# Black Hole Entropy, Finite Geometry and Mermin Squares

Péter Lévay

May 21, 2009

**QUANTUM INFORMATION THEORY$\leftrightarrow$ STRING THEORY**

**Multipartite Entanglement$\leftrightarrow$Black Hole solutions**

The main correspondence is between certain multipartite entanglement measures and the black hole entropy.

M. J. Duff, Phys. Rev. D**76**, 025017 (2007)
R. Kallosh and A. Linde, Phys. Rev. D**73**, 104033 (2006)

# The talk is based on a recent paper

Péter Lévay[1], Metod Saniga[2], Péter Vrana[1] and Petr Pracna[3]
**Physical Review D79, 084036 (2009)**

[1]Department of Theoretical Physics, Institute of Physics, Budapest University of Technology, Budapest, Hungary
[2]Astronomical Institute, Slovak Academy of Sciences Tatranská Lomnica, Slovak Republic
[3]J. Heyrovský Institute of Physical Chemistry, v.v.i., Academy of Sciences of the Czech Republic

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6. Finite subgroups of the $U$-duality group.
7. Conclusions.

# Black Hole Entropy in $D = 4$ and $D = 5$

The Bekenstein-Hawking entropy formula

$$S = k\frac{A}{4l_D^2}, \quad l_D^2 = \frac{\hbar G_D}{c^3}$$

for Reissner-Nordström type solutions arising from M-theory/String theory compactifications are described by **cubic** ($D = 5$) and **quartic** ($D = 4$) invariants as

$$S = \pi\sqrt{|I_3|}, \qquad S = \pi\sqrt{|I_4|}.$$

Here

$$48 I_3 = \mathrm{Tr}(\Omega\mathcal{Z}\Omega\mathcal{Z}\Omega\mathcal{Z})$$

$$64 I_4 = \mathrm{Tr}(\mathcal{Z}\overline{\mathcal{Z}})^2 - \frac{1}{4}(\mathrm{Tr}\mathcal{Z}\overline{\mathcal{Z}})^2 + 4(\mathrm{Pf}\mathcal{Z} + \mathrm{Pf}\overline{\mathcal{Z}}).$$

$$\mathcal{Z}_{AB} = -(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}, \quad \mathcal{Z}_{AB} = -\mathcal{Z}_{BA}, \quad A, B, I, J = 1, \ldots 8.$$

# Charges and U-duality groups

1. In $D = 5$ we have 27 charges transforming as the **27** of $E_{6(6)}$.

The groups $E_{6(6)}$ and $E_{7(7)}$ are the symmetry groups of the corresponding classical supergravities. In the quantum theory the black hole/string charges become integer-valued the U-duality groups are in this case broken to $E_{6(6)}(\mathbf{Z})$ and $E_{7(7)}(\mathbf{Z})$ accordingly.

1. In $D = 5$ we have 27 charges transforming as the **27** of $E_{6(6)}$.
2. In $D = 4$ we have 56 charges transforming as the **56** of $E_{7(7)}$.

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# Cubic Jordan algebras and entropy formulas in $D = 5$

The charge configurations describing electric black holes and magnetic black strings of the $N = 2$, $D = 5$ ($N = 8$, $D = 5$) magic supergravities are described by cubic Jordan algebras over a division algebra $\mathbf{A}$ (or its split cousin $\mathbf{A}_s$).

$$J_3(Q) = \begin{pmatrix} q_1 & Q^v & \overline{Q^s} \\ \overline{Q^v} & q_2 & Q^c \\ Q^s & \overline{Q^c} & q_3 \end{pmatrix} \qquad q_i \in \mathbf{R}, \qquad Q^{v,s,c} \in \mathbf{A}$$

The black hole entropy is given by the cubic invariant

$$I_3(Q) = q_1 q_2 q_3 - (q_1 Q^s \overline{Q^s} + q_2 Q^c \overline{Q^c} + q_3 Q^v \overline{Q^v}) + 2\mathrm{Re}(Q^c Q^s Q^v)$$

as

$$S = \pi \sqrt{|I_3(Q)|}.$$

# U-duality groups

The groups preserving $I_3$ are the ones $SL(3,\mathbf{R})$, $SL(3,\mathbf{C})$, $SU^*(6)$ and $E_{6(-26)}$.
For the split octonions we have

$$Q\overline{Q} = (Q_0)^2 + (Q_1)^2 + (Q_2)^2 + (Q_3)^2 - (Q_4)^2 - (Q_5)^2 - (Q_6)^2 - (Q_7)^2,$$

and the group preserving $I_3$ is $E_{6(6)}$.
The groups $E_{6(-26)}$ and $E_{6(6)}$ are the symmetry groups of the corresponding classical supergravities. In the quantum theory the black hole/string charges become integer-valued and the relevant $3 \times 3$ matrices are defined over the *integral* octonions and *integral* split octonions, respectively. Hence, the U-duality groups are in this case broken to $E_{6(-26)}(\mathbf{Z})$ and $E_{6(6)}(\mathbf{Z})$ accordingly.

# Finite generalized quadrangles $GQ(s, t)$

A *finite generalized quadrangle* of order $(s, t)$, is an incidence structure $S = (P, B, \mathrm{I})$, where $P$ and $B$ are disjoint (non-empty) sets of objects, called respectively points and lines, and where I is a symmetric point-line incidence relation satisfying the following axioms:

1. each point is incident with $1 + t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line

In what follows, we shall be uniquely concerned with generalized quadrangles having lines of size *three*, $\mathrm{GQ}(2, t)$. From a theorem of Feit and Higman it follows that we have the unique possibilities $t = 1, 2, 4$.

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1. each point is incident with $1 + t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line
2. each line is incident with $1 + s$ points $(s \geq 1)$ and two distinct lines are incident with at most one point
3. if $x$ is a point and $L$ is a line not incident with $x$, then there exists a unique pair $(y, M) \in P \times B$ for which $x\mathrm{I}M\mathrm{I}y\mathrm{I}L$

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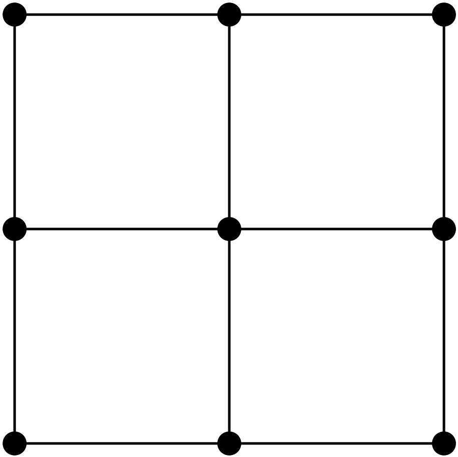
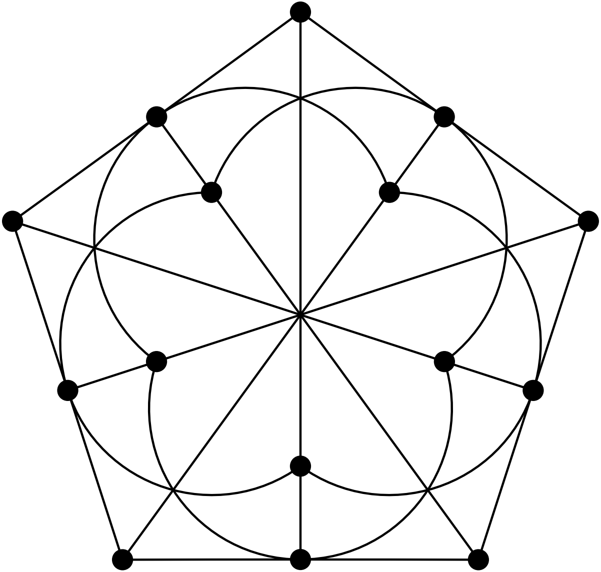
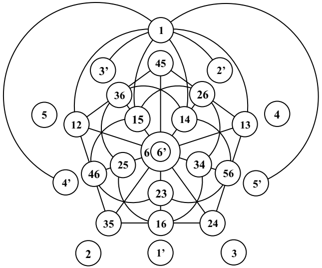
# A Grid, $GQ(2,1)$

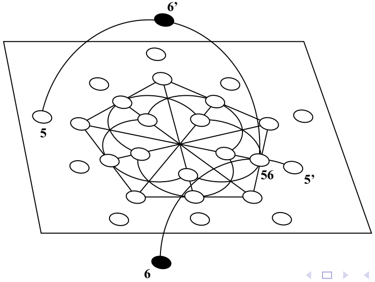


# The Doily, $GQ(2,2)$



# The Duad construction of $GQ(2,4)$



# Summary of patterns found for $D = 5$

**Generalized quadrangles**

1. $GQ(2,1)$ (grid) **9 points** and **6 lines**.

**Jordan algebras** (Charge configurations)

1. $J_3(\mathbf{C})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{2} = \mathbf{9}$.

**Cubic invariants** (Black Hole entropy)

1. $I_3(\mathbf{C})$ Number of terms: **6**. (Determinant)

# Summary of patterns found for $D = 5$

**Generalized quadrangles**

1. $GQ(2,1)$ (grid) **9 points** and **6 lines**.
2. $GQ(2,2)$ (doily) **15 points** and **15 lines**.

**Jordan algebras** (Charge configurations)

1. $J_3(\mathbf{C})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{2} = \mathbf{9}$.
2. $J_3(\mathbf{H})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{4} = \mathbf{15}$.

**Cubic invariants** (Black Hole entropy)

1. $I_3(\mathbf{C})$ Number of terms: **6**. (Determinant)
2. $I_3(\mathbf{H})$ Number of terms: **15**. (Pfaffian)

# Summary of patterns found for $D = 5$

**Generalized quadrangles**

1. $GQ(2,1)$ (grid) **9 points** and **6 lines**.
2. $GQ(2,2)$ (doily) **15 points** and **15 lines**.
3. $GQ(2,4)$ **27 points** and **45 lines**.

**Jordan algebras** (Charge configurations)

1. $J_3(\mathbf{C})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{2} = \mathbf{9}$.
2. $J_3(\mathbf{H})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{4} = \mathbf{15}$.
3. $J_3(\mathbf{O})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{8} = \mathbf{27}$.

**Cubic invariants** (Black Hole entropy)

1. $I_3(\mathbf{C})$ Number of terms: **6**. (Determinant)
2. $I_3(\mathbf{H})$ Number of terms: **15**. (Pfaffian)
3. $I_3(\mathbf{O})$ Number of terms: **45**.

$$E_{6(6)} \supset SL(2) \times SL(6)$$

under which

$$\mathbf{27} \to (\mathbf{2}, \mathbf{6}') \oplus (\mathbf{1}, \mathbf{15}).$$

This decomposition is displaying nicely its connection with the duad construction of GQ(2, 4). Under this decomposition $I_3$ factors as

$$I_3 = \mathrm{Pf}(A) + u^T A v,$$

where $u$ and $v$ are two six-component vectors and for the $6 \times 6$ antisymmetric matrix $A$ we have

$$\mathrm{Pf}(A) \equiv \frac{1}{3!2^3} \varepsilon_{ijklmn} A^{ij} A^{kl} A^{mn}.$$

We also have the decomposition

$$E_{6(6)} \supset SL(3,\mathbf{R})_A \times SL(3,\mathbf{R})_B \times SL(3,\mathbf{R})_C$$

under which

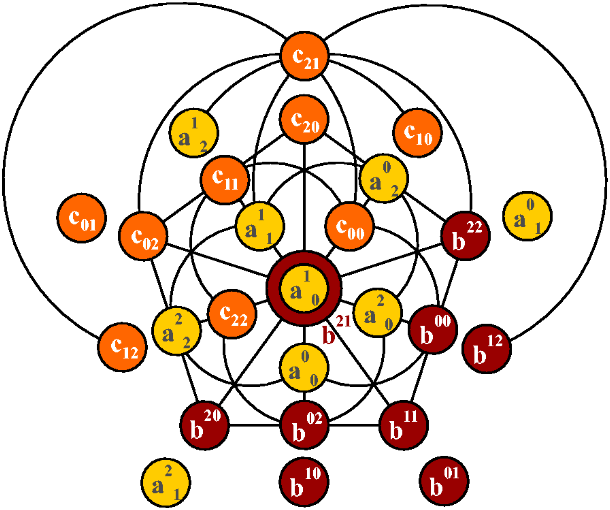
$$\mathbf{27} \rightarrow (\mathbf{3}',\mathbf{3},\mathbf{1}) \otimes (\mathbf{1},\mathbf{3}',\mathbf{3}') \otimes (\mathbf{3},\mathbf{1},\mathbf{3}).$$

The above-given decomposition is related to the "bipartite entanglement of three-qutrits" interpretation of the **27** of $E_6(\mathbf{C})$. (S. Ferrara and M. J. Duff, Phys. Rev. D**76**, 124023 (2007))
In this case we have

$$I_3 = \mathrm{Det} a + \mathrm{Det} b + \mathrm{Det} c - \mathrm{Tr}(abc),$$

where $a, b, c$ are $3 \times 3$ matrices transforming accordingly.

# The qutrit labelling of $GQ(2,4)$



# Truncations

1. Truncations to 36 possible **doilies** ("quaternionic magic" with 15 charges).

Perp-sets are obtained by selecting an arbitrary point and considering all the points collinear with it. A decomposition which corresponds to perp-sets is of the form

$$E_{6(6)} \supset SO(5,5) \times SO(1,1)$$

under which

$$\mathbf{27} \to \mathbf{16_1} \oplus \mathbf{10_{-2}} \oplus \mathbf{1_4}.$$

This is the usual decomposition of the $U$-duality group into $T$ duality and $S$ duality.

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2. Truncations to 120 possible **grids** ("complex magic" with 9 charges).
3. Truncations to 27 possible **perp sets** (with 11 charges).

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# Sign problems

What happened to the *signs* of the terms in the cubic invariant? Indeed, our labelling only produces the terms of the cubic invariant $I_3$ *up to a sign*. One could immediately suggest that we should also include a special distribution of signs to the points of $\mathrm{GQ}(2,4)$. However, it is easy to see that no such distribution of signs exists. We have a triple of grids inside our quadrangle corresponding to the three different two-qutrit states. Truncation to any of such states yields the cubic invariant $I_3(a) = \mathrm{Det}(a)$. The structure of this determinant is encapsulated in the structure of the corresponding grid. We can try to arrange the 9 amplitudes in a way that the 3 plus signs for the determinant should occur along the rows and the 3 minus signs along the columns. But this is impossible since multiplying all of the nine signs "row-wise" yields a plus sign, but "column-wise" yields a minus one. $\mapsto$**MERMIN SQUARES?!**

The real matrices of the Pauli group

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Three-qubit operations acting on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ e .g.

$$ZYX \equiv Z \otimes Y \otimes X = \begin{pmatrix} Y \otimes X & 0 \\ 0 & -Y \otimes X \end{pmatrix} = \begin{pmatrix} 0 & X & 0 & 0 \\ -X & 0 & 0 & 0 \\ 0 & 0 & 0 & -X \\ 0 & 0 & X & 0 \end{pmatrix}.$$

1. Operators containing an **even** number of $Y$s are **symmetric** e.g. $ZYY$.

# The Pauli group

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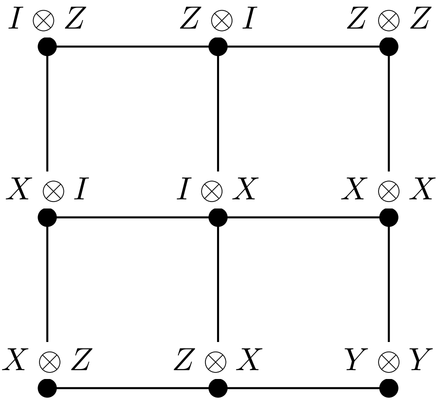
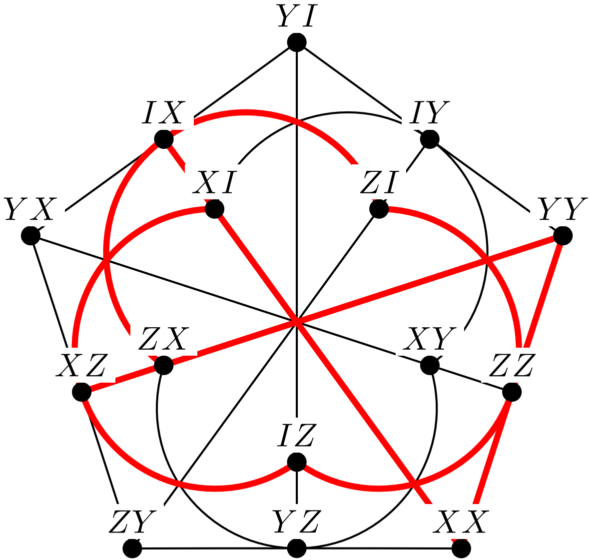
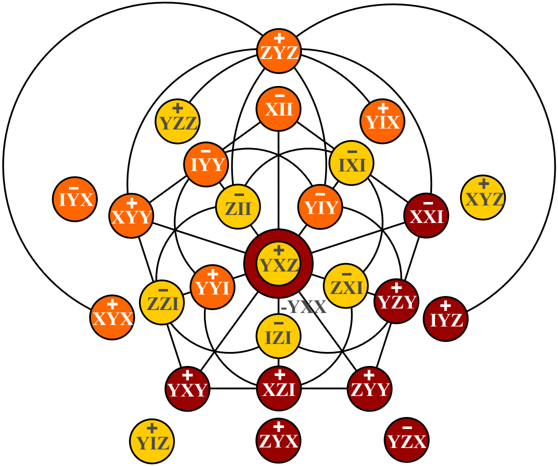
1. Operators containing an **even** number of $Y$s are **symmetric** e.g. $ZYY$.
2. Operators containing an **odd** number of $Y$s are **antisymmetric** e.g. $ZYX$.


$I \otimes Z$
$Z \otimes I$
$Z \otimes Z$
$X \otimes I$
$I \otimes X$
$X \otimes X$
$X \otimes Z$
$Z \otimes X$
$Y \otimes Y$

# The Doily with the Mermin square inside



# A labelling of $GQ(2,4)$ with three qubit Pauli operators



# The origin of the noncommutative labelling for $GQ(2,4)$

Interestingly the labelling taking care of the 120 Mermin squares living inside $GQ(2,4)$ and describing the structure of the **5D** Black Hole Entropy can be understood by using results on the structure of the **4D** Black Hole Entropy $S = \pi\sqrt{|I_4|}$ with

$$64 I_4 = \mathrm{Tr}(\mathcal{Z}\overline{\mathcal{Z}})^2 - \frac{1}{4}(\mathrm{Tr}\mathcal{Z}\overline{\mathcal{Z}})^2 + 4(\mathrm{Pf}\mathcal{Z} + \mathrm{Pf}\overline{\mathcal{Z}}).$$

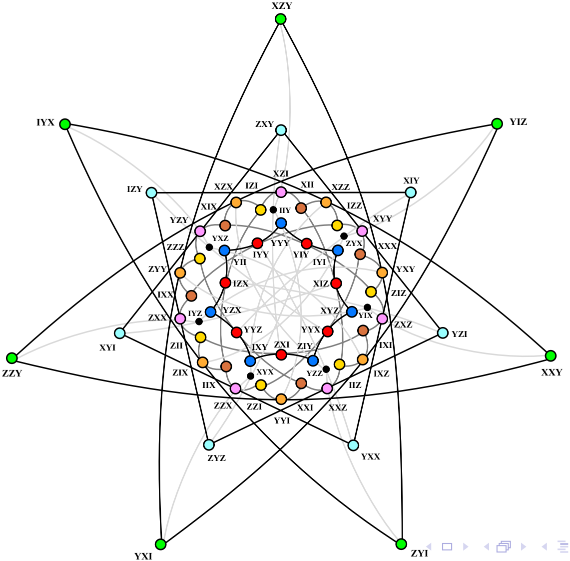
$$\mathcal{Z}_{AB} = -(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}, \quad \mathcal{Z}_{AB} = -\mathcal{Z}_{BA}, \quad A, B, I, J = 0, \dots 7.$$

Here $\Gamma^{0k} = \Gamma_k$, and $\Gamma^{kl} = \frac{1}{2}[\Gamma_k, \Gamma_l]$ with

$$\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7\} = \{IIY, ZYX, YIX, YZZ, XYX, IYZ, YXZ\}$$

$$\Gamma_j\Gamma_k + \Gamma_k\Gamma_j = -2\delta_{jk}\mathbf{1}, \qquad \mathbf{1} \equiv III, \qquad j, k = 1, 2, \dots, 7.$$

These $7 \oplus 21$ antisymmetric three-qubit operators are living within the **Split Cayley Hexagon of order two**. See: P. Lévay et.al. Phys. Rev. D**78**, 124022 (2008).


XZY
IYX
YIZ
ZXY
IZY
XIY
XZX
IZI
XZI
XII
XZZ
XIX
IIY
IZZ
YZY
XYY
YXZ
YYY
ZYX
ZZZ
IYY
YIY
XXX
YII
IYI
ZYY
YXY
IZX
XIZ
IXX
ZIZ
IYZ
YZX
XYZ
YIX
ZXX
ZXZ
YYZ
YYX
XYI
YZI
ZII
IXY
ZXI
ZIY
IXI
ZZY
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ZIX
XYX
YZZ
IXZ
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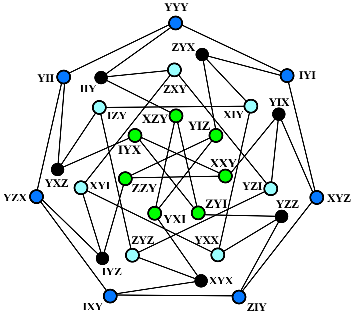
ZYI

# A subgeometry of the Hexagon. The Coxeter graph



# Klein's group $PSL_2(7)$

A presentation of this group of order 168 related to the automorphism group of the Coxeter graph and its complement is

$$PSL_2(7) \equiv \{\alpha, \beta, \gamma \mid \alpha^7 = \beta^3 = \gamma^2 = \alpha^{-2}\beta\alpha\beta^{-1} = (\gamma\beta)^2 = (\gamma\alpha)^3 = 1\}.$$

Let us define

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we can define an $8 \times 8$ representation acting on the three-qubit Pauli group by conjugation as follows:

# An $8 \times 8$ representation of Klein's group

$$\mathcal{D}(\alpha) = (C_{12}C_{21})(C_{12}C_{31})C_{23}(C_{12}C_{31}) \equiv \begin{pmatrix} P & Q & 0 & 0 \\ 0 & 0 & Q & P \\ 0 & 0 & QX & PX \\ PX & QX & 0 & 0 \end{pmatrix}$$

$$\mathcal{D}(\beta) = C_{12}C_{21} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}$$

$$\mathcal{D}(\gamma) = C_{21}(I \otimes I \otimes Z) = \begin{pmatrix} Z & 0 & 0 & 0 \\ 0 & 0 & 0 & Z \\ 0 & 0 & Z & 0 \\ 0 & Z & 0 & 0 \end{pmatrix}$$

# The $N = 2$, $D = 4$ STU truncation

By virtue of the $PSL(2,7)$ symmetry of the Coxeter graph we can identify **seven** subsectors with 8 charges each. These correspond to seven **three-qubit states** $a_\mu, b_\mu \dots g_\mu, \quad \mu = 0, 1, \dots 7$ with integer amplitudes. This gives rise to the **tripartite entanglement of seven qubits** interpretation of the 56 of $E_7$.

S. Ferrara and M. J. Duff, Phys. Rev. D**76**, 025018 (2007)
P. Lévay, Phys. Rev. D**75**, 024024 (2007)

The correspondence is based on the rotation of the pattern:

$$-a_7 - ia_0 \leftrightarrow IIY, \quad a_4 + ia_4 \leftrightarrow ZZY, \quad a_2 + ia_5 \leftrightarrow ZIY, \quad a_1 + ia_6 \leftrightarrow IZY.$$

related to

$$E_7 \supset SL(2)_a \times SL(2)_b \times \dots SL(2)_g$$

under which

$$\mathbf{56} \to \mathbf{2212111} \oplus \mathbf{1221211} \oplus \cdots \oplus \mathbf{2121112}.$$

# Cayley's hyperdeterminant, and the three qubit state of one of the **seven** $N = 2$ truncations

$$
\begin{aligned}
|\mathbf{a}\rangle &= a_0|0\rangle + a_1|1\rangle + \ldots a_7|7\rangle \\
&= a_{000}|000\rangle + a_{001}|001\rangle + \ldots a_{111}|111\rangle
\end{aligned}
$$

$$|ijk\rangle \equiv |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \in \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$$

$$
\begin{aligned}
D(\mathbf{a}) &= (a_0a_7)^2 + (a_1a_6)^2 + (a_2a_5)^2 + (a_3a_4)^2 \\
&- 2(a_0a_7)[(a_1a_6) + (a_2a_5) + (a_3a_4)] \\
&- 2[(a_1a_6)(a_2a_5) + (a_2a_5)(a_3a_4) + (a_3a_4)(a_1a_6)] \\
&+ 4a_0a_3a_5a_6 + 4a_1a_2a_4a_7
\end{aligned}
$$

$$S = \pi\sqrt{|D(\mathbf{a})|}.$$

# Geometric hyperplanes and the Wootters spin flip operation

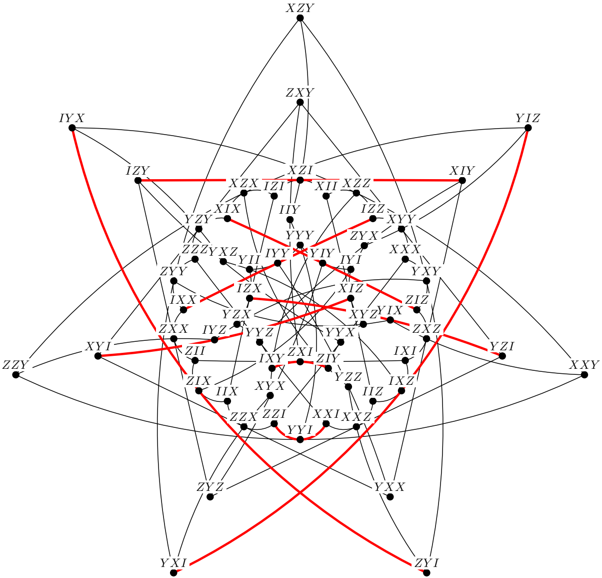
A **geometric hyperplane** $H$ of a point-line incidence geometry $\Gamma(P, L)$ is a proper subset of $P$ such that each line of $\Gamma$ meets $H$ in one or all points.

The complement of the Coxeter graph is a geometric hyperplane of the hexagon with automorphism group $PSL(2, 7)$. Are there other interesting ones?
For an $8 \times 8$ matrix we define the Wootters spin-flip operation as

$$\tilde{M} \equiv -(Y \otimes Y \otimes Y)M^T(Y \otimes Y \otimes Y).$$

If $M \in \mathcal{P}_3$ then we can consider from the 63 operators the Wootters **self-dual** ones for which $\tilde{M} = M$. It turns out that we have **27** self dual ones consisting of **12** antisymmetric and **15** symmetric operators. One can then prove that these **27** operators form a geometric hyperplane of the hexagon. $YYY \mapsto IIY$ gives another hyperplane e.t.c. altogether 28 ones!

# The hyperplane of the Hexagon with 27 points



# A $D = 4$ interpretation

Note that the decomposition

$$E_{7(7)} \supset E_{6(6)} \times SO(1,1) \quad (1)$$

under which

$$\mathbf{56} \to \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{27}' \oplus \mathbf{1}' \quad (2)$$

describes the relation between the $D = 4$ and $D = 5$ duality groups.
Notice that Wootters self-duality in the $N = 8$ language means that

$$\mathrm{Tr}(\Omega\mathcal{Z}) = 0, \qquad \overline{\mathcal{Z}} = \Omega\mathcal{Z}\Omega^T \quad \Omega = YYY.$$

The usual choice for $N = 8$ supergravity is $\Omega = IIY = \Gamma_1$ . With this choice one can prove that

$$\Omega\mathcal{Z} = \mathcal{S} + i\mathcal{A} \equiv \frac{1}{2}x^{jk}\Gamma_{1jk} + i(y_{0j}\Gamma_{1j} - y_{1j}\Gamma_j), \quad (3)$$

(summation for $j, k = 2, 3, \ldots, 7$).

# Connecting different forms of the cubic invariant.

Hence, with the notation

$$A^{jk} \equiv x^{j+1k+1}, \qquad u_j \equiv y_{0j+1}, \qquad v_j \equiv y_{1j+1}, \qquad j,k = 1,2,\dots,6,$$

we get

$$I_3 = \frac{1}{48}\mathrm{Tr}(\Omega\mathcal{Z}\Omega\mathcal{Z}\Omega\mathcal{Z}) = \mathrm{Pf}(A) + u^T A v.$$

Notice that the operators

$$\Gamma_j, \quad \Gamma_{1j}, \quad \Gamma_{1jk} \quad j,k = 2,3\dots 7$$

give rise to our noncommutative labelling, where

$$\{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4,\Gamma_5,\Gamma_6,\Gamma_7\} = \{IIY, ZYX, YIX, YZZ, XYX, IYZ, YXZ\}.$$

Hence the connection between the $D = 4$ and $D = 5$ is related to a one between the structures of GQ(2,4) and one of the geometric hyperplanes of the hexagon.

# The action of $W(E_6)$ of order 51840 on $GQ(2,4)$

Let us consider the correspondence

$$I \mapsto (00), \qquad X \mapsto (01), \qquad Y \mapsto (11), \qquad Z \mapsto (10).$$

For example, $XZI$ is taken to the 6-component vector (011000). Knowing that $W(E_6) \cong O^-(6,2)$,

$$O^-(6,2) = \langle c, d | c^2 = d^9 = (cd^2)^8 = [c, d^2]^2 = [c, d^3cd^3] = 1\rangle.$$

For the action of $c$

$$IXI \leftrightarrow XZI, \qquad ZYX \leftrightarrow YIX, \qquad IZI \leftrightarrow XXI$$

$$ZYZ \leftrightarrow YIZ, \qquad ZII \leftrightarrow YYI, \qquad ZYY \leftrightarrow YIY,$$

the remaining 15 operators are left invariant. For the action of $d$ we get

$$IXI \mapsto YXZ \mapsto YZX \mapsto YIX \mapsto XYZ \mapsto IYZ \mapsto YXX \mapsto ZZI \mapsto YXY \mapsto$$

$$IZI \mapsto ZYY \mapsto XII \mapsto YZY \mapsto XYX \mapsto XYY \mapsto YIY \mapsto YIZ \mapsto IYY \mapsto$$

$$IYX \mapsto ZXI \mapsto ZYZ \mapsto ZYX \mapsto YYI \mapsto YZZ \mapsto ZII \mapsto XZI \mapsto XXI \mapsto$$

It has been known for a long time that the maximal supergravity in $D$ dimensions obtained by Kaluza-Klein dimensional reduction from $D = 11$ has a $E_{n(n)}(\mathbf{R})$ symmetry where $n = 11 - D$. It is conjectured that the **infinite** discrete subgroup $E_{n(n)}(\mathbf{Z})$ is an exact symmetry of the corresponding string theory, known as $U$-duality group. It is useful to identify a **finite** subgroup of the $U$-duality group that maps the fundamental quantum states of string theory among themselves. (See e.g. H. Lü, C. N. Pope and K. S. Stelle: Nucl. Phys. B476,89 1996). This group is $W(E_{n(n)})$. Here motivated by some of the techniques of quantum information theory and finite geometry we have obtained an explicit realization of $W(E_6)$ acting on the charges ($U(1)$ gauge fields. (A similar construction holds also for $W(E_7)$.) Notice that

$$\mathcal{C}_3{}' = \mathbf{Z}_2^6 \rtimes W'(E_7), \qquad \mathcal{B}_3{}' = \mathbf{Z}_2^6 \rtimes W'(E_6).$$

Where $\mathcal{C}'$ and $\mathcal{B}'$ are the central quotients of the three-qubit Clifford and Bell groups.

# Conclusions

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6. Partial success in identifying the underlying finite geometric structures of the $4D-5D$ lift.