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D I P L O M A R B E I T

Hayman Admissible Functions and Generalisations with Applications in Combinatorics

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Contents

Introduction

An important and powerful technique for solving combinatorial enumeration problems is the generating function approach. A generating function is a formal power series in one or more indeterminates where the coefficients equal the number of objects in a given combinatorial class having certain characteristics (e.g., the coefficient of z^n in $B(z)$ defined by $B(z) = 1 + zB(z)^2$ equals the number of binary trees having n internal nodes). The generating function approach can be summarised as follows. Using combinatorial arguments we derive bijections between or decompositions of certain sets of composite combinatorial structures. These bijections are then reduced to functional relationships between formal power series. Our enumeration problem can now be solved by extracting the coefficients of the formal power series occurring. In the case of simple relationships between known power series we can immediately find the solution to our problem and in some other cases we can apply the Inversion Theorem of Lagrange. But most often the situation is not so easy and we have to resort to other methods.

In Chapter 1 we give a short overview on how to transform certain given types of decompositions into relations between formal power series. For a detailed discussion of this step we refer to the books of Goulden and Jackson [GJ04], Wilf [Wil90] and Flajolet and Sedgewick [FS].

Fortunately many formal power series occurring in combinatorial enumeration can be identified with analytic functions. Hence we can use methods from complex analysis for extracting the coefficients. This fact also makes it possible to find asymptotic expressions for the coefficients since we can express them by means of complex contour integrals and asymptotically evaluate these integrals. In Chapter 2 we discuss two methods yielding asymptotic expressions for the coefficients in question, namely singularity analysis and the saddlepoint method. The third method presented in this chapter is the analytic version of Lagrange's Inversion Theorem.

The central analytic method in this work is the saddlepoint method. It can be successfully applied to functions which are large for positive real arguments and satisfy sufficient decay conditions for nonreal arguments. Hayman [Hay56] defined classes of analytic functions which satisfy all requirements necessary for successfully applying the saddlepoint method and proved an asymptotic expression for the coefficients of such functions. In accordance to the literature these functions will be called H-admissible functions. In his work Hayman also proved certain closure properties satisfied by the classes of H-admissible functions. And in view of the decompositions mentioned above it is exactly the existence of these closure properties which makes Hayman's concept a very comfortable tool for combinatorial enumeration.

A detailed presentation of Hayman's results is given in Chapter 3.

In Chapter 4 we discuss some generalisations of Hayman's work. Harris and Schoenfeld [HS68] tightened Hayman's conditions and obtained complete asymptotic expansions for the coefficients but did not provide any closure properties. Some simple closure properties have later been provided by Odlyzko and Richmond [OR85] and Müller [Mül97].

Mutafchiev [Mut92] proposed a univariate generalisation of H-admissibility, called GH-admissibility, where the asymptotics required by H-admissibility are replaced with weaker conditions. Mutafchiev's goal was a concept that can be used to establish local limit theorems in combinatorial classes. Unfortunately none of the examples presented in his paper [Mut92] constitute valid applications of GH-admissibility as will be shown in this work. Valid applications of this concept can be found in [Mut97] where Mutafchiev proved some weak convergence results for the number of distinct component sizes.

We also present two multivariate extensions of Hayman's work which give answer to questions dealing with the distribution of some parameters on combinatorial classes (e.g. the distribution of the number of classes of a partition of a set of size n as $n \to \infty$). Bender and Richmond [BR96] presented a multivariate generalisation which yields local limit theorems for the parameters considered. They also proved some closure properties satisfied by their classes. In this work we present an additional simple closure property which seems to be new (see Section 4.3.1). Drmota, Gittenberger, and Klausner [DGK05] stated a concept for bivariate functions in the spirit of Hayman's concept and obtained central limit theorems for the parameter considered. An important fact to note on this last concept is the existence of many simple algebraic closure properties. This makes their concept (besides Hayman's concept itself) the only concept amenable to automated membership testing.

In Chapter 5 we apply the methods of Chapters 3 and 4 to some combinatorial problems concerning the number of components a randomly chosen combinatorial structure consists of. Some general remarks on this class of combinatorial problems can be found in [BBCR00] and [BCOR99].

Besides Canfield's [Can77] results, we present some examples taken from [BR96], [DGK05], and [GJ04]. The examples are chosen such as to show applicability as well as limitations of these methods.

Chapter 1

Generating Functions

Combinatorial structures consist of a finite set of atoms together with some relations between them (e.g. graphs consist of nodes that are related to others via edges). In some cases all atoms are considered equal while in others they are considered distinguishable by attached labels. In the former case the structures are called unlabelled combinatorial structures while in the latter case they are called labelled combinatorial structures. It proves convenient to use ordinary generating functions (ogf) in the unlabelled case and exponential generating functions (egf) in the labelled case.

The decompositions of combinatorial structures considered here can all be reduced to a number of disjoint-sum-operations and product-operations. While the disjoint-sum is essentially the same for labelled and unlabelled structures the product is a different one in these cases. The reason for this is that in the labelled case we have to consider all possible distributions of the set of labels over the factors.

In this chapter we show how to reduce given decompositions of the type described above to functional relationships between generating functions. This step is also known as "the symbolic method" and is extensively discussed in [FS], [GJ04] and [Wil90] using a very different notation. We have adopted the notation of [FS].

In Sections 1 and 2 we present the symbolic method for unlabelled and labelled combinatorial structures assuming that we are only interested in the total number of structures of a given size. If we want to keep track of more than one parameter we have to use multivariate generating functions (mgf). In this situation we can use a simple modification of the symbolic method described in the first two sections which is presented in Section 3. The last section contains some definitions concerning limiting distributions needed in later chapters.

All power series considered in this chapter will be treated as formal power series and all operations are performed in the ring of formal power series (see [GJ04] for necessary definitions).

1.1 Unlabelled Constructions

An example of a class of unlabelled structures is the class of all binary trees where the nodes in each tree are indistinguishable. The size of a tree can for example be defined as the number of its internal nodes. This example is an instance of

Definition 1.1. *A pair* $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ *is called an* unlabelled combinatorial class *if and only if*

(i) $|\cdot|_{\mathcal{A}}$ *is a function* $|\cdot|_{\mathcal{A}} : \mathcal{A} \to \mathbb{N}$ *and*

(ii) for each $n \in \mathbb{N}$ *the set* $\{\alpha \in \mathcal{A} |\, |\alpha|_{\mathcal{A}} = n\}$ *is finite.*

For each $\alpha \in \mathcal{A}$ *the nonnegative integer* $|\alpha|_{\mathcal{A}}$ *is called the* size of α. *The sequence* $(\mathrm{card}\{\alpha \in \mathcal{A} |\, |\alpha|_{\mathcal{A}} = n\})_{n\in\mathbb{N}}$ *is called the* counting sequence *of* $(\mathcal{A}, |\cdot|_{\mathcal{A}})$.

Remark. *As a consequence of (i) and (ii), the set* $\mathcal{A}$ *is at most denumerable.*

Remark. *The following naming convention will be adopted: If the unlabelled combinatorial class is called* $(\mathcal{A}, |\cdot|_{\mathcal{A}})$, *then its counting sequence is denoted by* $(a_n)_{n\in\mathbb{N}}$ *and the corresponding ordinary generating function is denoted by* $a(z) = \sum_{n\geq 0} a_n z^n$ *(analogous for* $(\mathcal{B}, |\cdot|_{\mathcal{B}})$, $(b_n)_{n\in\mathbb{N}}$ *and* $b(z)$*).*

Definition 1.2. *Two unlabelled combinatorial classes* $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ *and* $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ *are said to be isomorphic if and only if their counting sequences are identical:*

$$(\mathcal{A}, |\cdot|_{\mathcal{A}}) \cong (\mathcal{B}, |\cdot|_{\mathcal{B}}) \iff (a_n)_{n\in\mathbb{N}} = (b_n)_{n\in\mathbb{N}}.$$

Definition 1.3. *For a given unlabelled combinatorial class* $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ *, the subclass consisting of all elements of size* $\leq n$, $n \in \mathbb{N}$, *is denoted by* $(\mathcal{A}, |\cdot|_{\mathcal{A}})^{[n]} = \left(\mathcal{A}^{[n]}, |\cdot|_{\mathcal{A}^{[n]}}\right)$:

$$\alpha \in \mathcal{A}^{[n]} \iff \alpha \in \mathcal{A} \ \wedge \ |\alpha|_{\mathcal{A}} \leq n.$$

Definition 1.4. *Given two unlabelled combinatorial classes* $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ *and* $(\mathcal{B}, |\cdot|_{\mathcal{B}})$, *the Cartesian product* $(\mathcal{C}, |\cdot|_{\mathcal{C}}) = (\mathcal{A}, |\cdot|_{\mathcal{A}}) \times (\mathcal{B}, |\cdot|_{\mathcal{B}})$ *is defined as the class*

$$\mathcal{C} = \mathcal{A} \times \mathcal{B} \qquad \forall(\alpha, \beta) \in \mathcal{A} \times \mathcal{B} : \ |(\alpha, \beta)|_{\mathcal{C}} = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}\,.$$

The Cartesian product constitutes a combinatorial construction since the resulting set is an unlabelled combinatorial class. It is associative in the sense that (for any possible placement of the parenthesis) all resulting classes are isomorphic.

For $n \geq 1$, the n-th power of a class $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ (Cartesian product of n copies of $(\mathcal{A}, |\cdot|_{\mathcal{A}})$) is denoted by $(\mathcal{A}, |\cdot|_{\mathcal{A}})^n = (\mathcal{A}^n, |\cdot|_{\mathcal{A}^n})$. The 0-th power is defined as the class consisting of one structure of size 0.

Definition 1.5. *Given two unlabelled combinatorial classes* $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ *and* $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ *and two different structures* ϵ_1 *and* ϵ_2 *of size* 0, *the disjoint sum* $(\mathcal{D}, |\cdot|_{\mathcal{D}}) = (\mathcal{A}, |\cdot|_{\mathcal{A}}) + (\mathcal{B}, |\cdot|_{\mathcal{B}})$ *is defined as the class*

$$\mathcal{D} = \left(\bigcup_{\alpha\in\mathcal{A}} \{\epsilon_1\} \times \{\alpha\}\right) \cup \left(\bigcup_{\beta\in\mathcal{B}} \{\epsilon_2\} \times \{\beta\}\right).$$

The size of the objects remains unchanged.

The disjoint sum is an associative combinatorial construction in the same sense as the Cartesian product.

Theorem 1.1. *Let $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ and and $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ denote two unlabelled combinatorial classes. Then the following holds:*

(i) $(\mathcal{C}, |\cdot|_{\mathcal{C}}) \cong (\mathcal{A}, |\cdot|_{\mathcal{A}}) \times (\mathcal{B}, |\cdot|_{\mathcal{B}})$ *if and only if* $c(z) = a(z)b(z)$.

(ii) $(\mathcal{D}, |\cdot|_{\mathcal{D}}) \cong (\mathcal{A}, |\cdot|_{\mathcal{A}}) + (\mathcal{B}, |\cdot|_{\mathcal{B}})$ *if and only if* $d(z) = a(z) + b(z)$.

Proof. (i) For any $n \in \mathbb{N}$, the number of structures of size n in $(\mathcal{A}, |\cdot|_{\mathcal{A}}) \times (\mathcal{B}, |\cdot|_{\mathcal{B}})$ is given by

$$c_n = \sum_{n_1+n_2=n} a_{n_1} b_{n_2}$$

which is equal to $[z^n]\big(a(z)b(z)\big)$.

(ii) Any object of size n in $(\mathcal{A}, |\cdot|_{\mathcal{A}}) + (\mathcal{B}, |\cdot|_{\mathcal{B}})$ has either the form (ϵ_1, α) or (ϵ_2, β) where $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ are structures of size n. Thus, the total number of elements of size n in $(\mathcal{A}, |\cdot|_{\mathcal{A}}) + (\mathcal{B}, |\cdot|_{\mathcal{B}})$ is given by $a_n + b_n$ which is equal to $[z^n]\big(a(z) + b(z)\big)$.

□

The class of all finite sequences of a given unlabelled combinatorial class $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ satisfies Definition 1.1 if and only if $\mathcal{A}$ does not contain any structures of size 0. In case of existence, this structure is denoted by $\mathrm{seq}\big((\mathcal{A}, |\cdot|_{\mathcal{A}})\big)$.

Under the same restriction, the class of all finite subsets and the class of all finite multisets of $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ exist as unlabelled combinatorial classes and are denoted by $\mathrm{set}\big((\mathcal{A}, |\cdot|_{\mathcal{A}})\big)$ and $\mathrm{multiset}\big((\mathcal{A}, |\cdot|_{\mathcal{A}})\big)$, respectively.

As a consequence of the last theorem, one gets

Theorem 1.2. *Let $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ denote an unlabelled combinatorial class not containing any objects of size 0. Then the following holds*

(i) $(\mathcal{B}, |\cdot|_{\mathcal{B}}) \cong \mathrm{seq}\big((\mathcal{A}, |\cdot|_{\mathcal{A}})\big)$ *if and only if*

$$b(z) = \sum_{n\in\mathbb{N}} \big(a(z)\big)^n = \frac{1}{1-a(z)}.$$

(ii) $(\mathcal{C}, |\cdot|_{\mathcal{C}}) \cong \mathrm{set}\big((\mathcal{A}, |\cdot|_{\mathcal{A}})\big)$ *if and only if*

$$c(z) = \exp\left(\sum_{k\geq 1} (-1)^{k+1} \frac{a(z^k)}{k}\right).$$

(iii) $(\mathcal{D}, |\cdot|_{\mathcal{D}}) \cong \mathrm{multiset}\big((\mathcal{A}, |\cdot|_{\mathcal{A}})\big)$ *if and only if*

$$d(z) = \exp\left(\sum_{k\geq 1} \frac{a(z^k)}{k}\right).$$

Proof. (i) Since all elements of $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ have size ≥ 1, all sequences of length $k \geq 0$ of elements of $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ have size $\geq k$. Therefore $(\mathcal{B}, |\cdot|_{\mathcal{B}}) \cong \text{seq}(\,(\mathcal{A}, |\cdot|_{\mathcal{A}})\,)$ implies for any $m \geq 0$

$$[z^m]b(z) \quad = \quad [z^m]\sum_{n=0}^{m} a(z)^n \quad = \quad [z^m]\frac{1}{1-a(z)}.$$

(ii) Sets of elements of $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ of size $k \geq 0$ have cardinality of at most k. Let ϵ denote a structure of size 0. The class of all finite subsets of $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ satisfies

$$\left(\text{set}\,(\mathcal{A}, |\cdot|_{\mathcal{A}})\,\right)^{[k]} \cong \left(\prod_{\alpha \in \mathcal{A}^{[k]}} (\{\epsilon\} \times \{\alpha\}, |\cdot|\right)^{[k]}$$

for all $k \geq 0$. Therefore $(\mathcal{C}, |\cdot|_{\mathcal{C}}) \cong \text{set}(\,(\mathcal{A}, |\cdot|_{\mathcal{A}})\,)$ implies for all $m \geq 0$

$$\begin{aligned}
[z^m]c(z) \quad &= \quad [z^m]\prod_{n=0}^{m}(1+z^n)^{a_n} \quad = \quad [z^m]\exp\left(\sum_{n=0}^{m} a_n \ln\left(1+z^n\right)\right) \\
&= \quad [z^m]\exp\left(\sum_{j\geq 1}\sum_{n=0}^{m} a_n \frac{z^{jn}}{j}(-1)^{j+1}\right) \quad = \\
&\overset{a_0=0}{=} \quad [z^m]\exp\left(\sum_{j\geq 1}\sum_{n=1}^{\infty} a_n \frac{z^{jn}}{j}(-1)^{j+1}\right) \quad = \\
&= \quad [z^m]\exp\left(\sum_{j\geq 1}(-1)^{j+1}\frac{a(z^j)}{j}\right).
\end{aligned}$$

(iii) The class of all multisets of $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ satisfies

$$\left(\text{multiset}\,(\mathcal{A}, |\cdot|_{\mathcal{A}})\,\right)^{[k]} \cong \left(\prod_{\alpha \in \mathcal{A}^{[k]}} \text{seq}(\{\alpha\}, |\cdot|)\right)^{[k]}$$

for all $k \geq 0$. Therefore $(\mathcal{D}, |\cdot|_{\mathcal{D}}) \cong \text{multiset}\,(\mathcal{A}, |\cdot|_{\mathcal{A}})$ implies for all $m \geq 0$

$$\begin{aligned}
[z^m]d(z) \quad &= \quad [z^m]\prod_{n=0}^{m}(1-z^n)^{-a_n} \quad = \quad [z^m]\exp\left(-\sum_{n=1}^{m} a_n \log(1-z^n)\right) \\
&= \quad [z^m]\exp\left(\sum_{n\geq 1}\frac{a^{\leq m}(z^n)}{n}\right) \quad = \quad [z^m]\exp\left(\sum_{n\geq 1}\frac{a(z^n)}{n}\right).
\end{aligned}$$

□

The next example demonstrates the application of the results in this section. Once the class of interest is defined in terms of simple classes combined using Cartesian products and disjoint sums, the corresponding counting sequence is obtained in a rather mechanical way.

Example 1. *The class* $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ *of binary trees. A binary tree is either an external node or it consists of an internal node with two binary trees (the left and the right subtree) attached. The size of a tree is defined as the number of internal nodes.*

This can be formalised using the class $(\mathcal{E}, |\cdot|_{\mathcal{E}})$ *consisting of one structure, the external node, of size* 0 *and the class* $(\mathcal{N}, |\cdot|_{\mathcal{N}})$ *consisting of one structure, the internal node, of size* 1. *The class of all binary trees is then given by*

$$(\mathcal{B}, |\cdot|_{\mathcal{B}}) \cong (\mathcal{E}, |\cdot|_{\mathcal{E}}) + \big((\mathcal{N}, |\cdot|_{\mathcal{N}}) \times (\mathcal{B}, |\cdot|_{\mathcal{B}}) \times (\mathcal{B}, |\cdot|_{\mathcal{B}}) \big).$$

Since $E(z) = 1$ *and* $N(z) = z$, *the ogf* $B(z)$ *of the counting sequence of the class of binary trees satisfies*

$$B(z) = 1 + zB(z)^2.$$

One root of the equation above involves a negative power of z. *Therefore,* $B(z)$ *is uniquely determined by*

$$\begin{aligned} B(z) &= \frac{1-\sqrt{1-4z}}{2z} = -\frac{1}{2z}\sum_{j\geq 1}\binom{1/2}{j}(-4z)^j = \\ &= \frac{1}{2z}\sum_{j\geq 1}\frac{2}{j}\binom{2(j-1)}{j-1}z^j = \sum_{j\geq 0}\frac{1}{j+1}\binom{2j}{j}z^j. \end{aligned}$$

Thus, the number b_n *of binary trees of size* n *is equal to* $\frac{1}{n+1}\binom{2n}{n}$.

1.2 Labelled Constructions

This section deals with structures consisting of a finite number of different atoms each of which bears a label different from all others. Examples are labelled graphs (or trees) and permutations. For simplicity, all labels are assumed to be integers.

Since the disjoint sum and the labelled product (as defined below) do not depend on the way the labels are connected, the following definition can be used:

Definition 1.6. *A pair* (S, f) *is called* labelled combinatorial structure of size n, $n \in \mathbb{N}$, *if and only if*

(i) S *is a set of cardinality* n *(the set of atoms) and*

(ii) $f : S \to \mathbb{Z}$ *is an injective function (the labelling).*

For each $a \in S$, *the number* $f(a)$ *is called the* label of a. *The size of* (S, f) *is denoted by* $|(S, f)|$.

(S, f) *is called* well labelled *if and only if* $f(S) = \{0, 1, \dots n-1\}$.

Definition 1.7. *Let* (S, f) *be a labelled combinatorial structure and let* $g : S \to \mathbb{Z}$ *be an injective function. Then the pair* (S, g) *is a labelled combinatorial structure, too, and the function* $g \circ f^{-1}$ *is called* relabelling.

The relabelling $g \circ f^{-1} : \mathbb{Z} \to \mathbb{Z}$ *is called* admissible *if and only if it is order preserving.*

Remark 1.1. *For any labelled combinatorial structure* (S, f) *and a set* $L \subseteq \mathbb{N}$ *of cardinality* $|(S, f)|$ *there exists exactly one function* $g : S \to L$ *such that* $g \circ f^{-1}$ *is an admissible relabelling.*

Definition 1.8. *A set* $\mathfrak{C}$ *is called a* labelled combinatorial class *if and only if*

(i) $\mathfrak{C}$ *is a set consisting of well labelled combinatorial structures only and*

(ii) for every $n \in \mathbb{N}$ *the set* $\mathfrak{C}_n = \{\gamma \in \mathfrak{C} | |\gamma| = n\}$ *is finite.*

The sequence $(c_n)_{n\in\mathbb{N}}$ *satisfying* $c_n = \text{card}(\mathfrak{C}_n)$ *is called the* counting sequence of $\mathfrak{C}$.

Remark 1.2. *For any sequence* $(a_n)_{n\in\mathbb{N}}$ *the corresponding* exponential generating function *is denoted by* $\hat{a}(z) = \sum a_n z^n/n!$.

As in the last section, two labelled combinatorial classes $\mathfrak{C}$ and $\mathfrak{D}$ are said to be isomorphic, $\mathfrak{C} \cong \mathfrak{D}$, if and only if their counting sequences are identical.

The *disjoint sum* can be defined similar as in the case of unlabelled structures:

$$\mathfrak{C} + \mathfrak{D} \cong \left(\mathfrak{C} \star \{\epsilon_0\}\right) \cup \left(\mathfrak{D} \star \{\epsilon_1\}\right) \tag{1.1}$$

where ϵ_0 and ϵ_1 denote two different structures of size 0.

Definition 1.9. *Let* $\alpha = (S_\alpha, f_\alpha)$ *and* $\beta = (S_\beta, f_\beta)$ *denote two labelled structures where* $S_\alpha \cap S_\beta = \emptyset$.

Then $\alpha \star \beta$ *is defined as the set of well labelled structures satisfying*

$$(S_\alpha \cup S_\beta, g) \in \alpha \star \beta \iff g \circ f_\alpha^{-1} \text{ and } g \circ f_\beta^{-1} \text{ are admissible relabellings.}$$

The set $\alpha \star \beta$ *is called the* labelled product of α and β.

As a consequence of Remark 1.1 the function g in the last definition is uniquely defined if $g(S_\alpha) \subseteq \{0, 1, \ldots, |\alpha| + |\beta| - 1\}$ is known. This set can be chosen in $\binom{|\alpha|+|\beta|}{|\alpha|}$ ways, thus

$$\text{card}(\alpha \star \beta) = \binom{|\alpha| + |\beta|}{|\alpha|}. \tag{1.2}$$

The labelled product constitutes an associative and commutative operation. Equation (1.2) can be extended to $\text{card}(\alpha_1 \star \alpha_2 \star \cdots \star \alpha_m) = \binom{|\alpha_1|+|\alpha_2|+\cdots+|\alpha_m|}{|\alpha_1|,|\alpha_2|,\ldots,|\alpha_m|}$.

The labelled product of two labelled combinatorial classes $\mathfrak{A}$ and $\mathfrak{B}$ is defined as

$$\mathfrak{A} \star \mathfrak{B} = \bigcup_{\alpha\in\mathfrak{A}} \bigcup_{\beta\in\mathfrak{B}} \alpha \star \beta.$$

Theorem 1.3. *Let* $\mathfrak{A}$,$\mathfrak{B}$ *and* $\mathfrak{C}$ *denote labelled combinatorial classes with corresponding counting sequences* $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ *and* $(c_n)_{n\in\mathbb{N}}$. *Then the following holds:*

(i) $\mathfrak{C} \cong \mathfrak{A} + \mathfrak{B}$ *if and only if* $\hat{c}(z) = \hat{a}(z) + \hat{b}(z)$.

(ii) $\mathfrak{C} \cong \mathfrak{A} \star \mathfrak{B}$ *if and only if* $\hat{c}(z) = \hat{a}(z)\hat{b}(z)$.

Proof. (i) This is true since the counting sequences satisfy

$$(c_n)_{n\in\mathbb{N}} = (a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}.$$

(ii) From (1.2) it follows that the number of elements of size n in $\mathfrak{A} \star \mathfrak{B}$ is given by $\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ which is equal to $[z^n/n!]\hat{a}(z)\hat{b}(z)$:

$$\hat{a}(z)\hat{b}(z) = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{a_k}{k!}\frac{b_{n-k}}{(n-k)!}z^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}\right)\frac{z^n}{n!}.$$

□

For any labelled combinatorial class $\mathfrak{A}$ containing no structure of size 0 the sets of all finite sequences and finite sets of $\mathfrak{A}$ exist as labelled combinatorial classes and are denoted by $\mathsf{seq}(\mathfrak{A})$ and $\mathsf{set}(\mathfrak{A})$, respectively.

Let ϵ denote the empty sequence. The class of all cycles of elements of $\mathfrak{A}$ is denoted by $\mathsf{cyc}(\mathfrak{A})$ and is defined as

$$\mathsf{cyc}(\mathfrak{A}) \cong \big(\mathsf{seq}(\mathfrak{A}) - \{\epsilon\}\big)/_{\equiv} \tag{1.3}$$

where $\beta \equiv \gamma$, $\beta, \gamma \in \big(\mathsf{seq}(\mathfrak{A}) - \{\epsilon\}\big)$, if and only if β can be transformed into γ using a cyclic shift.

Theorem 1.4. *Let $\mathfrak{A}$ denote a labelled combinatorial class that contains no element of size 0. Then the following holds*

(i) $\mathfrak{B} \cong \mathsf{seq}(\mathfrak{A})$ *if and only if*

$$\hat{b}(z) = \frac{1}{1-\hat{a}(z)}$$

(ii) $\mathfrak{C} \cong \mathsf{cyc}(\mathfrak{A})$ *if and only if*

$$\hat{c}(z) = \log\left(\frac{1}{1-\hat{a}(z)}\right)$$

(iii) $\mathfrak{D} \cong \mathsf{set}(\mathfrak{A})$ *if and only if*

$$\hat{d}(z) = \exp\left(\hat{a}(z)\right)$$

Proof. (i) All elements of $\mathsf{seq}(\mathfrak{A})$ of size $\leq k$ must be sequences of length $\leq k$ since $\mathfrak{A}$ does only contain elements of size ≥ 1. Therefore $\mathfrak{B} \cong \mathsf{seq}(\mathfrak{A})$ implies for all $m \geq 0$

$$[z^m/m!]\hat{b}(z) \quad = \quad [z^m/m!]\sum_{k=0}^{m}\hat{a}(z)^k \quad = \quad [z^m/m!]\frac{1}{1-\hat{a}(z)}.$$

(ii) Every sequence of length $k \geq 1$ of elements of $\mathfrak{A}$ consists of k different components since their set of labels are pairwise disjoint. Therefore each cycle of length $k \geq 1$ can be associated with exactly k different sequences (of length k). Hence $\mathfrak{C} \cong \mathsf{cyc}(\mathfrak{A})$ implies for all $m \geq 0$

$$[z^m/m!]\hat{c}(z) \quad = \quad [z^m/m!]\sum_{n=1}^{m}\frac{\hat{a}(z)^n}{n} \quad = \quad [z^m/m!]\ln\frac{1}{1-\hat{a}(z)}.$$

(iii) Every set of size $k \geq 0$ can be arranged in $k!$ ways and thus can be associated with $k!$ sequences of length k. Therefore, $\mathfrak{D} \cong \mathsf{set}(\mathfrak{A})$ implies for all $m \geq 0$

$$[z^m/m!]\hat{d}(z) \quad = \quad [z^m/m!]\sum_{n=0}^{m}\frac{\hat{a}(z)^n}{n!} \quad = \quad [z^m/m!]\exp\left(\hat{a}(z)\right).$$

□

Example 2. *The Bell numbers. The n-th Bell number b_n, $n \geq 0$, is defined as the number of partitions of a set of cardinality n ($b_0 = 1$).*

Since the interesting parameter of a given set is its cardinality, the size of the set, viewed as labelled structure, should be equal to its cardinality. Thus in this context the class $\mathfrak{S}$ of all finite sets (viewed as labelled structures) can be modelled as

$$\mathfrak{S} \cong \mathsf{set}(\{\alpha\})$$

where α is a structure of size 1. The egf of its counting sequence is given by

$$\hat{s}(z) \quad = \quad \sum_{n\geq 0}\frac{z^n}{n!} \quad = \quad \exp(z).$$

Let ϵ denote the empty set. Every partition can be viewed as a set of non-empty set. Therefore the class $\mathfrak{B}$ of all partitions satisfies

$$\mathfrak{B} \quad \cong \quad \mathsf{set}(\mathfrak{S} - \{\epsilon\}) \quad \cong \quad \mathsf{set}\Big(\mathsf{set}(\{\alpha\}) - \{\epsilon\}\Big).$$

This is readily translated into the language of exponential power series:

$$\hat{b}(z) \quad = \quad \exp\left(e^z - 1\right)$$

Differentiation yields

$$\frac{d}{dz}\hat{b}(z) \quad = \quad e^z\hat{b}(z).$$

and comparison of the coefficients of $z^n/n!$ on both sides results in the well-known recurrence relation

$$b_{n+1} \quad = \quad \sum_{k=0}^{n}\binom{n}{k}b_k \qquad n \geq 0.$$

1.3 Parameters

Often, one is not only interested in the number of objects of size n, but would also like to keep track of certain other parameters (e.g. the number of components) and obtain probabilistic information on these parameters.

Definition 1.10. *Let $\mathcal{A}$ denote a combinatorial class. A* parameter *on $\mathcal{A}$ is a map $\mathcal{A} \to \mathbb{N}^d$, $d \geq 1$.*

The symbolic approach described in the last two sections can be nicely adapted in order to fulfil these needs for a certain type of parameter:

Definition 1.11. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ denote combinatorial classes (labelled or unlabelled) with parameters χ, ξ and ζ, resp., all of which have the same dimension. Then, the parameter χ is said to be* inherited from *ξ,ζ in the following cases:*

- *$\mathcal{A} = \mathcal{B} + \mathcal{C}$ and*
$$\chi(\alpha) = \begin{cases} \xi(\alpha) & \text{if } \alpha \in \mathcal{B} \\ \zeta(\alpha) & \text{if } \alpha \in \mathcal{C} \end{cases} \qquad \forall\, \alpha \in \mathcal{A}$$
- *$A = \mathcal{B} \otimes \mathcal{C}$ and*
$$\chi(\langle \alpha, \beta \rangle) = \xi(\alpha) + \zeta(\beta) \qquad \forall\, \langle \alpha, \beta \rangle \in \mathcal{A}$$
where $\otimes$ and $\langle \alpha, \beta \rangle$ denote either $\star$ and $\alpha \star \beta$ (labelled case) or $\times$ and (α, β) (unlabelled case).

This is extended for all finite sums and products and combinations thereof.

Remark 1.3. *For simplicity, we will adopt the following conventions: Vectors will be denoted by bold variables (e.g. $\mathbf{v}$). If $\mathbf{z}$ and $\mathbf{u}$ denote two vectors of dimension $d+1$, we use the abbreviation*
$$\mathbf{z}^{\mathbf{u}} \quad := \quad z_0^{u_0} z_1^{u_1} \cdots z_d^{u_d}.$$
If $\mathbf{z}$ is a vector of dimension $d+1$ and r an arbitrary complex number then we set
$$r^{\mathbf{z}} = (r^{z_0}, r^{z_1}, \dots, r^{z_d}).$$

Remark 1.4. *For the combinatorial class $\mathcal{A}$ with parameter ξ, we define*
$$A_{n,\mathbf{k}} \quad := \quad \left\{ \alpha \in \mathcal{A} \mid |\alpha|_{\mathcal{A}} = n \wedge \xi(\alpha) = \mathbf{k} \right\}$$
and
$$a_{n,\mathbf{k}} \quad := \quad \mathrm{card}(A_{n,\mathbf{k}}).$$
(For the class $\mathcal{B}$, it would be $B_{n,\mathbf{k}}$ and $b_{n,\mathbf{k}}$.)

Definition 1.12. *(i) For the unlabelled combinatorial class $\mathcal{A}$ with parameter ξ, the (formal) power series*
$$\sum_{\alpha \in \mathcal{A}} z^{|\alpha|_{\mathcal{A}}} \mathbf{u}^{\xi(\alpha)},$$
where $\mathbf{u}$ is a vector of indeterminates of same dimension as ξ, is called the multivariate generating function for the combinatorial class $\mathcal{A}$ with parameter ξ.

(ii) For the labelled combinatorial class $\mathcal{B}$ *with parameter* ζ*, the (formal) power series*

$$\sum_{\beta \in \mathcal{B}} \frac{z^{|\beta|_{\mathcal{B}}}}{|\beta|_{\mathcal{B}}!} \mathbf{u}^{\zeta(\beta)}$$

is called the multivariate generating function for the combinatorial class $\mathcal{B}$ with parameter ζ.

With this definitions, one can prove the same translation rules as stated in the Theorems 1.2 and 1.4 using analogous arguments.

As can be seen in the following example, one has to find an appropriate symbolic description of the class(es) in question before translating it into an equation between multivariate power series.

Example 3. *The combinatorial class* $\mathcal{G}$ *of planar unlabelled trees is defined by*

$$\mathcal{G} \quad \cong \quad \mathcal{N} \times \mathsf{seq}(\mathcal{G})$$

where $\mathcal{N}$ *denotes the class consisting of one element of size* 1 *(a node).*

If we want to introduce the parameter ξ *on* $\mathcal{G}$ *which counts the number of leafs, the recursive definition above cannot be used. But the definition above can be rephrased as*

$$\mathcal{G} \quad \cong \quad \mathcal{N} + \mathcal{Z} \times (\mathsf{seq}(\mathcal{G}) - \epsilon)$$

where ϵ *denotes the empty sequence and* $\mathcal{Z}$ *any class satisfying* $\mathcal{Z} \cong \mathcal{N}$*. Now,* $\mathcal{N}$ *plays the role of an external node and* $\mathcal{Z}$ *that of an internal node.*

It is found, that the parameter ξ *is identical with the parameter inherited from the parameter on* $\mathcal{N}$ *with constant value* 1 *and the parameter on* $\mathcal{Z}$ *with constant value* 0*. The classes* $\mathcal{N}$ *and* $\mathcal{Z}$ *thus have the mgf* uz *and* z*, resp.*

This leads to the equation

$$G(z,u) \quad = \quad zu + \frac{zG(z,u)}{1 - G(z,u)}$$

where $G(z,u)$ *denotes the mgf of the class* $\mathcal{G}$ *with* u *marking the number of leafs.*

1.4 Limiting Distributions

Given a combinatorial class $\mathcal{A}$ (labelled or unlabelled) with parameter ξ of dimension $d \geq 1$. The corresponding multivariate generating function $F(z, \mathbf{u})$ can be directly related to a sequence of random variables $(X_n : n \in I)$, $I \subseteq \mathbb{N}$, on $\mathbb{N}^d$ defined by

$$\mathbb{P}\{X_n = \mathbf{k}\} \quad = \quad \frac{[z^n \mathbf{u}^{\mathbf{k}}]F(z, \mathbf{u})}{[z^n]F(z, \mathbf{1})} \qquad \forall n \in I,\ \forall \mathbf{k} \in \mathbb{N}^d$$

where I consists of those numbers for which $[z^n]F(z, \mathbf{1}) > 0$.

In the case $d = 1$, the m-*th factorial moment* of X_n can simply be expressed as

$$\mathbb{E}(X_n(X_n-1)\cdots(X_n-m+1)) = \frac{1}{[z^n]F(z,1)}[z^n]\frac{\partial^m}{\partial u^m}F(z,u)\bigg|_{u=1}$$

which gives, denoting the partial derivatives w.r.t u with F_u and F_{uu}, the expressions

$$\begin{aligned}
\mathbb{E}X_n &= \frac{[z^n]F_u(z,1)}{[z^n]F(z,1)} \\
\mathbb{V}X_n &= \frac{[z^n]F_{uu}(z,1)}{[z^n]F(z,1)} + \frac{[z^n]F_u(z,1)}{[z^n]F(z,1)} - \left(\frac{[z^n]F_u(z,1)}{[z^n]F(z,1)}\right)^2
\end{aligned}$$

for the *mean* and *variance*, resp.

For applications in later chapters, we give the following definitions.

Definition 1.13. *A sequence* $(X_n : n \geq 0)$ *of random variables is called* asymptotically concentrated *if and only if there exists a sequence* (μ_n) *such that*

$$\forall\, \varepsilon > 0: \qquad \lim_{n\to\infty} \mathbb{P}\left\{1-\varepsilon < \frac{X_n}{\mu_n} < 1+\varepsilon\right\} = 1.$$

Definition 1.14. *A sequence* $(X_n : n \geq 0)$ *of random variables satisfies a* central limit theorem *if and only if there exist sequences* (μ_n) *and* (σ_n) *such that*

$$\lim_{n\to\infty} \sup_x \left| \mathbb{P}\left\{\frac{X_n-\mu_n}{\sigma_n} \leq x\right\} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}dt \right| = 0.$$

Definition 1.15. *A sequence* $(X_n : n \geq 0)$ *of random variables satisfies a* local limit theorem on S *if and only if there exist sequences* (μ_n) *and* (σ_n) *such that*

$$\lim_{n\to\infty} \sup_{x\in S} \left| \sigma_n \mathbb{P}\left\{\frac{X_n-\mu_n}{\sigma_n} = \lfloor \sigma_n x + \mu_n \rfloor\right\} - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0.$$

Chapter 2

Asymptotic Methods

The methods of the last chapter result in equations satisfied by the generating function in question. For certain types of equations one can immediately extract the desired information about the coefficients of the gf (see Example 1). But this is not always possible.

It is often the case that the generating function considered is analytic in some domain. One can then use methods of complex analysis in order to obtain exact or asymptotic representations for the coefficients in question.

The Lagrange Inversion Theorem can be applied to certain equations and results in an exact representation. We note that an analogous theorem holds for the larger class of formal power series (see [GJ04] for details).

A source of asymptotic information about the coefficients are the singularities on the circle of convergence of the generating function. The asymptotic behaviour can then often be determined by an application of Cauchy's theorem using an appropriate path and estimating the resulting integral. In the case of small singularities of an algebraic-logarithmic type Flajolet and Odlyzko's [FO90] so-called singularity analysis applies. If the dominant singularities are large singularities, that is the function is growing exponentially near these singularities, or in the case of entire functions one can often apply the saddle point method.

2.1 Lagrange Inversion

Consider a class $\mathcal{T}$ of trees with generating function $F(z)$. $\mathcal{T}$ is called simply generated if and only if there exists a power series $\Phi(u)$ such that

$$F(z) = z\Phi(F(z)), \qquad \Phi(0) > 0. \tag{2.1}$$

Lagrange proved a theorem that gives the power series expansion of $F(z)$ in terms of z. The following formulation is taken from [WW96, p.133] where one can also find a proof for it.

Theorem 2.1 (Lagrange Inversion Theorem). *Let $\Phi(z)$ be analytic on and inside a contour γ surrounding a point a. If $t \in \mathbb{C}$ is such that $|t\Phi(z)| < |z - u|$ for all $z \in \gamma$, then we have:*

(i) *The equation*

$$\zeta = u + t\Phi(\zeta)$$

has exactly one root $\zeta(t)$ *in the interior of* γ.

(ii) *If* $f(z)$ *is analytic on and inside* γ, *we have the expansion*

$$f(\zeta(t)) = f(u) + \sum_{n\geq 1} \frac{t^n}{n!} \frac{d^{n-1}}{dz^{n-1}} \left(f'(z)\Phi(z)^n\right)\bigg|_{z=u}. \tag{2.2}$$

A reformulation of equation (2.2) gives

$$[t^n]f(\zeta(t)) = \frac{1}{n}[(z-u)^{n-1}] \left(f'(z)\Phi(z)^n\right) \tag{2.3}$$

for $n \geq 1$.

Theorem 2.1 can be applied to the inversion problem (2.1) using $f(z) = z$ and $u = 0$. We apply formula (2.3) to the generating functions of some families of trees.

Example 4 (plane rooted trees). *The family* $\mathcal{T}$ *of plane rooted trees is defined by the equation*

$$\mathcal{T} = \mathcal{N} \times \mathsf{seq}(\mathcal{T}).$$

Thus, the ordinary generating function $F(z)$ *and the exponential generating function* $G(z)$ *satisfy*

$$F(z) = \frac{z}{1 - F(z)} \qquad \text{and} \qquad G(z) = ze^{G(z)}.$$

An application of Theorem 2.1 using $\Phi(z) = (1-z)^{-1}$ *and* $u = 0$ *gives*

$$[z^n]F(z) = \frac{1}{n}[t^{n-1}](1-t)^{-n} = \frac{(-1)^{n-1}}{n}\binom{-n}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

and setting $\Phi(z) = e^z$ *and* $u = 0$ *yields*

$$[z^n]G(z) = \frac{1}{n}[t^{n-1}]e^{nt} = \frac{n^{n-1}}{n!}.$$

Since $G(z)$ *is an egf, the actual number of labelled trees of size* n *is given by* n^{n-1}.

Example 5 (t-ary trees). *The family* $\mathcal{T}$ *of* t*-ary trees is defined by*

$$\mathcal{T} = \mathcal{E} + \mathcal{N} \times \mathcal{T}^t.$$

Thus, the ordinary generating function $F(z)$ *satisfies*

$$F(z) = 1 + F(z)^t.$$

By Langrange's Theorem, the number of t*-ary trees of size* $n \geq 1$ *is given by (setting* $\phi(z) = z^t$ *and* $u = 1$*)*

$$[z^n]F(z) = \frac{1}{n}[(w-1)^{n-1}]\big((w-1)+1\big)^{nt} = \frac{1}{n}\binom{nt}{n-1}.$$

2.2 Singularity Analysis

We consider functions having a unique dominant algebraic-logarithmic singularity on their circle of convergence. By normalisation, we may always assume that this singularity occurs at $z = 1$.

Flajolet and Odlyzko [FO90] considered a special class $\mathcal{S}$ of functions as described above and proved an asymptotic expansion for their Taylor coefficients (Theorem 2.3). Having this class at hand, they obtained asymptotic expansions for the Taylor coefficients of functions $f(z)$ of the form

$$f(z) = h_0(z) + h_1(z) + \cdots + h_k(z) + O(g(z)), \qquad z \to 1, \tag{2.4}$$

where $h_0(z), \ldots, h_k(z), g(z) \in \mathcal{S}$ and $h_0(z) \gg \cdots \gg h_k(z) \gg g(z)$ as $z \to 1$. Moreover, the so-called transfer-theorems (Theorem 2.2) proved in [FO90] which guarantee that

$$[z^n]O(g(z)) = O([z^n]g(z)), \qquad g(z) \in \mathcal{S},$$

show that the coefficients corresponding to (2.4) satisfy

$$f_n = h_{0,n} + \cdots + h_{k,n} + O(g_n)$$

and that $h_{0,n} \gg \cdots \gg h_{k,n} \gg g_n$.

We continue by stating the results mentioned above. The proofs are mainly based on Cauchy's theorem using a Hankel-contour as integration path and can be found in [FO90].

Theorem 2.2. *Assume that* $f(z)$ *is analytic in*

$$\Delta = \Delta(\phi, \eta) = \left\{ z \;\middle|\; |z| \le 1 + \eta \text{ and } |\arg(z-1)| \ge \phi \right\}$$

where $\eta > 0$ *and* $0 < \phi < \frac{\pi}{2}$ *and that* $z = 1$ *is a singularity of* $f(z)$. *Set* $L(z) = (\log z)^\beta (\log\log z)^\delta$. *If*

$$f(z) = O\left((1-z)^\alpha L\left(\frac{1}{1-z}\right)\right), \qquad z \in \Delta,\ z \to 1$$

for some real numbers α, β, δ *then*

$$[z^n]f(z) = O\left(n^{-\alpha-1} L(n)\right), \qquad n \to \infty.$$

Analogous results hold for o *and* $\sim$ *instead of* O.

Theorem 2.3. *Let* $\alpha, \beta, \delta \in \mathbb{C} - \{0, 1, 2, \ldots\}$ *and define*

$$f(z) = (1-z)^\alpha \left(\frac{1}{z}\log\frac{1}{1-z}\right)^\beta \left(\frac{1}{z}\log\left(\frac{1}{z}\log\frac{1}{1-z}\right)\right)^\delta. \tag{2.5}$$

Then we have

$$[z^n]f(z) \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} (\log n)^\beta (\log\log n)^\delta \left(1 + \sum_{k\ge1} \frac{e_k(\log\log n)}{(\log n \log\log n)^k}\right) \tag{2.6}$$

where

$$e_k(z) = \Gamma(-\alpha)E_k(z)\,\frac{d^k}{ds^k}\frac{1}{\Gamma(-s)}\bigg|_{s=\alpha}$$

with

$$\sum_{k\geq 0} E_k(z)u^k = (1-uz)^\beta\left(1-\frac{1}{z}\log(1-uz)\right)^\delta.$$

Remark 2.1. *Note that the functions considered in Theorems 2.2 and 2.3 differ by* z^{-1}. *The reason for introducing this factor is the fact that* $\left(-z^{-1}\log(1-z)\right)^\beta$ *is analytic at* $z=0$ *even for non-integral* β. *Noting that*

$$\frac{1}{z}\log\frac{1}{1-z} = \log\frac{1}{1-z} + O\left((1-z)\log\frac{1}{1-z}\right)$$

and applying Theorem 2.2 we see that

$$[z^n]\left(\frac{1}{z}\log\frac{1}{1-z} - \log\frac{1}{1-z}\right) = O\left(\frac{\log n}{n^2}\right).$$

Hence we may replace $-z^{-1}\log(1-z)$ *with* $-\log(1-z)$ *in (2.5) without destroying the truth of (2.6).*

Example 6. *The generating function for the harmonic numbers* $(H_n)_{n\geq 1}$ *is known to be*

$$H(z) = \sum_{n\geq 1} H_n z^n = \frac{1}{1-z}\log\frac{1}{1-z}.$$

Adopting the notation of Theorem 2.3 we have $\alpha=-1, \beta=1$ *and* $\delta=0$ *as well as*

$$E_k(z) = \begin{cases} -z & k=1 \\ 0 & k>1 \end{cases} \qquad e_k(z) = \begin{cases} \gamma & k=1 \\ 0 & k>1 \end{cases}$$

where $\gamma = 0.577\ldots$ *is Euler's constant and therefore obtain*

$$H_n = \log n + \gamma + o(1) \qquad n\to\infty.$$

Example 7. *An undirected labelled graph is said to be 2-regular if and only if all its nodes have degree 2. All connected 2-regular graphs are given by undirected cycles having at least 3 nodes. Hence, the egf for connected 2-regular graphs is*

$$c(z) = \sum_{n\geq 3}\frac{(n-1)!}{2}\frac{z^n}{n!} = \frac{1}{2}\left(\log\frac{1}{1-z} - z - \frac{z^2}{2}\right).$$

Applying Theorem 1.4 we see that the egf for 2-regular graphs is

$$\begin{aligned} f(z) &= e^{c(z)} = \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}} \\ &= e^{-3/4}\left(\frac{1}{\sqrt{1-z}} + \sqrt{1-z} + O(1-z)^{3/2}\right), \qquad z\to 1. \end{aligned}$$

Now, Theorem 2.2 shows that

$$[z^n]O(1-z)^{3/2} = O(n^{-5/2}), \qquad n \to \infty,$$

and from Stirling's formula for $\Gamma(z)$ *we obtain*

$$\begin{aligned}
[z^n](1-z)^{-1/2} &= \left(\frac{1}{n\pi}\right)^{1/2}\left(1+\frac{1}{8n}+O(n^{-2}))\right) \\
[z^n](1-z)^{1/2} &= -\frac{1}{2}\left(\frac{1}{n^3\pi}\right)^{1/2}\left(1+O(n^{-1})\right)
\end{aligned}$$

as $n \to \infty$ *(note that Theorem 2.3 would have given only the main term of the last two expansions).*

The number f_n *of 2-regular graphs having* n *nodes therefore satisfies*

$$f_n = \frac{e^{-3/4}}{\sqrt{\pi}}\left(\frac{1}{\sqrt{n}} - \frac{3}{8n^{3/2}} + O(n^{-5/2})\right), \qquad n \to \infty.$$

2.3 The Saddle Point Method

The saddle point method is a heuristic, that often yields good approximations to integrals of the form

$$\int_{\gamma(t)} F(z,t)dz \qquad \text{as } t \to \infty \tag{2.7}$$

where t is a real parameter, F is analytic w.r.t. z in some domain $G(t) \subseteq \mathbb{C}$ and $\gamma(t)$ is a path that entirely lays in $G(t)$.

In this section, we will only give a rough sketch and a sample application of the saddle point method and refer to [dB81, ch.5,6] for a thorough discussion.

Essentially, the method relies on the Theorem of Cauchy and the Method of Laplace (see [dB81, ch.4] for details) and can roughly be summarised as follows:

(i) Substitute the path of integration $\gamma(t)$ with another one, $\sigma(t)$ say, without changing the value of the integral such that along $\sigma(t)$ $|F(z,t)|$ has some sharp peaks and is small everywhere else and

(ii) apply the Method of Laplace:

   (a) choose neighbourhoods of these peaks so large that the main contribution to the value of the integral is being captured,

   (b) in these neighbourhoods, substitute the integrand with simpler functions and

   (c) asymptotically estimate the resulting integrals.

The name "saddle point method" stems from the way of obtaining an appropriate path: By the maximum modulus theorem, $|F(z,t)|$ does not have any maxima or minima in the interior of $G(t)$ (except for zeros). Thus, the only points where $\frac{d}{dz}|F(z,t)| = 0$ and $F(z,t) \neq 0$ are saddle points.

Suppose that $\zeta \in G(t)$ is a saddle point of $F(z) = F(z,t)$, that is $F(\zeta) \neq 0$, $F'(\zeta) = \cdots = F^{(k-1)}(\zeta) = 0$ and $F^{(k)}(\zeta) \neq 0$ for some $k \geq 2$.

Suppose that $k = 2$. Then, for $|z-\zeta|$ small enough, the function $\log F(z)$ can be expanded as

$$\log F(z) = \log F(\zeta) + \frac{F''(\zeta)}{F(\zeta)} \frac{(z-\zeta)^2}{2} + O((z-\zeta)^3). \tag{2.8}$$

Since $|F(z)| \quad = \quad e^{\Re \log F(z)}$, $|F(z)|$ is of fastest decrease for

$$z \quad = \quad \zeta + t \exp\left(\frac{i}{2}\left(\pi - \arg\frac{F''(\zeta)}{F(\zeta)}\right)\right), \qquad t \in \mathbb{R} \tag{2.9}$$

where the second addend of (2.8) is real and negative. The straight line (2.9) is called the *axis of the saddle point* or the *direction of steepest descent.*

In the case $k > 2$ there are several directions of steepest decent. See [FS] for some remarks on this.

So, for appropriate functions, the path σ in step (i) should be chosen such that the highest points of $|F(z)|$ along σ are also saddle points of $|F(z)|$ and in small neighbourhoods of these saddle points, σ approximates its axis.

2.3.1 An Illustrating Example

The goal of this section is an estimation of the number of permutations of n elements having only cycles of length ≤ 2 as $n \to \infty$. The corresponding egf will be denoted by F.

Each permutation can be represented as a set of (labelled) cycles. In this case, all cycles are of length 1 or 2 (corresponding to fixed points and transpositions, resp.). Thus, the egf is given by

$$F(z) \quad = \quad \sum_{n \geq 0} F_n \frac{z^n}{n!} \quad = \quad \exp\left(z + \frac{z^2}{2}\right).$$

The starting point for the asymptotic analysis is the residue theorem which reads

$$\frac{F_n}{n!} \quad = \quad \frac{1}{2\pi i} \oint \frac{e^{z+z^2/2}}{z^{n+1}} dz, \qquad n \geq 0, \tag{2.10}$$

where the path of integration encircles the origin exactly once (counterclockwise).

Rewriting the integrand of (2.10) as

$$e^{h(z)} \quad := \quad \exp\left(z + \frac{z^2}{2} - (n+1)\log z\right)$$

we see that there are two saddle points, determined by the equation $z^2 + z = n + 1$, namely

$$-\frac{1}{2} \pm \sqrt{\frac{5}{4} + n} \quad = \quad -\frac{1}{2} \pm \sqrt{n}\left(1 + \frac{5}{8} n^{-1} + O(n^{-2})\right).$$

First, consider the saddle point at

$$\zeta_n = -\frac{1}{2} + \sqrt{\frac{5}{4} + n}, \qquad n \geq 0. \tag{2.11}$$

The power series expansion of h centred in ζ_n is given by

$$h(z) = h(\zeta_n) + \left(1 + \frac{n+1}{\zeta_n^2}\right)\frac{(z-\zeta_n)^2}{2} + \sum_{k\geq 3}(-1)^k \frac{n+1}{k}\left(\frac{z-\zeta_n}{\zeta_n}\right)^k \tag{2.12}$$

which is convergent for $|z-\zeta_n| < \zeta_n$. The coefficient of $(z-\zeta)^2$ is real and therefore the axis of the saddle point ζ_n is perpendicular to the real line.

The other saddle point is negligible as will be seen below. Furthermore, we will show that the path $\gamma = \gamma_1 + \gamma_2$ given by

$$\begin{aligned} \gamma_1 &:= \left\{z \ : \ z = \zeta_n + it, \ -\delta \leq t \leq \delta\right\} \\ \gamma_2 &:= \left\{z \ : \ |z|^2 = \zeta_n^2 + \delta^2, \ \arg(\zeta_n + i\delta) \leq \arg(z) \leq 2\pi - \arg(\zeta_n + i\delta)\right\}, \end{aligned}$$

where $\delta \in \mathbb{R}^+$ has yet to be chosen, can be used to estimate the integral (2.10).

For successfully replacing $\int_{\gamma_1} e^{h(z)}dz$ by a complete Gaussian integral, δ has to be chosen such that for $z \in \gamma_1$ we have

(i) $h(z) \sim h(\zeta_n) + h''(\zeta_n)(z-\zeta_n)^2$ and

(ii) $h''(\zeta_n)\delta^2 \to \infty$

as $n \to \infty$.

The last sum of (2.12) can be rewritten as

$$-(n+1)\left(\frac{z-\zeta_n}{\zeta_n}\right)^3 \sum_{k\geq 0} \frac{(-1)^k}{k+2}\left(\frac{z-\zeta_n}{\zeta_n}\right)^k.$$

Since $\zeta_n \sim \sqrt{n}$ we have

$$(n+1)(z-\zeta_n)^3\zeta_n^{-3} \quad \sim \quad (n^{-1/2} + n^{-3/2})(z-\zeta_n)^3$$

and therefore condition (i) is satisfied if δ is chosen so small that

$$\delta^3 = o(\sqrt{n}) \qquad \text{as } n \to \infty. \tag{2.13}$$

The quantity $h''(\zeta_n) = \frac{1}{2} + \frac{n+1}{\zeta_n^2}$ tends to $\frac{1}{2}$ from above as $n \to \infty$ since $\zeta_n^2 \sim n$. Thus, in oder to satisfy condition (ii), δ has to be chosen such that

$$\delta^2 \to \infty \qquad \text{as } n \to \infty. \tag{2.14}$$

A possible choice satisfying (2.13) and (2.14) is

$$\delta = n^{1/8}$$

which will be used in the sequel.

For $z \in \gamma_1$, we have

$$\begin{aligned}
\frac{1}{i}\int_{\gamma_1} e^{h(z)}dz &= \int_{-\delta}^{\delta} e^{h(\zeta_n+it)}dt \\
&= \int_{-\delta}^{\delta} e^{h(\zeta_n)-h''(\zeta_n)t^2/2}(1+O(n\delta^3\zeta_n^{-3}))\,dt \\
&= \frac{e^{h(\zeta_n)}(1+O(n^{-1/8}))}{\sqrt{h''(\zeta_n)}} \int\limits_{-\delta\sqrt{h''(\zeta_n)}}^{\delta\sqrt{h''(\zeta_n)}} e^{-u^2/2}du \\
&= \frac{e^{h(\zeta_n)}(1+O(n^{-1/8}))}{\sqrt{h''(\zeta_n)}} \left(\int_{-\infty}^{\infty} e^{-u^2/2}du + O\left(e^{-\delta^2/2}\right)\right) \\
&= \frac{e^{h(\zeta_n)}\sqrt{2\pi}}{\sqrt{h''(\zeta_n)}}(1+O(n^{-1/8}))
\end{aligned}$$

For $z \in \gamma_2$, we have

$$\begin{aligned}
\left|e^{h(z)}\right| &= e^{\Re h(z)} \quad\leq\quad \exp\left(\zeta_n + \frac{\zeta_n^2}{2} - (n+1)\log\zeta_n\right) \\
&= \exp\left((n+1)(1-\log\zeta_n) - \frac{\zeta_n^2}{2}\right) \quad=\quad \left(\frac{e}{\zeta_n}\right)^{n+1} e^{-\zeta_n^2/2}
\end{aligned}$$

and therefore

$$\begin{aligned}
\left|\int_{\gamma_2} e^{h(z)}dz\right| &\leq 2\pi\sqrt{\zeta_n^2+\delta^2}\left(\frac{e}{\zeta_n}\right)^{n+1} e^{-\zeta_n^2/2} \\
&= 2\pi\sqrt{1+\left(\frac{\delta}{\zeta_n}\right)^2}\left(\frac{e}{\zeta_n}\right)^{n} e^{-\zeta_n^2/2+1}
\end{aligned}$$

which tends to zero as $n \to \infty$.

From (2.11) we get $\zeta_n^2 = n - \sqrt{n} + \frac{3}{2} + O(n^{-1/2})$ and since

$$\begin{aligned}
\zeta_n^{n+1} &= (n+1)\log\left(\sqrt{n}\left(1 - \frac{1}{2}\sqrt{n} + \frac{5}{8}n^{-1} + O(n^{-2})\right)\right) \\
&= (n+1)\log\sqrt{n} - (n+1)\left(\frac{1}{2\sqrt{n}} - \frac{5}{8n} + O(n^{-2}) + \frac{1}{2}\left(\frac{1}{4n} + O(n^{-3/2})\right)\right) \\
&= (n+1)\log\sqrt{n} - \left(\frac{\sqrt{n}}{2} - \frac{1}{2} + O(n^{-1})\right)
\end{aligned}$$

we have

$$\begin{aligned}
e^{h(\zeta_n)} &= e^{\zeta_n+\zeta_n^2/2}\zeta_n^{-(n+1)} \quad=\quad \frac{e^{n/2+\sqrt{n}/2+1/4+O(n^{-1/2})}}{n^{(n+1)/2}e^{-\sqrt{n}/2+1/2+O(n^{-1})}} \\
&= \sqrt{n}^{n+1}\exp\left(\frac{n}{2}+\sqrt{n}-\frac{1}{4}\right)(1+O(n^{-1/2})).
\end{aligned}$$

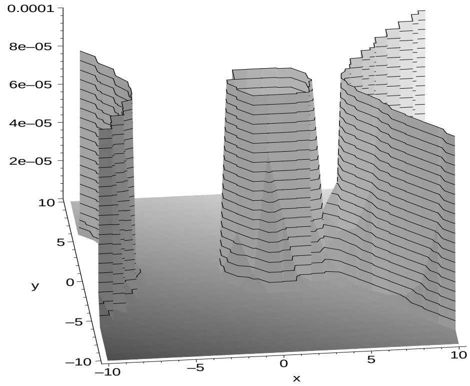

Figure 2.1: Surface of $|e^{z+z^2/2}z^{-16}|$ with the dominant saddlepoint at $z = -\frac{1}{2} + \sqrt{\frac{5}{4} + 15}$

This gives the intermediate result

$$\frac{F_n}{n!} \sim \frac{1}{2\pi} \frac{e^{h(\zeta_n)}\sqrt{2\pi}}{\sqrt{h''(\zeta_n)}} \sim \frac{1}{2\sqrt{\pi n^{n+1}}} \exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right)$$

Using Stirling's formula we finally obtain

$$F_n \sim \frac{n^{n/2}}{\sqrt{2}} \exp\left(-\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right) \qquad \text{as } n \to \infty.$$

Chapter 3

Hayman-admissible Functions

3.1 H-admissibility

Hayman [Hay56] stated conditions of power series $\sum_{n\geq 0} f_n z^n, c_n \in \mathbb{C}$, which ensure the determination of the behaviour of f_n as $n \to \infty$ (in the sense of $\sim$) using the saddle point method.

A notable fact about these functions is the existence of certain closure properties. This often simplifies the task of establishing these conditions (see below) for a given function and can also be used for an automation of this process.

Definition 3.1 (H-Admissibility). *Let $f(z)$ denote a function regular for $|z| < R$, $0 < R \leq \infty$. Assume further that there exists $R_0 < R$ such that $f(r) > 0$ for all $r \in [R_0, R)$.*

Define for $r \in [R_0, R)$ the two functions

$$a(r) = r\frac{f'(r)}{f(r)}$$

and

$$b(r) = ra'(r) = r\frac{f'(r)}{f(r)} + r^2\frac{f''(r)}{f(r)} - r^2\left(\frac{f'(r)}{f(r)}\right)^2.$$

Then f is said to be Hayman-admissible in $|z| < R$ *(or* H-admissible*) if and only if it satisfies*

(i) $b(r) \to \infty$ *as* $r \to R$ *and*

there exists a function $\delta : [R_0, r) \to (0, \pi)$ such that the following holds

(ii) Uniformly for $\phi \leq \delta(r)$ we have

$$f(re^{i\phi}) \sim f(r)\exp\left(i\phi a(r) - \frac{1}{2}\phi^2 b(r)\right) \tag{3.1}$$

as $r \to R$ and

(iii) uniformly for $\delta(r) \leq \phi$ *we have*

$$f(re^{i\phi}) = \frac{o(f(r))}{\sqrt{b(r)}} \tag{3.2}$$

as $r \to R$.

Remark 3.1. *As a consequence of conditions (ii) and (iii) we have for* $\delta = \delta(r)$

$$\frac{|f(re^{i\delta})|}{f(r)} \quad \sim \quad e^{-\delta^2 b(r)} \quad = \quad o(b(r))^{-1/2}$$

which is $o(1)$ *as* $r \to R$ *by condition (i). Thus, we have*

$$\delta^2 b(r) \to \infty \qquad \text{as } r \to R. \tag{3.3}$$

Remark 3.2. *Without loss of generality, we may assume that*

$$\delta(r) \leq \sqrt{2\frac{\log b(r)}{b(r)}} \tag{3.4}$$

since otherwise we have for $\sqrt{2 \log b(r)/b(r)} \leq |\phi| \leq \delta(r)$*, applying (3.1),*

$$\frac{|f(re^{i\phi})|}{f(r)} \sim \exp\left(-b(r)\frac{\phi^2}{2}\right) \leq \frac{1}{b(r)}$$

which implies (3.2).

Theorem 3.1. *Let* $f(z) = \sum_{n\geq 0} f_n z^n$ *be H-admissible in* $|z| < R$ *and define* $f_n = 0$ *for* $n < 0$*. Then we have*

$$f_n r^n \quad = \quad \frac{f(r)}{\sqrt{2\pi b(r)}}\left(\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right) + o(1)\right) \tag{3.5}$$

uniformly for all integers n *as* $r \to R$.

Proof. The claim can be proved by an estimation of Cauchy's Integral

$$f_n r^n \quad = \quad \frac{1}{2\pi}\left(\int_{-\delta}^{\delta} + \int_{\delta}^{2\pi-\delta}\right)\frac{f(re^{i\phi})}{e^{in\phi}}d\phi$$

where $\delta = \delta(r)$.

From (3.2), we have

$$\left|\int_{\delta}^{2\pi-\delta}\frac{f(re^{i\phi})}{e^{in\phi}}d\phi\right| \quad \leq \quad 2(\pi-\delta)\max_{\delta\leq\phi\leq 2\pi-\delta}\left|f(re^{i\phi})\right| \quad = \quad \frac{o(f(r))}{\sqrt{b(r)}}$$

uniformly in n as $r \to R$.

Equations (3.1) and (3.3) give

$$\begin{aligned}
\int_{-\delta}^{\delta} \frac{f(re^{i\phi})}{e^{in\phi}} d\phi &= f(r) \int_{-\delta}^{\delta} (1+o(1)) \exp\left(i\phi(a(r)-n) - \frac{1}{2}\phi^2 b(r)\right) d\phi \\
&= f(r) \left(\int_{-\delta}^{\delta} e^{i\phi(a(r)-n)-\frac{1}{2}\phi^2 b(r)} d\phi + o\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}b(r)\phi^2} d\phi \right) \right) \\
&= f(r) \left(\int_{-\delta}^{\delta} e^{i\phi(a(r)-n)-\frac{1}{2}\phi^2 b(r)} d\phi + o(b(r))^{-1/2} \right)
\end{aligned}$$

as $r \to R$.

An application of Cauchy's Theorem gives

$$\int_{-\infty+iW}^{\infty+iW} e^{-t^2/2} dt = \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}, \qquad W \in \mathbb{R},$$

and thus, by noting that

$$i\phi(a(r)-n) - \frac{1}{2}b(r)\phi^2 = -\frac{1}{2}\left(\phi\sqrt{b(r)} - i\frac{a(r)-n}{\sqrt{b(r)}}\right)^2 - \frac{(a(r)-n)^2}{2b(r)},$$

we get

$$\begin{aligned}
\int_{-\delta}^{\delta} e^{i\phi(a(r)-n)-\frac{1}{2}\phi^2 b(r)} d\phi &= \frac{1}{\sqrt{b(r)}} \exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right) \int_{-\delta\sqrt{b(r)}}^{\delta\sqrt{b(r)}} e^{-t^2/2} dt \\
&= \sqrt{\frac{2\pi}{b(r)}} \exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right) (1+o(1))
\end{aligned}$$

where the last equality follows from (3.3).

Combining the two estimates gives

$$\begin{aligned}
f_n r^n &= \frac{f(r)}{2\pi}\sqrt{\frac{2\pi}{b(r)}} \exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right)(1+o(1)) + \frac{o(f(r))}{\sqrt{b(r)}} \\
&= \frac{f(r)}{\sqrt{2\pi b(r)}} \left(\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right) + o(1) \right).
\end{aligned}$$

□

Corollary 3.1. *We have*

$$a(r) \to \infty, \quad \textit{as } r \to R. \tag{3.6}$$

Furthermore, there exists an $R_1 < R$ *such that* $a(r)$ *is strictly monotonic increasing in* $[R_1, R)$.

As a consequence thereof, we have

$$b(r) = o(a(r))^2, \quad \textit{as } r \to R. \tag{3.7}$$

Proof. Since $b(r) = ra'(r) \to \infty$ as $r \to R$, we know that $a(r)$ is strictly monotonic increasing in some range $R_1 < r < R$, $R_1 < R$.

Putting $n = -1$ in equation 3.5 yields

$$\exp\left(-\frac{(a(r)+1)^2}{b(r)}\right) = o(1), \quad \text{as } r \to R,$$

and as a consequence we get

$$\frac{(a(r)+1)^2}{b(r)} \to \infty, \quad \text{as } r \to R,$$

which proves (3.6) since $b(r) \to \infty$ as $r \to R$. □

In particular, the last corollary shows, that for $n \in \mathbb{N}$ large enough the equation $a(r) = n$ has a unique solution r_n that satisfies $r_n \to R$ as $n \to \infty$. This observation leads to

Corollary 3.2. *Let* $f(z) = \sum_{n\geq 0} f_n z^n$ *be H-admissible in* $|z| < R$ *and let* r_n *denote the unique solution of* $a(r) = n$. *The coefficients satisfy*

$$f_n \sim \frac{f(r_n)}{r_n^n\sqrt{2\pi b(r)}}, \quad \textit{as } n \to \infty. \tag{3.8}$$

Corollary 3.3. *Assume that* $f(z) = \sum_{n\geq 0} f_n z^n$ *is H-admissible in* $|z| < R$.

For any $n \in \mathbb{Z}$, *we have*

$$\frac{f(r)}{r^n} \to \infty, \quad \textit{as } r \to R \tag{3.9}$$

and for any $\varepsilon > 0$, *we have*

$$a(r) = O(f(r))^\varepsilon \quad \textit{and} \quad b(r) = O(f(r))^\varepsilon, \quad \textit{as } r \to R. \tag{3.10}$$

Proof. From (3.5) it follows that $f_n > 0$ if n is sufficiently large. Also, if r is sufficiently near to R, (3.5) gives

$$\frac{f(r)}{r^n} > \frac{1}{2} f_n \sqrt{2\pi b(r)}.$$

Since $b(r) \to \infty$ as $r \to R$ we obtain 3.9.

Clearly, $b(r) = O(f(r))^\varepsilon$ follows from $a(r) = O(f(r))^{\varepsilon/2}$ and (3.7).

For proving the remaining claim, we suppose on contrary that there exists $\varepsilon > 0$ such that

$$a(r) = r\frac{f'(r)}{f(r)} \geq f(r)^{\varepsilon}$$

for $r \in [R_1, R)$ and some $0 \leq R_1 < R$. This gives, for $R_1 < \rho < R$,

$$\int_{R_1}^{\rho} \frac{f'(r)}{f(r)^{1+\varepsilon}} dr \geq \int_{R_1}^{\rho} \frac{dr}{r}$$

and therefore

$$\frac{f(R_1)^{-\varepsilon} - f(\rho)^{-\varepsilon}}{\varepsilon} \geq \log \frac{\rho}{R_1}.$$

If $R = \infty$, we obtain a contradiction by letting $\rho \to \infty$ since the left hand side of the last equation remains finite. The proof for $R < \infty$ involves some additional technicalities and will be omitted (see [Hay56]). □

The next lemma gives sufficient conditions for (3.1).

Lemma 3.1. *Let $f(z)$ be analytic and not zero in $|z - r| < 2\eta r$ for some $r \in \mathbb{R}^+$ and $0 < \eta \leq \frac{1}{2}$ and set*

$$a(z) = z\frac{f'(z)}{f(z)} \qquad \text{and} \qquad b(z) = za'(z).$$

If $b(z)$ satisfies

$$|b(z)| < Cb(r), \qquad |z - r| < 2\eta r, \tag{3.11}$$

for some constant $C \in \mathbb{R}^+$ then we have the expansion

$$\log f(re^{\zeta}) = \log f(r) + a(r)\zeta + b(r)\frac{\zeta^2}{2} + \epsilon(r, \zeta)$$

where

$$|\epsilon(r, \zeta)| < \frac{Cb(r)|\zeta|^3}{2\eta}$$

for $|\zeta| < \eta$.

Proof. The function $b(re^{\zeta}) = \frac{\partial^2}{\partial \zeta^2} \log f(re^{\zeta})$ is analytic for $|\zeta| < \eta$ since we have

$$\left|re^{\zeta} - r\right| = r\left|e^{\zeta} - 1\right| \leq r\left(e^{|\zeta|} - 1\right) \leq r(e^{\eta} - 1) < 2\eta r$$

for $\eta \leq \frac{1}{2}$ and thus has a power series development

$$b(re^{\zeta}) = \sum_{n \geq 0} c_n \zeta^n.$$

Cauchy's inequality together with (3.11) yields

$$|c_n| \leq \frac{Cb(r)}{\eta^n}.$$

Integrating twice w.r.t. ζ gives

$$\log f(re^{\zeta}) = \log f(r) + a(r)\zeta + b(r)\frac{\zeta^2}{2} + \sum_{n\geq 1} \frac{c_n}{(n+1)(n+2)}\zeta^{n+2}$$

and noting that

$$\sum_{n\geq 1} \frac{c_n}{(n+1)(n+2)}\zeta^{n+2} \leq \frac{Cb(r)|\zeta|^3}{\eta} \sum_{n\geq 1} \frac{1}{(n+1)(n+2)} = \frac{Cb(r)|\zeta|^3}{2\eta}$$

completes the proof. □

The remaining part of this section is devoted to the study of the behaviour of H-admissible functions. Theorem 3.2 shows how H-admissible functions and their derivatives behave for real arguments, Lemma 3.2 gives a better approximation than (3.1) for arguments near the real line and finally, Theorem 3.3 shows that H-admissible functions attain their maximum on $\{z \in \mathbb{C} : |z| = r\}$, r properly chosen, at $z = r$. The technical proofs for the Theorems 3.2 and 3.3 will be omitted and can be found in [Hay56].

Theorem 3.2. *Let* $f(z)$ *be H-admissible in* $|z| < R$. *Then*

- *for any fixed positive constant* K *and* $|h| < K/a(r)$ *we have*

$$a(re^h) \sim a(r) \tag{3.12}$$

  uniformly as r, re^h *tend to* R *from below;*

- *for any fixed* $\kappa \in \mathbb{R}$

$$f\left(r + \frac{\kappa r}{a(r)}\right) \sim e^k f(r), \quad \text{as } r \to R; \tag{3.13}$$

- *for any fixed* $k \in \mathbb{N}, k > 0$,

$$f^{(k)}(r) \sim f(r)\left(\frac{a(r)}{r}\right)^k, \quad \text{as } r \to R; \tag{3.14}$$

Lemma 3.2. *Assume that* $f(z)$ *is H-admissible in* $|z| < R$.
We have, uniformly for $|\phi| < a(r)^{-1}$,

$$f(re^{i\phi}) = f(r) + i\phi f'(r) - \frac{\phi^2}{2}\big(rf'(r) + r^2 f''(r)\big) + O\big(\phi^3 f(r) a(r)^3\big) \tag{3.15}$$

as $r \to R$.

Proof. We obtain the result by applying Lemma 3.1 to the function $F(z) = \exp(f(z))$. Set

$$A(z) = z\frac{F'(z)}{F(z)} = zf'(z) \qquad \text{and} \qquad B(z) = zA'(z) = zf'(z) + z^2 f''(z).$$

By (3.14), we have

$$B(r) \sim f(r)a(r)^2, \qquad \text{as } r \to R.$$

$B(z)$ has, by definition, only a finite number of negative coefficients in its power series expansion. Therefore,

$$|B(\zeta)| \leq B\left(r + \frac{2r}{a(r)}\right) + O(r^N)$$

for $|\zeta - r| \leq \frac{2r}{a(r)}$ and some $N \geq 0$. Now we have, by (3.12) and (3.13),

$$B\left(r + \frac{2r}{a(r)}\right) \sim f\left(r + \frac{2r}{a(r)}\right) a\left(r + \frac{2r}{a(r)}\right)^2 \sim e^2 f(r)a(r)^2.$$

Thus, if r is sufficiently near to R, we may apply Lemma 3.1 to $F(z)$:

$$f(re^{i\phi}) = f(r) + i\phi A(r) - \frac{\phi^2}{2}B(r) + \epsilon(r, \phi)$$

where, for $|\phi| < a(r)^{-1}$ and some $C > 0$,

$$\epsilon(r, \phi) < Cf(r)a(r)^3|\phi|^3.$$

$\square$

Theorem 3.3. *Suppose that* $f(z)$ *is H-admissible in* $|z| < R$.

There exists $R_1 < R$ *such that for* $R_1 < r < R$ *and* $f(r)^{-2/5} \leq |\phi| \leq \pi$ *we have*

$$|f(re^{i\phi})| \leq f(r) - f(r)^{1/7} \tag{3.16}$$

and there exists $R_0 < R$ *such that for* $R_0 < r < R$ *and* $0 < |\phi| < \pi$ *we have*

$$|f(re^{i\phi}) < f(r). \tag{3.17}$$

Example 8. *The function* e^z *is H-admissible in* $\mathbb{C}$ *with* $\delta(r) = r^{-2/5}$ *and the functions of Definition 3.1 associated with* e^z *are given by*

$$a(r) = b(r) = r, \quad r \in \mathbb{R}_+.$$

Thus, the unique positive real solution to $a(r) = n$ *is given by* $r_n = n$. *Corollary 3.2 now yields the main term of Stirling's approximation to* $n!$, *viz.*

$$[z^n]e^z = \frac{1}{n!} \sim \left(\frac{e}{n}\right)^n (2\pi n)^{-1/2}.$$

3.2 Classes of H-admissible Functions

For given $R > 0$ define the class

$$\mathcal{H}_R := \{f : \mathbb{C} \to \mathbb{C} \mid f(z) \text{ is H-admissible in } |z| < R\}.$$

In this section we will prove certain closure properties satisfied by $\mathcal{H}_R$: Theorems 3.5 and 3.4 show how to construct new H-admissible functions from given ones and Theorems 3.6 and 3.7 show that "small perturbations" do not destroy H-admissibility.

These facts can often be used to simplify the task of establishing H-admissibility for a given function. Starting from a basic set of H-admissible functions, one can establish H-admissibility for many functions by use of the theorems below without having to check the conditions of Definition 3.1.

Theorem 3.4. $f_1(z), f_2(z) \in \mathcal{H}_R \implies f_1(z)f_2(z) \in \mathcal{H}_R$.

Proof. Set $f(z) = f_1(z)f_2(z)$. The corresponding functions $\delta(r), a(r), b(r)$ of Definition 3.1 will be denoted using no subscript, subscript 1 or 2, resp. These functions satisfy $a(r) = a_1(r) + a_2(r)$ and $b(r) = b_1(r) + b_2(r)$.

We show that the function $f(z)$ satisfies the conditions for H-admissibility with

$$\delta(r) = \min\left(\delta_1(r), \delta_2(r)\right).$$

The only thing that needs to be shown is condition (iii)(b) of H-admissibility all other properties immediately follow from H-admissibility of f_1 and f_2.

Suppose that in $R_0 < r < R$ we have $b_1(r) > e$ and $b_2(r) > e$ and that with $\varepsilon < \frac{1}{2}$ we have

$$\frac{|f_1(re^{i\phi})|}{f_1(r)} \le \frac{\varepsilon}{b_1(r)^{1/2}}, \qquad \delta_1(r) \le |\phi| \le \pi, \tag{3.18}$$

$$\frac{|f_2(re^{i\phi})|}{f_2(r)} \le \frac{\varepsilon}{b_2(r)^{1/2}}, \qquad \delta_2(r) \le |\phi| \le \pi. \tag{3.19}$$

Now, consider those r for which $b_1(r) \ge b_2(r)$. We have to show that (3.18) is valid for $\delta(r) \le |\phi| \le \pi$. If $\delta_1(r) \le \delta_2(r)$ there is nothing to prove. If $\delta_2(r) < \delta_1(r)$ then we have by (3.1)

$$\frac{|f_1(re^{i\phi})|}{f_1(r)} \sim e^{-b_1(r)\phi^2/2} \le e^{-b_1(r)\delta_2(r)^2/2}, \qquad r \to R, \tag{3.20}$$

as well as

$$\frac{|f_2(re^{i\delta_2(r)})|}{f_2(r)} \sim e^{-b_2(r)\delta_2(r)^2/2}, \qquad r \to R.$$

This gives for r sufficiently near to R

$$e^{-b_2(r)\delta_2(r)^2/2} < \frac{2\varepsilon}{b_2(r)^{1/2}}.$$

The function $\sqrt{t}e^{-at}$ is decreasing for $a > 0$ and $t \in \left[\frac{1}{2a}, \infty\right)$. Thus, for $b_2(r) \geq \delta_2(r)^{-2}$, the last equation implies

$$e^{-b_1(r)\delta_2(r)^2/2} < \frac{2\varepsilon}{b_1(r)^{1/2}}.$$

Hence (3.20) yields for r sufficiently near to R

$$\frac{|f_1(re^{i\phi})|}{f_1(r)} < \frac{3\varepsilon}{b_1(r)^{1/2}}. \tag{3.21}$$

Note that by (3.16) we have for r sufficiently near to R

$$\frac{|f_2(re^{i\phi})|}{f_2(r)} < 1, \qquad 0 < |\phi| < \pi. \tag{3.22}$$

Since $b(r) = b_1(r) + b_2(r) \leq 2b_1(r)$ we finally obtain (3.2) for f by multiplying the relations (3.21) and (3.22). ☐

Theorem 3.5. $f(z) \in \mathcal{H}_R \implies \exp(f(z)) \in \mathcal{H}_R$.

Proof. We show that $F(z) = \exp(f(z))$ is H-admissible with $\delta(r) = f(r)^{-2/5}$. The functions of Definition 3.1 read

$$A(z) = z\frac{F'(z)}{F(z)} = zf'(z) \quad \text{and} \quad B(z) = zA'(z) = zf'(z) + z^2f''(z).$$

First, note that, by (3.14),

$$B(r) \sim f(r)a(r)^2 \to \infty, \qquad r \to R.$$

We have $f(r)^{-2/5} = o(a(r)^{-1})$ as $r \to R$ by (3.10) and may therefore apply Lemma 3.2 for $|\phi| \leq f(r)^{-2/5}$ to $F(z)$ which yields

$$\log F(re^{i\phi}) = \log F(r) + i\phi A(r) - \frac{\phi^2}{2}B(r) + O(f(r))^{-1/5}a(r)^3$$

which gives (3.1) because of (3.10). Finally, for $f(r)^{-2/5} \leq |\phi| \leq \pi$ we have by (3.16) and (3.10) for r sufficiently near R

$$|F(re^{i\phi})| \leq F(r)\exp\left(-f(r)^{-1/7}\right) \leq F(r)\exp\left(-B(r)^{1/8}\right)$$

which gives (3.2). ☐

Theorem 3.6. *If* $f(z) \in \mathcal{H}_R$ *and* $p(z) = b_m z^m + \cdots + b_0 \in \mathbb{R}[z]$ *is a polynomial satisfying*

- $p(R) > 0$ *if* $R < \infty$ *or*
- $b_m > 0$ *if* $R = \infty$

then we have $p(z)f(z) \in \mathcal{H}_R$.

Proof. Set $a(r) = rf'(r)/f(r)$ and $b(r) = ra'(r)$ and let $f(z)$ satisfy (3.1) and (3.2) with a function $\delta(r)$ that satisfies (3.4).

Since $\delta(r) \to 0$ as $r \to R$ we have, for any R,

$$\frac{P(re^{i\phi})}{P(r)} \sim 1, \qquad r \to R,$$

uniformly for $|\phi| \le \delta(r)$. Hence, we get

$$\frac{f(re^{i\phi})}{f(r)} \frac{P(re^{i\phi})}{P(r)} \sim e^{i\phi a(r) - \phi^2 b(r)/2}, \qquad r \to R, \tag{3.23}$$

uniformly for $|\phi| \le \delta(r)$.

For any R, we have

$$\frac{P(r^{i\phi})}{P(r)} = O(1), \qquad r \to R$$

for any ϕ and thus, we deduce

$$\frac{f(re^{i\phi})}{f(r)} \frac{P(re^{i\phi})}{P(r)} = o(b(r))^{-1/2}, \qquad r \to R, \tag{3.24}$$

uniformly for $\delta(r) \le |\phi| \le \pi$.

In order to complete the proof we have to show that we may replace $a(r)$, $b(r)$ by

$$a(r) + r\frac{P'(r)}{P(r)}, \qquad b(r) + r\frac{d}{dr} r\frac{P'(r)}{P(r)}$$

in (3.23) and (3.24). But this follows immediately since $rP'(r)/P(r)$ and its derivative remains bounded, while $a(r)$,$b(r)$ tend to infinity as $r \to R$. ☐

Theorem 3.7. *If* $f(z) \in \mathcal{H}_R$ *and* $h(z)$ *is a function, regular in* $|z| < R$ *and real for real* z*, such that for some* $\eta > 0$ *we have*

$$\max_{|z|=r} |h(z)| = O(f(r))^{1-\eta} \qquad \textit{as } r \to R$$

then we have $f(z) + h(z) \in \mathcal{H}_R$.

Proof. Again, let $a(r), b(r), \delta(r)$ denote the functions of Definition 3.1 corresponding to $f(z)$. Assuming that $\delta(r)$ satisfies (3.4) we deduce that

$$|f(re^{i\phi})| \sim f(r)e^{-\phi^2 b(r)/2} \ge \frac{f(r)}{b(r)} > f(r)^{1-\frac{\eta}{2}}$$

uniformly for $|\phi| \le \delta(r)$ as $r \to R$. Hence we have, for $|\phi| \le \delta(r)$,

$$(f+h)(re^{i\phi}) \sim (f+h)(r)e^{i\phi a(r) - \phi^2 b(r)/2}, \qquad r \to R \tag{3.25}$$

and similarly, for $\delta(r) \le |\phi| \le \pi$,

$$(f+h)(re^{i\phi}) = \frac{o((f+h)(r))}{b(r)^{-1/2}}. \tag{3.26}$$

Set

$$A(r) = f\frac{(f+h)'(r)}{(f+h)(r)}, \qquad B(r) = rA'(r).$$

We have to show that $A(r) = a(r) + o(1)$ and $B(r) = b(r) + o(1)$ since we may then replace $a(r)$,$b(r)$ by $A(r)$,$B(r)$ in (3.25) and (3.26). Suppose that $|r - z| < ra(r)^{-1}$. Then we have

$$h(z) = O\left(f(\left(r + \frac{r}{a(r)}\right)\right) = O(f(r))^{1-\eta}$$

by (3.13). Thus for every fixed $k \in \mathbb{N}$

$$|h^{(k)}(r)| \le k!\left(\frac{a(r)}{r}\right)^k \max_{|z-r|\le r/a(r)} |h(z)| = \frac{O(f(r))^{1-\eta/2}}{r^k}$$

by Cauchy's inequality. Hence

$$\begin{aligned} A(r) &= r\frac{(f'+h')(r)}{(f+h)(r)} = r\frac{f'(r)}{h'(r)}\left(1 + O\left(\frac{h(r)}{f(r)}\right)\right)\left(1 + O\left(\frac{h'(r)}{f'(r)}\right)\right) \\ &= a(r)\left(1 + O(f(r))^{-\eta/2}\right), \quad r \to R. \end{aligned}$$

Similarly, $B(r) = b(r) + o(1)$. □

3.3 Examples

Without proof we quote three theorems that give a basic set of H-admissible functions and close this chapter with some simple examples.

Theorem 3.8. *Let* $P(z) = b_k z^k + \ldots + b_1 z + b_0$, $b_k \neq 0$, $k \geq 1$, *be a real polynomial and set*

$$f(z) = \sum_{k \ge 0} f_n z^n = e^{P(z)}.$$

Then the following four conditions are equivalent.

(i) $f(z)$ *is H-admissible in* $\mathbb{C}$.

(ii) For all sufficiently large r *we have*

$$|f(re^{i\phi})| < f(r), \quad 0 < |\phi| \le \pi.$$

(iii) For every integer $d > 1$, *there exists an integer* m, *such that* d *is not a factor of* m *and* $b_m \neq 0$. *Further if* $m = m(d)$ *is the largest such integer, then* $b_{m(d)} > 0$.

(iv) $a_n > 0$ *for all sufficiently large positive integers* n.

Theorem 3.9. *Suppose that* $f(z)$ *is an integral function of genus zero, positive for large positive* z*, having for some positive* δ *at most a finite number of zeros in the angle* $|\arg z| \le \pi/2 + \delta$*, and such that*

$$b(r) = r\frac{d}{dr}r\frac{d}{dr}\log f(r) \to \infty, \qquad r \to R.$$

Then $f(z)$ *is admissible in the plane.*

Theorem 3.10. *Suppose that* $f(z) = \sum_{n\ge 0} f_n z^n$ *is regular in* $|z| < 1$*. Furthermore, there exist positive constants* α, $\beta < 1$ *and* $R_0 < 1$ *and a positive function* $C(r)$, $0 < r < 1$ *such that*

- *as* $r \to 1$ *we have*

$$(1-r)\frac{C'(r)}{C(r)} \to 0;$$

- *uniformly for* $|\arg z| \le \beta(1-r)$ *we have*

$$\log f(z) \sim C(|z|)(1-z)^{-\alpha}, \qquad z \to 1;$$

- *for* r *sufficiently near* 1 *we have*

$$|f(re^{i\phi})| \le |f(re^{i\beta(1-r)})|, \qquad \beta(1-r) \le |\phi| \le \pi.$$

Then $f(z)$ *is H-admissible in* $|z| < 1$.

Example 9. *(i)* e^z*;*

(ii) The function of Section 2.3.1, $e^{z+z^2/2}$*, is H-admissible by Theorem 3.8.*

(iii) The function $e^z - 1$ *is H-admissible by Theorem 3.7 and Theorem 3.5 shows that the generating function for the Bell numbers,* $\exp(e^z - 1)$*, is H-admissible, too.*

Chapter 4

Generalisations and Related Concepts

We consider some concepts strongly related to H-admissibility.

Harris and Schoenfeld [HS68] defined a class of functions $f : \mathbb{C} \to \mathbb{C}$, called HS-admissible, and proved an asymptotic expansion for their coefficients. Their method is closely related to Hayman's but unfortunately their result is rather hard to apply. Odlyzko and Richmond [OR85] and Müller [Mül97] provided theorems which establish HS-admissibility for certain classes of functions.

Mutafchiev [Mut92] generalised the local limit result implied by H-admissibility (Corollary 3.2) by weakening the restrictions on the asymptotic behaviour of H-admissible functions. While Hayman requires that only the terms up to order 2 are significant, Mutafchiev only assumes that the function occurring in the asymptotic is the characteristic function of an infinitely divisible and absolutely integrable distribution with finite variance.

Bender and Richmond [BR96] stated a rather general analogue of H-admissibility for functions $f : \mathbb{C}^n \to \mathbb{C}$, $n \geq 1$, and proved an asymptotic formula for the coefficients of such functions. Their concept is useful for establishing local limit theorems for various combinatorial structures. In their paper they also proved some theorems which simplify the task of establishing this so-called BR-admissibility. Unfortunately it is not easy to use this concept for automatically obtaining asymptotic formulae for the coefficients in question and proving limit theorems for combinatorial structures.

In view of this, Drmota, Gittenberger, and Klausner [DGK05] stated an analogue of H-admissibility for functions $f : \mathbb{C}^2 \to \mathbb{C}$ and proved a central limit theorem for their coefficients. These so-called e-admissible classes satisfy various closure properties of an algebraic type. With this concept at hand, computers can, given a description of the combinatorial class as in Chapter 1, automatically prove central limit theorems. In their paper they also presented a Maple-implementation demonstrating this concept.

4.1 HS-admissibility

Definition 4.1. *Let* $f(z) = \sum_{n \geq 0} f_n z^n$ *be a function analytic in* $|z| < R, 0 < R \leq \infty$ *and real for real* z.

f *is said to be HS-admissible if and only if*

(i) There exists $R_0 \in (0, R)$ *and a function* $d : (R_0, R) \to (0, 1)$ *such that for* $r \in (R_0, R)$ *we have*

$$1 + d(r) < \frac{R}{r}$$

and

$$|z - r| \leq r d(r) \implies f(z) \neq 0.$$

(ii) For $k \geq 1$ *and* $r \in (R_0, R)$ *set*

$$A(z) = \frac{f'(r)}{f(r)}, \quad B_k(z) = \frac{z^k}{k!} \left(\frac{d}{dz}\right)^{k-1} A(z), \quad B(z) = \frac{z}{2} B_1'(z).$$

We have $B(r) > 0$ *for* $r \in (R_0, R)$ *and* $B_1(r) \to \infty$ *as* $r \to R-$.

(iii) For $R_1 < R$ *and* $n \in \mathbb{N}$ *suitably large the equation* $B_1(r) = n + 1$ *has a unique solution* $u_n \in (R_1, R)$. *Define*

$$C_j(z, r) = -\frac{B_{j+2}(z) + \frac{(-1)^j}{j+2} B_1(r)}{B(r)}$$

and suppose that for a certain fixed $N \geq 0$ *there exist non-negative numbers* D_n, E_n *and* n_0 *such that for all* $n \geq n_0$ *and for* $1 \leq j \leq 2N + 1$ *we have*

$$|C_j(u_n, u_n)| \leq E_n D_n^j.$$

In addition, we have for all $n \geq n_0$ *that either*

(a) $|C_j(u_n, u_n)| \leq E_n D_n^j$ *for all* $j \geq 2N + 2$ *or*

(b) $|C_{2N+2}(u_n + i\rho u_n, u_n)| \leq E_n D_n^{2N+2}$ *for* $\rho \in [-d(u_n), d(u_n)]$.

(iv) As $n \to \infty$, *we have*

$$B(u_n) d(u_n)^2 \to \infty, \quad D_n E_n B(u_n) d(u_n)^3 \to 0, \quad D_n d(u_n) \to 0.$$

We can now state the main theorem for HS-admissible functions. Using the abbreviations

- $\beta_n = B(u_n)$;
- $\gamma_j(n) = C_j(u_n, u_n)$;
- $Q(r)$ is the path consisting of the line segment L from $r + i r d(r)$ to $r\sqrt{1 - d(r)^2} + i r d(r)$ and the circular arc C from the last point to ir to $-r$ (see figure 4.1).

- $\lambda(r;d)$ is the maximum value of $|f(z)/f(r)|$ on $Q(r)$;
- $$\mu(r;d) = \max\left(\lambda(r;d)\sqrt{B(r)},\ \frac{\exp\left(-B(r)d(r)^2\right)}{d(r)\sqrt{B(r)}}\right);$$
- $E'_n = \min(1, E_n)$ and $E''_n = \max(1, E_n)$;
- $h_N(n;d) = \max\left(\mu(u_n;d),\ E'_n(D_nE''_n/\sqrt{\beta_n})^{2N+2}\right)$;
- $$F_k(n) = \frac{(-1)^k}{\sqrt{\pi}}\sum_{m=1}^{2k}\frac{\Gamma(m+k+\frac{1}{2})}{m!}\sum_{\substack{j_1+\cdots+j_m=2k\\ j_1,\ldots j_m\geq 1}}\gamma_{j_1}(n)\cdots\gamma_{j_m}(n)$$

we have

Theorem 4.1. *Let $f(z) = \sum_{n\geq 0} f_nz^n$ be HS-admissible (with either (iii)(a) or (iii)(b)). Then for the given N we have, as $n\to\infty$, the expansion*

$$f_n = \frac{f(u_n)}{u_n^n\sqrt{\pi\beta_n}}\left(1+\sum_{k=1}^{N}\frac{F_k(n)}{\beta_n^k}+O(h_N(n;d))\right) \tag{4.1}$$

In case of (iii)(a) equation (4.1) is valid for all $N\geq 0$.

Remark 4.1. *The detailed proof can be found in [HS68]. It mainly consists of an application of the saddle point-method to the function $f(z)z^{-n-1}$ which has the dominant saddle point at u_n. The path of integration used in the proof of Theorem 4.1 is depicted in Figure 4.1. It consists of a vertical line running through the dominant saddle point of the integrand, a circular arc of radius u_n and centre 0 and two horizontal lines with constant imaginary part $u_nd(u_n)$.*

Harris and Schoenfeld [HS68] gave two reasons for preferring the integration path described above over the circle $|z| = u_{n-1}$ as used in Hayman's proof:

- *The quantities $B_k(z)$ arising in Definition 4.1 are usually easier to determine than the quantities $(z\frac{d}{dz})^kA(z)$ which would arise when using the circle $|z| = u_{n-1}$.*
- *In applications considered by Harris and Schoenfeld it turns out that the concept as presented above produces better numerical results than the alternative concept based on the circle $|z| = u_{n-1}$.*

Remark 4.2. *Theorem 4.1 does not necessarily give a meaningful asymptotic result for every N. If we apply the Theorem as stated above to $f(z) = e^z$ and make the (optimal) choice $d(r) = \sqrt{2\log r}/\sqrt{r}$, then for all $N\geq 0$ we merely obtain $h_N(n;d) = O(\log n/\sqrt{n})$ which is independent of N. Noting that $F_k(n)/\beta_n^k \asymp n^{-k}$ as $n\to\infty$ for any fixed $k\geq 1$*

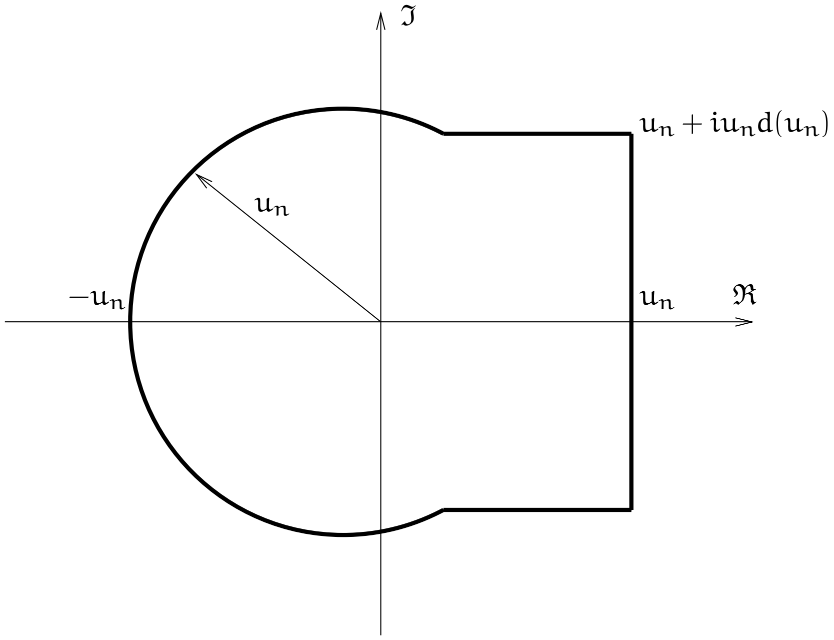

Figure 4.1: The integration path used in the proof of Theorem 4.1.

we see that for $N > 0$ *Theorem 4.1 does not give a better approximation to* $1/n!$ *than for* $N = 0$.

For this reason, Harris and Schoenfeld [HS68] stated an alternative form of Theorem 4.1 based on the circle $|z| = u_{n-1}$. *Details can be found in [HS68].*

Example 10. *Let* T_n *denote the set of functions mapping the set* $\{1, 2, \dots, n\}$ *into itself and let* $\circ$ *denote the composition of functions. Then,* $\langle T_n, \circ \rangle$ *is a semigroup. We are interested in the number* U_n *of idempotent elements in* $\langle T_n, \circ \rangle$, *i.e., functions* $f \in T_n$ *such that* $f \circ f = f$. *Harris and Schoenfeld [HS67] showed that*

$$1 + \sum_{n \geq 1} U_n \frac{z^n}{n!} = e^{ze^z}$$

and in [HS68] they showed that this functions is HS-admissible. In this case, $R = \infty$,

$$A(z) = (z+1)e^z, \quad B_k(z) = \frac{z^k}{k!}(z+k)e^z, \quad B(z) = \frac{z}{2}(z^2 + 3z + 1)e^z,$$

and the function $d(r)$ *is chosen such that* $d(r) = e^{-2r/5}$. *A calculation yields*

$$h_N(n; d) = O\left(\frac{\log n}{n}\right)^{N+1}, \qquad n \to \infty,$$

and

$$\frac{F_k(n)}{\beta_n^k} = \frac{P_k(u_n)}{C_n^{3k}(n+1)^k}$$

where u_n is the positive solution of $u(u+1)e^u = n+1$, $C_n = u_n^2 + 3u_n + 1$, *and* $P_k(u)$ *is a polynomial of degree* $7k$. *Hence, by Theorem 4.1,*

$$U_n = \sqrt{\frac{u_n+1}{2\pi(n+1)C_n}}\frac{n!}{u_n^n}e^{(n+1)/(u_n+1)}\left(1+\sum_{k=1}^{N}\frac{P_k(u_n)}{C_n^{3k}(n+1)^k}+O\left(\frac{\log n}{n}\right)^{N+1}\right)$$

as $n \to \infty$ *for any fixed* $N \geq 0$.

In general it is quite hard to check the conditions of HS-admissibility. However, Odlyzko and Richmond[OR85] showed that for a special type of function HS-admissibility can be established using H-admissibility only:

Theorem 4.2. *If* $f(z)$ *is H-admissible in* $|z| < R$ *then* $\exp(f(z))$ *is HS-admissible in* $|z| < R$. *Furthermore, the error term* $h_N(n; d)$ *of equation (4.1) is* $o(\beta_n^{-N})$ *as* $n \to \infty$ *for every fixed* $N \geq 0$.

Müller[Mül97] proved HS-admissibility for exponentials of certain polynomials which has an interesting implication on the relation between HS-admissibility and H-admissibility (see next remark).

Theorem 4.3. *Let* $P(z) = \sum_{k=1}^{m} p_k z^k$ *be a polynomial of degree* $m \geq 1$ *with complex coefficients* p_k *and let* $f(z) = \exp(P(z))$. *Then the following assertions are equivalent:*

(i) $f(z)$ *is HS-admissible in* $\mathbb{C}$.

(ii) $P(z) \in \mathbb{R}[z]$ *and* $c_m > 0$.

Müller[Mül97] also gave an upper bound for the error term for a more special class of polynomials.

Theorem 4.4. *Suppose that* $P(z) = \sum_{k=1}^{m} p_k z^k$ *is a polynomial of degree* ≥ 2 *and that* $p_1 > 0$ *and* $p_k \geq 0$ *for* $1 < k \leq m$.

Then the function $f(z) = \exp(P(z))$ *is HS-admissible and the auxiliary function* $d(r)$ *can be chosen in such a way that for each fixed* $N \geq 0$

$$h_N(r, d) = O\left(n^{-N-1}\right), \qquad n \to \infty.$$

Remark 4.3. *Theorem 4.3 shows that HS-admissibility does not imply H-admissibility. In fact, Theorem 4.3 shows that the function* $e^{z(z-1)}$ *is HS-admissible. But the function is not H-admissible by Theorem 3.8 since its power series expansion at* $z = 0$ *has infinitely many negative coefficients.*

Note that the functions of Theorem 4.4 are H-admissible.

Example 11. *The function of Section 2.3.1,*

$$F(z) = \sum_{n\geq 0} F_n \frac{z^n}{n!} = \exp\left(z + \frac{z^2}{2}\right)$$

is HS-admissible in $\mathbb{C}$ by Theorem 4.3 with the auxiliary functions

$$A(z) = 1 + z, \qquad B_k(z) = \begin{cases} z + z^2 & \textit{if } k = 1 \\ \frac{z^2}{2} & \textit{if } k = 2 \\ 0 & \textit{else} \end{cases},$$

$$B(z) = \frac{z}{2} + z^2, \qquad C_j(z, r) = \frac{(-1)^{j+1}}{j+2}\left(1 + \frac{1}{1+2z}\right).$$

The unique positive solution of $B_1(u) = n + 1$ is given by

$$\begin{aligned} u_n &= \frac{-1 + \sqrt{4n+5}}{2} = \sqrt{n}\left(-\frac{1}{2\sqrt{n}} + \sqrt{1 + \frac{5}{4n}}\right) \\ &= \sqrt{n}\left(1 - \frac{1}{2}n^{-1/2} + \frac{5}{8}n^{-1} - \frac{25}{128}n^{-2} + O(n^{-3})\right). \end{aligned}$$

Hence

$$\begin{aligned} C_j(u_n, u_n) &= \frac{(-1)^{j+1}}{j+2}\left(1 + \frac{1}{\sqrt{4n+5}}\right) \\ &= \frac{(-1)^{j+1}}{j+2}\left(1 + \frac{1}{2}n^{-1/2} - \frac{5}{16}n^{-3/2} + O(n^{-2})\right) \\ B(u_n) &= \frac{u_n}{2} + u_n^2 = n + 1 - \frac{u_n}{2} \\ &= n + 1 + \frac{\sqrt{n}}{2}\left(1 - \frac{1}{2\sqrt{n}} + \frac{5}{8n} + O(n^{-2})\right) \\ &= n\left(1 + \frac{1}{2}n^{-1/2} + \frac{3}{4}n^{-1} + \frac{5}{16}n^{-3/2} + O(n^{-2})\right) \end{aligned}$$

Theorems 4.1 and 4.4 now give (putting $N = 1$) after some simplifications

$$\frac{F_n}{n!} = \frac{e^{u_n + u_n^2/2}}{2u_n^n\sqrt{\pi B(u_n)}}\left(1 + \frac{7}{24}n^{-1} + \frac{5}{96}n^{-3/2} + O(n^{-2}))\right).$$

Calculating

$$\begin{aligned} u_n^{-n} &= n^{-n/2}\exp\left(-n\log\left(1 - \frac{1}{2\sqrt{n}} + \frac{5}{8n} - \frac{25}{128}n^{-2} + O(n^{-3})\right)\right) \\ &= n^{n/2}\exp\left(\frac{\sqrt{n}}{2} - \frac{1}{2} - \frac{13}{48}n^{-1/2} - \frac{13}{128}n^{-1} + O(n^{-2})\right) \\ u_n + \frac{u_n^2}{2} &= n + 1 - \frac{u_n^2}{2} \\ &= n + 1 - \frac{n}{2}\left(1 - n^{-1/2} + \frac{3}{2}n^{-1} - \frac{5}{8}n^{-3/2} + \frac{25}{128}n^{-5/2} + O(n^{-3})\right) \\ &= \frac{n + \sqrt{n}}{2} + \frac{1}{4} + \frac{5}{8\sqrt{n}} - \frac{25}{256}n^{-3/2} + O(n^{-2}). \end{aligned}$$

and

$$\begin{aligned} \frac{u_n}{2} + u_n^2 &= n + 1 - \frac{u_n}{2} \\ &= n\left(1 - \frac{1}{2}n^{-1/2} + \frac{5}{4}n^{-1} - \frac{5}{16}n^{-3/2} + O(n^{-2})\right) \\ \left(\frac{u_n}{2} + u_n^2\right)^{-1/2} &= \frac{1}{\sqrt{n}}\left(1 + \frac{1}{4}n^{-1/2} - \frac{17}{32}n^{-1} - \frac{35}{128}n^{-3/2} + O(n^{-2})\right) \end{aligned}$$

we obtain

$$\frac{F_n}{n!} = \frac{\exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right)}{2\sqrt{\pi n^{n-1}}}\left(1 + \frac{55}{48}n^{-1/2} + \frac{2245}{4608}n^{-1} + O(n^{-3/2})\right)$$

and an application of Stirling's formula gives the final result

$$\begin{aligned} F_n &= \frac{n^{n/2}}{\sqrt{2}}\exp\left(-\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right) \times \\ &\quad \times \left(1 + \frac{55}{48}n^{-1/2} + \frac{2629}{4608}n^{-1} + \frac{19061}{663552}n^{-3/2} + O(n^{-2})\right). \end{aligned} \tag{4.2}$$

4.2 GH-admissibility

Mutafchiev [Mut92] proposed a generalisation of Hayman's concept of admissibility, called GH-admissibility, with the asymptotics (3.1) and (3.2) of Definition 3.1 replaced by weaker conditions.

In the following we present Mutafchiev's definition and results but follow Hayman more closely and prove an analogue of Theorem 3.1 for GH-admissible functions. We will first list some important facts concerning characteristic functions of distribution functions. Then we state a slightly modified definition of GH-admissibility followed by some remarks. We then proceed stating the main results of this sections and show how GH-admissibility can be used to infer local limit theorems for the number of components of combinatorial structures.

Mutafchiev [Mut92] also applied his concept to three combinatorial problems already known to satisfy a local limit theorem. Unfortunately, as we will show in this section, the generating functions of the problems considered by Mutafchiev are not GH-admissible and therefore his paper [Mut92] does not contain any valid applications of the concept of GH-admissibility.

Remark 4.4. *We summarise some important properties of absolutely integrable and infinitely divisible characteristic functions* $h(t)$. *Details and proofs of these facts can be found in [Luk70]. Let* $H(x)$ *denote the probability distribution function corresponding to* $h(t)$. *We have*

- $H(x)$ *is absolutely continuous w.r.t. the Lebesgue-measure with continuous density* $H'(x)$*;*

- *The probability density* $H'(x)$ *satisfies*

$$H'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} h(t) dt, \qquad -\infty < x < \infty. \tag{4.3}$$

Definition 4.2. *Suppose that* $f(z)$ *is analytic in* $|z| < R$, $R > 0$, *and real for real z. Assume further that there exists* R_0, $0 \le R_0 < R$, *such that* $f(r) > 0$ *for* $R_0 \le r < R$. *For this range define the functions*

$$a(r) = r\frac{f'(r)}{f(r)} \qquad \text{and} \qquad b(r) = ra'(r) = r\frac{f'(r)}{f(r)} + r^2\frac{f''(r)}{f(r)} - \left(r\frac{f'(r)}{f(r)}\right)^2.$$

Then $f(z)$ *is said to be GH-admissible in* $|z| < R$ *if and only if there exists a function* $\delta : [0, R) \to [0, \pi]$ *and an absolute integrable and infinitely divisible characteristic function* $h(t)$ *defined by a non-degenerate probability distribution with finite variance such that*

(i) $b(r) \to \infty$ *as* $r \to R$*;*

(ii) the probability density function $H'(x)$ *corresponding to* $h(t)$ *satisfies* $H'(0) > 0$*;*

(iii) $f(re^{i\phi}) \sim f(r)e^{i\phi a(r)} h(\phi\sqrt{b(r)})$ *as* $r \to \infty$ *uniformly for* $|\phi| < \delta(r)$*;*

(iv) $\displaystyle\int_{\delta(r)\le|\phi|\le\pi} s(r,\phi) f(re^{i\phi}) d\phi = \frac{o(f(r))}{\sqrt{b(r)}}$ *as* $r \to R$ *for any complex valued function* $s(r,\phi)$ *satisfying* $s(r,\phi) = O(1)$ *as* $r \to R$ *uniformly for* $\delta(r) \le |\phi| \le \pi$.

Remark 4.5. *Instead of Condition (iv) in Definition 4.2 Mutafchiev originally required that*

(iv') $\displaystyle\int_{\delta(r)\le|\phi|\le\pi} s(r,\phi) f(re^{i\phi}) e^{-iZ\phi\sqrt{b(r)}} d\phi = \frac{o(f(r))}{\sqrt{b(r)}}$ *as* $r \to R$ *for any real* $Z \neq 0$ *and any complex valued function* $s(r,\phi)$ *satisfying* $s(r,\phi) = O(1)$ *as* $r \to R$ *uniformly for* $\delta(r) \le |\phi| \le \pi$.

Conditions (iv) and (iv') are equivalent since $s(r,\phi)e^{-iZ\phi\sqrt{b(r)}}$ *also satisfies the assumptions imposed on* $s(r,\phi)$. *Hence the factor* $e^{-iZ\phi\sqrt{b(r)}}$ *need not be mentioned explicitly.*

Remark 4.6. *We adopt the notation of the last definition.*

Consider the power-series distributed random variable $\xi(r)$ *whose distribution is determined by*

$$\mathbb{P}(\xi(r) = k) = \frac{f_k r^k}{f(r)}, \qquad k \ge 0, \ 0 < r < R.$$

Then we have $\mathbb{E}\xi(r) = a(r)$ *and* $\mathbb{V}\xi(r) = b(r)$ *and the characteristic function of the normalised random variable* $(\xi(r) - a(r))/\sqrt{b(r)}$ *is seen to be*

$$\alpha(t; r) = \frac{f\left(re^{it/\sqrt{b(r)}}\right)}{f(r)} \exp\left(-it\frac{a(r)}{\sqrt{b(r)}}\right).$$

Setting $\phi = t/\sqrt{b(r)}$ *in Definition 4.2, Condition (iii), we see that*

$$\alpha(t;r) \to h(t), \qquad r \to R,$$

whenever $|t| \le \delta(r)\sqrt{b(r)}$.

H-admissible functions are GH-admissible with $h(t) = e^{-t^2/2}$*, i.e., the coefficients of H-admissible functions satisfy a normal limit law (see Corollary 3.2.).*

As for H-admissible functions, we have

Lemma 4.1. *We adopt the notation of Definition 4.2. Then*

$$\delta(r)\sqrt{b(r)} \to \infty, \qquad r \to R.$$

The proof of Lemma 4.1 can be found in [Mut92]. We only note that

Theorem 4.5. *Suppose that* $f(z) = \sum_{n\ge 0} f_n z^n$ *is GH-admissible in* $|z| < R$ *with characteristic function* $h(t)$ *and corresponding probability distribution function* $H(x)$ *and define* $f_n = 0$ *for* $n \in \mathbb{Z}_-$*. Then we have uniformly for all integers* n

$$f_n = \frac{f(r)}{r^n\sqrt{b(r)}}\left(H'\left(-\frac{a(r)-n}{\sqrt{b(r)}}\right) + o(1)\right), \qquad r \to R. \tag{4.4}$$

Proof. The starting point is Cauchy's formula, viz.

$$f_n r^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{i\phi})\frac{d\phi}{e^{in\phi}}.$$

We set

$$I_1 := \frac{1}{2\pi}\int_{-\delta(r)}^{\delta(r)} f(re^{i\phi})\frac{d\phi}{e^{in\phi}} \qquad \text{and} \qquad I_2 := \frac{1}{2\pi}\int_{\delta(r)}^{2\pi-\delta(r)} f(re^{i\phi})\frac{d\phi}{e^{in\phi}}.$$

Putting $s(r,\phi) = \exp(-in\phi)$ in Condition (iv) of Definition 4.2 we see that

$$I_2 = \frac{o(f(r))}{\sqrt{b(r)}}, \qquad r \to R,$$

and Condition (iii) together with the absolute integrability of $h(t)$ gives uniformly in n

$$I_1 = \frac{f(r)}{2\pi}\left(\int_{-\delta(r)}^{\delta(r)} h\left(\phi\sqrt{b(r)}\right) e^{i\phi(a(r)-n)}\, d\phi + o(b(r))^{-1/2}\right).$$

Finally, substituting $t = \phi\sqrt{b(r)}$ and using Condition (i) as well as the absolute integrability of $h(t)$ and (4.3) we obtain as $r \to R$

$$\begin{aligned}\int_{-\delta(r)}^{\delta(r)} h\left(\phi\sqrt{b(r)}\right) e^{i\phi(a(r)-n)}\, d\phi &= \frac{1+o(1)}{\sqrt{b(r)}} \int_{-\infty}^{\infty} h(t) \exp\left(it\frac{a(r)-n}{\sqrt{b(r)}}\right) dt \\ &= \frac{2\pi f(r)}{\sqrt{b(r)}} H'\left(-\frac{a(r)-n}{\sqrt{b(r)}}\right) + \frac{o(f(r))}{\sqrt{b(r)}}\end{aligned}$$

which completes the proof. □

Corollary 4.1. *We have*

$$a(r) \to \infty, \qquad r \to R. \tag{4.5}$$

Furthermore, there exists $R_1 < R$ *such that* $a(r)$ *is strictly monotonically increasing for* $R_1 \le r < R$ *and*

$$b(r) = O(a(r))^2, \qquad r \to R. \tag{4.6}$$

Proof. We have $b(r) = ra'(r) \to \infty$ as $r \to R$ which shows that $a(r)$ is finally strictly monotonically increasing as $r \to R$.

Putting $n = -1$ in (4.4) yields

$$H'\left(-\frac{a(r)+1}{\sqrt{b(r)}}\right) = o(1), \qquad r \to R.$$

Since $H'(x)$ is continuous and $H'(0) \neq 0$ by Definition 4.2 there exists $M \in \mathbb{R}_+$ such that

$$\left|\frac{a(r)-1}{\sqrt{b(r)}}\right| \ge M, \qquad r \to R.$$

From this and the fact that $b(r) \to \infty$ as $r \to R$ it follows that $a(r) \to \infty$ as $r \to R$ and $b(r) = O(a(r))^2$. □

Corollary 4.2. *For* n *large enough the equation* $a(r) = n$ *has a unique solution* r_n *which satisfies* $r_n \to R$ *as* $n \to \infty$. *Furthermore*

$$f_n \sim \frac{f(r_n)H'(0)}{r_n^n\sqrt{b(r_n)}}, \qquad n \to \infty. \tag{4.7}$$

Proof. From (4.5) and the fact that $a(r)$ is finally strictly monotonic increasing we see that the equation $a(r) = n$ has, at least for n large enough, a unique positive solution r_n. This solution satisfies $r_n \to R$ as $n \to \infty$. Putting $r = r_n$ in (4.4) proves the asymptotic for f_n. □

We now turn to the problem of determining the distribution of the number of components in combinatorial classes.

Suppose that $\mathcal{G}$ and $\mathcal{F}$ are two labelled combinatorial classes related via $\mathcal{G} = \mathrm{set}\mathcal{F}$. The corresponding exponential generating functions

$$g(z) = \sum_{n\geq 0} g_n z^n/n! \qquad \text{and} \qquad f(z) = \sum_{n\geq 1} f_n z^n/n!$$

then satisfy

$$g(z) = e^{f(z)}. \tag{4.8}$$

Now let $\xi_1(r), \xi_2(r), \ldots$ denote a sequence of i.i.d. random variables generated by $f(z)$, that is

$$\mathbb{P}\{\xi_1 = k\} = \frac{f_k r^k}{f(r)k!}, \qquad 0 < r < R,\ k = 0, 1, \ldots, \tag{4.9}$$

where R denotes the radius of convergence of $f(z)$, define the sequence of random variables

$$S_N(r) = \xi_1(r) + \cdots + \xi_N(r), \qquad N = 1, 2, \ldots \tag{4.10}$$

and let $X = X(\omega)$ denote the number of components of $\omega \in \mathcal{G}$. Finally, let $\mathbb{P}_n$ and $\mathbb{E}_n$ denote the uniform probability measure and expectation defined on the set of structures of size n of $\mathcal{G}$. Then we have

$$\mathbb{P}_n\{X = N\} = \frac{f(r)^N n!}{g_n r^n N!}\mathbb{P}\{S_N(r) = n\}, \qquad N = 1, 2, \ldots, n,\ n = 1, 2, \ldots$$

and

$$\sum_{n\geq 0} (\mathbb{E}_n X)\frac{g_n z^n}{n!} = f(z)g(z).$$

Proofs for these facts can be found in [Kol86] and [Com87].

We can now state Mutafchiev's results concerning the distribution of X.

Theorem 4.6. *Suppose that $f(z)$ and $g(z)$ are power series with nonnegative coefficients satisfying (4.8) and let $R > 0$ denote their common radius of convergence. Moreover, let $g(z)$ be GH-admissible with distribution function $H(x)$ and functions $a(r)$ and $b(r)$ as defined in Definition 4.2 such that*

$$\sqrt{b(r)} \sim d g(r), \qquad r \to R,$$

for some $0 < d < \infty$.

Then

(i) If $N = f(r)(1 + o(1))$ as $r \to R$, then the distribution of the sums $S_N(r)$ defined by (4.10) satisfy

$$\mathbb{P}\{S_N(r) = k\} = \frac{H'(\gamma) + o(1)}{\sqrt{b(r)}}, \qquad r \to R,$$

where $\gamma = (a(r) - k)/\sqrt{b(r)}$. This convergence is uniform w.r.t. γ belonging to an arbitrary compact set.

(ii) If r_n *is defined by* $a(r_n) = n$ *then*

$$\mathbb{P}_n\{X = N\} \sim \frac{1}{\sqrt{2\pi f(r_n)}} \exp\left(-\frac{(N - f(r_n))^2}{2f(r_n)}\right), \qquad n \to \infty,$$

uniformly w.r.t. N *such that* $(N - f(r_n))/f(r_n)^{7/12}$ *belongs to an arbitrary compact set.*

Remark 4.7. *The last theorem can be applied to all GH-admissible functions which are not H-admissible (see [Mut92, Remark 2.2] for details).*

Theorem 4.7. *Suppose that* $f(z)$ *and* $g(z)$ *are analytic for* $|z| < R$ *and satisfy (4.8). If* $g(z)$ *is GH-admissible then*

$$\mathbb{E}_n X \sim f(r_n), \qquad n \to \infty,$$

where r_n *is defined by* $a(r_n) = n$*.*

Proofs for the last two theorems can be found in [Mut92].

We close this section with a comment on the examples given by Mutafchiev in his paper [Mut92].

Remark 4.8. *Mutafchiev [Mut92] considered the egf for the number of permutations of length* $n \geq 0$ *given by*

$$f(z) = e^{-\log(1-z)} = \frac{1}{1-z}.$$

We have

$$a(r) = \frac{r}{1-r}, \qquad b(r) = \frac{r}{(1-r)^2}.$$

Let $\delta : [0, 1) \to [0, \pi]$ *be such that*

$$(1-r)^{2/3} \ll \delta(r) \ll (1-r)^{1/2}, \qquad r \to 1.$$

Then we have, uniformly for $|\phi| \leq \delta(r)$*, as* $r \to 1$

$$\begin{aligned} \frac{f(re^{i\phi})}{f(r)} &= \frac{1-r}{1-re^{i\phi}} = \frac{1}{1 - a(r)\,(e^{i\phi} - 1)} \\ &\sim \frac{1}{1 - i\phi a(r)} \sim e^{i\phi a(r)} \frac{e^{-i\phi\sqrt{b(r)}}}{1 - i\phi\sqrt{b(r)}}. \end{aligned}$$

If $f(z)$ *would be GH-admissible in* $|z| < 1$ *then the corresponding characteristic function would be given by*

$$h(t) = \frac{e^{-it}}{1 - it}, \qquad -\infty < t < \infty,$$

which corresponds to a shifted Gamma density, viz.

$$H'(x) = \begin{cases} e^{-x-1} & \text{if } x \geq -1 \\ 0 & \text{if } x < -1. \end{cases}$$

The function $f(z)$ *cannot be GH-admissible for two reasons:*

- *The probability density function* $H'(x)$ *corresponding to* $h(t) = e^{-it}/(1 - it)$ *has a discontinuity at* $x = -1$. *Hence* $h(t)$ *cannot be absolutely integrable (see [Luk70, Theorem 3.2.2]).*
- *Condition (iv) of Definition 4.2 is not satisfied by* $f(z)$. *To see this, we choose the function* $s(r;\phi) = \exp\left(-i \arg f(r^{i\phi})\right)$. *Then*

$$\begin{aligned}\int\limits_{\delta(r)\le|\phi|\le\pi} s(r;\phi) f(re^{i\phi}) d\phi &= \int\limits_{\delta(r)\le|\phi|\le\pi} |f(re^{i\phi})| d\phi \\ &= \int\limits_{\delta(r)\le|\phi|\le\pi} \frac{d\phi}{\sqrt{1 + r^2 - 2r\cos\phi}} \\ &\ge \int\limits_{\delta(r)\le|\phi|\le\pi} \frac{d\phi}{\sqrt{1 + r^2}}\end{aligned}$$

  which is not $o(1)$ *as* $r \to 1$.

The generating functions

$$g(z) = \frac{1 + z}{\sqrt{1 - z}}$$

and

$$G(z) = \frac{1}{1 - T(z)} \qquad \textit{where} \qquad T(z) = ze^{T(z)}$$

considered in the remaining two examples of [Mut92], again, lead to characteristic functions which are not absolutely integrable since the corresponding probability distributions have got discontinuities. Hence, the functions $g(z)$ *and* $G(z)$ *cannot be GH-admissible.*

Remark 4.9. *Mutafchiev [Mut97] considered the distribution of the parameter 'number of distinct component sizes' on the set of combinatorial structures of size* n *in certain combinatorial classes as* $n \to \infty$. *Using the concept of GH-admissibility Mutafchiev [Mut97] established weak convergence results to a convolution of two distributions, where one of them is always Gaussian.*

In his paper [Mut97] he also presented three examples. In case of the first two examples, namely set-partitions and integer-partitions, the generating function is not only GH-admissible but even H-admissible. Unfortunately, the third example constitutes an invalid application of his results since the generating function for the number of functions from $\{1, 2, \ldots, n\}$ *into itself is given by* $G(z)$ *of Remark 4.8 which is not GH-admissible.*

4.3 BR-admissibility

In this section we consider a multivariate generalisation of the notion of H-admissibility which is due to Bender and Richmond [BR96]. They proved a theorem analogous to Theorem 3.1 which can be used to obtain local limit theorems for a variety of combinatorial problems.

First, we need some definitions concerning the notation:

Definition 4.3. *For the* d*-variable function* $f(z) = \sum_{n \geq 0} a_n z^n$ *we define:*

- Λ_f *is the* $\mathbb{Z}$*-module spanned by differences of those* n *for which* $a_n \neq 0$.
- $d(\Lambda_f)$ *denotes the absolute value of the determinant of a* $\mathbb{Z}$*-basis of* Λ_f.
- Λ_f^* *is the polar lattice of* Λ_f,

$$\Lambda_f^* = \left\{ \mathbf{x} \in \mathbb{R}^d \ : \ \forall \mathbf{n} \in \Lambda_f \text{ we have } \mathbf{x} \cdot \mathbf{n} \in \mathbb{Z} \right\}$$

  where $\mathbf{x} \cdot \mathbf{n}$ *denotes the usual scalar product.*
- *Let* $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ *be a* $\mathbb{Z}$*-basis of* Λ_f^*. *Then we define the* fundamental region of f *as the parallelepiped*

$$\Phi_f = \{c_1 \mathbf{v}_1 + \ldots + c_d \mathbf{v}_d \mid \forall\, 1 \leq k \leq d: \ -\pi \leq c_k \leq \pi\}. \tag{4.11}$$

We will also make use of the following asymptotic notation:

Definition 4.4. *We write* $f(z) = o_{u(z)}(g(z))$ *for* z *in some set* S, *if there exists a function* $\lambda(t)$ *such that* $\lambda(t) \to 0$ *as* $t \to \infty$ *and* $|f(z)/g(z)| \leq \lambda(|u(z)|)$ *for* $z \in S$.

Now, we can state the central definition and the main theorem of [BR96]. Remarks will be given right after the proof of this theorem.

Definition 4.5 (BR-Admissibility). *Let* $f(\mathbf{z})$ *be a* d*-variable function analytic at the origin having a fundamental region* Φ_f. *If* Λ_f *is* d*-dimensional, then we say that* $f(\mathbf{z})$ *is* BR-admissible in $\mathcal{R} \subseteq \mathbb{R}_+^d$ with angles Θ *if there exist functions*

- $\Theta : R \to \{S \subseteq \Phi_f \mid 0 \in S \text{ and } S \text{ is an open set}\}$,
- $a : \mathbb{C}^d \to \mathbb{C}^d$ *and*
- $B : \mathbb{C}^d \to \mathbb{C}^{d \times d}$

such that (we write o_B *for* $o_{\det B(\mathbf{r})}$*)*

(i) $f(\mathbf{z})$ *is analytic whenever* $\mathbf{r} \in \mathcal{R}$ *and* $|z_i| \leq r_i$ *for* $1 \leq i \leq d$,

(ii) $B(\mathbf{r})$ *is positive definite for* $\mathbf{r} \in \mathcal{R}$,

(iii) the diameter of $\Theta(\mathbf{r})$ *is* $o_B(1)$,

(iv) for $\mathbf{r} \in R$ *and* $\theta \in \Theta(\mathbf{r})$, *we have*

$$f(\mathbf{r}e^{i\theta}) = f(\mathbf{r})(1 + o_B(1)) \exp\left(i a(\mathbf{r})'\theta - \theta' B(\mathbf{r})\theta/2\right) \tag{4.12}$$

(v) and for $\mathbf{r} \in R$ *and* $\theta \in \Phi_f - \Theta(\mathbf{r})$, *we have*

$$f(\mathbf{r}e^{i\theta}) = \frac{o_B(f(\mathbf{r}))}{\sqrt{|B(\mathbf{r})|}}. \tag{4.13}$$

The function $f(z)$ *is called* BR-super-admissible *if condition (4.13) can be replaced by*

$$f(\mathbf{r}e^{i\theta}) = \frac{o_B(f(\mathbf{r}))}{|B(\mathbf{r})|^t} \tag{4.14}$$

for arbitrary $t \in \mathbb{R}$, *where* $o_B(f(\mathbf{r}))$ *may depend on* t.

Theorem 4.8. *Let* $f(z) = \sum_{\mathbf{n}\geq 0} a_{\mathbf{n}} z^{\mathbf{n}}$ *be a* d*-variable function that is BR-admissible in* $\mathcal{R}$ *with angles* Θ *and let* $\mathbf{k} \in \mathbb{N}^d$ *be such that* $[z^{\mathbf{k}}]f(z) \neq 0$ *and set* $\mathbf{v} = a(\mathbf{r}) - \mathbf{n}$. *Then we have*

$$[z^{\mathbf{n}}]f(z) = \frac{d(\Lambda_f) f(\mathbf{r})\mathbf{r}^{-\mathbf{n}}}{(2\pi)^{d/2}\sqrt{\det \mathbf{B}(\mathbf{r})}} \left(\exp\left(-\mathbf{v}'\mathbf{B}^{-1}(\mathbf{r})\mathbf{v}/2\right) + o_B(1)\right) \tag{4.15}$$

for $\mathbf{r} \in \mathcal{R}$ *and* $\mathbf{n} - \mathbf{k} \in \Lambda_f$.

Proof. For brevity, we omit the arguments of $B(\mathbf{r})$ and $\Theta(\mathbf{r})$.

By Cauchy's theorem, we have

$$a_{\mathbf{n}}\mathbf{r}^{\mathbf{n}} = \frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} f(\mathbf{r}e^{i\theta}) e^{-i\mathbf{n}'\theta} d\theta. \tag{4.16}$$

Assume that $a_{\mathbf{n}} \neq 0$ and $\mathbf{c} \in \Lambda_f^*$. By definition of Λ_f^*, the integrand remains unchanged if θ is replaced with $\theta + 2\pi\mathbf{c}$. Thus, we can write

$$a_{\mathbf{n}}\mathbf{r}^{\mathbf{n}} = \frac{d(\Lambda_f)}{(2\pi)^d} \int\limits_{\Phi_f} f(\mathbf{r}e^{i\theta}) e^{-i\mathbf{n}'\theta} d\theta. \tag{4.17}$$

Now, let $\Theta^\star = \Theta^\star(\mathbf{r})$ denote the greatest star-shaped region contained in Θ, that is

$$\theta \in \Theta^\star \iff \forall\, 0 < \rho < 1:\ \rho\theta \in \Theta.$$

We can work with $\Theta^\star$ instead of Θ since

- The interior of $\Theta^\star$ is contained in Θ.
- The boundary of $\Theta^\star$ is contained in $\Phi_f - \Theta$ and therefore we have

  $$\exp(-\theta' B\theta/2) = \frac{o_B(1)}{\sqrt{\det B}}$$

  on the boundary of $\Theta^\star$.
- For every θ, there is a $\kappa = \kappa(\mathbf{r})$ such that $\kappa\theta \in \Theta^\star$ because $0 \in \Theta^\star$.

- B is positive definite, that is $(\rho\theta)'B(\rho\theta) > \theta'B\theta$ for $\rho > 1$. Therefore we have

$$\exp\left(-\theta'B\theta/2\right) = \frac{o_B(1)}{\sqrt{\det B}} \qquad \text{for all } \theta \notin \Theta^\star. \tag{4.18}$$

Now, the integral (4.17) can be rewritten as

$$a_n r^n = \frac{d(\Lambda_f)}{(2\pi)^d} \int_{\Theta^\star} f(re^{i\theta}) \exp\left(-in'\theta\right) d\theta + \frac{o_B(f(r))}{\sqrt{\det B}}.$$

B is a positive definite matrix and thus, there exists a real $d \times d$ matrix S such that $B = S'S$. Using the abbreviations $y = S\theta$ and $w^2 = w'w$, we have

$$\begin{aligned} iv'\theta - \theta'B\theta/2 &= iv'S^{-1}y - y^2/2 = \\ &= -(S'^{-1}v)^2/2 - (y - iS'^{-1}v)^2/2 = \\ &= -v'B^{-1}v/2 - (y - iS'^{-1}v)^2/2. \end{aligned}$$

Hence, using equation (4.12), we get

$$\begin{aligned} \int_{\Theta^\star} e^{iv'\theta - \theta'B\theta/2} d\theta &= \frac{e^{-v'B^{-1}v/2}}{\sqrt{\det B}} \int_{S\Theta^\star} e^{-(y - iS'^{-1}v)^2/2} dy = \\ &= \frac{e^{-v'B^{-1}v/2}}{\sqrt{\det B}} \int_{\mathbb{R}^d} e^{-(y - iS'^{-1}v)^2/2} dy + \\ &\quad + \frac{O(1)e^{-v'B^{-1}v/2}}{\sqrt{\det B}} \int_T e^{-x^2/2} dx. \end{aligned}$$

As in the proof of Theorem 3.1 we get

$$\int_{\mathbb{R}^d} e^{-(y - iS'^{-1}v)^2/2} dy = \left(\int_{\mathbb{R}} e^{-(x - ic)^2/2} dx \right)^d = (2\pi)^{d/2}.$$

For $x \in T$ we have, by equation (4.18), $e^{-x^2} = o_B(1)/\sqrt{\det B}$ and therefore, we get

$$\int_T e^{-x^2/2} dx = o_B(1)$$

by essentially the same argument as in the proof of Theorem 3.1. Combining these results we obtain equation (4.15). ☐

Remark 4.10. *(i) Usually, one can let $a(z)$ and $B(z)$ be the first and second logarithmic derivatives of $f(z)$.*

(ii) *Definition 4.5 does not demand the unboundedness of* $\det B(\mathbf{r})$ *in* $\mathcal{R}$*. So, the* o_B*-terms in the equations (4.12) and (4.13) need not be small anywhere in* $\mathcal{R}$*. Hence, if* f *is BR-admissible in some set* $\mathcal{R}$*, then* f *is BR-admissible in every set* $\mathcal{R}_1 \subseteq \mathcal{R}$*, too.*

In applications, however, one is usually interested in maximal regions or at least regions large enough for $\det B$ *being unbounded.*

(iii) *If the function* $f(z)$ *is H-admissible for* $|z| < R$*, then* $f(z)$ *is BR-admissible in every set* $\mathcal{R} \subseteq (0, R)$*.*

(iv) *Let* $f(\mathbf{z})$ *be BR-admissible in* $\mathcal{R}$*. For every* $\mathbf{k} \in \mathbb{N}, k_i \geq 1$*, the function* $g(\mathbf{z}) := f(\mathbf{z}^{\mathbf{k}})$ *is BR-admissible, too.*

As a simple example take e^z*, which is H-admissible and thus BR-admissible in* $\mathbb{R}$*. This function has a fundamental region* $[-\pi, \pi]$ *and we can set* $\Theta(r) = [0, \delta(r)]$ *where* $\delta(r)$ *is the function of Definition 3.1 (in this case, we can use* $\delta(r) = r^{-2/5}$*). The function* e^{z^2} *is BR-admissible (but not H-admissible) with fundamental region* $[-\pi/2, \pi/2]$ *and* $\Theta(r) = [0, \delta(r)/2]$.

(v) *Besides* $\mathcal{R} \subseteq \mathbb{R}_+^d$*, Definition 4.5 does not impose any restriction on the set* $\mathcal{R}$*.*

Thus, one has to verify that BR-admissibility holds in a region $\mathcal{R}$ *having the right shape before applying this concept. In Section 5.2.3, we consider a function which is BR-admissible only in regions which cannot be used for proving an asymptotic normal distribution.*

(vi) *Theorem 4.8 allows one to compute asymptotics for the coefficients of BR-admissible functions if one has got sufficiently good estimates for the solution* $\mathbf{r_n}$ *of* $a(\mathbf{r}) = \mathbf{n}$ *as well as* $f(\mathbf{r_n})$ *and* $\mathbf{r_n^n}$*.*

(vii) *In many cases, BR-admissibility can be used to establish local limit theorems: Suppose that* $f(\mathbf{z}, \mathbf{u})$ *is BR-admissible and ordinary in* $\mathbf{u}$*. Partition all vectors and matrices into block form according to the two sets of variables* $\mathbf{x}$ *and* $\mathbf{y}$*. Solve* $\mathbf{a}(\mathbf{r}, \mathbf{1}) = (\mathbf{n}, \mathbf{k}^*)$ *for* $\mathbf{r}$ *asymptotically in terms of* $\mathbf{n}$ *and use this to compute* $\mathbf{k}^*$ *and* $\mathbf{B}(\mathbf{r}, \mathbf{1})$ *asymptotically in terms of* $\mathbf{n}$*. Let* $\mathbf{n} \to \infty$ *in a way such that* $(\mathbf{r}, \mathbf{1}) \in \mathcal{R}$ *and* $\det \mathbf{B}(\mathbf{r}, \mathbf{1}) \to \infty$*. It follows that* $[\mathbf{z^n u^k}]f(\mathbf{z}, \mathbf{u})$ *satisfies a local limit theorem with means vector asymptotic to* $\mathbf{k}^*$ *and covariance matrix asymptotic to*

$$(\mathbf{B}_{2,2} - \mathbf{B}_{1,2}'(\mathbf{B}_{1,1})^{-1}\mathbf{B}_{1,2})^{-1}$$

where

$$\mathbf{B}(\mathbf{z}, \mathbf{u}) = \begin{pmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{1,2}' & \mathbf{B}_{2,2} \end{pmatrix}$$

is the block form according to the variable sets $\mathbf{x}$ *and* $\mathbf{y}$*.*

If z *and* $\mathbf{u}$ *are* 1*-dimensional then the variance is given by* $\det \mathbf{B}/B_{1,1}$.

Note, that BR-admissibility does not necessarily entail a local limit theorem. See section 5.2.3 for a counter example.

If we want to combine functions having different sets of variables, we have to extend these functions and the definitions of $\mathcal{R}$, Θ, a and B to include all occurring variables.

Remark 4.11. *Let* $f(z)$ *be BR-admissible in* $\mathcal{R}$ *with angles* Θ *and let* y *be a variable not appearing in* f. *Set*

$$\bar{\mathcal{R}} = \mathcal{R} \times (0, \infty) \quad \textit{and} \quad \bar{\Theta}(\mathbf{r}, \rho) = \Theta(\mathbf{r}) \times [-\pi, \pi]$$

for $\mathbf{r} \in \mathcal{R}$ *and* $\rho \in (0, \infty)$. *The functions* a *and* B *are extended to* $\bar{a}$ *and* $\bar{B}$ *by adding entries of zeroes. Note that* $\det \bar{B} = 0$.

Generalisation to more variables is straight forward.

From now on, we will always assume that the functions are properly extended as described above.

Theorem 4.9. *Let* f *and* g *be BR-super-admissible in* $\mathcal{R}_f$ *with angles* Θ_f *and in* $\mathcal{R}_g$ *with angles* Θ_g, *resp. Assume that* $\det(\bar{B}_f + \bar{B}_g)$ *is unbounded in* $\mathcal{R} = \bar{\mathcal{R}}_f \cap \bar{\mathcal{R}}_g$.

If there are constants C *and* k *such that*

$$\det(\bar{B}_f(\mathbf{r}) + \bar{B}_g(\mathbf{r})) \leq C \min(\det(B_f(\mathbf{r}))^k, \det(B_g(\mathbf{r}))^k), \qquad \mathbf{r} \in \mathcal{R}, \tag{4.19}$$

then fg *is BR-super-admissible in* $\mathcal{R}$ *with angles* $\Theta = \bar{\Theta}_f \cap \bar{\Theta}_g$ *and we may take*

$$a_{fg} = \bar{a}_f + \bar{a}_g \quad \textit{and} \quad B_{fg} = \bar{B}_f + \bar{B}_g.$$

Furthermore, we have $\Lambda = \bar{\Lambda}_f + \bar{\Lambda}_g$.

Theorem 4.10. *Let* f *be BR-(super-)admissible in* $\mathcal{R}$ *with angles* Θ. *If the function* $g(\mathbf{r}e^{i\theta})$ *is analytic for* $\mathbf{r} \in \mathcal{R}$ *and for some functions* a_g *and* B_g, g *satisfies*

(i) $\Lambda_g \subseteq \Lambda_f$;

(ii) for $\mathbf{r} \in \mathcal{R}$ *and* $\theta \in \Theta$

$$(\mathbf{r}e^{i\theta}) = g(\mathbf{r}) \exp\left(i a_g' \theta - \theta' B_g \theta + o_B(1)\right);$$

(iii) there is a constant C *such that* $|g(\mathbf{r}e^{i\theta}| \leq C g(\mathbf{r})$ *for* $\mathbf{r} \in \mathcal{R}$;

(iv) there is a constant K *such that* $\det(\mathbf{B}_f + \bar{\mathbf{B}}_g) \leq K \det \mathbf{B}_f$ *for* $\mathbf{r} \in \mathcal{R}$.

Then fg *is BR-(super-)admissible in* $\mathcal{R}$ *with angles* Θ *and we may take*

$$a_{fg} = a_f + a_g \quad \textit{and} \quad \mathbf{B}_{fg} = \mathbf{B}_f + \mathbf{B}_g.$$

Theorem 4.11. *Let* $f(z) = \sum a_{\mathbf{n}} z^{\mathbf{n}}$ *be BR-(super-)admissible. For any sublattice* Λ *of* Λ_f *define*

$$g(z) = \sum_{\mathbf{n} \in \Lambda} a_{\mathbf{k}+\mathbf{n}} z^{\mathbf{k}+\mathbf{n}}$$

where $\mathbf{k}$ *is such that* $a_{\mathbf{k}} \neq 0$. *Then the function* g *is BR-(super-)admissible with*

$$\Lambda_g = \Lambda, \quad a_g = a_f, \quad B_g = B_f, \quad \mathcal{R}_g = \mathcal{R}_f, \quad \textit{and } \Theta_g = \Theta_f.$$

Theorem 4.12. *Assume that*

- $f(z) = \sum_{n\geq 1} a_n z^n$ *is H-admissible in* $|z| < R$*;*
- S *is a subset of* $\{0, 1, \dots, m-1\}$*;*
- *for* $0 \leq k < m$*, choose* $\lambda_k \in \mathbb{R}_+$ *such that* $\lambda_k > 0$ *if and only if* $k \in S$*;*
- *for* $n > m$*, define* $\lambda_n = \lambda_k$ *whenever* $n \equiv k \mod m$*;*
- *define*
$$g(z) = \sum_{n=0}^{\infty} \lambda_n a_n z^n$$
and $\bar{\lambda} = m^{-1} \sum_{k=0}^{m-1} \lambda_k$.

Then

(i) For some $R_0 < R$*, the function* $h(z) = e^{g(z)}$ *is BR-super-admissible in* $\mathcal{R} = (R_0, R)$ *with angles* $\Theta(r) = \{\theta : |\theta| < g(r)^{-1/3-\varepsilon}\}$ *and the functions* a *and* B*, provided* $\varepsilon > 0$ *is sufficiently small.*

(ii) For some $R_0 < R$ *and all* $\delta > 0$*, the function* $h(x, y) = e^{yg(x)}$ *is BR-super-admissible in*
$$\mathcal{R} = \left\{(r, s)\middle| R_0 < r < R \text{ and } g(r)^{\delta-1} < s < g(r)^{1/\delta}\right\}$$
with angles
$$\Theta(r, s) = \left\{\theta\middle||\theta_k| < (sg(r))^{-1/3-\varepsilon}\right\}$$
and functions a *and* B*, provided* $\varepsilon > 0$ *is sufficiently small.*

Example 12. *Consider the function*
$$F(z, u, w) = e^{u(\cosh z-1)} e^{w \sinh z}.$$
The coefficient of $z^n u^k w^m/n!$ *in* $F(z, u, w)$ *equals the number of partitions of a set of size* n *having* k *blocks of even size and* m *blocks of odd size. We show that* $F(z, u, w)$ *is BR-admissible.*

We consider the two functions
$$f(z, u) = e^{u(\cosh z-1)} \qquad \text{and} \qquad h(z, w) = e^{w \sinh z}.$$

Theorem 4.12, applied to the H-admissible function $e^z - 1$ *with* $m = 2$*,* $S = \{0\}$ *and* $\lambda_0 = 1$ *shows that* $f(z, u)$ *is BR-super-admissible in*
$$\mathcal{R}_f = \left\{(r, s) \;\middle|\; R_0 < r \text{ and } (\cosh r - 1)^{\delta-1} < s < (\cosh r - 1)^{1/\delta}\right\}$$
with angles
$$\Theta_f(r, s) = \left\{(\theta_1, \theta_2) \;\middle|\; |\theta_k| < (s(\cosh r - 1))^{-\epsilon_1 - 1/3},\ k = 1, 2\right\}$$

and matrix

$$B_f(r,s) = s \begin{pmatrix} r^2\cosh r + r\sinh r & r\sinh r \\ r\sinh r & \cosh r - 1 \end{pmatrix}$$

for any $\delta > 0$, *any* $R_0 > 0$ *and* ϵ_1 *sufficiently small. Putting* $S = \{1\}$ *and* $\lambda_1 = 1$ *instead, we see that* $h(z,w)$ *is BR-super-admissible in*

$$\mathcal{R}_h = \left\{(r,t) \;\middle|\; R_0 < r \text{ and } (\sinh r)^{\delta-1} < t < (\sinh r)^{1/\delta}\right\}$$

with angles

$$\Theta_h(r,t) = \left\{(\theta_1,\theta_2) \;\middle|\; |\theta_k| < (t\sinh r)^{-\epsilon_2-1/3},\ k=1,2\right\}$$

and matrix

$$B_h(r,t) = t \begin{pmatrix} r^2\sinh r + r\cosh r & r\cosh r \\ r\cosh r & \sinh r \end{pmatrix}$$

for any $\delta > 0$, *any* $R_0 > 0$ *and* ϵ_2 *sufficiently small.*

We now extend the functions f *and* h *and the corresponding matrices to include all variables occurring as described in Remark 4.11 and set* $\mathbf{B}(r,s,t) = \bar{\mathbf{B}}_f(r,s,t) + \bar{\mathbf{B}}_h(r,s,t)$, *that is*

$$\mathbf{B}(r,s,t) = \begin{pmatrix} r(rs+t)\cosh r + r(rt+s)\sinh r & rs\sinh r & rt\cosh r \\ rs\sinh r & s(\cosh r - 1) & 0 \\ rt\cosh r & 0 & t\sinh r \end{pmatrix}.$$

Furthermore, we set $\mathcal{R} = \bar{\mathcal{R}}_f \cap \bar{\mathcal{R}}_h$.

The determinants of the matrices $\mathbf{B}_f$, $\mathbf{B}_h$ *and* $\mathbf{B}$ *are given by*

$$\begin{aligned} \det \mathbf{B}_f(r,s) &= rs(\sinh r - r)(\cosh r - 1) \\ \det \mathbf{B}_h(r,t) &= rt(\cosh r \sinh r - r) \\ \det \mathbf{B}(r,s,t) &= (\sinh r)\det \mathbf{B}_f(r,s) + (\cosh r - 1)\det \mathbf{B}_h(r,t). \end{aligned}$$

For brevity we set $\det \mathbf{B}_f = \det \mathbf{B}_f(r,s)$ *and* $\det \mathbf{B}_h = \det \mathbf{B}_h(r,t)$. *We have*

$$\det \mathbf{B}_f \sim s\frac{re^{2r}}{4} \qquad \text{and} \qquad \det \mathbf{B}_h \sim t\frac{re^{2r}}{4} \qquad \text{as } r \to \infty$$

and therefore

$$\frac{\det \mathbf{B}_f}{\det \mathbf{B}_h} \sim \frac{s}{t}, \qquad t \neq 0,\ r \to \infty.$$

Now if $(r,s,t) \in \mathcal{R}$ *then*

$$\frac{(\cosh r - 1)^{\delta-1}}{(\sinh r)^{1/\delta}} < \frac{s}{t} < \frac{(\cosh r - 1)^{1/\delta}}{(\sinh r)^{\delta-1}}$$

which gives

$$\frac{e^{-r(1-\delta+1/\delta)}}{2^{1-\delta}}(1+o(1)) < \frac{s}{t} < \frac{e^{r(1-\delta+1/\delta)}}{2^{1/\delta}}(1+o(1)), \qquad r \to \infty.$$

If we let $R_0 > 0$ *be so large that*

$$\frac{e^{-r(1-\delta+1/\delta)}}{2^{2-\delta}} < \frac{s}{t} < \frac{e^{r(1-\delta+1/\delta)}}{2^{1/\delta-1}}, \qquad r \geq R_0,$$

as well as

$$s\frac{re^{2r}}{8} < \det \mathbf{B}_f < s\frac{re^{2r}}{2} \qquad \textit{and} \qquad t\frac{re^{2r}}{8} < \det \mathbf{B}_h < t\frac{re^{2r}}{2}$$

we see that there exist constants $C > 0$ *and* $\eta > 1$ *such that*

$$\det \mathbf{B}_f < K(\det \mathbf{B}_h)^{\eta} \qquad \textit{and} \qquad \det \mathbf{B}_h < K(\det \mathbf{B}_f)^{\eta}.$$

Since $\sinh r \sim \cosh r - 1 \sim e^r/2$ *as* $r \to \infty$ *we get*

$$\det \mathbf{B} \leq K\left((\det \mathbf{B}_f)^{\kappa}, (\det \mathbf{B}_h)^{\kappa}\right)$$

for some constants $K > 0$ *and* $\kappa > 1$.

Hence, by noting that $\det \mathbf{B}(r,s,t)$ *is unbounded in* $\mathcal{R}$ *we see that all assumptions of Theorem 4.9 are satisfied and we conclude that* $F(z,u,w)$ *is BR-super-admissible in* $\mathcal{R}$ *with angles*

$$\begin{aligned} \Theta(r,s,t) &= \bar{\Theta}_f(r,s) \cap \bar{\Theta}_h(r,t) \\ &= \left[-(s(\cosh r - 1))^{-\epsilon_1 - 1/3}, (s(\cosh r - 1))^{-\epsilon_1 - 1/3}\right]^2 \times \\ &\quad \times \left[-(t\sinh r)^{-\epsilon_2 - 1/3}, (t\sinh r)^{-\epsilon_2 - 1/3}\right] \end{aligned}$$

and matrix $\mathbf{B}(r,s,t)$.

4.3.1 Perturbation of BR-admissible Functions

If $f(z^2)$ is BR-admissible then $g(z) = f(z^2) + z$ is not. The reason for this lies in the definition of the corresponding lattices. We have $\Lambda_f = 2\mathbb{Z}$ and $\Lambda_g = \mathbb{Z}$, but clearly $[z^{2k+1}]g(z) = 0$ for $k > 0$. Bender and Richmond [BR96] remarked that the definition of BR-admissibility could be modified to include $g(z)$. But then the product rules for BR-admissible functions would not hold anymore.

In this section we show that if $f(\mathbf{z})$ is BR-admissible then $(f+g)(\mathbf{z})$ is BR-admissible, too, provided that $g(\mathbf{z})$ is sufficiently small and that $\Lambda_{f+g} \subseteq \Lambda_f$. Our reasoning will essentially be the same as in the proof of Theorem 3.7 for H-admissible functions. As a matter of fact the proof for this theorem is a bit shorter than the corresponding one for H-admissible functions. For details see the remark following the proof.

Remark 4.12. *Suppose that* $f : \mathbb{C}^n \to \mathbb{C}$ *is BR-admissible in* $\mathcal{R}$ *with angles* Θ *and functions* $\mathbf{a} = \mathbf{a}(\mathbf{r})$ *and* $\mathbf{B} = \mathbf{B}(\mathbf{r})$.

We may suppose that for $\mathbf{r} \in \mathcal{R}$

$$\Theta(\mathbf{r}) \subseteq \left\{\theta \in [-\pi,\pi]^n \mid \theta' \mathbf{B} \theta \leq 2 \log m(\mathbf{r})\right\} \tag{4.20}$$

where

$$m(\mathbf{r}) = \max(1, \det \mathbf{B})$$

since otherwise we should have for $\theta' \mathbf{B} \theta > 2 \log m(\mathbf{r})$

$$\exp\left(-\theta' \mathbf{B} \theta / 2\right) \quad < \quad \exp\left(-\log m(\mathbf{r})\right) \quad = \quad m(\mathbf{r})^{-1}$$

which implies implies (4.13) for $f(z)$.

Theorem 4.13. *Let* $f(z)$ *be BR-super-admissible in* $\mathcal{R}$ *with angles* Θ *and functions* $\mathbf{a}$ *and* $\mathbf{B}$ *and let* $g(z)$ *be a function such that*

(i) $g(z)$ *is analytic for* $|z| \in \mathcal{R}$;

(ii) $\Lambda_g \subseteq \Lambda_f$.

(iii) for any $t > 0$ *we have*

$$\max_{|z|=\mathbf{r}} |g(z)| = \frac{o_B(f(\mathbf{r}))}{(\det \mathbf{B}(\mathbf{r}))^t}, \qquad \mathbf{r} \in \mathcal{R}; \tag{4.21}$$

Then $f + g$ *is BR-super-admissible in* $\mathcal{R}$ *with angles* Θ *and functions* $\mathbf{a}$ *and* $\mathbf{B}$.

If f *is BR-admissible and* g *satisfies condition (iii) with* $t = 1$, *then* $f + g$ *is BR-admissible.*

Proof. We adopt the notation of Remark 4.12 and assume that $\Theta(\mathbf{r})$ satisfies (4.20).

First, we note that

$$f(\mathbf{r}) \quad = \quad f(\mathbf{r}) + g(\mathbf{r}) - g(\mathbf{r}) \quad = \quad (f+g)(\mathbf{r})(1 + o_B(1)) \tag{4.22}$$

by (4.21).

Suppose now that $\theta \notin \Theta(\mathbf{r})$. Since $f(z)$ is BR-super-admissible we have

$$f(\mathbf{r}e^{i\theta}) \quad = \quad \frac{o_B(f(\mathbf{r}))}{(\det \mathbf{B}(\mathbf{r}))^t}$$

for any $t > 0$. Now we get from this, (4.21) and (4.22)

$$\begin{aligned}
(f+g)\left(\mathbf{r}e^{i\theta}\right) \quad &= \quad \frac{o_B(f(\mathbf{r}))}{(\det \mathbf{B}(\mathbf{r}))^t} + \frac{o_B(f(\mathbf{r}))}{(\det \mathbf{B}(\mathbf{r}))^t} \\
&= \quad \frac{o_B(f(\mathbf{r}))}{(\det \mathbf{B}(\mathbf{r}))^t} \quad = \quad \frac{o_B((f+g)(\mathbf{r}))}{(\det \mathbf{B}(\mathbf{r}))^t}
\end{aligned}$$

uniformly for $\theta \notin \Theta(\mathbf{r})$ which implies (4.14) for $f + g$. In the same way we obtain (4.13) for BR-admissible f.

If $\theta \in \Theta(\mathbf{r})$ then we get from (4.20)

$$\left| f\left(\mathbf{r}e^{i\theta}\right) \right| \quad = \quad f(\mathbf{r})(1 + o_B(1)) e^{-\theta' \mathbf{B} \theta / 2} \quad \geq \quad \frac{f(\mathbf{r})}{m(\mathbf{r})}(1 + o_B(1))$$

where $m(\mathbf{r}) = \max(1, \det \mathbf{B})$. As a consequence of this and (4.21) we obtain

$$\frac{g(\mathbf{r}e^{i\theta})}{f(\mathbf{r}e^{i\theta})} \leq g\left(\mathbf{r}e^{i\theta}\right)\frac{m(\mathbf{r})}{f(\mathbf{r})}(1+o_B(1)) = o_B(1)$$

uniformly for $\theta \in \Theta(\mathbf{r})$. Hence by BR-admissibility of f and (4.22) we get

$$\begin{aligned}(f+g)\left(\mathbf{r}e^{i\theta}\right) &= f(\mathbf{r}e^{i\theta})(1+o_B(1)) = f(\mathbf{r})(1+o_B(1))e^{i\mathbf{a}(\mathbf{r})\theta-\theta'\mathbf{B}\theta/2} \\ &= (f+g)(\mathbf{r})(1+o_B(1))e^{i\mathbf{a}(\mathbf{r})\theta-\theta'\mathbf{B}\theta/2}\end{aligned}$$

uniformly for $\theta \in \Theta(\mathbf{r})$. This proves (4.12) for $f+g$. □

Remark 4.13. *Definition 4.5 does only require the existence of functions $\mathbf{a}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ but does not assume that these are equal to the first and second logarithmic derivatives of $f(\mathbf{z})$.*

This results in a shorter proof compared to the proof of Theorem 3.7 where we also had to show that the asymptotics hold with the appropriate functions (namely the logarithmic derivatives).

Example 13. *Consider the function*

$$G(z,u,w) = \left(e^{u(\cosh z-1)} - 1 - u(\cosh z - 1)\right)e^{w\sinh z}$$

which is a simple modification of the function $F(z,u,w)$ considered in Example 12. $G(z,u,w)$ is the exponential generating function for set partitions having at least 2 blocks of even cardinality and the coefficient of $z^n u^k w^m/n!$ in $G(z,u,w)$ equals the number of partitions of size n having k blocks of even cardinality and m blocks of odd cardinality.

We adopt the notation of Example 12 and set

$$f(z,u) = e^{u(\cosh z-1)} \qquad \text{and} \qquad g(z,u) = 1 + u(\cosh z - 1)$$

We have

$$\max_{|z|=r,|u|=s} g(z,u) = g(r,s) = 1 + s(\cosh r - 1) < 1 + s\frac{e^r}{2}$$

and for $r \geq R_0$

$$\frac{f(r,s)}{\det B_f(r,s)} > 2\frac{f(r,s)}{rse^{2r}} > 2\frac{e^{rse^r/2}}{rse^{2r}}$$

which gives

$$\frac{g(r,s)}{f(r,s)/\det \mathbf{B}_f} < \frac{rs}{2}\frac{e^{2r}+se^{3r}/2}{e^{rse^r/2}} = o(1), \qquad r \to \infty.$$

The last asymptotic is sufficient for proving (4.21) since $\det \mathbf{B}_f$ remains bounded in $\mathcal{R}$ if r remains bounded. The other conditions of Theorem 4.13 are satisfied, too, and therefore we may conclude that $G(z,u,w)$ is BR-super-admissible in $\mathcal{R}$ with angles Θ as defined in Example 12.

4.4 E-admissibility

In this section we present a bivariate analogon of H-admissibility developed by Drmota, Gittenberger, and Klausner [DGK05]. In their paper [DGK05] they defined classes of bivariate functions $f(z,u)$, called extended-admissibility (e-admissible) functions, such that the random variable X defined by

$$\mathbb{P}_n\{X = k\} = \frac{[z^n u^k]f(z,u)}{[z^n]f(z,1)}, \qquad k \geq 0, \tag{4.23}$$

satisfies a central limit theorem. An important property of these classes of e-admissible functions is the existence of many simple closure properties which simplify the task of establishing e-admissibility and can be used to automatically prove a central limit theorem for X given only a deconstruction of the combinatorial class considered. For details concerning this automation we refer to their paper [DGK05] where they also present an implementation of this concept using Maple.

Roughly speaking, a bivariate gf $f(z,u)$ is e-admissible if it is

- H-admissible w.r.t. z for u in some real interval around $u = 1$ and
- satisfies Lemma 3.1 w.r.t. u for $z \in \mathbb{R}^+$.

Some additional requirements are needed in order to have simple algebraic closure properties. We have

Definition 4.6 (e-admissibility). *Let $f(z,u) = \sum_{n,k\geq 0} a_{nk} z^n u^k$ be a bivariate function analytic in the domain*

$$\Delta_{R,\eta} = \left\{(z,u) \in \mathbb{C}^2 \ : \ |z| < R, \ |u| < 1+\eta\right\}$$

for some $R > 0$ and $\eta > 0$. Assume further that there exists $R_0 < R$ such that

$$f(r,1) > 0, \quad R_0 < r < R.$$

Let $a, \bar{a}, b, \bar{b}, c$ denote the derivatives of $\log f(z,u)$ w.r.t. $\log z$ and $\log u$, that is

$$a(z,u) = z\frac{f_z(z,u)}{f(z,u)}, \quad \bar{a}(z,u) = u\frac{f_u(z,u)}{f(z,u)},$$

$$b(z,u) = z a_z(z,u) = z\frac{f_z(z,u)}{f(z,u} + z^2\frac{f_{zz}(z,u)}{f(z,u)} - z^2\left(\frac{f_z(z,u)}{fz,u}\right)^2,$$

$$\bar{b}(z,u) = u\bar{a}_u(z,u), \quad c(z,u) = u a_u(z,u).$$

The function $f(z,u)$ is called e-admissible *in $\Delta_{R,\eta}$ if and only if*

(i) Let $K > 0$ be an arbitrary constant and set

$$\epsilon(r) = K\left(\bar{b}(r,1) - \frac{c(r,1)^2}{b(r,1)}\right)^{-1/2}. \tag{4.24}$$

Then, for each choice of K *there exists a function* $\delta(r) : (R_0, R) \to (0, \pi)$ *such that uniformly for* $|\phi| < \delta(r)$ *and* $1 - \epsilon(r) \le u \le 1 + \epsilon(r)$ *we have*

$$f(re^{i\phi}, u) \sim f(r, u) \exp\left(i\phi a(r, u) - \frac{\phi^2}{2} b(r, u)\right), \quad \text{as } r \to R, \tag{4.25}$$

and uniformly for $\delta(r) \le |\phi| \le \pi$ *and* $1 - \epsilon(r) \le u \le 1 + \epsilon(r)$ *we have*

$$f(re^{i\phi}, u) = \frac{o(f(r, u))}{\sqrt{b(r, u)}}, \quad \text{as } r \to R. \tag{4.26}$$

(ii) $b(r, 1) \to \infty$ *as* $r \to R$*;*

(iii) Uniformly for $1 - \epsilon(r) \le u \le 1 + \epsilon(r)$ *we have* $b(r, u) \sim b(r, 1)$ *as* $r \to R$.

(iv) For $r \in (R_0, R)$ *and* $u \in [1 - \epsilon(r), 1 + \epsilon(r)]$ *we have*

$$a(r, u) = a(r, 1) + c(r, 1)(u - 1) + O(c(r, 1)(u - 1)^2). \tag{4.27}$$

(v) For all $r < R$ *and* u *in some arbitrary but fixed complex neighbourhood of* 1 *we have*

$$\bar{a}(r, u) = O(\bar{a}(r, 1)) \quad \text{and} \quad \bar{b}(r, u) = O(\bar{b}(r, 1)). \tag{4.28}$$

(vi) $\bar{b}(r, 1) - \frac{c(r,1)^2}{b(r,1)} \to \infty$ *as* $r \to R$*;*

(vii) $\epsilon(r)^3 \bar{b}(r, 1) \to 0$ *as* $r \to R$*;*

(viii) For every $\lambda > 0$ *we have, as* $r \to R$,

$$\bar{a}(r, 1) = O(f(r, 1)^\lambda) \quad \text{and} \quad \bar{b}(r, 1) = O(f(r, 1)^\lambda).$$

Remark 4.14. *The class of functions that are e-admissible in a domain* $\Delta_{R,\eta}$ *will be denoted by* $\mathcal{E}_R$.

Remark 4.15. *The definition of e-admissibility implies that for every* $f(z, u) \in \mathcal{E}_R$ *we have* $f(z, 1) \in \mathcal{H}_R$.

Theorem 4.14. *Let* $f(z, u)$ *be e-admissible for* $|z| < R$*. Then we have*

$$[z^n]f(z, u) = \frac{f(r, u)}{r^n \sqrt{2\pi b(r, u)}} \left(\exp\left(-\frac{(a(r, u) - n)^2}{2b(r, u)}\right) + o(1)\right)$$

uniformly in n *and* $u \in [1 - \varepsilon(r), 1 + \varepsilon(r)]$ *as* $r \to R$.

The same arguments as in the proof of Theorem 3.1 can be used to prove Theorem 4.14.

Theorem 4.15. *Let $f(z,u)$ be e-admissible in $\Delta_{R,\eta}$ such that for sufficiently large n all coefficients a_{nk} are nonnegative. Let (X_n) be the sequence of random variables related to $f(z,u)$ via (4.23). The positive solution of $a(r,1) = n$ will be denoted by r_n.*

For $n \geq 0$ set

$$\begin{aligned}
\mu_n &= \bar{a}(r_n, 1), \\
\sigma_n^2 &= \frac{|\det B(r_n, 1)|}{b(r_n, 1)} = \bar{b}(r_n, 1) - \frac{c(r_n, 1)^2}{b(r_n, 1)} \\
Y_n &= \frac{X_n - \mu_n}{\sigma_n}.
\end{aligned}$$

Then the following central limit theorem holds:

$$Y_n \xrightarrow{w} \mathcal{N}(0,1), \quad n \to \infty. \tag{4.29}$$

Furthermore, we have, as $n \to \infty$,

$$\mathbb{E}X_n = \mu_n + o(\sigma_n)^2 \tag{4.30}$$

and

$$\mathbb{V}X_n = \sigma_n(1 + o(1)). \tag{4.31}$$

Proof. First note that $f(z,1)$ is H-admissible and therefore r_n is uniquely determined at least for n sufficiently large and $r_n \to R$ as $n \to \infty$.

Next, consider the moment generating function of X_n,

$$m_n(t) = \frac{[z^n]f(z, e^t)}{[z^n]f(z, 1)}, \quad |t| < \epsilon(r_n).$$

We note that $t \to 0$ as $n \to \infty$. An application of Theorem 4.14 gives

$$\begin{aligned}
[z^n]f(z,1) &= \frac{f(r_n, 1)}{r_n^n\sqrt{2\pi b(r_n, 1)}}(1 + o(1)) \\
[z^n]f(z,e^t) &= \frac{f(r_n, e^t)(1 + o(1))}{r_n^n\sqrt{2\pi b(r_n, e^t)}} \exp\left(-\frac{(a(r_n, e^t) - a(r_n, 1))^2}{2b(r_n, e^t)}\right)
\end{aligned}$$

as $n \to \infty$. (4.28) ensures applicability of Lemma 3.1 to the second argument of $f(r, e^t)$, viz.

$$f(r, e^t) = f(r, 1)\exp\left(t\bar{a}(r, 1) + \frac{t^2}{2}\bar{b}(r, 1) + O(\bar{b}(r, 1)t^3)\right), \quad r \to R.$$

Conditions (iii) and (iv) of Definition 4.6 give, as $n \to \infty$ (and so $t \to 0$),

$$\begin{aligned}
\frac{(a(r_n, e^t) - a(r_n, 1))^2}{2b(r_n, e^t)} &= \frac{c(r_n, 1)(e^t - 1)^2 + O(c(r_n, 1)(e^t - 1)^3)}{2b(r_n, 1)(1 + O(t))} \\
&= \frac{c(r_n, 1)t^2}{2b(r_n, 1)}\left(1 + O\left(\frac{c(r_n, 1)^2}{b(r_n, 1)}t^3\right)\right).
\end{aligned}$$

Combining these results we obtain

$$m_n(t) \quad = \quad \exp\left(t\mu_n + \frac{t^2}{2}\sigma_n^2 + O\left(\frac{c(r_n,1)^2}{b(r_n,1)}t^3\right) + o(1)\right)$$

for $|t| < \epsilon(r_n)$ as $n \to \infty$. Condition (vi) shows that $c(r,1) < \bar{b}(r,1)b(r,1)$ and therefore condition (vii) finally gives

$$m_n(t) = \exp\left(t\mu_n + \frac{t^2}{2}\left(\bar{b}(r_n,1) - \frac{c(r_n,1)^2}{b(r_n,1)}\right)\right)(1+o(1)) \tag{4.32}$$

for $|t| < \epsilon(r_n)$ as $n \to \infty$.

Hence the moment generating function of Y_n is given by

$$M_n(s) = e^{-s\mu_n/\sigma_n} m_n\left(\frac{s}{\sigma_n}\right) = e^{s^2/2}(1+o(1)) \tag{4.33}$$

for $|s| < K$ which proves (4.29).

The convergence $M_n(s) \to e^{t^2/2}$, $n \to \infty$, is uniform in every compact set and therefore the sequence (Y_n) has an exponential tail (see [Flajolet,Soria,Sec.4] for details). This implies the convergence of all moments. In particular we have

$$\begin{aligned} \mathbb{E}X_n &= \mu_n + \sigma_n \mathbb{E}Y_n &&= \mu_n + o(\sigma_n) \\ \mathbb{V}X_n &= \sigma_n^2 \mathbb{V}Y_n &&= \sigma_n^2(1+o(1)). \end{aligned}$$

□

Without proof we state the following two theorems which establish various closure properties satisfied by $\mathcal{E}_R$. The proofs can be found in [DGK05].

Theorem 4.16. *The following classes of functions are e-admissible:*

- *Let* $P(z,u) = \sum_n p_n z^{k_n} u^{l_n}$ *be a polynomial in* z *and* u *with real coefficients and set* $P(z,1) = \sum_m b_m z^m$. *Define*

$$\begin{aligned} K &= \max E &&= \max\left\{k_i + k_j \;:\; \det\begin{pmatrix} k_i & l_i \\ k_j & l_j \end{pmatrix} \neq 0\right\} \\ I &= \{(i,j) \;:\; k_i + k_j = K\}. \end{aligned}$$

  Then $e^{P(z,u)} \in \mathcal{E}_\infty$ *if and only if the following conditions are satisfied:*

  (i) For every $d > 1$ *there exists an* $m \not\equiv 0 \mod d$ *such that* $b_m \neq 0$. *Moreover, for* $m_d = \max\{m \not\equiv 0 \mod d \;:\; b_m \neq 0\}$ *we have* $b_{m_d} > 0$.

  (ii) $E \neq \emptyset$ *and*

$$\sum_{(i,j)\in I} p_i p_j \left(\det\begin{pmatrix} k_i & l_i \\ k_j & l_j \end{pmatrix}\right)^2 > 0.$$

(iii) $\max\{k_j : p_j \neq 0\} < 3K/5$.

- *If* $f(z) \in \mathcal{H}_R$ *and* $g(u)$ *is analytic for* $|u| \leq 1 + \zeta$ *and satisfies* $g(1) > 0$ *as well as* $g'(1) + g''(1) > \frac{g'(1)^2}{g(1)}$, *then* $\exp(g(u)f(z)) \in \mathcal{E}_R$.

Theorem 4.17. *Suppose that* $f(z,u), g(z,u) \in \mathcal{E}_R$, $h(z) \in \mathcal{H}_R$ *and* $P(z,u)$ *is a polynomial with positive real coefficients. Then the following functions are in* $\mathcal{E}_R$, *too:*

- $f(z,u)g(z,u)$
- $h(z)f(z,u)$
- $P(z,u)f(z,u)$
- $e^{f(z,u)}$
- $e^{P(z,u)h(z)}$ *if* $P(z,u)$ *is not independent of* u
- $e^{P(z,u)+h(z)}$ *if* $R = \infty$ *and* $P(z,u)$ *is not independent of* u
- $f(z,u) + Q(z,u)$ *where* $Q(z,u)$ *is an arbitrary polynomial.*

Example 14. *Consider the bivariate gf for permutations having cycles of length* $\leq \ell$ *only with* u *marking the number of cycles, viz.*

$$g(z,u) = \exp\left(u \sum_{k=1}^{\ell} \frac{z^k}{k}\right).$$

In order to establish e-admissibility for $g(z,u)$ *we have to check the conditions of Theorem 4.16:*

We see that K *is well-defined if and only if* $\ell \geq 2$. *In this case* $K = 2\ell - 1$ *and* $I = \{(\ell-1,\ell),(\ell,\ell-1)\}$. *Conditions (i) and (ii) are clearly satisfied and from condition (iii) we see that* $g(z,u)$ *is e-admissible if and only if* $\ell > 3$.

The same argumentation shows that all functions of the form

$$\exp\left(u \sum_{k=1}^{\ell} \alpha_k z^k\right), \qquad \alpha_k > 0,$$

are e-admissible if and only if $\ell > 3$.

Chapter 5

Number of Components

Let $\mathcal{C}$ and $\mathcal{S}$ denote two labelled combinatorial classes with exponential generating functions $C(z) = \sum_{n\geq 1} C_n z^n$ and $S(z) = \sum_{n\geq 0} S_n z^n$ respectively. We assume that $\mathcal{C}$ does not contain objects of size zero and that every object of S can be uniquely represented as a disjoint union of objects in $\mathcal{C}$. The elements of $\mathcal{C}$ will be referred to as connected objects. Theorem 1.4 tells us that $C(z)$ and $S(z)$ are related via

$$S(z) = e^{C(z)}. \tag{5.1}$$

Bell, Bender, Cameron, and Richmond [BBCR00] investigated the possible behaviour of the sequence C_n/S_n and proved the following theorem.

Theorem 5.1. *Let $\mathcal{S}$, $\mathcal{C}$, $S(z)$ and $C(z)$ be as described above. Furthermore, assume that $C(z)$ is analytic in $|z| < R$, $0 \leq R < \infty$.*

Consider the sequence $\{\rho_n : n \in I\}$ where $\rho_n = C_n/S_n$ and $I = \{n \in \mathbb{N} : S_n \neq 0\}$ and set

$$\rho_{\inf} = \liminf_{n\in I} \rho_n, \qquad \rho_{\sup} = \limsup_{n\in I} \rho_n.$$

Then

(i) If $R = 0$ then $0 \leq \rho_{\inf} \leq \rho_{\sup} = 1$;

(ii) If $R > 0$ and $C(R)$ diverges then $0 = \rho_{\inf} \leq \rho_{\sup} \leq 1$;

(iii) If $R > 0$ and $C(R)$ converges then $(\rho_{\inf}, \rho_{\sup}) \in [0,1]^2 - \{(0,0),(1,1)\}$.

If $S(z)$ is an H-admissible function then the second implication of the last theorem applies. But in this case, we can say even more as was shown by Bender, Cameron, Odlyzko, and Richmond [BCOR99].

Theorem 5.2. *We adopt the notation of the last theorem. If $S(z)$ is H-admissible in $|z| < R$, $0 < R \leq \infty$, then $C(R)$ diverges and we have $\rho_n \to 0$ as $n \to \infty$. That is, the probability of connectedness tends to zero as the size of the structures considered tends to infinity.*

Proof. The H-admissibility of $S(z)$ implies that $S(r) \to \infty$ as $r \to R$ and this implies that $C(R) = \infty$ which means divergence.

Set $b(r) = \left(\frac{d}{d\log r}\right)^2 \log S(r)$. Then we have for any $0 < \varepsilon < 1$ and $M = (1-\varepsilon)^2/2$

$$S_n \quad \sim \quad \frac{e^{C(r_n)}}{r_n^n\sqrt{2\pi b(r_n)}} \quad > \quad \frac{e^{(1-\varepsilon)C(r_n)}}{r_n^n} \quad > \quad MC(r_n)\frac{C(r_n)}{r_n^n} \quad \geq \quad MC(r_n)C_n.$$

Hereby, the first relation is a consequence of H-admissibility (Corollary 3.2) and the second one follows from the fact that $b(r) = S(r)^\varepsilon$ for all $\varepsilon > 0$ (Corollary 3.3). The remaining inequalities hold since the sum of nonnegative terms is at least as large as a single term.

The proof is completed by noting that $C(r_n) \to \infty$ as $n \to \infty$. □

In the remaining of this section we consider some instances of the general setting described above.

5.1 The Polynomial Case

Canfield [Can77] established the normal limit law for polynomials $C(z)$ having real nonnegative coefficients.

Although all functions (except the degenerated case) considered in this section are BR-admissible, too, Canfield's assumptions can be checked more easily and a central limit theorem follows without much work.

Canfield's method of proof is similar to the one establishing the normal limit law for e-admissible functions (see Theorem 4.15): using Hayman's ideas, Canfield proved the limit law via pointwise convergence of the corresponding moment generating functions. The proofs will be omitted can can be found in [Can77].

Theorem 5.3. *Let* $C(z) = \sum_{k=1}^m c_k z^k$ *be a polynomial having real nonnegative coefficients and set*

$$S(z,u) = e^{uC(z)}.$$

Further define the functions

$$A(z) = zC'(z) \qquad \textit{and} \qquad B(z) = zA'(z)$$

and let $r(z)$ *denote the inverse function of* $A(z)$*. Without loss of generality, we assume that* $\gcd\{k : c_k \neq 0\} = 1$ *and* $m > 1$*.*

Then the random variable X_n *defined by*

$$\mathbb{P}\{X_n = \mathbf{k}\} \quad = \quad \frac{[z^n\mathbf{u}^\mathbf{k}]S(z,u)}{[z^n]S(z,1)} \qquad \forall n \in I,\ \forall \mathbf{k} \in \mathbb{N}^d$$

where I *consists of those numbers for which* $[z^n]S(z,1) > 0$ *is asymptotically normal with mean*

$$\mu_n = C(r(n))$$

and variance

$$\sigma_n^2 = \sqrt{\frac{C(r(n)) - A(r(n))^2}{B(r(n)}}.$$

Remark 5.1. *(i) If* $\gcd\{k : c_k \neq 0\} = d > 1$ *then we have* $C_n = 0$ *if* $d \not| n$ *and the polynomial* $C(z^{1/d})$ *satisfies the requirements of the last theorem.*

(ii) The case $m = 1$ *constitutes a degenerate case since then the whole probability mass is concentrated at* μ_n.

We continue with the example of Section 2.3.1. Note that this function is not e-admissible (see Section 4.4, Example 14).

Example 15. *Recall the egf* $P(z, u)$ *for the permutations having only cycles of length* 1 *and* 2 *(*u *marking the number of cycles):*

$$P(z, u) = e^{u(z+z^2/2)}.$$

Thus, we have

$$C(z) = z + \frac{z^2}{2}, \qquad A(z) = z + z^2, \qquad B(z) = z + 2z^2.$$

The inverse function $r(x)$ *of* $A(x)$ *for* $x \in [-1/4, \infty)$ *is given by*

$$r(x) = -\frac{1}{2} + \sqrt{\frac{1}{4} + x}.$$

Therefore, one sees, using Theorem 5.3, that the number of cycles is asymptotically normal with mean

$$\mu_n = \frac{n}{2} + \frac{1}{2}\sqrt{\frac{1}{4} + n} - \frac{1}{4}$$

and variance (setting $y = n + \frac{1}{4}$*)*

$$\sigma_n^2 = \sqrt{\frac{y^2 + y(1 - 2\sqrt{y})}{4(2y - \sqrt{y})}} = \sqrt{\frac{y}{8}}\sqrt{1 + \frac{1 - \frac{3}{2}\sqrt{y}}{y - \frac{1}{2}\sqrt{y}}} \sim \sqrt{\frac{n}{8} + \frac{1}{32}}.$$

This shows that the distribution is asymptotically concentrated.

5.2 Set Partitions

The exponential generating function for the number of partitions of a set of size n is given by

$$F(z) = \sum_{n \geq 0} \varpi_n \frac{z^n}{n!} = \exp\left(e^z - 1\right). \tag{5.2}$$

The number ϖ_n is known as the n-th Bell number.

Applying the concepts of Hayman [Hay56] and Harris and Schoenfeld [HS68] we obtain asymptotics for ϖ_n as $n \to \infty$. We will then study some parameters on the set of partitions as $n \to \infty$ and obtain limit laws for these parameters.

5.2.1 Total Number

The function $F(z)$ is seen to be H-admissible in $\mathbb{C}$ with the functions

$$a(r) = re^r \qquad \text{and} \qquad b(r) = (r+1)re^r.$$

Thus, we get a first approximation to ϖ_n by an application of Corollary 3.2, viz.

$$\varpi_n \sim \frac{n! \exp\left(e^{r_n} - 1\right)}{r_n^n \sqrt{2\pi n r_n}} \tag{5.3}$$

where r_n is the unique positive solution of the equation

$$re^r = n. \tag{5.4}$$

De Bruijn [dB81, sec. 2.4] shows that, for n large enough, the solution of (5.4) can be represented as

$$r_n = \log n - \log\log n + \sum_{k\geq 0}\sum_{m\geq 0} c_{km} \frac{(\log\log n)^{m+1}}{(\log n)^{k+m+1}}, \qquad c_{km} \in \mathbb{C}. \tag{5.5}$$

As $F(z)$ is the exponential of an H-admissible function it is HS-admissible, too, and we can get a full asymptotic expansion by an application of Theorem 4.1. The quantities of Definition 4.1 now read

$$\begin{aligned}
A(z) &= e^z - 1\\
B_k(z) &= \begin{cases} z(e^z-1) & \text{if } k = 1\\ \frac{z^k}{k!}e^z & \text{else}\end{cases}\\
B(z) &= \frac{ze^z}{2}\left(1 + z - e^{-z}\right)\\
C_m(u_n, u_n) &= -\frac{2u_n^{m+1}}{(m+2)!\,(1+u_n-e^{-u_n})} - \frac{(-1)^m\,(1-e^{-u_n})}{(m+2)\,(1+u_n-e^{-u_n})}
\end{aligned}$$

where u_n is the unique positive solution of

$$u(e^u - 1) = n + 1. \tag{5.6}$$

Thus we get the refinement

$$\varpi_n = \frac{n! e^{e^{u_n}-1}}{2u_n^n\sqrt{\pi B(u_n)}}\left(1 + \frac{1}{8}\frac{u_n^2}{e^{u_n}} + \frac{5}{24}\frac{u_n^3 + 2u_n(1-e^{-u_n})}{e^{u_n}(1+u_n-e^{-u_n})} + o(e^{-u_n})\right). \tag{5.7}$$

We close this section by showing that the expressions (5.7) and (5.3) are indeed asymptotically equal. For brevity set $u = u_n$ and $r = r_n$. Setting $u = r + w$ in (5.6) yields

$$n(e^w - 1) + we^{r+w} = u + 1. \tag{5.8}$$

From (5.6) we see that

$$u+1 \sim \log n - \log\log n, \qquad n \to \infty$$

and therefore we know that $w \to 0$ as $n \to \infty$. Moreover, since $e^r \sim n/\log n$ as $n \to \infty$ we have

$$n(e^w - 1) + we^{r+w} \sim n\left(e^w - 1\frac{we^w}{\log n}\right) \sim nw.$$

Combining these results we obtain

$$w = w_n \sim \frac{\log n}{n}, \qquad n \to \infty.$$

Hence u_n admits the same asymptotic expansion (5.5) as r_n. Now we have

$$\begin{aligned} u^n &= (r+w)^n = r^n\left(1+\frac{w}{r}\right)^n \\ &= r^n\left(1+\frac{1}{n}+o(n^{-1})\right)^n \sim er^n \\ e^u - e^r &= e^r\left(e^w - 1\right) \sim \frac{n}{\log n}\frac{\log n}{n} = 1 \\ B(u) &\sim \frac{u^2e^u}{2} \sim \frac{r^2e^r}{2} \sim \frac{nr_n}{2} \end{aligned}$$

which shows that (5.7) and (5.3) are asymptotically equal.

5.2.2 Stirling Numbers of the Second Kind

We determine the behaviour of the number of subsets of a randomly chosen partition of a set of size n consists of as $n \to \infty$.

The bivariate exponential generating function with u marking the parameter "number of subsets" is given by

$$G(z,u) = \exp\left(u\left(e^z - 1\right)\right) = \sum_{n\geq 0}\sum_{k\geq 0} S_{n,k}u^k\frac{z^n}{n!}$$

where the $S_{n,k}$ denote the Stirling numbers of the second kind. The first and second logarithmic derivatives read

$$\begin{aligned} \mathbf{a}(z,u) &= \begin{pmatrix} a(z,u) \\ \bar{a}(z,u) \end{pmatrix} = \begin{pmatrix} uze^z \\ u(e^z-1) \end{pmatrix} \\ \mathbf{B}(z,u) &= \begin{pmatrix} b(z,u) & c(z,u) \\ c(z,u) & \bar{b}(z,u) \end{pmatrix} = \begin{pmatrix} u(z+z^2)e^z & uze^z \\ uze^z & u(e^z-1) \end{pmatrix}. \end{aligned}$$

Setting $m=0$ and $\lambda_0 = 1$ in theorem 4.11 we see that $G(z,u)$ is BR-admissible in

$$\mathcal{R} = \left\{(r,s)\middle| R_0 < r \quad \text{and} \quad e^{r(\delta-1)} < s < e^{r/\delta}\right\}$$

with angles

$$\Theta(r,s) = \left\{\theta \middle| |\theta| < (s(e^r - 1))^{-1/3-\varepsilon}\right\}$$

for some $R_0 > 0$, any $\delta > 0$ and $\varepsilon > 0$ sufficiently small.

Let $(r_{n,k}, s_{n,k}) \in \mathcal{R}$ denote the solution to $\mathbf{a}(r,s) = (n,k)$. The solution of $re^{\kappa r} = n$, $\kappa > 0$, satisfies $\kappa r = \log n - \log\log n + \log\kappa + o(1)$ as $n \to \infty$ and we therefore see that

- $r_{n,k}$ satisfies, as $n \to \infty$,

  $$\frac{1+o(1)}{\delta}\log n \le r_{n,k} \le \frac{\delta(1+o(1))}{1+\delta}\log n$$

  and
- we have

  $$\frac{n}{k} = \frac{r_{n,k}}{1-e^{-r_{n,k}}} \sim r_{n,k}, \qquad n \to \infty.$$

Hence, for any positive constants c and C, Theorem 4.8 provides uniform asymptotics for $S_{n,k}$ when

$$\frac{cn}{\log n} < k < \frac{Cn}{\log n}.$$

The equation $\mathbf{a}(r,1) = (n,k^*)$ leads to

$$\begin{aligned} re^r &= n \\ e^r - 1 &= k^*. \end{aligned} \tag{5.9}$$

The solutions r_n and k_n^* of (5.9) satisfy

$$\begin{aligned} r_n &\sim \log n - \log\log n \sim \log n, \\ k_n^* &\sim \frac{n}{\log n} \end{aligned}$$

as $n \to \infty$. Calculating

$$\frac{\det \mathbf{B}(r_n,1)}{b(r_n,1)} = (e^{r_n} - 1) - \frac{(r_n e^{r_n})^2}{(r_n^2 + r_n)e^{r_n}} \sim \frac{e^{r_n}}{r_n} \sim \frac{n}{(\log n)^2}, \qquad n \to \infty,$$

we see that $S_{n,k}$ satisfies a local limit law with mean and variance asymptotic to $n/\log n$ and $n/(\log n)^2$, respectively.

The function $G(z,u)$ is seen to be e-admissible, too, by Theorem 4.17 and the $S_{n,k}$ therefore satisfy a central limit law with mean and variance

$$\begin{aligned} \bar{a}(r_n,1) &= k_n^* \sim \frac{n}{\log n}, \qquad n \to \infty, \\ \frac{\det B(r_n,1)}{b(r_n,1)} &\sim \frac{n}{(\log n)^2}, \qquad n \to \infty. \end{aligned}$$

5.2.3 Singleton Blocks

Now, we determine the asymptotic behaviour of the parameter "number of singleton blocks" as $n \to \infty$.

The bivariate gf $f(z,u)$, exponential w.r.t. z, with u marking the subsets of cardinality 1 is given by

$$f(z,u) = \exp\left(e^z - 1 - z + uz\right). \tag{5.10}$$

The first and second logarithmic derivatives are given by

$$\begin{aligned}
\mathbf{a}(z,u) &= \begin{pmatrix} a(z,u) \\ \bar{a}(z,u) \end{pmatrix} &&= \begin{pmatrix} ze^z - z + uz \end{pmatrix} \\
\mathbf{B}(z,u) &= \begin{pmatrix} b(z,u) & c(z,u) \\ c(z,u) & \bar{b}(z,u) \end{pmatrix} &&= \begin{pmatrix} z + z^2)e^z - z + uz & uz \\ uz & uz \end{pmatrix}
\end{aligned}$$

and we have

$$\det \mathbf{B}(z,u) = uz^2 e^z (z + 1 - e^{-z}). \tag{5.11}$$

Write

$$f(z,u) = \exp\left(e^z - 1\right) e^{uz - z}. \tag{5.12}$$

Theorem 3.5 shows that $\exp\left(e^z - 1\right)$ is H-admissible in $\mathbb{C}$ with $\delta(r) = (e^r - 1)^{-2/5}$.

Now, let k_1 and k_2 denote arbitrary reals satisfying $0 < k_1 < k_2 < \infty$ and set

$$\begin{aligned}
\mathcal{R} &= \left\{ (r,s) \in \mathbb{R}_+^2 \;\middle|\; k_1 \le \frac{s}{r} \le k_2 \right\}, \\
\Theta(r,s) &= [0, \delta(r)]^2, \qquad (r,s) \in \mathcal{R}.
\end{aligned}$$

In the following, we will also use the abbreviation $\boldsymbol{\alpha} = (\phi, \theta)'$. Let $(r,s) \in \mathcal{R}$. For $\boldsymbol{\alpha} \in \Theta(r,s)$ we have, as $r \to \infty$,

$$f\left(re^{i\phi}, se^{i\theta}\right) = f(r,s)(1 + o(1)) e^{i\mathbf{a}(r,s)'\boldsymbol{\alpha} - \boldsymbol{\alpha}'\mathbf{B}(r,s)\boldsymbol{\alpha}/2}. \tag{5.13}$$

If $\boldsymbol{\alpha} \in [-\pi,\pi]^2 - \Theta(r,s)$ then

$$\left|\exp\left(rse^{i(\phi+\theta)} - re^{i\phi}\right)\right| \le e^{rs - r} e^r \tag{5.14}$$

and from the proof of Theorem 3.5 we obtain

$$\left|\exp\left(e^{re^{i\phi}} - 1\right)\right| \le \exp\left(e^r - 1\right) \exp\left(-r^{1/4} e^{3r/8}\right) \tag{5.15}$$

as $r \to \infty$. Combining (5.15) and (5.14) yields

$$f\left(re^{i\phi}, se^{i\theta}\right) = f(r,s)\, o\left(\exp\left(-e^{r/4}\right)\right), \qquad r \to \infty. \tag{5.16}$$

Finally note that for $(r,s) \in \mathcal{R}$

$$B(r,s) \asymp r^3 e^r, \qquad r \to \infty. \tag{5.17}$$

Hence (5.13) and (5.16) imply (4.12) and (4.13) respectively and therefore $f(z,u)$ is BR-admissible in $\mathcal{R}$ with angles $\Theta(r,s)$ and functions $\mathbf{a}(r,s)$ and $\mathbf{B}(r,s)$. We may therefore apply Theorem 4.8 to $f(z,u)$. The combinatorial interpretation of $f(z,u)$ reveals that $\Lambda_f = \mathbb{Z}^2$, that is $d(\Lambda_f) = 1$, and the equation $\mathbf{a}(r,s) = (n,k)'$ reads

$$\begin{aligned} re^r - r + rs &= n, \\ rs &= k. \end{aligned} \tag{5.18}$$

This leads to

$$re^r\left(1 - e^{-r}\right) = n - k \tag{5.19}$$

and therefore we see that the solution $r_{n,k}$ has got the same asymptotic expansion as r_{n-k} in (5.5).

Let $(r_{n,k}, s_{n,k}) \in \mathcal{R}$ denote the solution to (5.18). Theorem 4.8 now yields

$$[z^n u^k] f(z,u) = \frac{f(r_{n,k}, s_{n,k}) r_{n,k}^{-n} s_{n,k}^{-k}}{2\pi\sqrt{r_{n,k}^3 e^{r_{n,k}}}}(1 + o_B(1)) \tag{5.20}$$

and from (5.18) we see that (5.20) is valid for (n,k) satisfying

$$k_1(1+o(1)) \le \frac{k}{(\log n)^2} \le k_2(1+o(1)), \qquad n \to \infty.$$

An interesting fact to note is that normality cannot be established using the concept of BR-admissibility. Adopting the notation of Definition 4.5, we have

Proposition 5.1. *If* $f(z,u)$ *is BR-admissible in* $R \subseteq \mathbb{R}_+^2$ *with angles* Θ*, then* $\det B(r,1)$ *is bounded in* R*.*

Proof. If r is bounded away from 0, it follows from (5.11) that

$$\det B(r,s) \asymp r^3 s e^r.$$

Therefore, we have

$$\sqrt{\det B(r,s)}\frac{|f(r,se^{i\theta})|}{f(r,s)} \asymp s^{1/2} r^{3/2} \exp\left(r(s\cos\theta - s + \frac{1}{2})\right). \tag{5.21}$$

Observing that the exponent of (5.21) is nonnegative for

$$\cos\theta \ge 1 - \frac{1}{2s} \tag{5.22}$$

and using the inequality $\cos x \ge 1 - \frac{x^2}{2}$, $x \in \mathbb{R}$, we see that

$$\operatorname{diam}\Theta(r,s) \ge \frac{1}{\sqrt{s}}. \tag{5.23}$$

$f(z,u)$ is BR-admissible and thus, we know that

$$\mathrm{diam}\Theta(r,s) = o_B(1), \qquad (r,s) \in R. \tag{5.24}$$

Combining (5.23) and (5.24) we see that $\det B(r,s)$ is bounded in R if s remains bounded. □

The function (5.10) is seen to be e-admissible by Theorem 4.17 and thus the parameter considered is asymptotically normal distributed. The solution r_n of $a(r,1) = re^r = n$ satisfies $r_n \sim \log n - \log\log n$ as $n \to \infty$ and therefore the asymptotic mean and variance are given by

$$\begin{aligned} \bar{a}(r_n,1) &\sim \log n \\ \frac{\det \mathbf{B}(r_n,1)}{b(r_n,1)} &= \frac{(r_n^3 + r_n^2)e^{r_n} - r_n^2}{r_n(r_n+1)e^{r_n}} \quad \sim \quad \log n \end{aligned}$$

respectively.

5.2.4 A Multivariate Limit Law

In Section 4.3, Example 12 we showed that the function

$$F(z,u,w) = e^{u(\cosh z - 1)} e^{w \sinh z}$$

is BR-admissible in

$$\mathcal{R} = \left\{ (r,s,t) \;\middle|\; R_0 \le r,\ (\cosh r - 1)^{\delta - 1} < s < (\cosh r - 1)^{1/\delta},\ (\sinh r)^{\delta-1} < t < (\sinh r)^{1/\delta} \right\}$$

with matrix

$$\mathbf{B}(r,s,t) = \begin{pmatrix} r(rs+t)\cosh r + r(rt+s)\sinh r & rs\sinh r & rt\cosh r \\ rs\sinh r & s(\cosh r - 1) & 0 \\ rt\cosh r & 0 & t\sinh r \end{pmatrix}.$$

The corresponding function $\mathbf{a}(r,s,t)$ is given by

$$\mathbf{a}(r,s,t) = \begin{pmatrix} r(s\sinh r + t\cosh r) \\ s(\cosh r - 1) \\ t\sinh r \end{pmatrix}.$$

In order to determine the local limit we proceed as described in Section 4.3, Remark 4.10.

The equation

$$\mathbf{a}(r,1,1) = \begin{pmatrix} re^r \\ \cosh r - 1 \\ \sinh r \end{pmatrix} = \begin{pmatrix} n \\ k^* \\ m^* \end{pmatrix}$$

leads to the solutions r_n and k_n^*, m_n^* asymptotically given by

$$\begin{aligned} r_n &\sim \log n - \log\log n \\ k_n^* &\sim \frac{n}{2\log n} \\ m_n^* &\sim \frac{n}{2\log n} \end{aligned}$$

as $n \to \infty$. Furthermore, we have

$$\mathbf{B}(r,1,1) \sim \begin{pmatrix} n\log n & \frac{n}{2} & \frac{n}{2} \\ \frac{n}{2} & \frac{n}{2\log n} & 0 \\ \frac{n}{2} & 0 & \frac{n}{2\log n} \end{pmatrix}, \qquad n \to \infty.$$

In this case we have

$$\mathbf{B}_{1,1} = n\log n, \qquad \mathbf{B}_{1,2} = \frac{1}{2}\begin{pmatrix} n \\ n \end{pmatrix}, \qquad \mathbf{B}_{2,2} = \frac{1}{2}\begin{pmatrix} \frac{n}{\log n} & 0 \\ 0 & \frac{n}{\log n} \end{pmatrix}$$

which gives

$$D = B_{2,2} - B_{1,2}(B_{1,1})^{-1}B_{1,2}' = \frac{1}{2}\begin{pmatrix} \frac{n}{2\log n} & -\frac{n}{2\log n} \\ -\frac{n}{2\log n} & \frac{n}{2\log n} \end{pmatrix}.$$

Now, let $(X_n : n \geq 0)$ denote the sequence of random variables associated with $F(z,u,w)$ as described in Section 1.4. If π is a partition of a set of size n, then $X_n(\pi) = (k,m)$ where k is the number of blocks of π of even cardinality and m is the number of blocks of odd cardinality. Furthermore,

$$\mathbb{P}\{X_n = (k,m)\} = \frac{[z^n u^k w^m/n!]F(z,u,w)}{[z^n]F(z,1,1)}.$$

It follows from the BR-admissibility of $F(z,u,w)$ that X_n satisfies a local limit theorem with means vector asymptotically equal to (k_n^*, m_n^*) and covariance matrix asymptotically equal to D as $n \to \infty$. So we can expect that a randomly chosen partition has about equally many blocks of even and odd size.

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