{|I_3|}, \quad S = \pi \sqrt{|I_4|}.$$ 

Here

$$48I_3 = \text{Tr}(\Omega Z \Omega Z \Omega Z)$$

$$64I_4 = \text{Tr}(Z \bar{Z})^2 - \frac{1}{4}(\text{Tr} Z \bar{Z})^2 + 4(\text{Pf} Z + \text{Pf} \bar{Z}).$$

$$Z_{AB} = -(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}, \quad Z_{AB} = -Z_{BA}, \quad A, B, I, J = 1, \ldots 8.$$
In $D = 5$ we have 27 charges transforming as the $27$ of $E_{6(6)}$. The groups $E_{6(6)}$ and $E_{7(7)}$ are the symmetry groups of the corresponding classical supergravities. In the quantum theory the black hole/string charges become integer-valued the U-duality groups are in this case broken to $E_{6(6)}(\mathbb{Z})$ and $E_{7(7)}(\mathbb{Z})$ accordingly.
1 In $D = 5$ we have 27 charges transforming as the $27$ of $E_6(6)$.

2 In $D = 4$ we have 56 charges transforming as the $56$ of $E_7(7)$.

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Cubic Jordan algebras and entropy formulas in $D = 5$

The charge configurations describing electric black holes and magnetic black strings of the $N = 2$, $D = 5$ ($N = 8$, $D = 5$) magic supergravities are described by cubic Jordan algebras over a division algebra $\mathbf{A}$ (or its split cousin $\mathbf{A}_s$).

$$J_3(Q) = \begin{pmatrix} q_1 & Q^v & \overline{Q^s} \\ \frac{Q^v}{Q^s} & q_2 & Q^c \\ \frac{Q^s}{Q^c} & \frac{Q^c}{q_3} \end{pmatrix}, \quad q_i \in \mathbb{R}, \quad Q^{v,s,c} \in \mathbf{A}$$

The black hole entropy is given by the cubic invariant

$$I_3(Q) = q_1 q_2 q_3 - (q_1 Q^s \overline{Q^s} + q_2 Q^c \overline{Q^c} + q_3 Q^v \overline{Q^v}) + 2\Re(Q^c Q^s Q^v)$$

as

$$S = \pi \sqrt{|I_3(Q)|}.$$
The groups preserving $I_3$ are the ones $SL(3, \mathbb{R})$, $SL(3, \mathbb{C})$, $SU^*(6)$ and $E_6(-26)$.
For the split octonions we have

$$Q \overline{Q} = (Q_0)^2 + (Q_1)^2 + (Q_2)^2 + (Q_3)^2 - (Q_4)^2 - (Q_5)^2 - (Q_6)^2 - (Q_7)^2,$$

and the group preserving $I_3$ is $E_6(6)$.
The groups $E_6(-26)$ and $E_6(6)$ are the symmetry groups of the corresponding classical supergravities. In the quantum theory the black hole/string charges become integer-valued and the relevant $3 \times 3$ matrices are defined over the integral octonions and integral split octonions, respectively. Hence, the U-duality groups are in this case broken to $E_6(-26)(\mathbb{Z})$ and $E_6(6)(\mathbb{Z})$ accordingly.
Finite generalized quadrangles $GQ(s, t)$

A finite generalized quadrangle of order $(s, t)$, is an incidence structure $S = (P, B, I)$, where $P$ and $B$ are disjoint (non-empty) sets of objects, called respectively points and lines, and where $I$ is a symmetric point-line incidence relation satisfying the following axioms:

1. each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line

In what follows, we shall be uniquely concerned with generalized quadrangles having lines of size three, $GQ(2, t)$. From a theorem of Feit and Higman it follows that we have the unique possibilities $t = 1, 2, 4$. 
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2. each line is incident with \(1 + s\) points \((s \geq 1)\) and two distinct lines are incident with at most one point

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1. each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line
2. each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point
3. if $x$ is a point and $L$ is a line not incident with $x$, then there exists a unique pair $(y, M) \in P \times B$ for which $xI M I y I L$

In what follows, we shall be uniquely concerned with generalized quadrangles having lines of size three, $GQ(2, t)$. From a theorem of Feit and Higman it follows that we have the unique possibilities $t = 1, 2, 4$. 
The Doily, $GQ(2, 2)$
The Duad construction of $GQ(2, 4)$
Summary of patterns found for $D = 5$

**Generalized quadrangles**
- $GQ(2, 1)$ (grid) **9 points** and **6 lines**.

**Jordan algebras** (Charge configurations)
- $J_3(C)$ Number of real numbers: $3 + 3 \cdot 2 = 9$.

**Cubic invariants** (Black Hole entropy)
- $l_3(C)$ Number of terms: **6**. (Determinant)
Summary of patterns found for $D = 5$

**Generalized quadrangles**
1. $GQ(2, 1)$ (grid) **9 points** and **6 lines**.
2. $GQ(2, 2)$ (doily) **15 points** and **15 lines**.

**Jordan algebras** (Charge configurations)
1. $J_3(C)$ Number of real numbers: $3 + 3 \cdot 2 = 9$.
2. $J_3(H)$ Number of real numbers: $3 + 3 \cdot 4 = 15$.

**Cubic invariants** (Black Hole entropy)
1. $I_3(C)$ Number of terms: **6**. (Determinant)
2. $I_3(H)$ Number of terms: **15**. (Pfaffian)
Summary of patterns found for $D = 5$

**Generalized quadrangles**
1. $GQ(2, 1)$ (grid) **9 points** and **6 lines**.
2. $GQ(2, 2)$ (doily) **15 points** and **15 lines**.
3. $GQ(2, 4)$ **27 points** and **45 lines**.

**Jordan algebras** (Charge configurations)
1. $J_3(C)$ Number of real numbers: $3 + 3 \cdot 2 = 9$.
2. $J_3(H)$ Number of real numbers: $3 + 3 \cdot 4 = 15$.
3. $J_3(O)$ Number of real numbers: $3 + 3 \cdot 8 = 27$.

**Cubic invariants** (Black Hole entropy)
1. $I_3(C)$ Number of terms: **6**. (Determinant)
2. $I_3(H)$ Number of terms: **15**. (Pfaffian)
3. $I_3(O)$ Number of terms: **45**.
The cubic invariant and the duad construction

\[ E_6(6) \supset SL(2) \times SL(6) \]

under which

\[ 27 \rightarrow (2, 6') \oplus (1, 15). \]

This decomposition is displaying nicely its connection with the duad construction of GQ(2, 4). Under this decomposition \( I_3 \) factors as

\[ I_3 = \text{Pf}(A) + u^T Av, \]

where \( u \) and \( v \) are two six-component vectors and for the \( 6 \times 6 \) antisymmetric matrix \( A \) we have

\[ \text{Pf}(A) \equiv \frac{1}{3!2^3} \varepsilon_{ijklmn} A^{ij} A^{kl} A^{mn}. \]
We also have the decomposition

\[ E_{6(6)} \supset SL(3, \mathbb{R})_A \times SL(3, \mathbb{R})_B \times SL(3, \mathbb{R})_C \]

under which

\[ 27 \rightarrow (3', 3, 1) \otimes (1, 3', 3') \otimes (3, 1, 3). \]

The above-given decomposition is related to the "bipartite entanglement of three-qutrits" interpretation of the 27 of \( E_6(\mathbb{C}) \).


In this case we have

\[ l_3 = \text{Det}a + \text{Det}b + \text{Det}c - \text{Tr}(abc), \]

where \( a, b, c \) are \( 3 \times 3 \) matrices transforming accordingly.
The qutrit labelling of $GQ(2, 4)$
Truncations

1. Truncations to 36 possible doilies ("quaternionic magic" with 15 charges).

Perp-sets are obtained by selecting an arbitrary point and considering all the points collinear with it. A decomposition which corresponds to perp-sets is of the form

\[ E_{6(6)} \supset SO(5,5) \times SO(1,1) \]

under which

\[ 27 \rightarrow 16_1 \oplus 10_{-2} \oplus 1_4. \]

This is the usual decomposition of the \( U \)-duality group into \( T \) duality and \( S \) duality.
Truncations

1. Truncations to 36 possible doilies ("quaternionic magic" with 15 charges).
2. Truncations to 120 possible grids ("complex magic" with 9 charges).

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Truncations

1. Truncations to 36 possible **doilies** ("quaternionic magic" with 15 charges).
2. Truncations to 120 possible **grids** ("complex magic" with 9 charges).
3. Truncations to 27 possible **perp sets** (with 11 charges).

Perp-sets are obtained by selecting an arbitrary point and considering all the points collinear with it. A decomposition which corresponds to perp-sets is of the form

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This is the usual decomposition of the *U*-duality group into *T* duality and *S* duality.
What happened to the signs of the terms in the cubic invariant? Indeed, our labelling only produces the terms of the cubic invariant $l_3$ up to a sign. One could immediately suggest that we should also include a special distribution of signs to the points of GQ(2, 4). However, it is easy to see that no such distribution of signs exists. We have a triple of grids inside our quadrangle corresponding to the three different two-qutrit states. Truncation to any of such states yields the cubic invariant $l_3(a) = \text{Det}(a)$. The structure of this determinant is encapsulated in the structure of the corresponding grid. We can try to arrange the 9 amplitudes in a way that the 3 plus signs for the determinant should occur along the rows and the 3 minus signs along the columns. But this is impossible since multiplying all of the nine signs “row-wise” yields a plus sign, but “column-wise” yields a minus one. $\mapsto$ MERMIN SQUARES?!
The Pauli group

The real matrices of the Pauli group

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Three-qubit operations acting on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) e.g.

\[ ZYX \equiv Z \otimes Y \otimes X = \begin{pmatrix} Y \otimes X & 0 \\ 0 & -Y \otimes X \end{pmatrix} = \begin{pmatrix} 0 & X & 0 & 0 \\ -X & 0 & 0 & 0 \\ 0 & 0 & 0 & -X \\ 0 & 0 & X & 0 \end{pmatrix}. \]

Operators containing an even number of \( Y \)s are symmetric e.g. \( ZYY \).
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1. Operators containing an even number of \( Y \)s are symmetric e.g. \( ZYY \).
2. Operators containing an odd number of \( Y \)s are antisymmetric e.g. \( ZYX \).
A Mermin square for two qubits

\[
\begin{array}{ccc}
I \otimes Z & Z \otimes I & Z \otimes Z \\
X \otimes I & I \otimes X & X \otimes X \\
X \otimes Z & Z \otimes X & Y \otimes Y
\end{array}
\]
The Doily with the Mermin square inside

Péter Lévay  Black Hole Entropy, Finite Geometry and Mermin Squares
A labelling of $GQ(2, 4)$ with three qubit Pauli operators

Péter Lévay
Black Hole Entropy, Finite Geometry and Mermin Squares
The origin of the noncommutative labelling for $GQ(2, 4)$

Interestingly the labelling taking care of the 120 Mermin squares living inside $GQ(2, 4)$ and describing the structure of the 5D Black Hole Entropy can be understood by using results on the structure of the 4D Black Hole Entropy $S = \pi \sqrt{|I_4|}$ with

$$64|I_4| = \text{Tr} (\bar{Z} \bar{Z})^2 - \frac{1}{4} (\text{Tr} \bar{Z} \bar{Z})^2 + 4 (\text{Pf} Z + \text{Pf} \bar{Z}).$$

$$Z_{AB} = -(x^{IJ} + iy^{IJ})(\Gamma^{IJ})_{AB}, \quad Z_{AB} = -Z_{BA}, \quad A, B, I, J = 0, \ldots 7.$$ Here $\Gamma^{0k} = \Gamma_k$, and $\Gamma^{kl} = \frac{1}{2} [\Gamma_k, \Gamma_l]$ with

$$\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7\} = \{IIIY, ZYX, YIX, YZZ, XYX, IYZ, YXZ\}$$

$$\Gamma_j \Gamma_k + \Gamma_k \Gamma_j = -2 \delta_{jk} 1, \quad 1 \equiv III, \quad j, k = 1, 2, \ldots, 7.$$ These 7 $\oplus$ 21 antisymmetric three-qubit operators are living within the **Split Cayley Hexagon of order two**. See: P. Lévay et.al. Phys. Rev. D78, 124022 (2008).
The split Cayley hexagon of order two
A subgeometry of the Hexagon. The Coxeter graph
A presentation of this group of order 168 related to the automorphism group of the Coxeter graph and its complement is

$$PLS_2(7) \equiv \{\alpha, \beta, \gamma \mid \alpha^7 = \beta^3 = \gamma^2 = \alpha^{-2}\beta\alpha\beta^{-1} = (\gamma\beta)^2 = (\gamma\alpha)^3 = 1\}.$$ 

Let us define

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then we can define an $8 \times 8$ representation acting on the three-qubit Pauli group by conjugation as follows:
An 8 × 8 representation of Klein’s group

\[ D(\alpha) = (C_{12}C_{21})(C_{12}C_{31})C_{23}(C_{12}C_{31}) \equiv \begin{pmatrix} P & Q & 0 & 0 \\ 0 & 0 & Q & P \\ 0 & 0 & QX & PX \\ PX & QX & 0 & 0 \end{pmatrix} \]

\[ D(\beta) = C_{12}C_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]

\[ D(\gamma) = C_{21}(I \otimes I \otimes Z) = \begin{pmatrix} Z & 0 & 0 & 0 \\ 0 & 0 & 0 & Z \\ 0 & 0 & Z & 0 \\ 0 & Z & 0 & 0 \end{pmatrix} \]
The $N = 2, D = 4$ STU truncation

By virtue of the $PSL(2, 7)$ symmetry of the Coxeter graph we can identify seven subsectors with 8 charges each. These correspond to seven three-qubit states $a_\mu, b_\mu \ldots g_\mu, \quad \mu = 0, 1, \ldots 7$ with integer amplitudes. This gives rise to the tripartite entanglement of seven qubits interpretation of the 56 of $E_7$.


The correspondence is based on the rotation of the pattern:

$-a_7 - ia_0 \leftrightarrow IiY, \quad a_4 + ia_4 \leftrightarrow ZZY, \quad a_2 + ia_5 \leftrightarrow ZiY, \quad a_1 + ia_6 \leftrightarrow IzY$.

related to

$E_7 \supset SL(2)_a \times SL(2)_b \times \ldots SL(2)_g$

under which

$56 \rightarrow 2212111 \oplus 1221211 \oplus \cdots \oplus 2121112$. 
Cayley’s hyperdeterminant, and the three qubit state of one of the seven $N = 2$ truncations

$$|a\rangle = a_0|0\rangle + a_1|1\rangle + \ldots a_7|7\rangle$$
$$= a_{000}|000\rangle + a_{001}|001\rangle + \ldots a_{111}|111\rangle$$

$$|ijk\rangle \equiv |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$D(a) = (a_0a_7)^2 + (a_1a_6)^2 + (a_2a_5)^2 + (a_3a_4)^2$$
$$- 2(a_0a_7)[(a_1a_6) + (a_2a_5) + (a_3a_4)]$$
$$- 2[(a_1a_6)(a_2a_5) + (a_2a_5)(a_3a_4) + (a_3a_4)(a_1a_6)]$$
$$+ 4a_0a_3a_5a_6 + 4a_1a_2a_4a_7$$

$$S = \pi \sqrt{|D(a)|}.$$
A geometric hyperplane $H$ of a point-line incidence geometry $\Gamma(P, L)$ is a proper subset of $P$ such that each line of $\Gamma$ meets $H$ in one or all points.

The complement of the Coxeter graph is a geometric hyperplane of the hexagon with automorphism group $PSL(2, 7)$. Are there other interesting ones?

For an $8 \times 8$ matrix we define the Wootters spin-flip operation as

$$\tilde{M} \equiv -(Y \otimes Y \otimes Y)M^T(Y \otimes Y \otimes Y).$$

If $M \in P_3$ then we can consider from the 63 operators the Wootters self-dual ones for which $\tilde{M} = M$. It turns out that we have 27 self-dual ones consisting of 12 antisymmetric and 15 symmetric operators. One can then prove that these 27 operators form a geometric hyperplane of the hexagon. $YYY \mapsto I/Y$ gives another hyperplane e.t.c. altogether 28 ones!
The hyperplane of the Hexagon with 27 points
A $D = 4$ interpretation

Note that the decomposition

$$E_{7(7)} \supset E_{6(6)} \times SO(1, 1)$$  \hspace{1cm} (1)

under which

$$56 \rightarrow 1 \oplus 27 \oplus 27' \oplus 1'$$  \hspace{1cm} (2)

describes the relation between the $D = 4$ and $D = 5$ duality groups.

Notice that Wootters self-duality in the $N = 8$ language means that

$$\text{Tr}(\Omega \mathcal{Z}) = 0, \quad \mathcal{Z} = \Omega \mathcal{Z} \Omega^T \quad \Omega = Y Y Y.$$

The usual choice for $N = 8$ supergravity is $\Omega = I I Y = \Gamma_1$. With this choice one can prove that

$$\Omega \mathcal{Z} = \mathcal{S} + i\mathcal{A} \equiv \frac{1}{2} \epsilon^{j k} \Gamma_{1 j k} + i(y_{0 j} \Gamma_{1 j} - y_{1 j} \Gamma_j),$$  \hspace{1cm} (3)

(summation for $j, k = 2, 3, \ldots, 7$).
Connecting different forms of the cubic invariant.

Hence, with the notation

\[ A^{jk} \equiv x^{j+1k+1}, \quad u_j \equiv y_{0j+1}, \quad v_j \equiv y_{1j+1}, \quad j, k = 1, 2, \ldots, 6, \]

we get

\[ I_3 = \frac{1}{48} \text{Tr}(\Omega Z \Omega Z \Omega Z) = \text{Pf}(A) + u^T Av. \]

Notice that the operators

\[ \Gamma_j, \quad \Gamma_{1j}, \quad \Gamma_{1jk} \quad j, k = 2, 3 \ldots 7 \]

give rise to our noncommutative labelling, where

\[ \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7\} = \{IIY, ZYX, YIX, YZZ, XYX, IYZ, YXZ\}. \]

Hence the connection between the \( D = 4 \) and \( D = 5 \) is related to a one between the structures of GQ(2, 4) and one of the geometric hyperplanes of the hexagon.
The action of $W(E_6)$ of order 51840 on $GQ(2, 4)$

Let us consider the correspondence

\[
I \mapsto (00), \quad X \mapsto (01), \quad Y \mapsto (11), \quad Z \mapsto (10).
\]

For example, $XZI$ is taken to the 6-component vector $(011000)$. Knowing that $W(E_6) \cong O^-(6, 2)$,

\[
O^-(6, 2) = \langle c, d | c^2 = d^9 = (cd^2)^8 = [c, d^2]^2 = [c, d^3 cd^3] = 1 \rangle.
\]

For the action of $c$

\[
IXI \leftrightarrow XZI, \quad ZYX \leftrightarrow YIX, \quad IZI \leftrightarrow XXI
\]

\[
ZYZ \leftrightarrow YIZ, \quad ZII \leftrightarrow YYI, \quad ZYY \leftrightarrow YIY,
\]

the remaining 15 operators are left invariant. For the action of $d$ we get

\[
IXI \leftrightarrow YXZ \leftrightarrow YZX \leftrightarrow YIX \leftrightarrow XYZ \leftrightarrow IYZ \leftrightarrow YXX \leftrightarrow ZZI \leftrightarrow YXY \leftrightarrow
\]

\[
IZI \leftrightarrow ZYY \leftrightarrow XII \leftrightarrow YZY \leftrightarrow XYX \leftrightarrow XYY \leftrightarrow YIY \leftrightarrow YIZ \leftrightarrow IYY \leftrightarrow
\]

\[
IYX \leftrightarrow ZXI \leftrightarrow ZYZ \leftrightarrow ZYX \leftrightarrow YYI \leftrightarrow YZZ \leftrightarrow ZII \leftrightarrow XZI \leftrightarrow XXI \leftrightarrow
\]
It has been known for a long time that the maximal supergravity in $D$ dimensions obtained by Kaluza-Klein dimensional reduction from $D = 11$ has a $E_{n(n)}(\mathbb{R})$ symmetry where $n = 11 - D$. It is conjectured that the infinite discrete subgroup $E_{n(n)}(\mathbb{Z})$ is an exact symmetry of the corresponding string theory, known as $U$-duality group. It is useful to identify a finite subgroup of the $U$-duality group that maps the fundamental quantum states of string theory among themselves. (See e.g. H. Lü, C. N. Pope and K. S. Stelle: Nucl. Phys. B476,89 1996). This group is $W(E_{n(n)})$. Here motivated by some of the techniques of quantum information theory and finite geometry we have obtained an explicit realization of $W(E_6)$ acting on the charges ($U(1)$ gauge fields. (A similar construction holds also for $W(E_7)$.) Notice that

$$C_3' = \mathbb{Z}_2^6 \rtimes W'(E_7), \quad B_3' = \mathbb{Z}_2^6 \rtimes W'(E_6).$$

Where $C'$ and $B'$ are the central quotients of the three-qubit Clifford and Bell groups.
The finite geometric structure $GQ(2, 4)$ of the $D = 5$ black hole entropy is revealed.
Conclusions

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2. Truncations in string theory are related to restriction to restriction to hyperplanes.
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5. An interesting role of finite discrete subgroups ($W(E_n)$, $PSL(2, 7)$) of the $U$-duality group within the context of the Black Hole Analogy was established.
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2. Truncations in string theory are related to restriction to hyperplanes.

3. A noncommutative labelling based on three-qubit Pauli operators was given.

4. A connection to Mermin squares was established,

5. An interesting role of finite discrete subgroups ($W(E_n)$, $PSL(2, 7)$) of the $U$-duality group within the context of the Black Hole Analogy was established.

6. Partial success in identifying the underlying finite geometric structures of the $4D - 5D$ lift.