FROM PAULI GROUPS TO STRINGY BLACK HOLES*
(Part I: Projective (Near)Ring Lines)

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AN OVERVIEW OF THE TALK

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INTRODUCTION

Projective ring lines turned out to be a very important concept in unveiling the intricate geometrical nature of the structure of finite-dimensional Hilbert spaces [11–13].

It was our working on these intriguing physical applications when we discovered novel, and rather unexpected, properties of the fine structure of the projective lines not so far discussed by either physicists or mathematicians.

Hence, as the general theory does not exist yet, the purpose of the talk is simply to outline, in a rather illustrative way, the main findings and briefly address their possible applications.

BASIC DEFINITIONS AND NOTATION [1,2]

R: a finite associative ring with unity (1); we shall specifically refer to a ring as being of X/Y type, where X is the cardinality of R and Y the number of its zero-divisors.

$R(a,b)$: a (left) cyclic submodule of R^2,
$R(a,b) = \big\{(\alpha a, \alpha b) | (a,b) \in R^2, \alpha \in R\big\}$;
a cyclic submodule $R(a,b)$ is called *free* if the mapping $\alpha \mapsto (\alpha a, \alpha b)$ is injective, i. e., if all $(\alpha a, \alpha b)$ are distinct.

Admissibility: a pair/vector $(a,b) \in R^2$ is called admissible, if it is the first row of an invertible 2×2 matrix over R

Unimodularity: a pair/vector $(a,b) \in R^2$ is called unimodular, if there exist $x, y \in R$ such that $ax + by = 1$

For the rings under consideration, admissibility and unimodularity mean the same

PROJECTIVE RING LINE [3–10]

$P(R)$, the projective line over R,
$P(R) = \big\{R(a,b) \subset R^2 | (a,b) \text{ admissible}\big\}$

Crucial property: if (a,b) is admissible, then $R(a,b)$ is free; there, however, are also rings in which there exist free cyclic submodules containing no admissible pairs!

Distant/Neighbour relation: Two distinct points $A =: R(a,b)$ and $B =: R(c,d)$ of $P(R)$ are called distant if the 2×2 matrix with the first row a,b and the second row c,d is invertible; otherwise, they are called neighbor.

It can easily be shown that any two distant points of $P(R)$ have only the pair $(0,0)$ in common. As this pair lies on any cyclic submodule, the distant/neighbour condition can be rephrased as follows:

Theorem 1: Two distinct points $A =: R(a,b)$ and $B =: R(c,d)$ of $P(R)$ are distant if $|R(a,b) \cap R(c,d)| = 1$ and neighbor if $|R(a,b) \cap R(c,d)| > 1$.

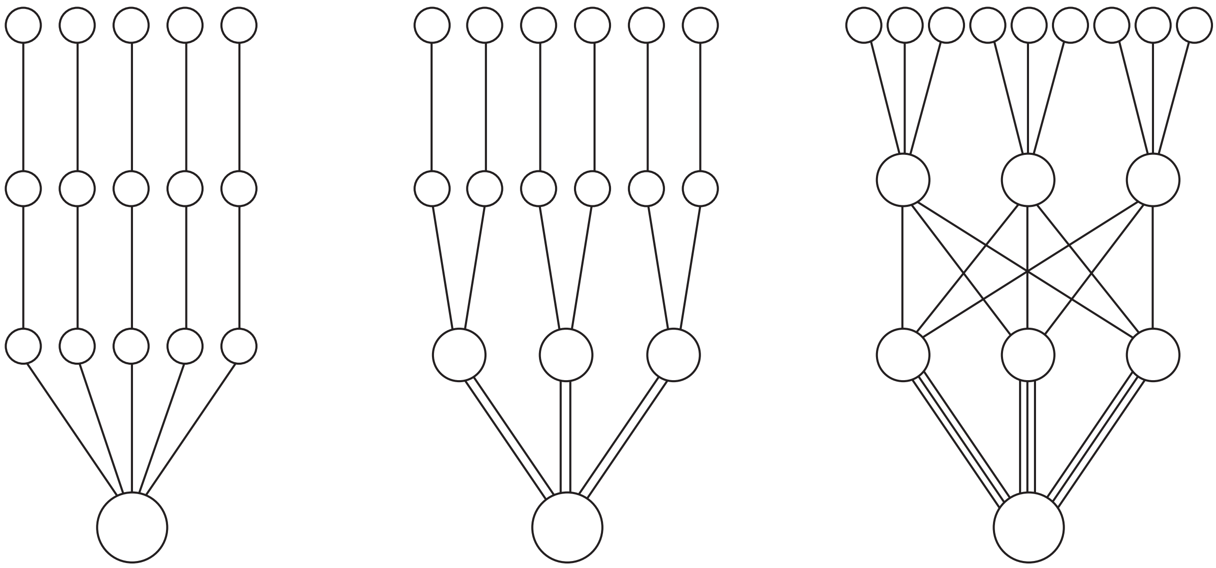
VISUALISING THE STRUCTURE OF PRL IN TERMS OF FCS'S

The structure of $P(R)$ can be visualized in terms of a "tree" comprising all the FCSs generated by admissible pairs. Any such tree consists of the "corolla" (α being units of R) and the "trunk" (α being zero-divisors of R) and in the figures that follow it is illustrated in the following way:

$\hookrightarrow$ a pair/vector of R^2 is represented by a circle whose size is proportional to the number of FCSs containing this pair and

$\hookrightarrow$ the fact that two different pairs/vectors lie on a FCS is indicated by a line segment joining the corresponding circles.

The finest traits of the structure of the line pertain uniquely to the trunk — this fact is already fairly obvious from the examples of projective lines defined over (all) unital rings of order four (Figure 1)



GF(2)[x]/<x^2+x+1> $\cong$ **GF(4)**

GF(2)[x]/<x^2> or Z(4)

GF(2)[x]/<x(x+1)> $\cong$ **GF(2) x GF(2)**

Figure 1: The forms of the trees representing the projective lines defined over rings with unity of order four; we see that (left-to-right) as the number of zero-divisors (and maximal ideals) of the ring increases, the trunk becomes more pronounced and intricate.

REFINEMENT OF THE NEIGHBOR RELATION

From THEOREM 1 it follows that one can refine the neighbor relation by introducing the degree of the "neighborness" between any two neighbor points in terms of the number of shared pairs/vectors by their representing FCSs. This is illustrated in Figure 2 on an example of the projective line defined over $Z_{12} \cong Z_3 \times Z_4$.

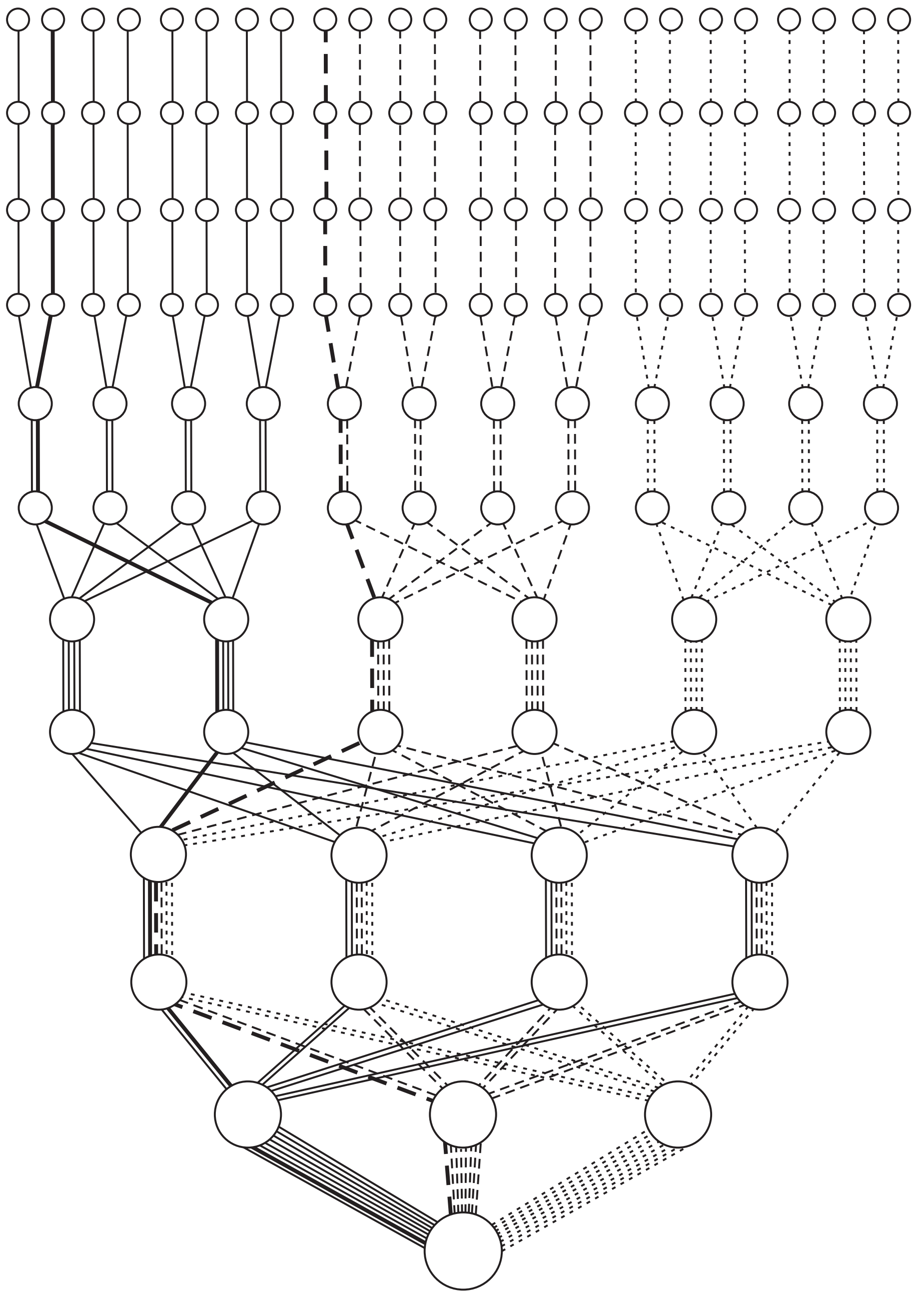


Figure 2:

EXISTENCE OF "OUTLIERS"

Outlier: a pair/vector of R^2 not belonging to any FCS generated by an admissible pair/vector.

Smallest order where they occur are some rings of 8/4 type (Figure 3, right) and the non-commutative 8/6 ring (Figure 4, right); many more are found in the case of commutative 16/8 rings (Figure 5, bottom and top right). Also all non-commutative rings of type 16/8 and 16/12 feature outliers, as well as the non-commutative ring of type 16/14; interestingly, the line over the full two-by-two matrix ring with Z_2-valued coefficients has no outliers.

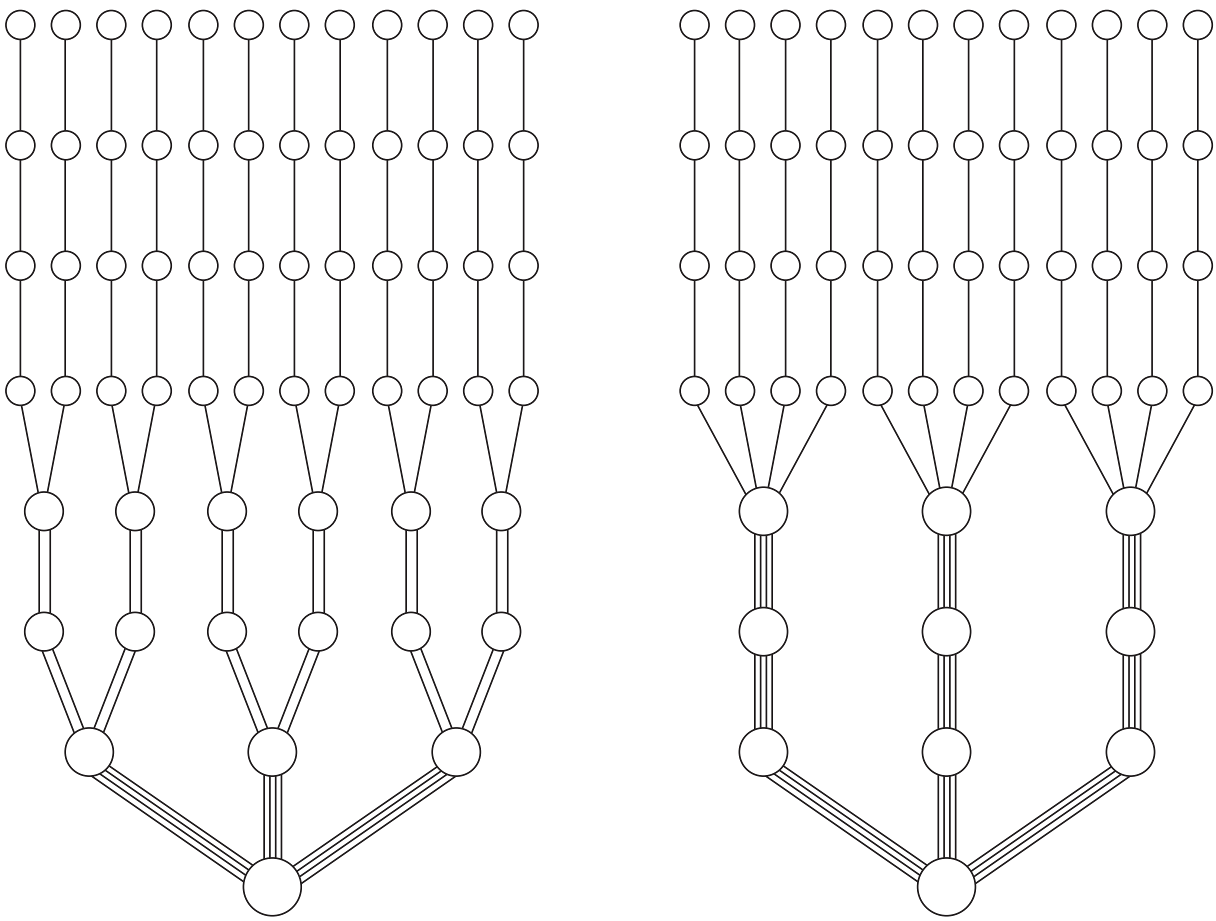


Figure 3: A generic shape of the trees representing projective lines defined over local rings of 8/4 type: *left* – lines featuring no outliers (three distinct kinds of non-isomorphic rings, including Z_8 and $Z_2[x]/\langle x^3\rangle$), *right* – lines featuring six outliers (two kinds of non-isomorphic rings).

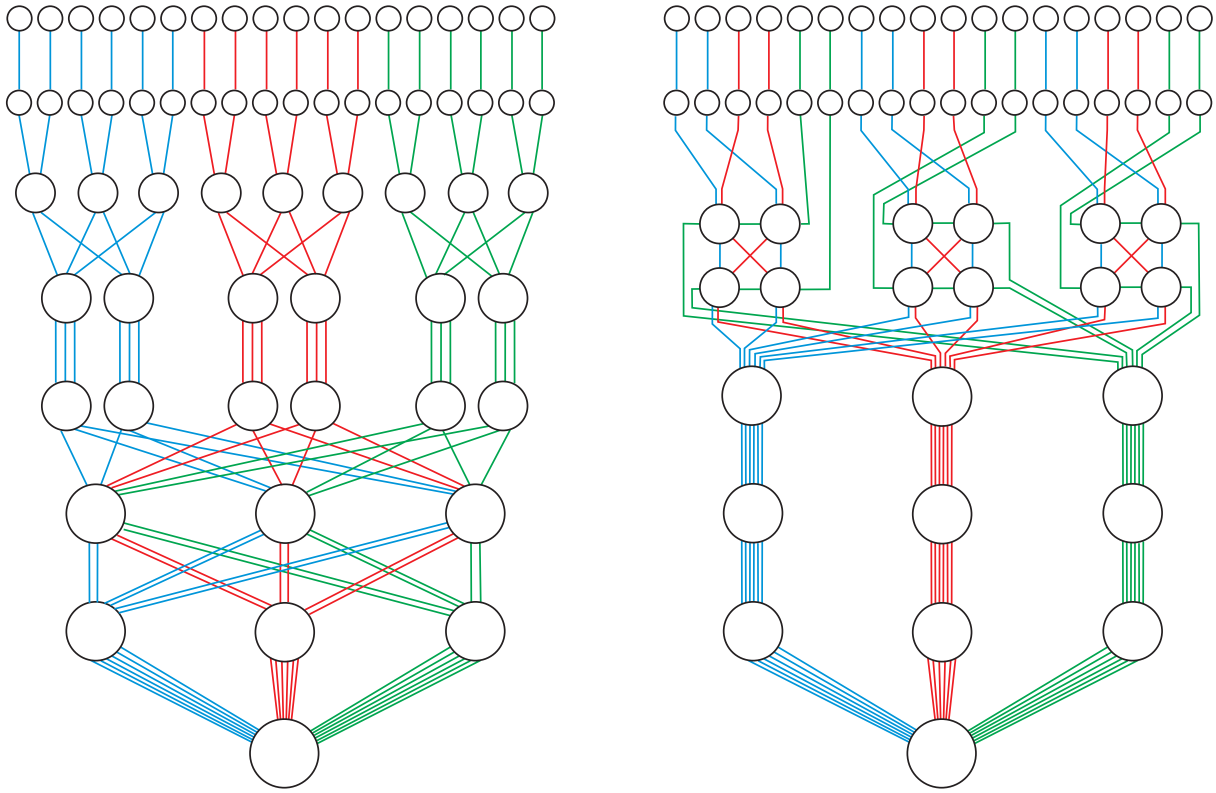


Figure 4: The trees of the projective lines defined over the rings of 8/6 type: *left* – the commutative ring $Z_2 \times Z_4$ (line features no outliers), *right* – the non-commutative ring of ternions (i. e., ring of upper/lower triangular matrices over Z_2 – six outliers).

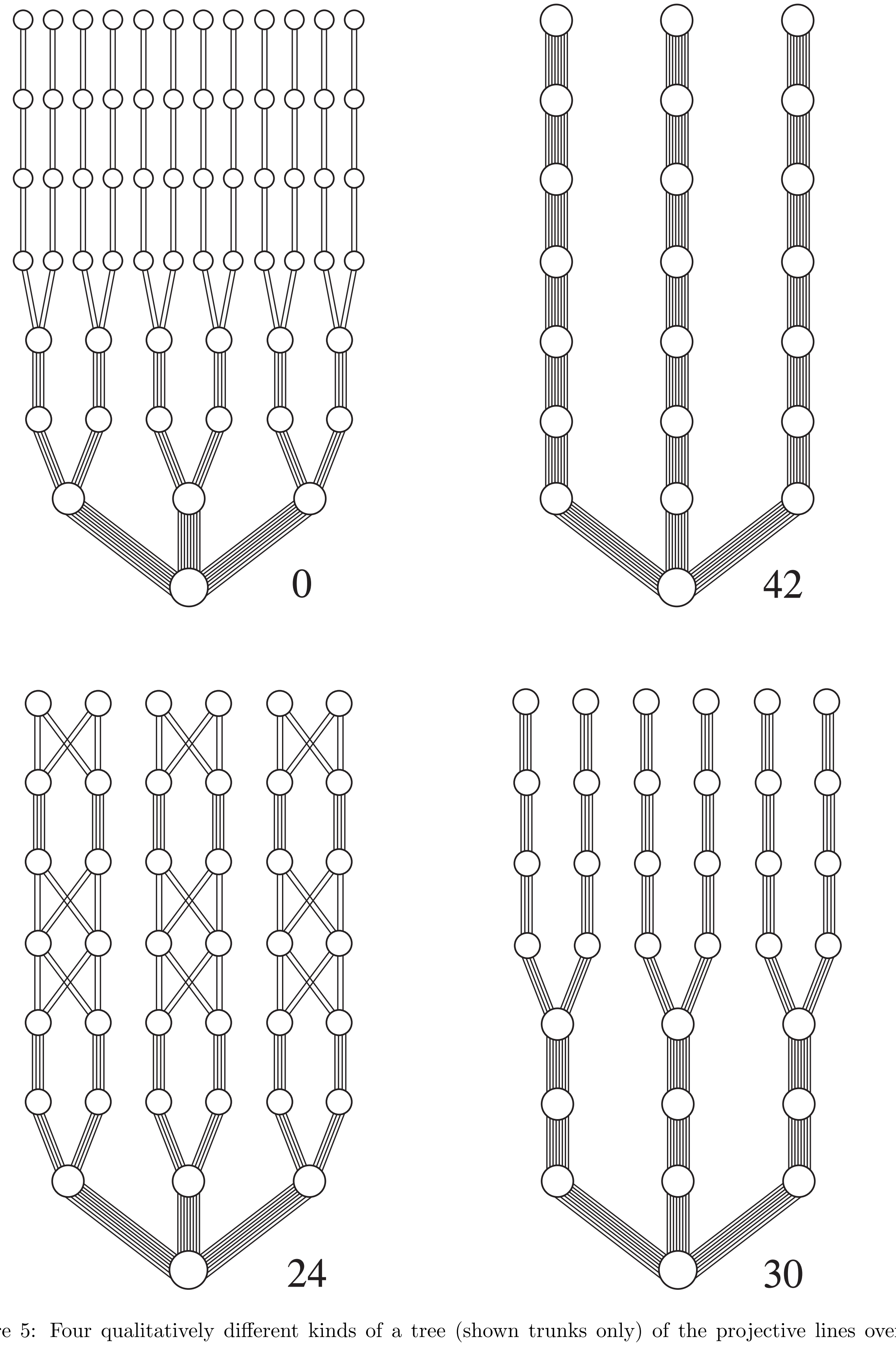


Figure 5: Four qualitatively different kinds of a tree (shown trunks only) of the projective lines over local commutative rings of 16/8 type: *top left* – no outliers (four distinct kinds of non-isomorphic rings, including Z_{16} and $Z_2[x]/\langle x^4 \rangle$), *bottom left* – 24 outliers (5 rings; this is also the tree exhibited by projective lines defined over all the four non-commutative rings of the same type), *bottom right* – 30 outliers (4 rings) and *top right* – 42 outliers (2 rings).

OUTLIERS GENERATING FCS'S

The smallest order where they appear is 8/6 non-commutative (Figure 6).

They are also found in all but one non-commutative rings of type 16/12 and in the non-commutative ring of type 16/14.

No commutative example has been found among the rings so-far-analyzed.

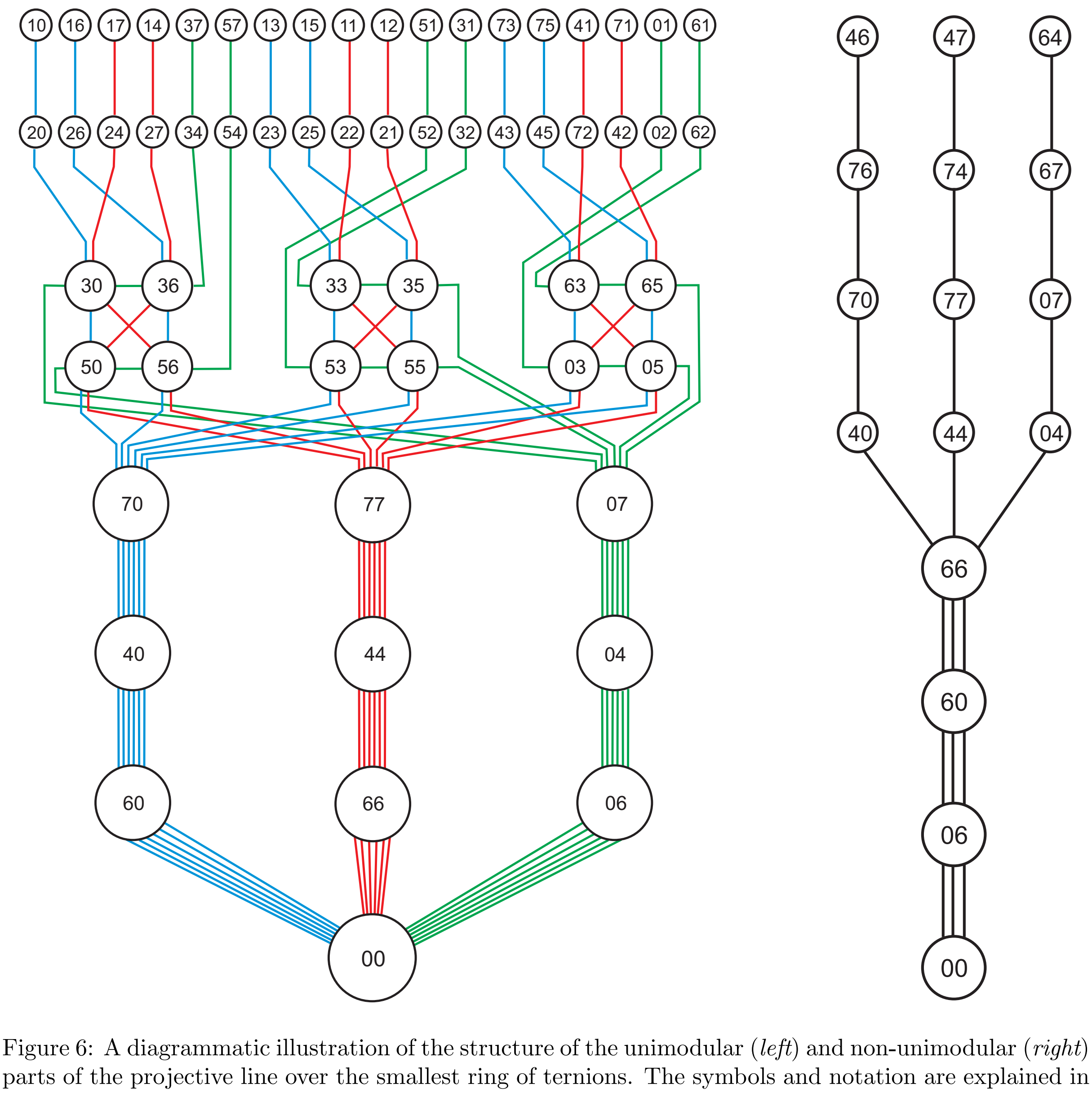


Figure 6: A diagrammatic illustration of the structure of the unimodular (*left*) and non-unimodular (*right*) parts of the projective line over the smallest ring of ternions. The symbols and notation are explained in the text.

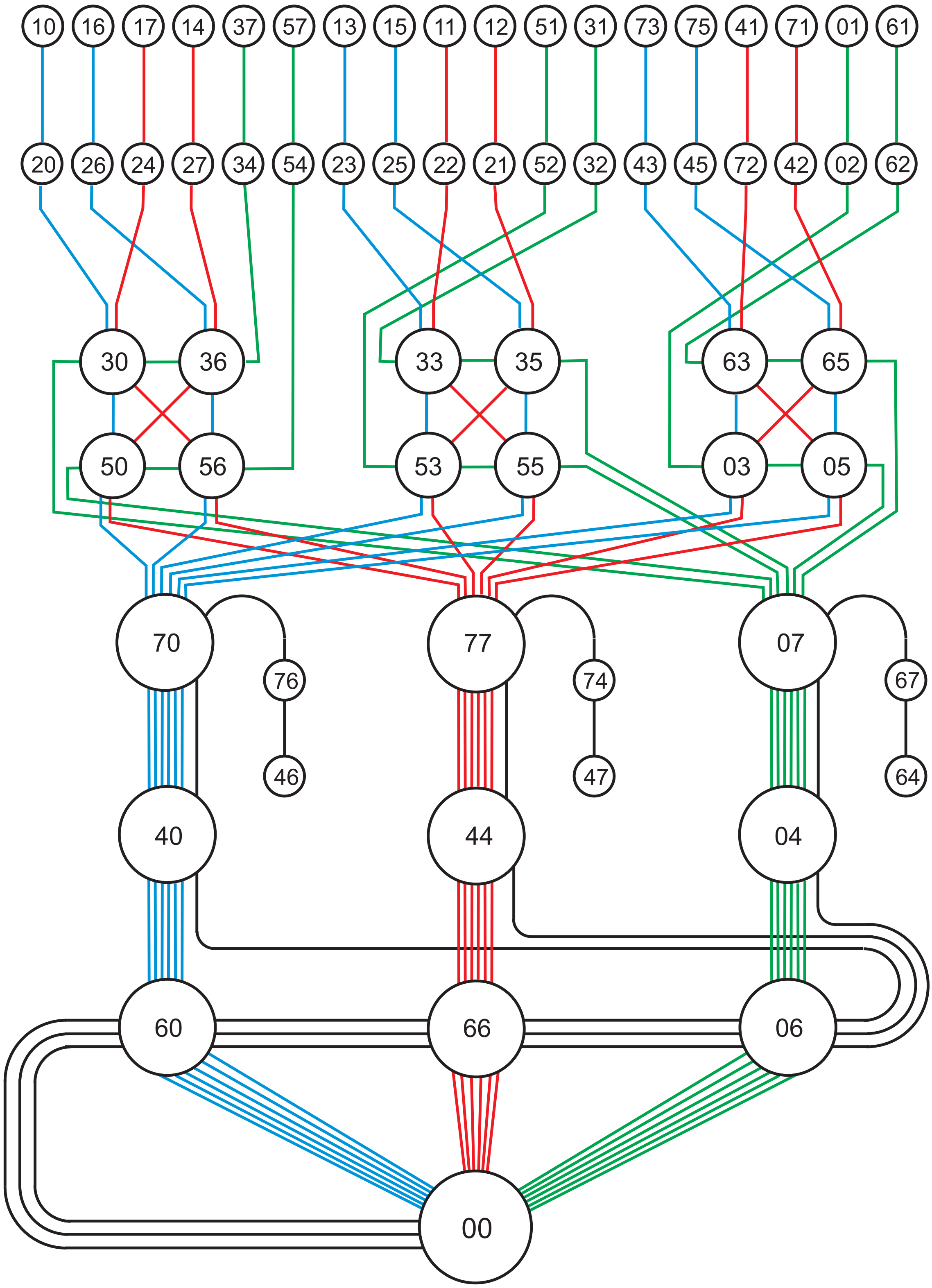


Figure 7: A diagrammatic sketch of the intricate link between the two parts of the line shown in the preceding figure.

GEOMETRY BEHIND OUTLIERS' GENERATED FCS'S

In each (non-commutative) example listed below, all outliers' generated FCSs share apart from the pair $(0,0)$ also several other pairs. In the 27/15 case the number of such additional pairs is eight, whereas in all the remaining cases it is three (cf. Figure 6 for the 8/6 case). This suggests to consider the "condensed" lines grouping nine (the former case) resp. four (the latter cases) different pairs on any outlier's generated FCS into a single entity and looking what the resulting "condensed" trees look like:

8/6	6/6	Z_2
16/12a	30/24	Z_4 or $Z_2[x]/\langle x^2\rangle$
16/12b	42/36	???
16/14	24/18	$Z_2 \times Z_2$
24/20	54/48	$Z_6 \simeq Z_2 \times Z_3$
27/15	48/48	Z_3

Here the first column gives the line type, the second column features the number of its outliers (total vs generating FCSs) and the last column lists the type of "condensed" line.

FINEST DIFFERENCE BETWEEN PRL'S OVER NON-LOCAL COMMUTATIVE RINGS

$Z_4 \times Z_4$ versus $Z_2 \times Z_8$:

They are both non-local of the same (16/12) type and they both feature no outliers; having identical corollas and all "macroscopic" characteristics (total number of points, cardinality of neighborhoods, intersections of neighborhoods of two distant points, number of Jacobson points and maximum number of pairwise distant points), they differ profoundly in the "microscopic" structure of their trunks (Figure 8).

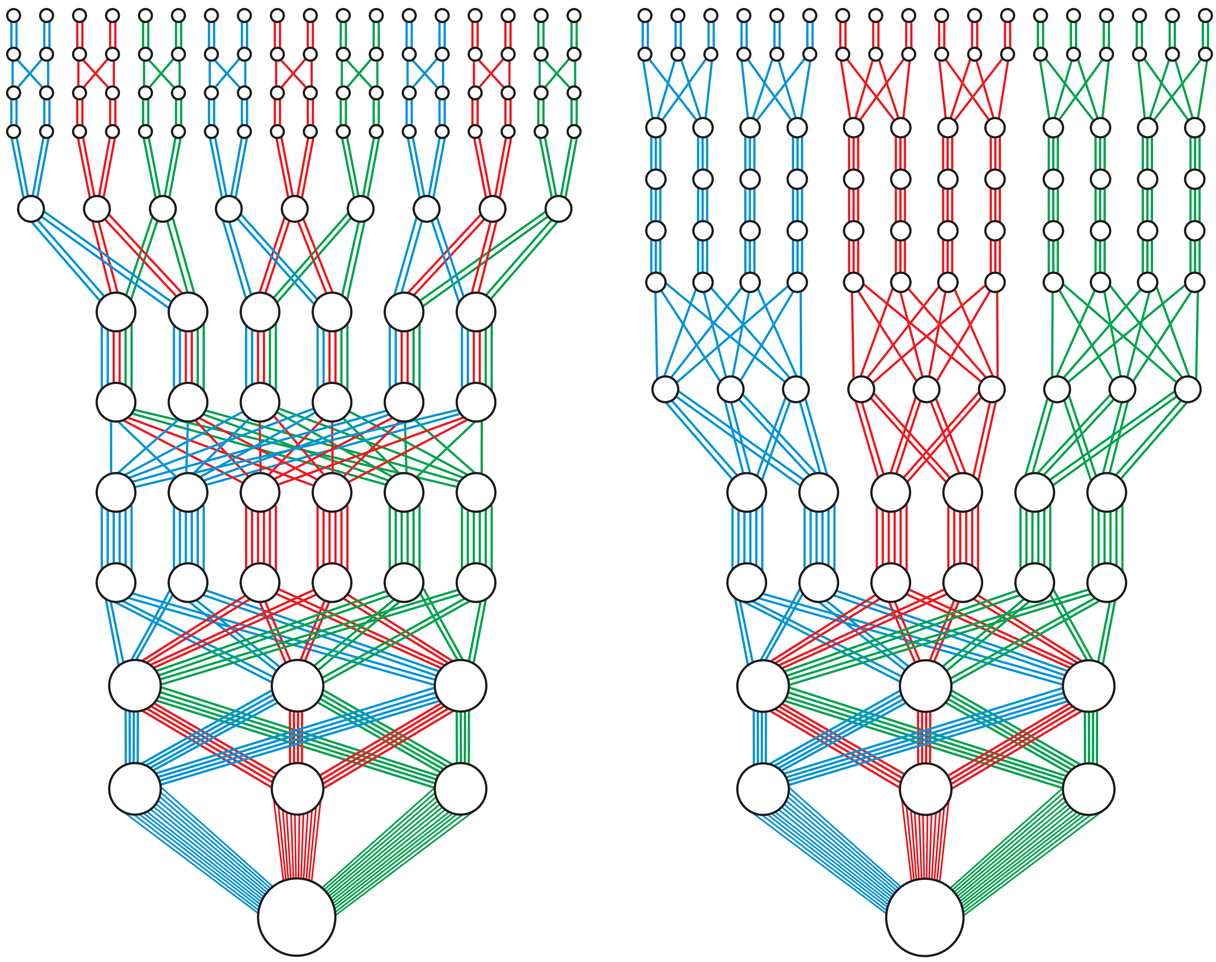


Figure 8: The tree of the projective line defined over $Z_4 \times Z_4$ (*left*) and that defined over $Z_2 \times Z_8$ (*right*).

FULL MATRIX RING $M_2(GF(2))$ AND ITS SUBRINGS

The projective line we are exclusively interested in here is the one defined over the full two-by-two matrix ring with $GF(2)$-valued coefficients, i. e.,

$$R = M_2(GF(2)) \equiv \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in GF(2) \right\}. \tag{1}$$

In an explicit form:

UNITS: Invertible matrices (i. e., matrices with non-zero determinant). They are of two distinct kinds: those which square to 1,

$$1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 9 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad 11 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{2}$$

and those which square to each other,

$$12 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad 13 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

ZERO-DIVISORS: Matrices with vanishing determinant. These are also of two different types: *nil*potent, i. e. those which square to zero,

$$3 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad 8 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 10 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad 0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4}$$

and *idem*potent, i. e. those which square to themselves,

$$4 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad 5 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad 6 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad 7 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

$$14 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad 15 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

The structure of this full matrix ring can be well understood from the accompanying colour figure featuring its most important subrings, namely those isomorphic to

- $GF(4)$ (yellow),
- $GF(2)[x]/\langle x^2\rangle$ (red),
- $GF(2) \otimes GF(2)$ (pink) and to
- the non-commutative ring of 8/6 type (green).

Irrespectively of colour, the dashed/dotted lines join elements represented by upper/lower triangular matrices, while the solid lines link elements represented by "diagonal parity preserving" matrices.

It is worth mentioning a very interesting symmetry of the picture. Namely, the "dpp" ring of 8/6 type (solid green) incorporates both the upper and lower triangular matrix rings isomorphic to $GF(2)\otimes GF(2)$, while, in turn, the "dpp" $GF(2)\otimes GF(2)$ ring (solid pink) is the intersection of the upper and lower triangular matrix rings of 8/6 type.

It is also to be noted that $GF(4)$ has only one representative, the "dpp" set, whereas each of the remaining types have *three* distinct (namely upper and lower triangular, and "dpp") representatives. The shaded circles denote a Jordan system.

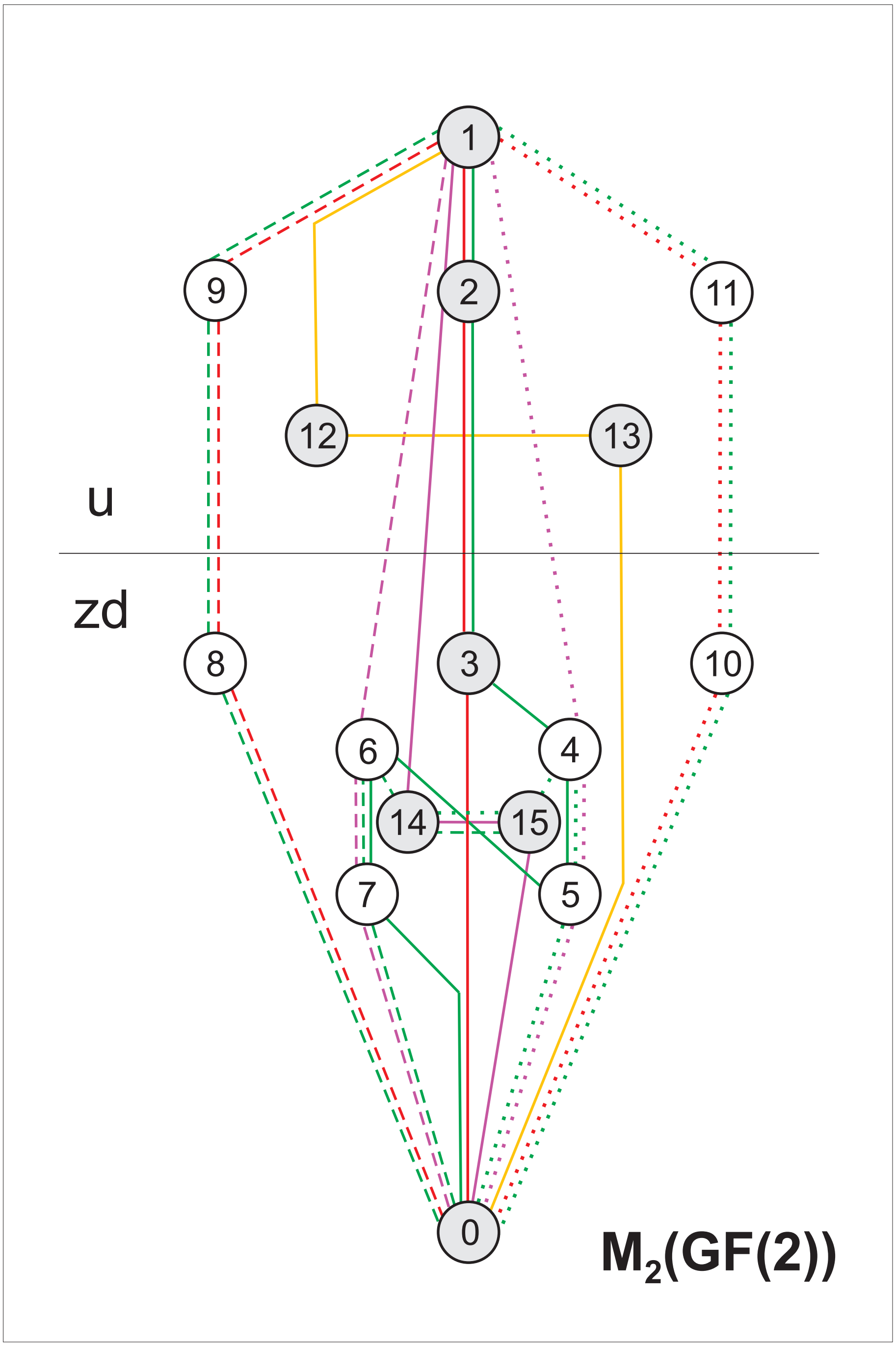


Figure 9: The subrings and a Jordan system of $M_2(GF(2))$.

Line Type	Cardinalities of Points							Representative Rings
	Tot	TpI	1N	∩2N	∩3N	Jcb	MD	
63/15	80	78	16	2	0	2	8	$GF(7) \otimes GF(9)$
63/27	96	90	32	6	0	14	4	$GF(7) \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
63/39	128	102	64	26	6	4	4	$GF(7) \otimes GF(3) \otimes GF(3)$
62/32	96	94	33	2	0	29	3	$GF(2) \otimes GF(31)$
61/1	62	62	0	0	0	0	62	$GF(61)$
60/36	120	96	59	24	6	5	4	$GF(3) \otimes GF(5) \otimes GF(4)$
60/44	144	104	83	40	12	15	3	$GF(3) \otimes GF(5) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
60/52	216	112	155	104	60	7	3	$GF(3) \otimes GF(5) \otimes GF(2) \otimes GF(2)$
59/1	60	60	0	0	0	0	60	$GF(59)$
58/30	90	88	31	2	0	27	3	$GF(2) \otimes GF(29)$
57/21	80	78	22	2	0	16	4	$GF(3) \otimes GF(19)$
56/14	72	70	15	2	0	1	8	$GF(7) \otimes GF(8)$
56/32	96	88	39	8	0	23	3	$GF(7) \otimes Z_8, GF(7) \otimes GF(2)[x]/\langle x^3 \rangle, \ldots$
56/38	120	94	63	26	6	17	3	$GF(7) \otimes GF(2) \otimes GF(4)$
56/44	144	100	87	44	12	11	3	$GF(7) \otimes GF(2) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
56/50	216	106	159	110	66	5	3	$GF(7) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
55/15	72	70	16	2	0	6	6	$GF(5) \otimes GF(11)$
54/28	84	82	29	2	0	25	3	$GF(2) \otimes GF(27)$
54/36	108	90	53	18	0	17	3	$GF(2) \otimes Z_{27}, GF(2) \otimes GF(3)[x]/\langle x^3 \rangle, \ldots$
54/38	120	92	65	28	6	15	3	$GF(2) \otimes GF(3) \otimes GF(9)$
54/42	144	96	89	48	18	11	3	$GF(2) \otimes GF(3) \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
54/46	192	100	137	92	54	7	3	$GF(2) \otimes GF(3) \otimes GF(3) \otimes GF(3)$
53/1	54	54	0	0	0	0	54	$GF(53)$
52/16	70	68	17	2	0	9	5	$GF(13) \otimes GF(4)$
52/28	84	80	31	4	0	23	3	$GF(13) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
52/40	126	92	73	34	6	11	3	$GF(13) \otimes GF(2) \otimes GF(2)$
51/19	72	70	20	2	0	14	4	$GF(3) \otimes GF(17)$
50/26	78	76	27	2	0	23	3	$GF(2) \otimes GF(25)$
50/30	90	80	39	10	0	19	3	$GF(2) \otimes [Z_{25}$ or $GF(5)[x]/\langle x^2 \rangle]$
50/34	108	84	57	24	6	15	3	$GF(2) \otimes GF(5) \otimes GF(5)$
49/1	50	50	0	0	0	0	50	$GF(49)$
49/7	56	56	6	0	0	6	8	$Z_{49}, GF(7)[x]/\langle x^2 \rangle$
49/13	64	62	14	2	0	0	8	$GF(7) \otimes GF(7)$
48/18	68	66	19	2	0	13	4	$GF(3) \otimes GF(16)$
48/24	80	72	31	8	0	7	4	$GF(3) \otimes [GF(4)[x]/\langle x^2 \rangle$ or $Z_4[x]/\langle x^2+x+1 \rangle]$
48/30	100	78	51	22	6	3	4	$GF(3) \otimes GF(4) \otimes GF(4)$
48/32	96	80	47	16	0	15	3	$GF(3) \otimes Z_{16}, GF(3) \otimes Z_4[x]/\langle x^2 \rangle, \ldots$
48/34	108	82	59	26	6	13	3	$GF(3) \otimes GF(2) \otimes GF(8)$
48/36*	120	84	71	36	12	11	3	$GF(3) \otimes GF(4) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
48/40	144	88	95	56	24	7	3	$GF(3) \otimes Z_4 \otimes Z_4, GF(3) \otimes GF(2) \otimes Z_8, \ldots$
48/42	180	90	131	90	54	5	3	$GF(3) \otimes GF(2) \otimes GF(2) \otimes GF(4)$
48/44	216	92	167	124	84	3	3	$GF(3) \otimes GF(2) \otimes GF(2) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
48/46	324	94	275	230	186	1	3	$GF(3) \otimes GF(2) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
47/1	48	48	0	0	0	0	48	$GF(47)$
46/24	72	70	25	2	0	21	3	$GF(2) \otimes GF(23)$
45/13	60	58	14	2	0	4	6	$GF(5) \otimes GF(9)$
45/21	72	66	26	6	0	8	4	$GF(5) \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
45/29	96	74	50	22	6	2	4	$GF(5) \otimes GF(3) \otimes GF(3)$
44/14	60	58	15	2	0	7	5	$GF(11) \otimes GF(4)$
44/24	72	68	27	4	0	19	3	$GF(11) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
44/34	108	78	63	30	6	9	3	$GF(11) \otimes GF(2) \otimes GF(2)$

Line Type	Cardinalities of Points							Representative Rings
	Tot	TpI	1N	∩2N	∩3N	Jcb	MD	
43/1	44	44	0	0	0	0	44	$GF(43)$
42/30	96	72	57	24	6	11	3	$GF(2)\otimes GF(3)\otimes GF(7)$
41/1	42	42	0	0	0	0	42	$GF(41)$
40/12	54	52	13	2	0	3	6	$GF(5)\otimes GF(8)$
40/24	72	64	31	8	0	15	3	$GF(5)\otimes Z_8, GF(5)\otimes GF(2)[x]/\langle x^3\rangle, \ldots$
40/28	90	68	49	22	6	11	3	$GF(5)\otimes GF(2)\otimes GF(4)$
40/32	108	72	67	36	12	7	3	$GF(5)\otimes GF(2)\otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
40/36	162	76	121	86	54	3	3	$GF(5)\otimes GF(2)\otimes GF(2)\otimes GF(2)$
39/15	56	54	16	2	0	10	4	$GF(3)\otimes GF(13)$
38/20	60	58	21	2	0	17	3	$GF(2)\otimes GF(19)$
37/1	38	38	0	0	0	0	38	$GF(37)$
36/12	50	48	13	2	0	5	5	$GF(4)\otimes GF(9)$
36/18	60	54	23	6	0	5	4	$GF(4)\otimes [Z_9 \text{ or } GF(3)[x]/\langle x^2\rangle]$
36/24a	80	60	43	20	6	1	4	$GF(4)\otimes GF(3)\otimes GF(3)$
36/20	60	56	23	4	0	15	3	$[Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]\otimes GF(9)$
36/24b	72	60	35	12	0	11	3	$[Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]\otimes [Z_9 \text{ or } GF(3)[x]/\langle x^2\rangle]$
36/28a	90	64	53	26	6	7	3	$GF(2)\otimes GF(2)\otimes GF(9)$
36/28b	96	64	59	32	12	7	3	$[Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]\otimes GF(3)\otimes GF(3)$
36/30	108	66	71	42	18	5	3	$GF(2)\otimes GF(2)\otimes [Z_9 \text{ or } GF(3)[x]/\langle x^2\rangle]$
36/32	144	68	107	76	48	3	3	$GF(2)\otimes GF(2)\otimes GF(3)\otimes GF(3)$
35/11	48	46	12	2	0	2	6	$GF(5)\otimes GF(7)$
34/18	54	52	19	2	0	15	3	$GF(2)\otimes GF(17)$
33/13	48	46	14	2	0	8	4	$GF(3)\otimes GF(11)$
32/1	33	33	0	0	0	0	33	$GF(32)$
32/11	45	43	12	2	0	0	5	$GF(4)\otimes GF(8)$
32/16	48	48	15	0	0	15	3	$Z_{32}, GF(2)[x]/\langle x^5\rangle, \ldots$
32/17	51	49	18	2	0	14	3	$GF(2)\otimes GF(16)$
32/18	54	50	21	4	0	13	3	$GF(8)\otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
32/20	60	52	27	8	0	11	3	$GF(4)\otimes Z_8, GF(2)\otimes GF(4)[x]/\langle x^2\rangle, \ldots$
32/23	75	55	42	20	6	8	3	$GF(2)\otimes GF(4)\otimes GF(4)$
32/24	72	56	39	16	0	7	3	$GF(2)\otimes Z_{16}, Z_4\otimes Z_8, \ldots$
32/25	81	57	48	24	6	6	3	$GF(2)\otimes GF(2)\otimes GF(8)$
32/26*	90	58	57	32	12	5	3	$GF(2)\otimes GF(4)\otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
32/28	108	60	75	48	24	3	3	$GF(2)\otimes GF(2)\otimes Z_8, GF(2)\otimes Z_4\otimes Z_4, \ldots$
32/29	135	61	102	74	48	2	3	$GF(2)\otimes GF(2)\otimes GF(2)\otimes GF(4)$
32/30	162	62	129	100	72	1	3	$GF(2)\otimes GF(2)\otimes GF(2)\otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
32/31	243	63	210	180	150	0	3	$GF(2)\otimes GF(2)\otimes GF(2)\otimes GF(2)\otimes GF(2)$
31/1	32	32	0	0	0	0	32	$GF(31)$
30/22	72	52	41	20	6	7	3	$GF(2)\otimes GF(3)\otimes GF(5)$
29/1	30	30	0	0	0	0	30	$GF(29)$
28/10	40	38	11	2	0	3	5	$GF(7)\otimes GF(4)$
28/16	48	44	19	4	0	11	3	$GF(7)\otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
28/22	72	50	43	22	6	5	3	$GF(7)\otimes GF(2)\otimes GF(2)$
27/1	28	28	0	0	0	0	28	$GF(27)$
27/9	36	36	8	0	0	8	4	$Z_{27}, GF(3)[x]/\langle x^3\rangle, \ldots$
27/11	40	38	12	2	0	6	4	$GF(3)\otimes GF(9)$
27/15	48	42	20	6	0	2	4	$GF(3)\otimes [Z_9 \text{ or } GF(3)[x]/\langle x^2\rangle]$
27/19	64	46	36	18	6	0	4	$GF(3)\otimes GF(3)\otimes GF(3)$
26/14	42	40	15	2	0	11	3	$GF(2)\otimes GF(13)$
25/1	26	26	0	0	0	0	26	$GF(25)$
25/5	30	30	4	0	0	4	6	$Z_{25}, GF(5)[x]/\langle x^2\rangle$
25/9	36	34	10	2	0	0	6	$GF(5)\otimes GF(5)$

Line Type	Cardinalities of Points							Representative Rings
	Tot	TpI	1N	∩2N	∩3N	Jcb	MD	
24/10	36	34	11	2	0	5	4	$GF(3) \otimes GF(8)$
24/16	48	40	23	8	0	7	3	$GF(3) \otimes Z_8, GF(3) \otimes GF(2)[x]/\langle x^3 \rangle, \ldots$
24/18	60	42	35	18	6	5	3	$GF(3) \otimes GF(2) \otimes GF(4)$
24/20	72	44	47	28	12	3	3	$GF(3) \otimes GF(2) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
24/22	108	46	83	62	42	1	3	$GF(3) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
23/1	24	24	0	0	0	0	24	$GF(23)$
22/12	36	34	13	2	0	9	3	$GF(2) \otimes GF(11)$
21/9	32	30	10	2	0	4	4	$GF(3) \otimes GF(7)$
20/8	30	28	9	2	0	1	5	$GF(5) \otimes GF(4)$
20/12	36	32	15	4	0	7	3	$GF(5) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
20/16	54	36	33	18	6	3	3	$GF(5) \otimes GF(2) \otimes GF(2)$
19/1	20	20	0	0	0	0	20	$GF(19)$
18/10	30	28	11	2	0	7	3	$GF(2) \otimes GF(9)$
18/12	36	30	17	6	0	5	3	$GF(2) \otimes [Z_9 \text{ or } GF(3)[x]/\langle x^2 \rangle]$
18/14	48	32	29	16	6	3	3	$GF(2) \otimes GF(3) \otimes GF(3)$
17/1	18	18	0	0	0	0	18	$GF(17)$
16/1	17	17	0	0	0	0	17	$GF(16)$
16/4	20	20	3	0	0	3	5	$Z_4[x]/\langle x^2+x+1 \rangle$, $GF(4)[x]/\langle x^2 \rangle$
16/7	25	23	8	2	0	0	5	$GF(4) \otimes GF(4)$
16/8	24	24	7	0	0	7	3	Z_{16}, $Z_4[x]/\langle x^2 \rangle$, $GF(2)[x]/\langle x^4 \rangle$, $\ldots$
16/9	27	25	10	2	0	6	3	$GF(2) \otimes GF(8)$
16/10*	30	26	13	4	0	5	3	$GF(4) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
16/12	36	28	19	8	0	3	3	$Z_4 \otimes Z_4$, $GF(2) \otimes Z_8$, $\ldots$
16/13	45	29	28	16	6	2	3	$GF(2) \otimes GF(2) \otimes GF(4)$
16/14	54	30	37	24	12	1	3	$GF(2) \otimes GF(2) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
16/15	81	31	64	50	36	0	3	$GF(2) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
15/7	24	22	8	2	0	2	4	$GF(3) \otimes GF(5)$
14/8	24	22	9	2	0	5	3	$GF(2) \otimes GF(7)$
13/1	14	14	0	0	0	0	14	$GF(13)$
12/6	20	18	7	2	0	1	4	$GF(3) \otimes GF(4)$
12/8	24	20	11	4	0	3	3	$GF(3) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
12/10	36	22	23	14	6	1	3	$GF(3) \otimes GF(2) \otimes GF(2)$
11/1	12	12	0	0	0	0	12	$GF(11)$
10/6	18	16	7	2	0	3	3	$GF(2) \otimes GF(5)$
9/1	10	10	0	0	0	0	10	$GF(9)$
9/3	12	12	2	0	0	2	4	Z_9, $GF(3)[x]/\langle x^2 \rangle$
9/5	16	14	6	2	0	0	4	$GF(3) \otimes GF(3)$
8/1	9	9	0	0	0	0	9	$GF(8)$
8/4	12	12	3	0	0	3	3	$Z_8, GF(2)[x]/\langle x^3 \rangle, \ldots$
8/5	15	13	6	2	0	2	3	$GF(2) \otimes GF(4)$
8/6	18	14	9	4	0	1	3	$GF(2) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
8/7	27	15	18	12	6	0	3	$GF(2) \otimes GF(2) \otimes GF(2)$
7/1	8	8	0	0	0	0	8	$GF(7)$
6/4	12	10	5	2	0	1	3	$GF(2) \otimes GF(3)$
5/1	6	6	0	0	0	0	6	$GF(5)$
4/1	5	5	0	0	0	0	5	$GF(4)$
4/2	6	6	1	0	0	1	3	Z_4, $GF(2)[x]/\langle x^2 \rangle$
4/3	9	7	4	2	0	0	3	$GF(2) \otimes GF(2)$
3/1	4	4	0	0	0	0	4	$GF(3)$
2/1	3	3	0	0	0	0	3	$GF(2)$

POSSIBLE PHYSICAL APPLICATIONS

There exists a *bijection* between the pairs/vectors (a, b) of the modular ring Z_d and the elements of the generalized Pauli group of the d-dimensional Hilbert space generated by the standard shift (X) and clock (Z) operators, $\omega^c X^a Z^b$.

Under this correspondence, the operators of the group commuting with a given operator form:

- $\hookrightarrow$ the *set-theoretic* union of the points of the projective line over Z_d which contain a given pair (Figure 10) if d is a product of *distinct* primes [11], and
- $\hookrightarrow$ the *span* of the points for any other values of d [12].

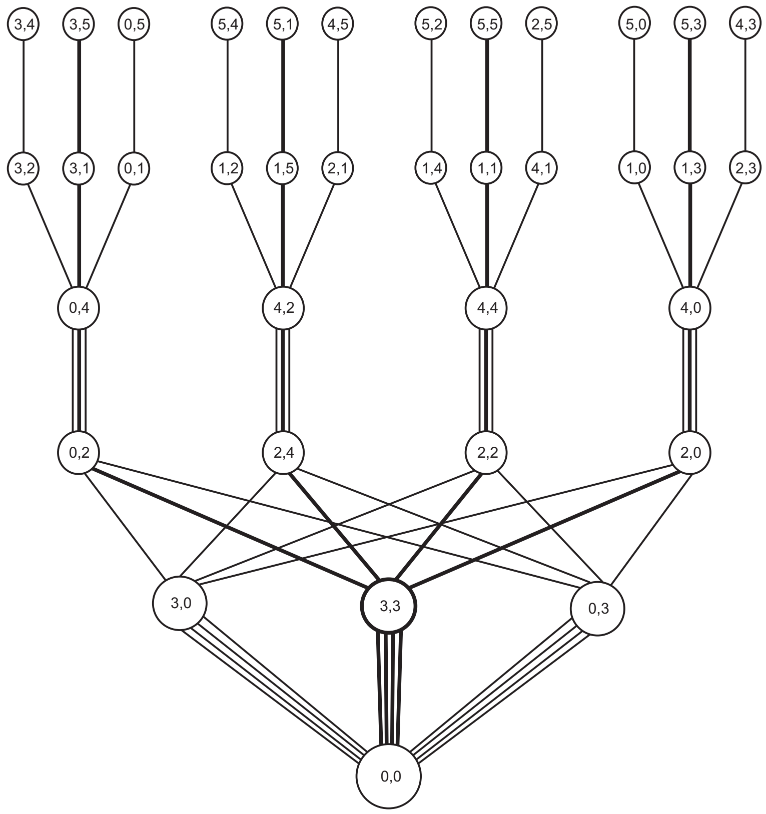


Figure 10: The projective line over $Z_6 \simeq Z_2 \times Z_3$; shown is the set-theoretic union of the points of the pair/vector (3, 3) (highlighted), which comprises all the pairs/vectors joined by heavy line segments.

As yet, there is no general theory for tensorial products of the above-defined operators, but some interesting particular cases have already been computer-analyzed [13].

NEXT MOVE: GEOMETRIES OVER NEARRINGS?

A (left) nearring $N(+,\cdot)$ meets all the axioms of a ring except that the group $N(+)$ is not necessarily commutative and only left distributive law holds, i. e. $x\cdot(y+z) = x\cdot y + x\cdot z$.

Fundamental difference with respect to rings [14]:
$\hookrightarrow$ $x\cdot 0 = 0$ for any x ($x\cdot 0 = x\cdot(0+0) = x\cdot 0 + x\cdot 0 \Rightarrow x\cdot 0 = 0$)
$\hookrightarrow$ whereas, in general, $0\cdot x \neq 0$.

As a group, N is the *semi-direct* product of N_Z and N_C, i. e., N_Z is a normal subgroup of N, $N = N_Z + N_C$ and $N_Z \cap N_C = \{0\}$, where
$N_Z \equiv \{r \in N | 0\cdot r = 0\}$ is called the *zero-symmetric* part of N, and
$N_C \equiv \{r \in N | 0\cdot r = r\}$
$= \{r \in N | \forall x \in N : x\cdot r = r\}$ is called the *constant* part of N.
N is called zero-symmetric if $N = N_Z$.

Let N be a nearring with unity 1; then its group of units factorizes as
$N^* = {N_Z}^*(N_C+1)$ with ${N_Z}^* \cap (N_C+1) = \{1\}$.

Towards defining geometries over N:
$N_l(a,b) = \{(\alpha a, \alpha b) | \alpha \in N\}$ is not a submodule of N^2, for, in general,
$(\alpha a, \alpha b) + (\beta a, \beta b) = (\alpha a + \beta a, \alpha b + \beta b) \neq ((\alpha+\beta)a, (\alpha+\beta)b)$
$N_r(a,b) = \{(a\alpha, b\alpha) | \alpha \in N\}$ is also to be checked if it meets all the requirements (I haven't done it yet).
Yet, irrespectively of these tasks, I shall call both the sets *free cyclic sets* (FCSs) if $|N_l(a,b)| = |N_r(a,b)| = |N|$.

Let's us have a look at what such sets look like for a few simplest cases of finite nearrings in order to get the feeling how beautiful geometries they can generate.

The non-zero-symmetric nearring of a particular type p^n/p^{n-1} (p being a prime and n a positive integer ≥ 2) such that all p^{n-1} non-units[1] are the constant elements of the nearring.

$\bullet N_l(a,b)$ (l.h.s. of the figures, trunks only):

$\hookrightarrow$ their number is always equal to $p^n + p^{n-1}$ and for $n = 2$ they can be regarded as the lines of the *affine* plane over $GF(p)$, $AG(n,p)$;

$\hookrightarrow$ the trunk of the tree consists solely of $p^{2(n-1)}$ pairs of the constant elements; for $n = 2$ these pairs can be viewed as the points of $AG(n,p)$.

$\bullet N_r(a,b)$ (r.h.s. of the figures, full trees):

$\hookrightarrow$ the trunk of the tree features only p^{n-1} "diagonal" pairs of constant elements, i.e., pairs of type (c,c);

$\hookrightarrow$ the number of FCSs equals the number of pairs in the corolla divided by the number of units in the nearring, viz. $(p^{2n} - p^{n-1})/(p^n - p^{n-1}) = (p^{n+1} - 1)(p - 1) = p^n + p^{n-1} + \cdots + p + 1$, that is to the number of points of the ordinary n-dimensional *projective* space over $GF(p)$, $PG(n,p)$; in fact, "condensing" p^{n-1} pairs into a single pair in an obvious way we get the tree of $PG(n,p)$;

$\hookrightarrow$ out of them, there are $p^{n-2} + p^{n-3} + \cdots + p + 1$ such which consist solely of "off-diagonal" pairs of constant elements (these are represented by bullets in accompanying figures).

[1] I deliberately avoid using here the term "zero-divisor" because in the case of a non-zero-symmetric nearring a unit can also be a (one-sided) zero-divisor.

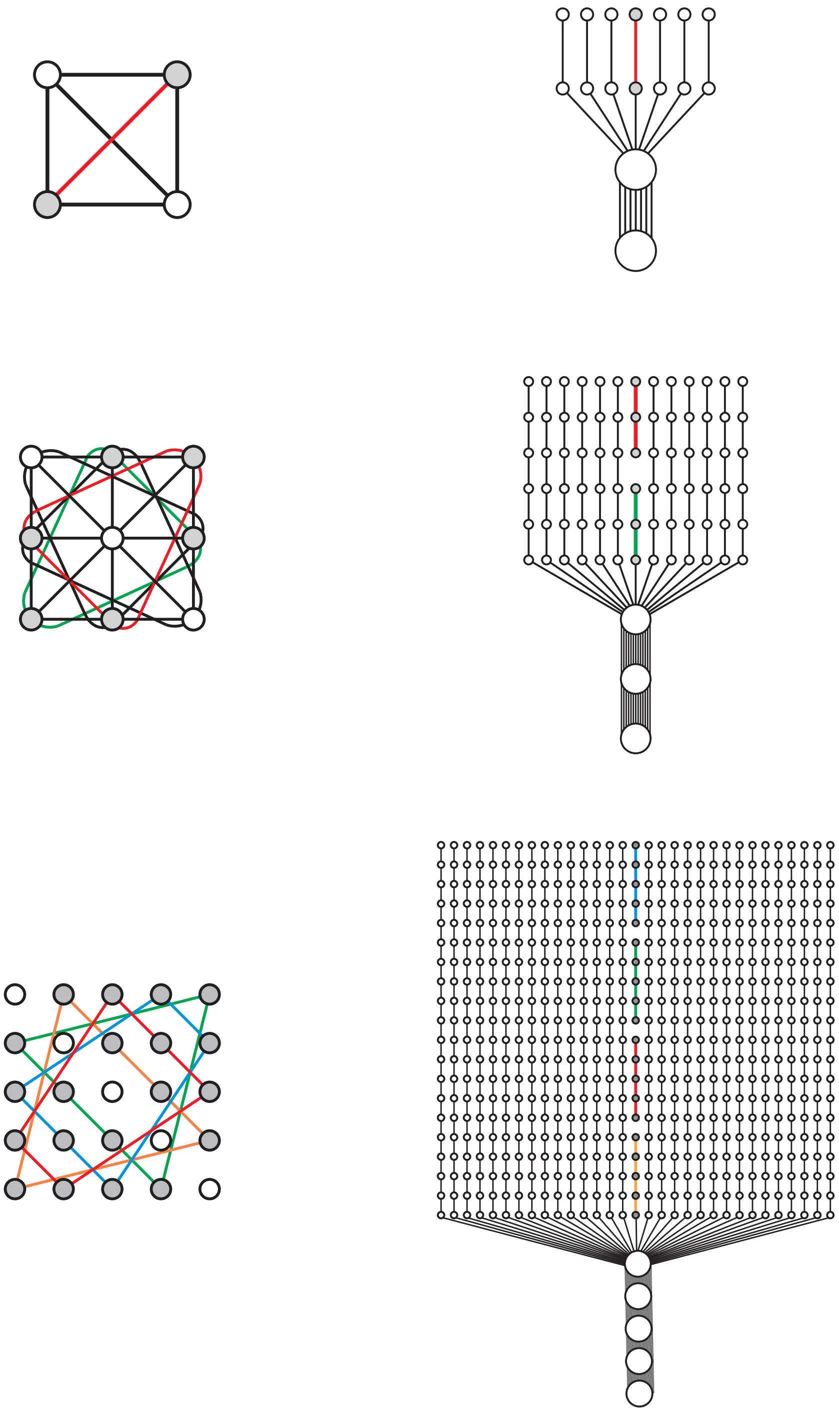


Figure 11: $n = 2$ and $p = 2$ (top), $p = 3$ (middle) and $p = 5$ (bottom)

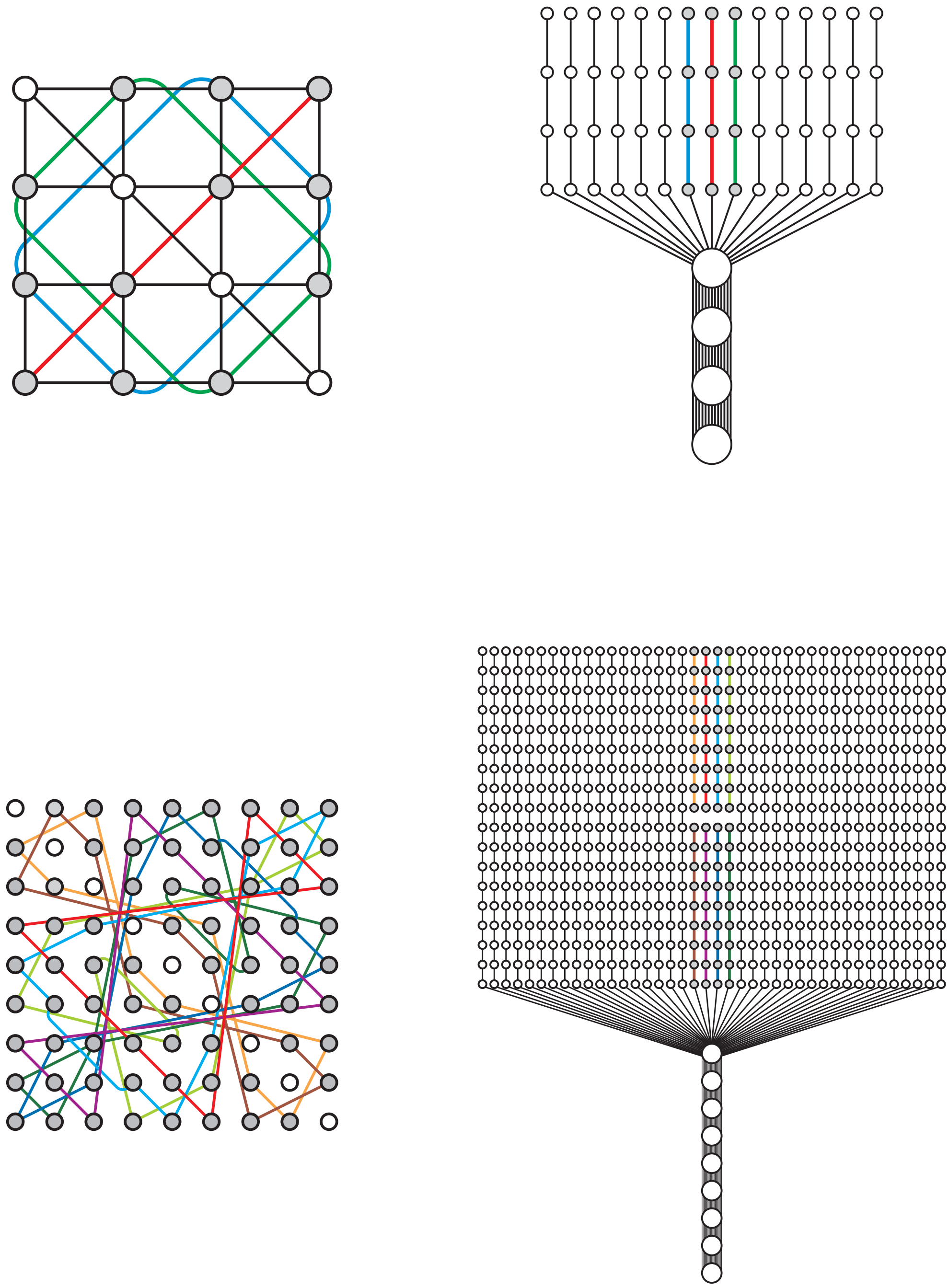


Figure 12: $n = 3$ and $p = 2$ (top) and $p = 3$ (bottom)

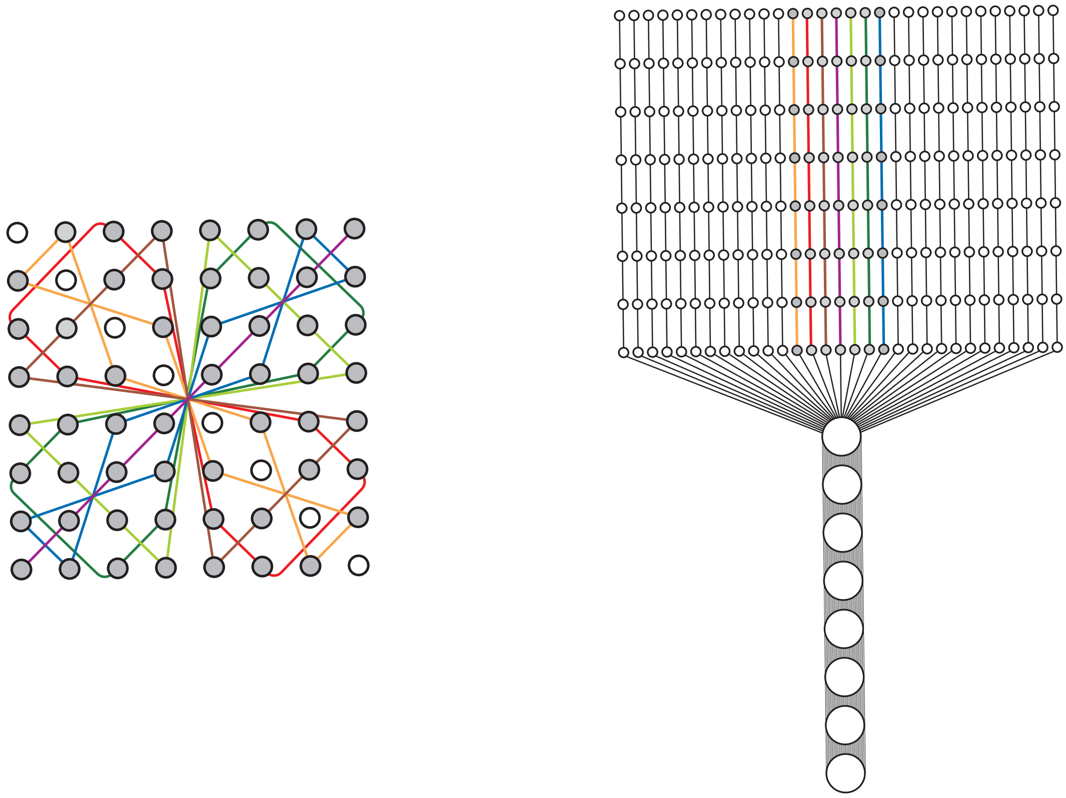


Figure 13: $n = 4$ and $p = 2$

$N_l(a,b)$ — Full trees over some small non-zero-symmetric nearrings; notice that each consists of four copies of the tree over the zero-symmetric part of the nearring in question and that exactly half of FCSs **do not pass through the $(0,0)$ pair**!!!

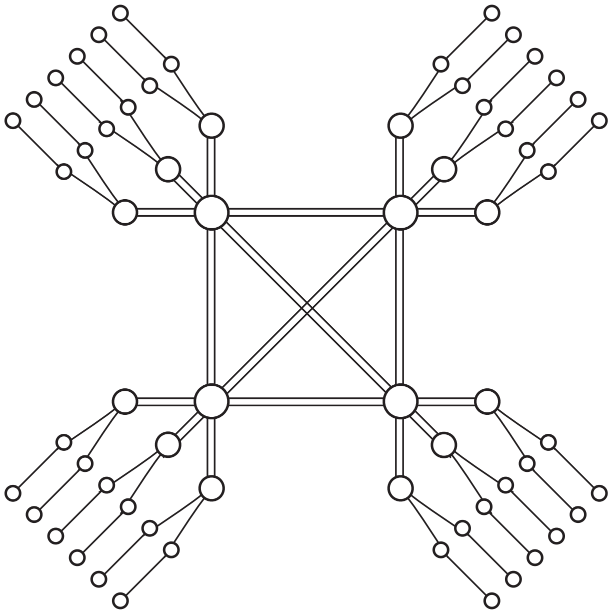


Figure 14: 8/4: $N_Z \simeq Z_4$

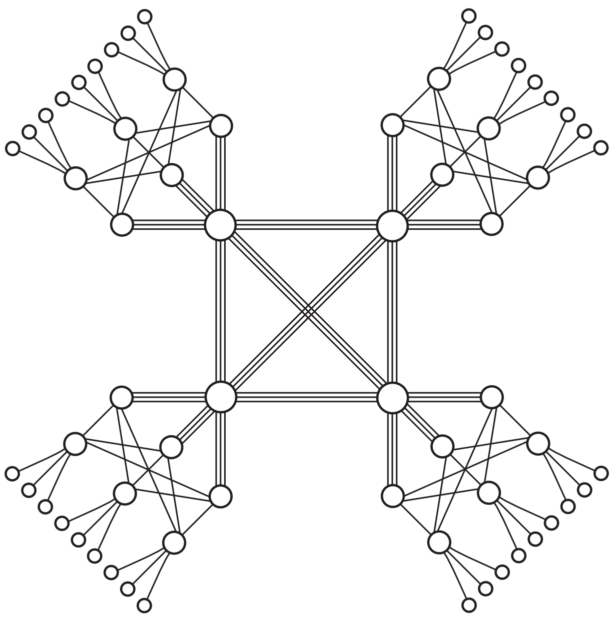


Figure 15: 8/6a: $N_Z \simeq Z_2 \times Z_2$

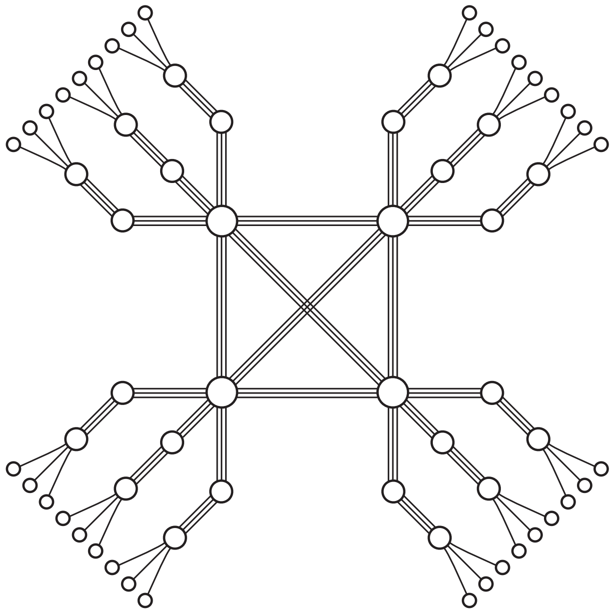


Figure 16: 8/6b: $N_Z \simeq$ proper nearring of 4/3 type

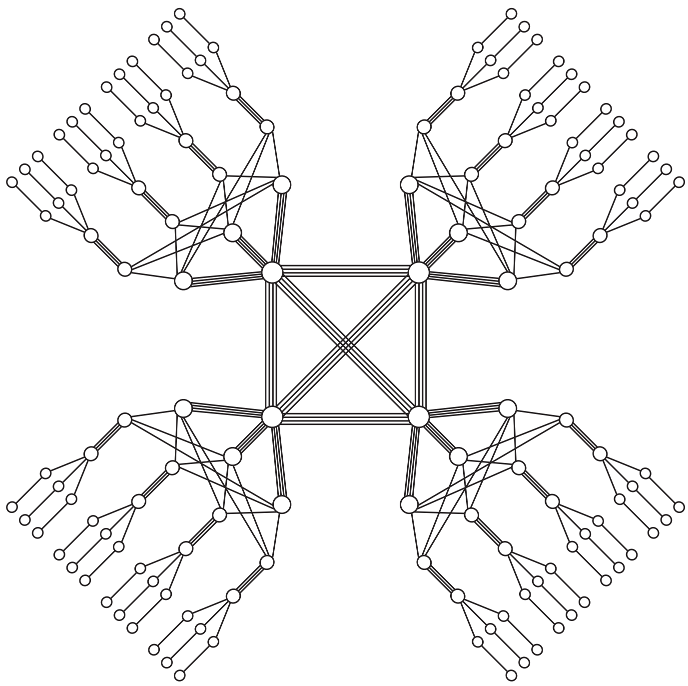


Figure 17: 12/8: $N_Z \simeq Z_6$

Last example — non-abelian, non-zero-symmetric nearring of 16/8 type; note that the left tree features four copies of the projective line over Z_8, whilst the right tree (of which only trunk is shown) comprises two copies of what seems not to be any line over (near)rings of order eight.

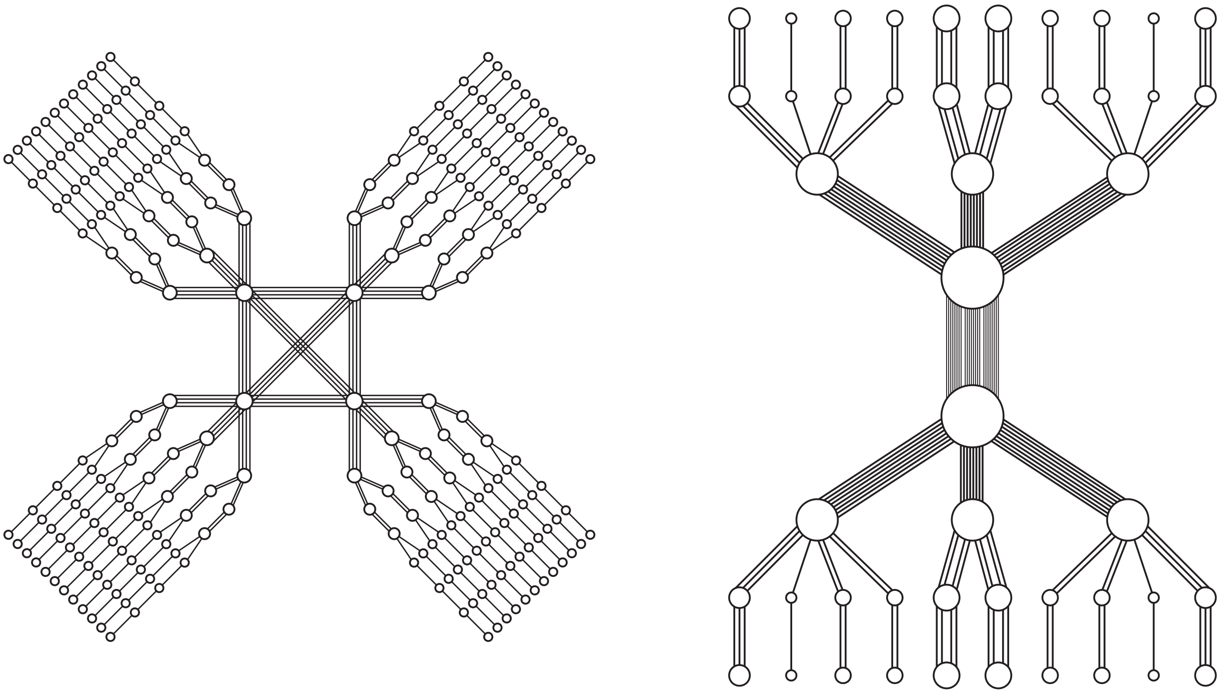


Figure 18: 16/8: non-abelian, $N_Z \simeq Z_8$

Interesting Fact:

If the cardinality of a nearring with identity is the product of distinct primes, then this nearring is a ring (Maxson, 1967)

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