On group theory, quantum gates and quantum coherence

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- **1. Geometry of commutation/anti-commutation** relations of (generalized) Pauli operators.
- **2. Finite group extensions**: a natural language for quantum computing: **error gates** from the Pauli group $\mathcal{P}$, and **stabilizing gates** within an extension group $\mathcal{C}$.
- Single qubit $\mathcal{C}_1$ and ... magic states.
- Two-qubit $\mathcal{C}_2$ and ... alt. group A_6, the *non-coherent* group U_6 (order 5760), Mathieu group M_{22}, alt. group A_5, the *coherent* group M_{20} (order 960)...
- Three-qubit coherence, A_5 and ...$SU(4,2)$.

On the Pauli graphs on N-qudits[1]

[1] M. Planat and M. Saniga, Quant Inf Comp 8, 127-146 (2008)

Glossary on finite geometries: 1

- FINITE GEOMETRY: a space $\mathcal{S} = \{P, L\}$ of points P and lines L such that certain conditions, or axioms, are satisfied.
- **A near linear space**/linear space: a space such that any line has at least two points and two points are **at most**/*exactly* on one line.
- **A projective plane**: a linear space in which any two lines meet and there exists a set of four points no three of which lie on a line. The projective plane axioms are **dual**. The smallest one is $PG(2,2)$: the Fano plane with 7 points and 7 lines.
- **A projective space**: a linear space such that any two-dimensional subspace of it is projective plane. The smallest one is three dimensional and binary: $PG(3,2)$.

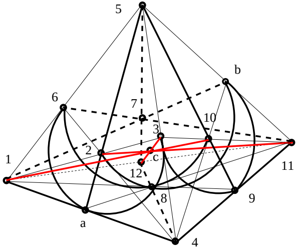
Glossary on finite geometries: 2

- **A generalized quadrangle**: a near linear space such that given a line L and a point P not on the line, there is exactly one line K through P that intersects L (in some point Q). A **finite** generalized quadrangle GQ is said to be of **order** (s, t) if every line contains $s + 1$ points and every point is in exactly $t + 1$ lines[2].
- **A geometric hyperplane** H: a set of points such that every line of the geometry either contains exactly one point of H, or is completely contained in H.
- **A polar space** $S = \{P, L\}$: a near-linear space such that for every point P not on a line L, the number of points of L joined to P by a line equals either one (as for a generalized quadrangle) or to the total number of points of the line.

[2] Properties: $\#P = (s + 1)(st + 1)$, $\#L = (t + 1)(st + 1)$, the incidence graph is strongly regular and the eigenvalues of the adjacency matrix are $k = s(t + 1)$, $r = s - 1$, $l = t - 1$; moreover r has multiplicity $f = st(s + 1)(t + 1)/(s + t)$.

Geometry of commuting/anti-commuting relations of the two-qubit system

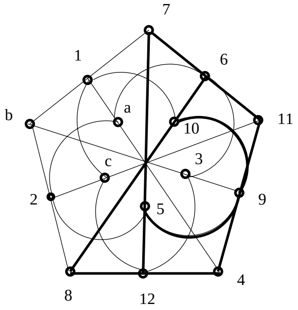
- Fifteen tensor products $\sigma_i \otimes \sigma_j$ of Pauli matrices $\sigma_i = (I_2, \sigma_x, \sigma_y, \sigma_z)$, where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_y = i\sigma_x\sigma_z$.
- Labels: $1 = I_2 \otimes \sigma_x$, $2 = I_2 \otimes \sigma_y$, $3 = I_2 \otimes \sigma_z$, $a = \sigma_x \otimes I_2$, $4 = \sigma_x \otimes \sigma_x \ldots$, $b = \sigma_y \otimes I_2, \ldots$, $c = \sigma_z \otimes I_2, \ldots$



Embedding of the generalized quadrangle $GQ(2)$ (and thus of the Pauli graph $\mathcal{G}_2$ into the projective space $PG(3,2)$).

Maximal commuting sets

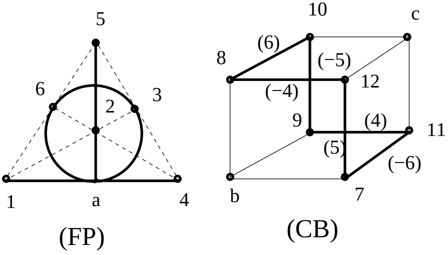
$$\{1, a, 4\}, \{2, a, 5\}, \{3, a, 6\}, \quad \{1, b, 7\}, \{2, b, 8\}, \{3, b, 9\}, \quad \{1, c, 10\}, \{2, c, 11\}, \{3, c, 12\},$$
$$\{4, 8, 12\}, \{5, 7, 12\}, \{6, 7, 11\}, \quad \{4, 9, 11\}, \{5, 9, 10\}, \{6, 8, 10\}$$



- GQ(2) as the *unique* underlying geometry of the two-qubit system. The Pauli operators correspond to the points and the base/maximally commuting subsets of them to the lines of the quadrangle. .

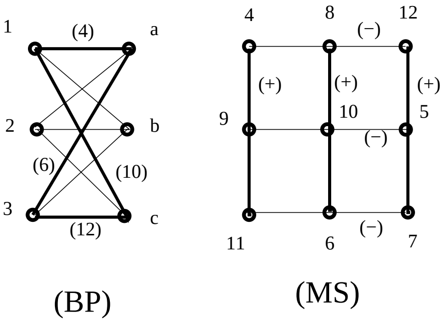
- two-qubit *geometry* GQ(2), *-graph* $\mathcal{G}_2$, *-group* $\mathcal{P}_2$
- GQ(2) as the two-qubit Pauli graph $\mathcal{G}_2$
- $\mathrm{Aut}(\mathrm{GQ}(2)) = S_6$
- $\mathcal{G}_2 = \hat{L}(K_6)$ generalizes Petersen graph $PG = \hat{L}(K_5)$
- There exists 6 maximal sets of 5 disjoint lines (MUBs)
- $\mathrm{Out}(S_6) = \mathcal{Z}_2 \times \mathcal{Z}_2$
- Later, I define $\mathcal{Z}_2 \wr A_6$ as $\mathrm{Aut}(\mathcal{P}_2)$

Basic partitionings: FP+CB



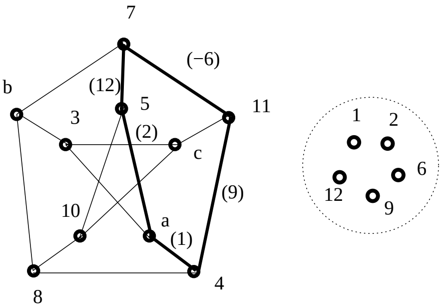
- Partitioning of $\mathcal{G}_2$ into a pencil of lines in the Fano plane (FP) and a cube (CB).

Basic partitionings: BP+MS



- Partitioning of $\mathcal{G}_2$ into an unentangled bipartite graph (BP) and a fully entangled Mermin square (MS). Operators on all six lines carry a base of entangled states. The graph is polarized.

Basic partitionings: I+PG



(PG) (I)

- The partitioning of $\mathcal{G}_2$ into a maximum independent set (I) and the Petersen graph (PG), *aka* its minimum vertex cover.).

Geometric hyperplanes of $GQ(2)$

A geometric hyperplane H: a set of points such that every line of the geometry either contains exactly one point of H, or is completely contained in H.

- A **perp-set** $(H_{cl}(X))$, i. e., a set of points collinear with a given point X, the point itself inclusive (there are 15 such hyperplanes). It corresponds to the pencil of lines in the Fano plane.
- A **grid** (H_{gr}) of nine points on six lines (there are 10 such hyperplanes). It is a Mermin square.
- An **ovoid** (H_{ov}), i. e., a set of (five) points that has exactly one point in common with every line (there are six such hyperplanes). An ovoid corresponds to a maximum independent set.

Group theory, quantum gates and quantum coherence[3]

[3] M. Planat and P. Jorrand, J Phys A:Math Theor 41 (2008)

Outline of group commutators and group extensions: 1

- A subgroup N of a group G is called a **normal subgroup** if it is invariant under conjugation: that is, for each n in N and each g in G, the conjugate element gng^{-1} still belongs to N.
- e.g. 1: the **center** $Z(G)$ of a group of G. The group $\tilde{G} = G/Z(G)$ is called the **central quotient** of G.
- e.g. 2: the subgroup G' of **commutators** (generated by all the commutators $[g, h] = ghg^{-1}h^{-1}$ of elements of G). The quotient group $H^{\mathrm{ab}} = G/G'$ is an abelian group called the **abelianization** of G and corresponds to its first homology group. The set $K(G)$ of all commutators of a group G may depart from G'.

Outline of group commutators and group extensions: 2

- e.g. 3: **group extensions**. Let $\mathcal{P}$ and $\mathcal{C}$ be two groups such that $\mathcal{P}$ is normal subgroup of $\mathcal{C}$. The group $\mathcal{C}$ is an extension of $\mathcal{P}$ by H if there exists a **short exact sequence** of groups

$$1 \rightarrow \mathcal{P} \xrightarrow{f_1} \mathcal{C} \xrightarrow{f_2} H \rightarrow 1,$$

i.e.
(i) $\mathcal{P} \cong$ a normal subgroup N of $\mathcal{C}$,
(ii) $H \cong \mathcal{C}/N$.
In an exact sequence $\mathrm{Im}(f_1) = \mathrm{Ker}(f_2)$, then the map f_1 is injective and f_2 is surjective.

Outline of group commutators and group extensions: 3

- Given any groups $\mathcal{P}$ and H the **direct product** of $\mathcal{P}$ and H is an extension of $\mathcal{P}$ by H,
- The **semidirect product** $\mathcal{P} \rtimes H$ of $\mathcal{P}$ and H:
  The group $\mathcal{C}$ is an extension of $\mathcal{P}$ by H and
  (i) H is isomorphic to a subgroup of $\mathcal{C}$,
  (ii) $\mathcal{C}$=$\mathcal{P}H$ and
  (iii) $\mathcal{P} \cap H = \langle 1 \rangle$.
  One says that the short exact sequence **splits**.
- The **wreath product** $M \wr H$ of a group M with a permutation group H acting on n points is the semidirect product of the normal subgroup M^n with the group H which acts on M^n by permuting its components.

Outline of group commutators and group extensions: 4

- **Icosahedral symmetry** and the "Mathieu group " M_{20}: Let $G = \mathcal{Z}_2 \wr A_5$, then G is a perfect group with order $2^5.60$. One has $G' \neq K(G)$. Let $H = Z_2^5 \rtimes A_5$, one can think of A_5 having a wreath action on Z_2^5. The group $G' = \tilde{H} = M_{20}$ is the smallest perfect group having its commutator subgroup distinct from the set of the commutators [4].
- M_{20} also corresponds to the derived subgroup W' of the Weyl group (also called hyperoctahedral group) $W = \mathcal{Z}_2 \wr S_5$ for the Lie algebra of type B_5.

[4] On commutators in groups. L C Kappe and R F Morse. available on line at http://faculty.evansville.edu/rm43/publications/commutatorsurvey.pdf

Group of automorphisms

- Given the group operation $*$ of a group G, a **group endomorphism** is a function ϕ from G to itself such that $\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$, for all $g_1, g_2 \in G$. If it is **bijective** it is called an **automorphism**.
- An automorphism of G that is induced by conjugation of some $g \in G$ is called inner. Otherwise it is called an outer automorphism. Under composition the set of all automorphisms defines a group denoted $\mathrm{Aut}(G)$. The **inner automorphisms** form a **normal subgroup** $\mathrm{Inn}(G)$ of $\mathrm{Aut}(G)$, that is isomorphic to the cental quotient of G. The quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the **outer** automorphism group.

Quantum computing: a few quantum gates

- The Hadamard gate: $H = 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
  and the phase shift gate $P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.
  superpositions: $H\left|0\right\rangle = 1/\sqrt{2}(\left|0\right\rangle + \left|1\right\rangle)$, $H\left|1\right\rangle = 1/\sqrt{2}(\left|0\right\rangle - \left|1\right\rangle)$.
- The Controlled not gate CNOT $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.
  entanglement: $CNOT(\alpha\left|0\right\rangle + \beta)\left|1\right\rangle)\left|0\right\rangle = \alpha\left|00\right\rangle + \beta\left|11\right\rangle$.
- The Toffoli gate $\text{TOF} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

A quantum computing challenge

- Correcting the errors in **quantum computing**: **quantum codes** or methods immune of **decoherence**.
- error group: the Pauli group $\mathcal{P}$
  $|\psi\rangle \longrightarrow^{\text{error}} g\,|\psi\rangle \longrightarrow^{\text{unitary evolution}} Ug\,|\psi\rangle = UgU^{\dagger}U\,|\psi\rangle$.
  Stabilizing the error $g \in \mathcal{P}$ requires $UgU^{\dagger} \in \mathcal{P}$.
- Error free operators are in the **Clifford group** $\mathcal{C}$
  e.g. H, P, $CNOT$.
- Since $U^{\dagger} = U^{-1}$, $\mathcal{P}$ is a **normal subgroup** of $\mathcal{C}$.

Generating the Clifford groups

- For a system of n qubits one denotes the **Pauli group** as $\mathcal{P}_n$ and the **Clifford group** as $\mathcal{C}_n$.
- $\mathcal{C}_1 = \langle H, P \rangle$, $\mathcal{C}_2 = \langle \mathcal{C}_1 \otimes \mathcal{C}_1, CZ \rangle$ with $CZ = \mathrm{Diag}(1, 1, 1, -1)$. Any gate in $\mathcal{C}_n$ is a circuit of gates from $\mathcal{C}_1$ and $\mathcal{C}_2$. [5].
- Clifford group $\mathcal{C}_n$ on n-qubits has order $|\mathcal{C}_n| = 2^{n^2+2n+3} \prod_{j=1}^{n} 4^j - 1$.
- e.g. **a MAGMA program** //Two-qubit Clifford group
  K$\langle w \rangle$:=CyclotomicField(8); r2:=w+ComplexConjugate(w);
  H:=Matrix(K,2,2,[$1/r2, 1/r2, 1/r2, w^4/r2$]);
  P:=Matrix(K,2,2,[$1, 0, 0, w^2$]); CZ:=DiagonalMatrix([$1, 1, 1, w^4$]);
  H2:=KroneckerProduct(H,H); HP:=KroneckerProduct(H,P);
  C2:=MatrixGroup$\langle 4, K | H2, HP, CZ \rangle$; Order(C2);

  192, **92 160**, 743 178 240

[5] Generalized Clifford groups and simulation of associated quantum circuits. S Clark, R Jozsa and N Linden. Quant Inf Comp 8, 106–26 (2008).

The Clifford group on a single qubit

- **One-qubit Clifford group** $\mathcal{C}_1 = \langle H, P \rangle$: $|\mathcal{C}_1| = 192$, $Z(\mathcal{C}_1) \cong \mathcal{Z}_8$, $\mathcal{C}_1' \cong SL(2,3)$, $\tilde{\mathcal{C}}_1 = S_4$ and $\mathcal{C}_1^{\mathrm{ab}} = \mathcal{Z}_4 \times \mathcal{Z}_2$.
- A split extension attached to the **commutator subgroup** $\mathcal{C}_1'$

$$1 \rightarrow SL(2,3) \rightarrow \mathcal{C}_1 \rightarrow \mathcal{Z}_2 \times \mathcal{Z}_3 \rightarrow 1.$$

- ... attached to the **magic group**[6] $\langle T, H \rangle$, where $T = \exp(i\pi/4)PH$

$$1 \rightarrow GL(2,3) \rightarrow \mathcal{C}_1 \rightarrow \mathcal{Z}_4 \rightarrow 1.$$

- ... attached to the **Pauli group**

$$1 \rightarrow \mathcal{P}_1 \rightarrow \mathcal{C}_1 \rightarrow D_{12} \rightarrow 1,$$

in which $D_{12} = \mathcal{Z}_2 \times S_3$ is the symmetry group of a *regular hexagon*.

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[6] Universal quantum computation with ideal Clifford gates and noisy ancillas. S Bravyi and A Kitaev. PRA71, 1-14 (2005).

The Clifford group on two qubits: 1

- **Two-qubit Pauli group**
  $\mathcal{P}_2 = \langle \sigma_x \otimes \sigma_x, \sigma_z \otimes \sigma_z, \sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_z, \sigma_z \otimes \sigma_x \rangle$, order 64,
  $Z(\mathcal{P}_2) = \{\pm 1, \pm i\}$.
  Two-qubit Clifford group $\mathcal{C}_2 = \langle H \otimes H, H \otimes P, CZ \rangle$, order 92160.
- $Z(\mathcal{C}_2) = \mathcal{Z}_8$, $\tilde{\mathcal{C}}_2$ such that

$$1 \rightarrow U_6 \rightarrow \tilde{\mathcal{C}}_2 \rightarrow \mathcal{Z}_2 \rightarrow 1.$$

- It turns out that the group $\tilde{\mathcal{C}}_2$ only contains **two normal subgroups** $\mathcal{Z}_2^{\times 4}$ and $\tilde{\mathcal{C}}_2' = U_6 = \mathcal{Z}_2^{\times 4} \rtimes A_6$. The group U_6, of order 5760, is a perfect group. $\mathrm{Out}(U_6) = \mathrm{Out}(A_6) = \mathcal{Z}_2 \times \mathcal{Z}_2$.
- $Aut(\mathcal{P}_2) = \mathcal{Z}_2 \wr A_6$, $U_6 = Aut(\mathcal{P}_2)'$.

$$\mathcal{C}_2/\mathcal{P}_2 = \mathcal{Z}_2 \times S_6.$$

The Clifford group on two qubits: 2

- U_6 is a maximal subgroup of several sporadic groups. The smallest one is M_{22}. It appears in relation to a subgeometry of M_{22} known as an **hexad**.
- A **Steiner system** $S(a, b, c)$ with parameters a, b, c, is a c-element set together with a set of b-element subsets of S (called *blocks*) with the property that each a-element subset of S is contained in exactly one block.
  M_{22} **stabilizes the Steiner system** $S(3, 6, 22)$ comprising 22 points and 6 blocks, each set of 3 points being contained exactly in one block.
  Any block in $S(3, 6, 22)$ is a **Mathieu hexad, stabilized by** the *general* alternating group U_6.

Two-qubit coherence and the anyons: 1

- **Topological quantum computing** based on anyons has been proposed as way of encoding quantum bits in non local observables that are immune of decoherence [7]. The basic idea is to use pairs $\left|v, v^{-1}\right\rangle$ of **"magnetic fluxes"** playing the roles of the **qubits** and permuting them within some large enough non abelian finite group G such as A_5. The "magnetic flux" carried by the (anyonic) quantum particle is labeled by an element of G, and **"electric charges"** are labeled by **irreducible representation** of G.
- The exchange within G modifies the quantum numbers of the fluxons according to the fundamental logical operation

$$\left|v_1, v_2\right\rangle \to \left|v_2, v_2^{-1} v_1 v_2\right\rangle,$$

a form of Aharonov-Bohm interactions (in a non abelian group).

[7] Fault tolerant quantum computation. J Preskill. in Introduction to Quantum Computation and Information. ed H K Lo, T Spiller, S Popescu (Singapore, World Scientific, 1998). Preprint quant-ph/9712048.

Two-qubit coherence and the anyons: 2

- This process can be shown to produce **universal** quantum computation. It is intimely related to topological entanglement, the braid group and unitary solutions of the **Yang-Baxter equation** [8]

$$(R \otimes I_2)(I_2 \otimes R)(R \otimes I_2) = (I_2 \otimes R)(R \otimes I_2)(I_2 \otimes R),$$

in which the operator R: $V \otimes V \to V \otimes V$ acts on the tensor product of the bidimensional vector space V. One elegant unitary solution of the Yang-Baxter equation is a universal quantum gate known as the Bell basis change matrix

$$R = 1/\sqrt{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

[8] Braiding operators are universal quantum gates. L H Kauffman and S J Lomonaco. New J Phys 6, 134 (2004).

Two-qubit coherence and the anyons: 3

- **Two-qubit topological quantum computing** and the **Bell subgroup** of the Clifford group, of order 15360

$$\mathcal{B}_2 = \langle H \otimes H, H \otimes P, R \rangle . \tag{1}$$

- $Z(\mathcal{B}_2) = \mathcal{Z}_8$, $\mathcal{B}_2' = \mathcal{Z}_2 \wr A_5$, and

$$1 \rightarrow \mathcal{Z}_2 \wr A_5 \rightarrow \mathcal{B}_2 \rightarrow \mathcal{Z}_2 \rightarrow 1.$$

- $\tilde{\mathcal{B}}_2$ only contains **two normal subgroups** $\mathcal{Z}_2^{\times 4}$ and $M_{20} = \mathcal{Z}_2^{\times 4} \rtimes A_5$.

  Relation between Bell and Pauli groups

$$\mathcal{B}_2/\mathcal{P}_2 = \mathcal{Z}_2 \times S_5$$

  S_5 is the the stabilizer of Petersen graph.

G	g_2	g_3	g_4	g_5	g_6
Aut(G)	$\mathcal{D}_8$	$\mathcal{Z}_2 \times S_4$	$\mathcal{Z}_2 \wr A_5$	$\mathcal{Z}_2^{\times 2} \wr A_5$	$\mathcal{Z}_2^{\times 3} \wr A_5$
$\lvert$Aut(G)$\rvert$	8	48	1920	61440	1966080

- Group structure of the maximal independent set **generating a complete set of MUBs**: $g_i = \langle m_1, m_2 \cdots m_i \rangle$.
- The wreath product $\mathcal{Z}_2 \wr S_5$ corresponds to the first known example of a **non-additive quantum code**.
- A_5 is the symmetry group of the **icosahedron**: S Benjamin, *Towards a fullerene-based quantum computer*, J Phys:Cond Matter 18, S867-83 (2006).

Three-qubit quantum coherence

- **Two-qubit system**
  $\tilde{\mathcal{C}}_2 = \mathcal{Z}_2^{\times 4} \rtimes S_6,\ S_6 = Sp(4,2)$ (order 720),
  $\tilde{\mathcal{B}}_2 = \mathcal{Z}_2^{\times 4} \rtimes S_5$
- **Tree-qubit system**
  Let $\mathcal{B}_3 = \langle H \otimes H \otimes P, H \otimes R, R \otimes H \rangle$.
  $\tilde{\mathcal{C}}_3 = \mathcal{Z}_2^{\times 6} \rtimes G_1,\ G_1 := Sp(6,2)$ (order 1 451 520),
  $\tilde{\mathcal{B}}_3 = \mathcal{Z}_2^{\times 6} \rtimes G_2$,
  with $G_2 = SU(4,2) \cong PSp(4,3)$ (order 25920).
- **Geometry**: G_1 (resp G_2) are the derived subgroups of the Weyl groups attached to *exceptional* Lie algebra of type E_7 (resp E_6).

Merging of several conceps?

- Quantum gates and the **Geometry of classical groups**
  * Tits systems (BN pairs) (see D. E. Taylor, 1992)
- Topological quantum computing
- Non-additive quantum codes
- Ring geometry [collaboration with M. Saniga (SK) and H. Havlicek(Austria)]