DIPLOMARBEIT

Increasingly labeled structures

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Kurzfassung

Diese Arbeit gibt einen Überblick über die kombinatorischen Aspekte von monoton wachsend markierten Strukturen mit einem Schwerpunkt auf Modellen aus der Informatik. Die Hauptobjekte dieser Untersuchung sind wachsende Bäume, Bucket-wachsende Bäume, wachsende Schaltkreise sowie drei verschiedene Modelle konkurrierender Prozesse. Die bestehende Literatur zu diesen Themen bildet die Grundlage dieser Masterarbeit. Die wesentlichen verwendeten Werkzeuge sind die Methode der Singularitätenanalyse sowie Grenzverteilungsresultate für eine Verallgemeinerung von Pólya–Eggenberger-Urnenmodellen. Für wachsende Bäume umfassen die Ergebnisse exakte und asymptotische Abzählresultate, ein asymptotisches multivariates Normalitätsgesetz für die Verteilung der Knotengrade in sehr einfachen Familien wachsender Bäume sowie eine Untersuchung der Höhe rekursiver Bäume. Die Abzählresultate werden auf Bucket-wachsende Bäume verallgemeinert. Es wird ein Phasenübergangssatz über die multivariate Grenzverteilung der Bucket-Kapazitäten für einige Familien von Bucket-wachsenden Bäumen angegeben, der das zuvor bekannte Resultat über Bucket-rekursive Bäume verallgemeinert, indem ein neues Monotonieergebnis verwendet wird. Für wachsende Schaltkreise wird eine asymptotische Normalität der Anzahl der Ausgabeknoten gezeigt, und die Tiefe zufälliger rekursiver Schaltkreise wird untersucht. Abschließend werden asymptotische Abzählresultate für wachsende Diamanten, wachsende „fork–join“ Prozesse und wachsende Bogenprozesse angegeben. Einige dieser Ergebnisse werden anschließend verwendet, um Abschätzungen für die durchschnittliche Anzahl zulässiger Ausführungen solcher Prozesse zu erhalten.

Abstract

This thesis gives an overview of the combinatorial aspects of increasingly labeled structures with a focus on models from computer science. The main objects of study are increasing trees, bucket increasing trees, increasing circuits, and three different models of concurrent processes. The existing literature on these topics provides the foundation of this thesis. The main tools used are the method of singularity analysis and distributional limit results for a generalization of Pólya–Eggenberger urn models. For increasing trees, results include exact and asymptotic enumeration results, an asymptotic joint normality law for the degree counts in so-called very simple families of increasing trees, and a study of the height of recursive trees. The enumeration results are then generalized to bucket increasing trees. A phase transition theorem about the joint limiting distribution of the bucket capacities for some families of bucket increasing trees is given, generalizing the previously known result about bucket recursive trees by employing a new monotonicity result. For increasing circuits, an asymptotic normality of the number of output nodes is shown, and the depth of random recursive circuits is studied. Lastly, asymptotic enumeration results for increasing diamonds, increasing fork–join processes, and increasing arch processes are given. Some of these results are then used in order to obtain estimates for the average number of admissible executions of such processes.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe. Hilfsmittel der künstlichen Intelligenz wurden nur verwendet, um sprachliche Formulierungen und Zeichensetzung Korrektur zu lesen und gegebenenfalls zu verbessern.

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Contents

1 Introduction

The main goal of this thesis is to give an overview of the combinatorial aspects of increasingly labeled structures. Throughout the thesis, these structures will be modeled via finite directed acyclic graphs. These graphs naturally induce a direction on their nodes and enable the term "increasingly labeled" to be defined meaningfully, namely by requiring that labels increase along any path in the graph. The results of this thesis include exact and asymptotic enumerations as well as asymptotic distributional results for some parameters of various classes of increasingly labeled directed acyclic graphs. In many cases, such as in Sections 3.3 and 4.2 and Chapter 5, the graphs of interest can be interpreted as a graph growing in time, where the labels determine the relative order when the specific nodes were added to the graph, and the edges indicate a dependency from which previously added nodes a new node arises. This interpretation also explains the relevance of this topic, as such objects are used to model evolutionary processes, dynamically evolving networks like the internet, and the execution of concurrent programs. While increasingly labeled structures have found important applications in phylogenetics, the present thesis concentrates on their role in computer science and related combinatorial models.

In Chapter 2, we will at first introduce the reader to some of the notation and conventions used within this thesis. Afterwards, we summarize the mathematical background required for the subsequent chapters, including several fundamental theorems. We then present urn-based methods, which constitute one of the main tools used in the analysis of increasingly labeled structures.

In Chapter 3, we define the notion of increasing trees and various tree families. We start with a systematic treatment of the enumeration of simple families of trees and then give some insight into the ideas of "Varieties of increasing trees" [2], which historically motivated a more systematic development of the theory. We then conclude the chapter with the study of degree counts and the height of tree families emerging from a growth process.

In Chapter 4, we will study the bucket increasing trees, a generalization of increasing trees, where a node may carry multiple labels. Some of the results from the previous chapter are generalized to this setting. The chapter then concludes with a study of the so-called bucket capacities, where a new monotonicity result is employed to extend a phase transition result of [23] to other families of bucket increasing trees.

In Chapter 5, we take a completely probabilistic approach to studying increasing circuits. This chapter is mainly dedicated to the number of output nodes and the depth of random circuits. The methodology applied for this part mostly parallels the situation of the node counts and height for the increasing trees.

In Chapter 6, we present enumerative results for three different models of concurrent processes. As an application, we obtain the average number of executions in some models.

Finally, in Chapter 7, the thesis concludes with a brief outlook on further results, and also some open problems are stated.

2 Preliminaries and methods

This chapter introduces the notation and conventions used throughout the thesis and summarizes the main tools required. We give selected results for the method of singularity analysis, a branch of analytic combinatorics, and review a fundamental result about analytic differential equations. These two sections are based on [10] and [29], respectively. Finally, we state two key theorems from [12, 24] about limit distributions for generalizations of Pólya–Eggenberger urns that will be used to obtain asymptotic normality results.

2.1 Notation and conventions

Graph theory

Let $G = (V, E)$ be a finite directed acyclic graph. We denote by $V(G) := V$ and $E(G) := E$ the sets of vertices and edges of G, respectively, and by $|G| := |V(G)|$ the size of G. For a node $v \in V(G)$, the out-degree is denoted by $d^+(v)$ and the in-degree by $d^-(v)$. A rooted tree will be modeled as a weakly connected directed acyclic graph, where the root has in-degree 0, and any other node has in-degree 1. Using this convention, any graph occurring throughout the thesis is a finite directed acyclic graph. In any figure, if no orientation of the edges is given, they shall all be oriented from top to bottom.

Probability theory

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. We denote the notions of convergence almost surely, convergence in distribution, and convergence in probability to the random variable X by $X_n \overset{a.s.}{\to} X$, $X_n \overset{D}{\to} X$, and $X_n \overset{P}{\to} X$, respectively. If X and Y are two random variables or, more generally, stochastic processes, then $X \overset{D}{=} Y$ denotes that X and Y have the same distribution.

Formal power series

For $n \in \mathbb{N}$, we denote the falling factorial and rising factorial by $x^{\underline{n}} := \prod_{j=0}^{n-1}(x-j)$ and $x^{\overline{n}} := \prod_{j=0}^{n-1}(x+j)$, respectively. If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then $\mathbb{K}[[x_1, \dots, x_n]]$ denotes the ring of formal power series in the variables $x_1, \dots, x_n$ with coefficients in $\mathbb{K}$. For $f \in \mathbb{K}[[x_1, \dots, x_n]]$, the coefficient of $x_1^{k_1} \dots x_n^{k_n}$ in f is denoted by $[x_1^{k_1} \dots x_n^{k_n}]f(x_1, \dots, x_n)$. For $f \in \mathbb{K}[[x]]$, any $p \in \mathbb{N}_{\geq 1}$ such that there is some $f^* \in \mathbb{K}[[x]]$ and $q \in \mathbb{N}$ with $f(x) = x^q f^*(x^p)$ is called a period of f. If $f \neq 0$, there is a unique maximum period $p \in \mathbb{N}_{\geq 1}$ of f. If the maximum period p satisfies $p > 1$, the formal power series f is called periodic, and otherwise f is called aperiodic.

For $f, g \in \mathbb{K}[[x]]$ with $f(x) = \sum_{n\geq 0} f_n x^n$ and $g(x) = \sum_{n\geq 0} g_n x^n$, we also use the notation $(f \odot g)(x) := \sum_{n\geq 0} f_n g_n x^n$ for the Hadamard product of f and g and $\mathcal{L}_c f(x) := \sum_{n=0}^{\infty} n! f_n x^n$ for the combinatorial Laplace transform of f.

Conventions for combinatorial structures

If $\mathcal{A}, \mathcal{B}, \ldots$ are combinatorial structures[1], we denote their corresponding generating function (GF) by $A(z), B(z), \ldots$, where ordinary generating functions (OGF) are taken for unlabeled objects and exponential generating functions (EGF) for labeled structures. Also $\mathcal{A}_n, \mathcal{B}_n, \ldots$ denote the subsets of $\mathcal{A}, \mathcal{B}, \ldots$ of all elements of size n and $A_n := |\mathcal{A}_n|, B_n := |\mathcal{B}_n|, \ldots$ denote the number of objects of size n. Regardless of the labeled or unlabeled case, we denote by $\mathcal{E}$ and $\mathcal{Z}$ the combinatorial structures composed of a single object of size 0 and 1, respectively. It is also possible to consider weighted combinatorial families, where any $a \in \mathcal{A}$ is associated with a positive weight $w(a)$. In this case, $A_n := \sum_{a\in\mathcal{A}_n} w(a)$ denotes the total weight of all objects of size n, and the classical case is obtained from this definition by setting all weights equal to 1. For $A_n \neq 0$, the natural probabilistic model of $\mathcal{A}_n$ is given by $\mathbb{P}(a) := \frac{w(a)}{A_n}$ for $a \in \mathcal{A}_n$. Unless stated otherwise, we will always assume the natural probabilistic model when talking about a random object $a \in \mathcal{A}_n$.

Combinatorial constructions for unlabeled structures

Let $\mathcal{A}$ and $\mathcal{B}$ be unlabeled combinatorial structures. The following combinatorial constructions for unlabeled structures will be used throughout this thesis. Definitions for the size function of the structure $\mathcal{C}$ and an explanation of the combinatorial interpretation of these constructions can be found in [10, Part A. I].

Name of construction	Formal equation	Functional equation
Disjoint union	$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$C(z) = A(z) + B(z)$
Cartesian product	$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(z) = A(z) \cdot B(z)$
Sequence operator[2]	$\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$	$C(z) = \frac{1}{1-A(z)}$

Table 1: Combinatorial constructions for unlabeled structures

Combinatorial constructions for labeled structures

Let $\mathcal{A}$ and $\mathcal{B}$ be labeled combinatorial structures. The following combinatorial constructions for labeled structures will be used throughout this thesis. Definitions for the size function of the structure $\mathcal{C}$ and an explanation of the combinatorial interpretation for most of these constructions can be found in [10, Part A. II]. The missing combinatorial constructions (which are the last two in the table) are discussed in [4].

[1] Likewise, if another symbol is used like $\tilde{\mathcal{A}}$, then we also write $\tilde{\mathcal{A}}_n, \tilde{A}(z)$ and $\tilde{A}_n$ for the corresponding objects associated to $\tilde{\mathcal{A}}$

[2] The operator is only defined if $A_0 = 0$.

Name of construction	Formal equation	Functional equation
Disjoint union	$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$C(z) = A(z) + B(z)$
Partition product	$\mathcal{C} = \mathcal{A} \star \mathcal{B}$	$C(z) = A(z) \cdot B(z)$
Sequence operator[3]	$\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$	$C(z) = \frac{1}{1-A(z)}$
Set operator[3]	$\mathcal{C} = \mathrm{SET}(\mathcal{A})$	$C(z) = \exp(A(z))$
Boxed product	$\mathcal{C} = \mathcal{A}^{\square} \star \mathcal{B}$ or $\mathcal{C} = \mathcal{A}^{\blacksquare} \star \mathcal{B}$	$C(z) = \int_0^z A'(y)B(y)\,\mathrm{d}y$
Ordered product	$\mathcal{C} = \mathcal{A} \boxed{\star} \mathcal{B}$	$\mathcal{L}_c C(z) = (\mathcal{L}_c A(z)) \cdot (\mathcal{L}_c B(z))$
Ordered set operator[3]	$\mathcal{C} = \mathrm{SET}^{\boxed{\star}}(\mathcal{A})$	$\mathcal{L}_c C(z) = \frac{1}{1-\mathcal{L}_c A(z)}$

Table 2: Combinatorial constructions for labeled structures. In the second variant of the boxed product, the largest label is constrained to belong to the $\mathcal{A}$-component. The ordered product $\boxed{\star}$ assigns the smaller labels to the $\mathcal{A}$-component and the larger labels to the $\mathcal{B}$-component. The ordered set operator forms sequences with increasing labels across components.

2.2 Singularity analysis

In this section, we revise some standard tools from [10, Part B] for extracting the asymptotic growth of a sequence by analyzing singularities of the associated generating function.

Definition 2.1. Let $R > 1$ and $\phi \in (0, \pi/2)$ be given. Define

$$\Delta(\phi, R) := \{z \in U_R^{\mathbb{C}}(0) \mid |\arg(z-1)| > \phi\}.$$

Any such domain is called **$\boldsymbol{\Delta}$-domain**.

Theorem 2.2 (Pringsheim's Theorem)**.** *[10, Theorem IV.6] Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be analytic at 0 with radius of convergence $\rho \in (0, \infty)$ and assume $f_n \geq 0$ for all $n \in \mathbb{N}$. Then $f(z)$ has a singularity at $z = \rho$.*

Lemma 2.3 (Daffodil Lemma)**.** *[10, Lemma IV.1] Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be analytic at 0 with radius of convergence ρ and assume $f_n \geq 0$ for all $n \in \mathbb{N}$. If $f(z)$ is not a monomial, then for $z = re^{i\theta}$ with $r \in (0, \rho)$ the following are equivalent:*

(i) $|f(z)| = f(|z|)$.

(ii) *There exists $p \in \mathbb{N}_{\geq 1}$ such that $\theta \in \frac{2\pi}{p}\mathbb{Z}$ and p is a period of $f(z)$.*

Theorem 2.4. *[10, Theorem VI.1] Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $f(z) = (1-z)^{-\alpha}$. Then the coefficients $f_n = [z^n]f(z)$ satisfy*

$$f_n = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \frac{\alpha(\alpha-1)}{2n} + O(n^{-2})\right).$$

[3] The operator is only defined if $A_0 = 0$.

Theorem 2.5. *[10, Theorem VI.2] Let* $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, \beta \in \mathbb{C}$ *and* $f(z) = (1-z)^{-\alpha}(\log(\frac{1}{1-z}))^{\beta}$*. Then the coefficients* $f_n = [z^n]f(z)$ *satisfy*

$$f_n = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + O\left(\frac{1}{\log(n)}\right)\right).$$

Theorem 2.6 (Transfer Theorem)**.** *[10, Theorem VI.5] Let*

$$\mathcal{S} := \{(1-z)^{-\alpha} \left(\log\left(\frac{1}{1-z}\right)\right)^{\beta} \mid \alpha, \beta \in \mathbb{C}\},$$

$\rho > 0$ *and* $f(z)$ *be analytic in* $|z| < \rho$ *with singularities* $\xi_1, \dots, \xi_p$ *on the circle* $|z| = \rho$*. Further, assume that there is a* Δ*-domain* Δ_0 *such that* $f(z)$ *is analytic in*

$$D := \bigcap_{j=1}^{p} \xi_j \Delta_0,$$

and functions $\sigma_1, \dots, \sigma_p \in \operatorname{span}(\mathcal{S}), \tau \in \mathcal{S}$*, such that*

$$f(z) = \sigma_j(z/\xi_j) + O(\tau(z/\xi_j)), \quad z \to \xi_j$$

for $z \in D$ *and* $j = 1, \dots, p$*. Then the coefficients* $f_n = [z^n]f(z)$ *satisfy*

$$f_n = \sum_{j=1}^{p} \xi_j^{-n} \sigma_{j,n} + O(\rho^{-n}\tau_n),$$

where $\sigma_{j,n} = [z^n]\sigma_j(z)$ *and* $\tau_n = [z^n]\tau(z)$*.*

2.3 Differential equations

In some cases, the study of increasingly labeled structures leads to an analytic ordinary differential equation for the generating function. The following theorem may then assist in proving analyticity in an appropriate domain.

Theorem 2.7. *[29] Let* $\Phi : \mathbb{C}^m \to \mathbb{C}$ *be an entire function and let* $(t_0, y_0) \in \mathbb{C} \times \mathbb{C}^m$*. Then there exists a domain* $U \subseteq \mathbb{C}$ *with* $t_0 \in U$ *and a uniquely determined analytic function* $y : U \to \mathbb{C}^m$ *with*

$$y'(t) = \Phi(y(t)) \quad \text{and} \quad y(t_0) = y_0.$$

Additionally, if $v \in \mathbb{C} \setminus \{0\}$ *is given, then there exists a maximal* $s_* \in [0, \infty]$ *such that* y *admits an analytic continuation along the ray* $\{t_0 + vs \mid s \in [0, s_*)\}$*. If* $s_* < \infty$*, the analytic continuation* y *satisfies* $\lim_{s \to s_*^-} |y(t_0 + sv)| = \infty$*.*

Proof sketch. The existence and uniqueness of an analytic solution is given by the usual Picard iteration argument as presented in [29, Section 4.1]. The conjectured blow-up phenomenon of the solution can then be dealt with as in the real case, which is given in [29, Corollary 2.16]. □

2.4 Limit laws

Several parameters considered in this thesis can be modeled via suitable urn models. In this section, we provide a formal framework for these models and two theorems establishing asymptotic normality for their ball counts.

Definition 2.8. Let $N \in \mathbb{N}_{\geq 1}$. By Δ_{N-1} we denote the simplex

$$\Delta_{N-1} := \{x \in [0,\infty)^N \mid \sum_{i=1}^{N} x_i = 1\}.$$

Definition 2.9. Let $A \in \mathbb{C}^{q\times q}$. A vector $u \in \mathbb{C}^q$ is called **left eigenvector**[4] of A, if u is an eigenvector of A^T.

Definition 2.10. A **generalized Pólya–Eggenberger urn model** $(U_n)_{n\in\mathbb{N}} = (X_n, \delta_n)_{n\in\mathbb{N}}$ with **transition matrix** $R \in \mathbb{R}^{K\times L}$ is defined as follows: There are L different types of balls and the entries of the random vector $X_n = (X_{n,1}, \dots, X_{n,L})$ denote the number[5] of balls of each type $k = 1, \dots, L$ in the urn at time n, where an **initial state** X_0 with non-negative entries and at least one positive entry is given. The matrix R and the random vector δ_n with values in the natural basis of $\mathbb{R}^K$ determine the replacement policy at step n according to the rule

$$X_{n+1} = X_n + R^T \delta_n.$$

We require $X_{n,k} \geq 0$ and $\sum_{k=0}^{L} X_{n,k} > 0$ almost surely for all $n \in \mathbb{N}$. Additionally, assume that for each n there is a **decision function** $G_n : \Delta_{L-1} \to \Delta_{K-1}$, such that the components $G_n^i : \Delta_{L-1} \to [0,1]$ of G_n satisfy

$$\mathbb{P}(\delta_{n+1} = e_i \mid \mathcal{F}_n) = G_n^i\left(\frac{X_n}{\sum_{k=1}^{L} X_{n,k}}\right), \quad \text{where} \quad \mathcal{F}_n = \sigma(U_i, 1 \leq i \leq n)$$

for $i = 1, \dots, K$, and that there is a limit function $G : \Delta_{L-1} \to \Delta_{K-1}$ with

$$\|G_n(x) - G(x)\| = O(n^{-\gamma})$$

uniformly for all $x \in \Delta_{L-1}$ and a constant $\gamma > \frac{1}{2}$.

The urn model is called **balanced** if the number of balls added in each step is constant, i.e., there exists $m \in [0,\infty)$ such that $m = \sum_{j=1}^{L} R_{i,j}$ for all $i = 1, \dots, K$.

The urn model is called **standard generalized Pólya–Eggenberger urn model** if $K = L$, $G_n(x) = x$ for all $n \in \mathbb{N}$, the model is balanced and any type $i = 1, \dots, L$ is **dominating**, that is if $X_0 = e_i$, then for any type $j = 1, \dots, L$ we have $\mathbb{P}(X_{n,j} > 0) > 0$ for some $n \in \mathbb{N}_0$.

Remark 2.11. The term "generalized Pólya–Eggenberger urn model" or "generalized Pólya urn model" is used in the literature for any generalization of the classical Pólya urn model. Our definition is a special case of the one in [30] and uses the prefix "standard" to obtain a

[4] For symmetry purposes, we will also refer to eigenvectors as **right eigenvectors.**
[5] Note that the numbers of balls need not be integers.

special case of the model described in [12, Remark 4.2]. In [12], each color is assigned an activity, and the probability of drawing a specific ball is proportional to the activity of the ball's color. In our setting, this additional weighting is not required, since all activities are equal to 1. We also note that [12] follows the convention that its transition matrix is the transpose of our transition matrix.

Theorem 2.12. *[12] Let $(X_n)_{n\in\mathbb{N}}$ be a standard generalized Pólya–Eggenberger urn model with transition matrix $A \in \mathbb{R}^{q\times q}$ and number of balls added in each step equal to $m > 0$. Let $\lambda_1, \ldots, \lambda_q$ be the eigenvalues of A, with $m = \lambda_1 > \Re\lambda_2 \geq \ldots \geq \Re\lambda_q$. Further assume that there is a left eigenvector $v \in \mathbb{R}^q$ of A corresponding to the eigenvalue λ_1 with strictly positive entries and such that $\sum_{j=1}^{q} v_j = 1$. Then the following holds:*

(i) $n^{-1}X_n \overset{a.s.}{\to} \lambda_1 v$.

(ii) *Assume $\Re\lambda_2 < \frac{1}{2}\lambda_1$. Let A_i denote the i-th row of A for $i = 1, \ldots, q$ and define*

$$B := \sum_{i=1}^{q} v_i A_i A_i^T, \quad \psi(s, A) := e^{sA} - \lambda_1 v(1, \ldots, 1) \int_0^s e^{tA}\,\mathrm{d}t, \quad \textit{and}$$
$$\Sigma := \int_0^\infty \psi(s, A) B \psi(s, A)^T e^{-\lambda_1 s} \lambda_1\,\mathrm{d}s - \lambda_1 2vv^T.$$

Then
$$n^{-1/2}(X_n - n\lambda_1 v) \overset{D}{\to} \mathcal{N}(0, \Sigma).$$

(iii) *Assume $\Re\lambda_2 > \frac{1}{2}\lambda_1$, $|\{j \mid \Re\lambda_j = \Re\lambda_2, \Im\lambda_j > 0\}| = 1$ and that there is no real eigenvalue λ with $\lambda = \Re\lambda_2$. Let $v_2 \in \mathbb{C}^q$ be a left eigenvector of A for the eigenvalue λ_2. Then there exists some random vector $\hat{W}$ with values in* $\mathrm{span}(v_2)$, *such that*

$$n^{-\Re(\lambda_2)/\lambda_1}(X_n - n\lambda_1 v) - \Re(n^{i(\Im(\lambda_2)/\lambda_1)}\hat{W}) \overset{a.s.}{\to} 0.$$

If additionally $X_0 = e_j$ for some $j \in \{1, \ldots, q\}$, then $\mathbb{E}(\hat{W}) = 2\frac{\Gamma(1/\lambda_1)}{\Gamma((1+\lambda_2)/\lambda_1)} \frac{v_2^T e_j}{\|v_2\|^2} v_2$.

Proof sketch. We show how to obtain the results with these assumptions from [12], using their terminology and notations. Firstly, we require the activity to be constant with value 1 and deterministic replacement rules and initial values. Since we only consider balanced processes, they never go extinct, and the removal of non-present balls is prohibited by definition. The assumptions regarding the eigenvalues and eigenvectors of A as in [12, Remark 4.2] are also assumed in this theorem except for the existence of a right eigenvector u corresponding to λ_1, where u has strictly positive entries. However, $u = (1, \ldots, 1)^T$ is implied by the fact that the process is balanced together with the assumption $m = \lambda_1$. The other assumptions (A2)-(A6) in [12, Remark 4.2] are either assumed or obvious consequences of our assumptions. By [12, Remark 4.2], the only missing part to apply the appropriate results in [12, Section 3] lies in the proof of [12, Lemma 9.7(iii)]. However, since our processes are balanced, the random variables[6] $u_1 \cdot \mathscr{Y}(t)$ and W can be computed explicitly, which

[6] The variable $\mathscr{Y}(t)$ is essentially obtained by embedding $(X_n)_{n=0}^\infty$ into a continuous process and then applying a norm.

is done in [12, Example 3.11], such that $W \sim \Gamma(\frac{1}{m}, m)$. This shows $\mathbb{P}(W = 0) = 0$ and therefore all necessary conditions as stated in [12, Remark 4.2] hold.

Thus, [12, Theorems 3.21, 3.22, and 3.24] give most of (i), (ii), and (iii), respectively. The only missing part is the additional claim in (iii) about the expectation of $\hat{W}$. Since the statement about $\mathbb{E}W_{\lambda,i}$ in [12, Theorem 9 (i)] is a direct consequence of [12, (9.19) and Lemma 9.5], it is also applicable in our setting, and we can conclude together with [12, Theorem 3.26]. □

Theorem 2.13. *[24] Let $(X_n, \delta_n)_{n\in\mathbb{N}}$ be a balanced generalized Pólya–Eggenberger urn model with transition matrix $R \in \mathbb{R}^{K\times 2}$, decision functions $G_n(x)$ with limit $G(x)$ and number of balls added in each step equal to $r > 0$. Let $F(x) = R^T G(x) - rx$ and $T_n = X_{n,1} + X_{n,2}$. Assume $G \in C^2$ and that there is a unique solution $u = (u_1, u_2) \in \Delta_1$ and a constant $a > 0$, such that $F(u) = 0$ and $(x - u)^T F(x) \leq -a\|x - u\|^2$ for $x \in \Delta_1$. Then*

$$\frac{1}{T_n} X_n \overset{a.s.}{\to} u.$$

If additionally $(1, 1)$ is a left eigenvector of the Jacobian $\mathrm{D}F(u)$ of F at u for some eigenvalue m and the other eigenvalue λ satisfies $\lambda < -\frac{r}{2}$ with left eigenvector $v^T = (v_1, v_2)$, then

$$n^{-1/2}(X_{n,1} - nru_1) \overset{D}{\to} \mathcal{N}(0, \sigma^2),$$

where $C = R^T \left(\mathrm{diag}(G(u)) - G(u)G(u)^T\right) R$ and $\sigma^2 = \frac{-rv^T Cv}{(2\lambda+r)(v_1-v_2)^2}$.

3 Increasing trees

The study of increasing trees allows for a variety of different methods. Firstly, the recursive decomposition of trees motivates the definition of simple families of increasing trees. Sections 3.1 and 3.2 follow the contents of [2] and [9, Sections 1.3 and 6.2] and will provide general methods for the study of simple families of increasing trees. In Section 3.3, we restrict our attention to increasing trees which arise from a probabilistic growth process. Such a growth process will allow for the application of further probabilistic methods. We will study the asymptotic distribution of degree counts for these families of increasing trees and the height of recursive trees. These parts are based on [11] and [26], respectively.

Definition 3.1. An **increasing tree** is a rooted labeled tree, such that the labels along any branch starting from the root are increasing.

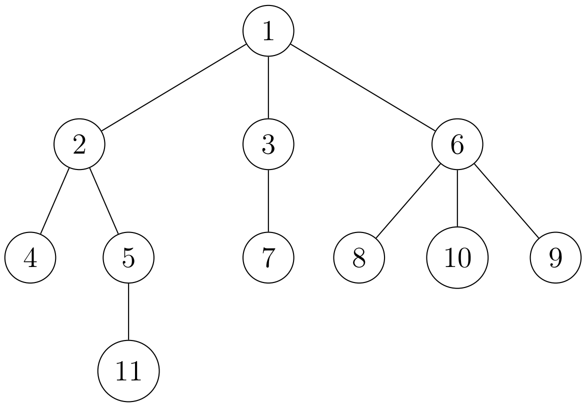

Figure 1: An example of an increasing tree of size 11.

3.1 Simple families of increasing trees

In this section, we define simple families of increasing trees and demonstrate how the most prominent classes of increasing trees are covered by this definition. Lemma 3.9 acts as a tool for obtaining the generating function for simple families of increasing trees, and the method of singularity analysis then enables the asymptotic enumeration for any polynomial family of increasing trees.

Definition 3.2. Let $(\varphi_k)_{k\in\mathbb{N}} \in [0,\infty)^{\mathbb{N}}$ with $\varphi_0, \varphi_k > 0$ for some $k \geq 2$. To any plane increasing tree τ we assign its **weight** $w(\tau) := \prod_{v\in V(\tau)} \varphi_{d^+(v)}$. The family $\mathcal{T}$ then consists of all plane increasing trees of positive weight together with their weights. A family $\mathcal{T}$ which can be constructed in this way is called **simple family of increasing trees**. The sequence $(\varphi_k)_{k\geq 0}$ and its generating function $\varphi(t) := \sum_{k\geq 0} \varphi_k t^k$ are a **degree-weight sequence** and **degree-weight generating function** of $\mathcal{T}$, respectively. A simple family of increasing trees is a **polynomial family of increasing trees**, if its degree-weight generating function is a polynomial. The **degree** of a polynomial family of increasing trees is given by the degree of $\varphi(t)$.

Remark 3.3. Although the underlying rooted unlabeled trees are assumed to be plane, this definition still accounts for the case of non-plane increasing trees. If $\mathcal{T}$ is a simple family of increasing trees with degree-weight sequence $(\varphi_k)_{k\in\mathbb{N}}$, where the trees modeled by $\mathcal{T}$ are plane, then the family modeling the corresponding non-plane increasing trees has the degree-weight sequence $(\frac{\varphi_k}{k!})_{k\in\mathbb{N}}$. The canonical plane representation of a non-plane increasing tree is obtained by ordering the children of every node from left to right in increasing order.

Remark 3.4. Let $\mathcal{T}$ be a simple family of increasing trees with degree-weight sequence $(\varphi_k)_{k\in\mathbb{N}}$ and degree-weight generating function $\varphi(t)$. Scaling φ_k by a factor of ab^k, or equivalently exchanging $\varphi(t)$ by $a\varphi(bt)$, for some $a, b > 0$, contributes a factor of $a^n b^{n-1}$ to $w(\tau)$ for any $\tau \in \mathcal{T}_n$. Therefore, the underlying natural probability distribution remains unchanged under this transformation.

Lemma 3.5. *A class of increasing trees $\mathcal{T}$ is a simple family of increasing trees iff it can be defined by the formal recursive equation*

$$\mathcal{T} = \mathcal{Z}^{\Box} \star \varphi(\mathcal{T}) \tag{3.1}$$

for some degree-weight generating function φ.

Proof. Decompose a tree τ into its root v with out-degree $d^+(v) = r$ and the r increasing trees[1] $\tau_1, \ldots, \tau_r$ attached to the root. The root contributes a factor of φ_r, therefore the weight satisfies $w(\tau) = \varphi_r \prod_{i=1}^{r} w(\tau_i)$. Since the root is necessarily labeled with the least label, we obtain the equivalent recursive definition

$$\mathcal{T} = \mathcal{Z}^{\Box} \star \sum_{r\geq 0} \varphi_r \mathcal{T}^r = \mathcal{Z}^{\Box} \star \varphi(\mathcal{T}). \qquad \Box$$

Definition 3.6. The simple family of increasing trees associated with the degree-weight generating function $\varphi(t)$ is called the family of

(i) **recursive trees** if $\varphi(t) = \exp(t)$,

(ii) **generalized[2] plane-oriented recursive trees** if $\varphi(t) = \frac{1}{(1-t)^{\alpha}}$ for some $\alpha > 0$,

(iii) ***d*-ary[3] increasing trees** if $\varphi(t) = (1+t)^d$ for some $d \in \mathbb{N}$ with $d \geq 2$.

[1] Technically, the trees $\tau_1, \ldots, \tau_r$ are not increasing trees. However, simply re-enumerating each one in a canonical order-preserving way fixes this problem.

[2] We will abbreviate this term with GPORT or α-PORT if we want to make the dependence on α explicit. In case of $\alpha = 1$, the class is called **plane-oriented recursive trees** (PORT).

[3] The special cases of 2-ary and 3-ary trees will be referred to as **binary** and **ternary** trees, respectively.

A **very simple family of increasing trees** is a simple family of increasing trees that falls into one of the above three categories.

Remark 3.7. The classes of recursive trees, PORT, and d-ary increasing trees also provide motivation for the definition of simple increasing trees via a degree-weight sequence. We will give an intuitive explanation via "node-types" for the role of the degree-weight φ_k in these cases. The weight of a tree then corresponds to the number of distinct interpretations of this tree according to its node types:

In a d-ary increasing tree, any node v has d positions for its children, and any position either has a node or remains empty. For $d^+(v) = k$ there are therefore $\varphi_k = \binom{d}{k}$ possible placements of the children of v. For PORT, there is only one node-type for the out-degree k, which justifies the weight $\varphi_k = 1$. Recursive trees are the non-plane counterpart of PORT, and therefore by Remark 3.4 we have $\varphi_k = \frac{1}{k!}$. The general case of GPORT does not provide an equally intuitive explanation and will be treated to some extent in Remark 3.26.

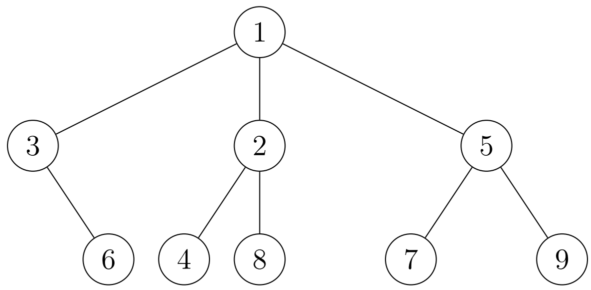

Figure 2: An example of a 3-ary increasing tree. The nodes with labels 2 and 5 have out-degree 2 but are of different node type.

Remark 3.8. In [25], the terms "very simple families of increasing trees" and "grown simple families of increasing trees" are used synonymously. However, our definition of "very simple families of increasing trees" essentially still captures the structure of these classes, which is justified by Theorem 3.25, while allowing for a more intuitive understanding, which is illustrated in Remark 3.7.

Lemma 3.9. *[2] Let $\mathcal{T}$ be a simple family of increasing trees with degree-weight generating function $\varphi(t)$. Then T satisfies the autonomous ordinary differential equation $T'(z) = \varphi(T(z))$, $T(0) = 0$ and is the implicit solution of*

$$\int_0^{T(z)} \frac{1}{\varphi(t)}\,\mathrm{d}t = z. \tag{3.2}$$

Proof. The formal equation (3.1) translates into

$$T'(z) = \varphi(T(z)), \quad T(0) = 0. \tag{3.3}$$

Dividing by $\varphi(T(z))$, integrating and then substituting for T shows

$$z = \int_0^z 1\,\mathrm{d}w = \int_0^z \frac{T'(w)}{\varphi(T(w))}\,\mathrm{d}w = \int_0^{T(z)} \frac{1}{\varphi(t)}\,\mathrm{d}t.$$

In contrast, differentiation shows that any solution of (3.2) satisfies (3.3). $\square$

Theorem 3.10. *[9, Section 6.2] Let $\mathcal{T}$ be a very simple family of increasing trees. Then the following formulas hold:*

(i) *Recursive trees:*

$$T(z) = \log\left(\frac{1}{1-z}\right)$$
$$T_n = (n-1)!$$

(ii) *α-PORT:*

$$T(z) = 1 - (1-(\alpha+1)z)^{1/(\alpha+1)}$$
$$T_n = -\binom{n-1/(\alpha+1)-1}{n}(\alpha+1)^n n!$$
$$T_n \sim -\frac{n^{-1-1/(\alpha+1)}}{\Gamma(-1/(\alpha+1))}(\alpha+1)^n n!$$

(iii) *d-ary increasing trees:*

$$T(z) = -1 + (1-(d-1)z)^{-\frac{1}{d-1}}$$
$$T_n = \binom{n+1/(d-1)-1}{n}(d-1)^n n!$$
$$T_n \sim \frac{n^{-1+1/(d-1)}}{\Gamma(1/(d-1))}(d-1)^n n!$$

Proof. The proof consists of solving the integral in Lemma 3.9 and rearranging terms. The expansions for the generating functions are well-known, such that the coefficients can be extracted. The corresponding asymptotic expressions can be obtained by singularity analysis. We only address the case of recursive trees and refer the interested reader to [9, Section 6.2] for a detailed treatment of all 3 cases:

$$z = \int_0^{T(z)} \frac{1}{e^t}\,\mathrm{d}t = 1 - e^{-T(z)} \Leftrightarrow e^{T(z)} = \frac{1}{1-z} \Leftrightarrow T(z) = \log\left(\frac{1}{1-z}\right) = \sum_{n\geq 1}\frac{z^n}{n}. \qquad \square$$

The following two lemmata constitute a generalized and strengthened version of [2, Lemma 1] and will also be used in Theorem 4.12.

Lemma 3.11. *Let $\psi : [0,\infty) \to \mathbb{R}$ be strictly increasing and analytic. Assume that there are $c, \varepsilon > 0$ such that $\psi(t) \geq c(\sqrt{t} + t^{1+\varepsilon})$ for all $t \in [0,\infty)$. Let ρ and T be (implicitly) defined by the following equations*

$$\rho = \int_0^{\infty} \frac{1}{\psi(t)}\,\mathrm{d}t \quad \textit{and} \quad z = \int_0^{T(z)} \frac{1}{\psi(t)}\,\mathrm{d}t.$$

Then T is analytic in $[0,\rho)$ and satisfies $\lim_{z\to\rho^-} T(z) = \infty$, thus T has a singularity in ρ.

Proof. By the lower bound of ψ, we see that ρ is well-defined. Define $z : [0,\infty) \to [0,\rho)$ by $z(t) := \int_0^t \frac{1}{\psi(s)}\,\mathrm{d}s$. Then z is analytic with $z' = \frac{1}{\psi} > 0$, thus z admits an analytic inverse function $T : [0,\rho) \to [0,\infty)$. Since z is strictly increasing, also T is strictly increasing such that $t^* := \lim_{z\to\rho^-} T(z) \in [0,\infty]$ is well-defined. If $t^* < \infty$ were true, the continuity of z would imply $z(t^*) = \rho$ contradicting $z(t^*) < \rho$. Therefore, $t^* = \infty$ and the claim holds. □

Lemma 3.12. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an entire function with $\varphi_n := [z^n]\varphi(z) \geq 0$ for all $n \in \mathbb{N}$. Let $m \in \mathbb{N}_{\geq 1}, T_0, \ldots, T_{m-1} \in [0,\infty)$ and T be the solution of the initial value problem*

$$T^{(m)}(z) = \varphi(T(z)), \quad T^{(j)}(0) = T_j \quad \text{for } j = 0, \ldots, m-1.$$

Assume that T is analytic in $[0,\rho)$ and singular at ρ for some $\rho \in (0,\infty)$. If the maximum period[4] of T is $p \geq 1$, then the dominant singularities of T are given by $\xi_j := \rho e^{i\theta_j}$, where $\theta_j = \frac{2j\pi}{p}$ for $j = 0, \ldots, p-1$.

Proof. By the assumption on the coefficients of φ and the initial values, we deduce that T has nonnegative coefficients, and since T cannot be a constant, T must be strictly increasing in $[0,\rho)$. By Theorem 2.2, $\xi_0 = \rho$ is a dominant singularity of T and by periodicity also $\xi_1, \ldots, \xi_{p-1}$ are dominant singularities. We will now show that T has no other dominant singularities.

Define $r : U_\rho^{\mathbb{C}}(0) \to [0,\infty), r(z) := T^{-1}(|T(z)|)$. If $z \neq |z|e^{i\theta_j}$ for $j = 0, \ldots, p-1$, from Lemma 2.3 we obtain $|T(z)| < T(|z|)$, resulting in $r(z) < |z|$. Fix $z_0 = r_0 e^{i\theta} \in U_\rho^{\mathbb{C}}(0)$ and let $r_1 := r(z_0)$. The function $\psi(z) := T(z - r_0 + r_1)$ satisfies $\psi^{(m)}(z) = \varphi(\psi(z))$. By Theorem 2.7, the function T can be analytically continued along the ray of angle θ until T gets unbounded. Since all coefficients of φ and also the initial values $T_0, \ldots, T_{m-1}$ are nonnegative, we obtain for $z = (r_0 + s)e^{i\theta}$ with $s > 0$ the inequality

$$|T((r_0 + s)e^{i\theta})| = |T(z)| \leq \psi(|z|) = T(r_1 + s).$$

Therefore T can be continued along the ray of angle θ for $|z| < \rho + r_0 - r(z_0)$ and no dominant singularities other than ξ_j for $j = 0, \ldots, p-1$ can occur. □

Theorem 3.13. *[2] Let $\mathcal{T}$ be a polynomial family of increasing trees of degree $d \geq 2$ with degree-weight generating function $\varphi(t) = \varphi_d t^d + \ldots + \varphi_1 t + \varphi_0$. Let $p \in \mathbb{N}_{\geq 1}$ be the maximum period[4] of φ and define*

$$\rho := \int_0^\infty \frac{1}{\varphi(t)}\,\mathrm{d}t, \quad \delta := \frac{1}{d-1}, \quad \text{and} \quad \eta := (\frac{\varphi_d \rho}{\delta})^\delta.$$

Then T_n satisfies

$$\frac{T_n}{n!} = \begin{cases} \frac{p}{\eta\Gamma(\delta)} n^{\delta-1}\rho^{-n} + O((n^{\delta-2} + n^{-\delta-1})\rho^{-n}) & \text{if } n \equiv 1 \mod p, \\ 0 & \text{else.} \end{cases}$$

[4]See page 2 for the definition.

Proof. From Lemma 3.9, we obtain

$$\rho - z = \int_{T(z)}^{\infty} \frac{1}{\varphi(t)} \,\mathrm{d}t.$$

Since $\deg(\varphi) \geq 2$, we observe that $H(t) := \int_0^t \frac{1}{s^2\varphi(\frac{1}{s})} \,\mathrm{d}s$ is analytic at 0 with $H(0) = 0$. Substituting $\frac{1}{s} = z$ in the previous integral shows $\rho - z = H(\frac{1}{T(z)})$. Extracting the leading order $\frac{1}{t^d}$ of the integrand for $t \to \infty$ gives

$$\begin{aligned}
\int_{T(z)}^{\infty} \frac{1}{\varphi(t)} \,\mathrm{d}t &= \int_{T(z)}^{\infty} \frac{1}{\varphi_d t^d + \ldots + \varphi_1 t + \varphi_0} \,\mathrm{d}t \\
&= \int_{T(z)}^{\infty} \frac{1}{\varphi_d t^d} + O(t^{1-d}) \,\mathrm{d}t \\
&= \frac{1}{\varphi_d (d-1)} T(z)^{1-d} + O(T(z)^{-d}).
\end{aligned}$$

Therefore H has a root of order $d-1$ at 0 with $[t^{d-1}]H(t) = \frac{1}{\varphi_d(d-1)}$. It follows, that H is of the form $H(t) = \frac{1}{\varphi_d(d-1)} t^{d-1} G(t)$ for a function G which is analytic at 0 with $G(0) = 1$. Therefore we obtain

$$\rho - z = \frac{1}{\varphi_d(d-1)} T(z)^{1-d} G\left(\frac{1}{T(z)}\right).$$

Due to $G(0) = 1 \neq 0$, also $G^{\delta}(t) := (G(t))^{1/(d-1)}$ is analytic at 0. Rearranging terms, substituting δ and η and then raising to the power of δ yields

$$\eta \left(1 - \frac{z}{\rho}\right)^{\delta} = \left(\frac{\varphi_d \rho}{\delta}\left(1 - \frac{z}{\rho}\right)\right)^{\delta} = \frac{1}{T(z)} G^{\delta}\left(\frac{1}{T(z)}\right).$$

Since $G^{\delta}(0) = 1 \neq 0$, the function $tG^{\delta}(t)$ admits an inverse function L, which is analytic at 0 and satisfies $L'(0) = 1$. Applying L to both sides and extracting $T(z)$ results in

$$T(z) = \frac{1}{L(\eta(1 - z/\rho)^{\delta})} = \frac{1}{\eta(1 - z/\rho)^{\delta}} + c + O((1 - z/\rho)^{\delta}), \quad z \to \rho$$

for some $c \in \mathbb{C}$.

From the maximum periodicity of φ, it is immediate that also T has maximum period p and is of the form $T(z) = zT^*(z^p)$ for some function T^* analytic at 0. Therefore, $T_n = 0$ if $n \not\equiv 1 \bmod p$ as well as $T(\omega_j z) = \omega_j T(z)$ and

$$T(z) = \frac{\omega_j}{\eta(1 - z/(\rho\omega_j))^{\delta}} + c\omega_j + O((1 - z(\rho\omega_j)^{\delta}), \quad z \to \rho\omega_j \quad ,$$

where $\omega_j = e^{i\frac{2j\pi}{p}}$ for $j = 0, \ldots, p-1$. Furthermore, by Lemmata 3.11 and 3.12, the dominant singularities of T are given by $\rho\omega_j$. From the previous local representations of T, we can deduce that T is analytic in a domain sufficient for Theorem 2.6. Let $n \equiv 1 \bmod p$ and use the transfer theorem to obtain the claim:

$$\frac{T_n}{n!} = \sum_{j=0}^{p-1} [z^n] \frac{\omega_j}{\eta(1 - z/(\omega_j\rho))^{\delta}} + O(n^{-\delta-1}\rho^{-n})$$

$$= \sum_{j=0}^{p-1} \frac{\omega_j^{1-n}}{\eta\Gamma(\delta)} n^{\delta-1}\rho^{-n} + O((n^{\delta-2} + n^{-\delta-1})\rho^{-n})$$
$$= \frac{1}{\eta\Gamma(\delta)} n^{\delta-1}\rho^{-n} \sum_{j=0}^{p-1} \omega_j^{1-n} + O((n^{\delta-2} + n^{-\delta-1})\rho^{-n})$$
$$= \frac{p}{\eta\Gamma(\delta)} n^{\delta-1}\rho^{-n} + O((n^{\delta-2} + n^{-\delta-1})\rho^{-n}). \qquad \square$$

Remark 3.14. Starting from the equation $\eta(1-\frac{z}{\rho})^\delta = \frac{1}{T(z)} G^\delta(T(z))$ in the above proof, an analogous calculation would show the same formula for $n \equiv 2 \mod p$, if $T(\omega_j z) = \omega_j^2 T(z)$ holds instead of $T(\omega_j z) = \omega_j T(z)$.

3.2 Additive parameters

Some parameters of interest admit a certain recursive description that can be captured by the definition of additive parameters. Theorems 3.18 and 3.19 give ordinary differential equations for the EGF of additive parameters. We will then use Theorem 3.18 to obtain explicit results for the total path length in very simple families of increasing trees. Although a similar treatment for the number of nodes of out-degree k would be possible via Theorem 3.19, we will take a different route. An improvement of this is stated in Theorem 3.23 and later extended in the next section.

Definition 3.15. Let s be a complex-valued function defined for all trees. Furthermore, assume that there exists a sequence $(f_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, such that one of the following two cases holds for all trees: If the root of τ has out-degree $r \geq 0$ and the r subtrees attached to the root are $\tau_1, \ldots, \tau_r$, then, recursively,

$$s(\tau) = f_{|\tau|} + \sum_{i=1}^{r} s(\tau_i)$$

or

$$s(\tau) = f_r + \sum_{i=1}^{r} s(\tau_i).$$

Such a function s is called **size-dependent additive parameter** in the first case and **root-dependent additive parameter** in the second case. The sequence $(f_n)_{n\in\mathbb{N}}$ is called the **toll function** of s. Given a simple family of increasing trees $\mathcal{T}$ we define the **exponential generating function** of s by the formula

$$S(z) := \sum_{\tau\in\mathcal{T}} w(\tau)s(\tau)\frac{z^{|\tau|}}{|\tau|!}.$$

Example 3.16. Some examples of size-dependent additive parameters are given by the tree size ($f_n = 1$), the number of leaves ($f_n = \delta_{n,1}$), and the total path length ($f_n = n-1$). The first two are also root-dependent additive parameters with toll functions $f_n = 1$ and $f_n = \delta_{n,0}$, respectively. More generally, also the number of nodes of out-degree k for fixed $k \in \mathbb{N}$ is a root-dependent additive parameter and its toll function is given by $f_n = \delta_{n,k}$.

Remark 3.17. It is also possible to consider an additive parameter s with a toll function $f(\tau)$ which is defined for all increasing trees τ. Without assumptions on f, any function s defined for all trees would be an additive parameter. However, with suitable growth conditions imposed on $f(\tau)$, it is possible to generate quite general results. In [27], the authors managed to prove central limit laws for additive parameters of GPORT and d-ary increasing trees within this general setting.

Theorem 3.18. *[2] Let $\mathcal{T}$ be a simple family of increasing trees and let s be a size-dependent additive parameter with toll function $(f_n)_{n\in\mathbb{N}}$ and EGF S. If F denotes the GF of $(f_n)_{n\in\mathbb{N}}$, then*

$$S(z) = T'(z)\int_0^z \frac{(F\odot T)'(y)}{T'(y)}\,\mathrm{d}y.$$

Proof. [5] Let $(\varphi_n)_{n\in\mathbb{N}}$ be a degree-weight generating sequence of $\mathcal{T}$. Starting from the definition $S(z)=\sum_{\tau\in\mathcal{T}} w(\tau)s(\tau)\frac{z^{|\tau|}}{|\tau|!}$, we use the recursive definition of $s(t)$, then recognize the application of the integral operator inside the second sum and exploit the symmetry of the summand to obtain

$$\begin{aligned}
S(z) &= \sum_{\tau\in\mathcal{T}} w(\tau)f_{|\tau|}\frac{z^{|\tau|}}{|\tau|!} + \sum_{r\geq 1}\varphi_r z \sum_{\tau_1,\dots,\tau_r\in\mathcal{T}} (s(\tau_1)+\ldots+s(\tau_r))\frac{w(\tau_1)z^{|\tau_1|}\cdot\ldots\cdot w(\tau_r)z^{|\tau_r|}}{(|\tau_1|+\ldots+|\tau_r|+1)!}\\
&= \sum_{n\geq 1}\frac{f_nT_n}{n!}z^n + \sum_{r\geq 1}\varphi_r\int_0^z \sum_{\tau_1,\dots,\tau_r\in\mathcal{T}} (s(\tau_1)+\ldots+s(\tau_r))\frac{w(\tau_1)y^{|\tau_1|}\cdot\ldots\cdot w(\tau_r)y^{|\tau_r|}}{(|\tau_1|+\ldots+|\tau_r|)!}\,\mathrm{d}y\\
&= (F\odot T)(z) + \int_0^z \sum_{r\geq 1} r\varphi_r \sum_{\tau_1,\dots,\tau_r\in\mathcal{T}} s(\tau_1)\frac{w(\tau_1)y^{|\tau_1|}\cdot\ldots\cdot w(\tau_r)y^{|\tau_r|}}{(|\tau_1|+\ldots+|\tau_r|)!}\,\mathrm{d}y\\
&= (F\odot T)(z) + \int_0^z \sum_{r\geq 1} r\varphi_r S(y)T(y)^{r-1}\,\mathrm{d}y\\
&= (F\odot T)(z) + \int_0^z S(y)\varphi'(T(y))\,\mathrm{d}y.
\end{aligned}$$

Differentiating shows that S is the solution of the linear differential equation

$$S'(z) = (F\odot T)'(z) + S(z)\varphi'(T(z))$$

with initial condition $S(0)=0$. The general solution to the homogeneous equation is given by $CT'(z)$, and the usual variation of constants method gives the claim. □

The following theorem is a generalization of [9, Lemma 6.7] and follows its proof.

Theorem 3.19. *[9, Lemma 6.7] Let $\mathcal{T}$ be a simple family of increasing trees and let s be a root-dependent additive parameter with toll function $(f_n)_{n\in\mathbb{N}}$ and EGF S. Further let*

$$T(z,u) := \sum_{\tau\in\mathcal{T}} w(\tau)u^{s(\tau)}\frac{z^{|\tau|}}{|\tau|!}$$

[5] The original proof of this theorem uses a flawed identity for the bivariate generating function. Since we were unable to correct this flaw, we instead opt for a more direct approach to obtain the differential equation for S.

and $F(z)$ denote the GF of $(f_n)_{n\in\mathbb{N}}$. Then

$$\frac{\partial T}{\partial z}(z,u) = \sum_{r\geq 0} \varphi_r u^{f_r}(T(z,u))^r, \quad T(0,u) = 0$$

and

$$S(z) = T'(z) \int_0^z \frac{(F \odot \varphi)(T(y))}{T'(y)}\,\mathrm{d}y.$$

Proof. The recursive description of $\mathcal{T}$ in (3.1) together with the additive definition of s immediately yields the formal equation

$$\frac{\partial T}{\partial z}(z,u) = \sum_{r\geq 0} \varphi_r u^{f_r}(T(z,u))^r \quad \text{and} \quad T(0,u) = 0.$$

By differentiating with respect to u, substituting $u = 1$ and using $\frac{\partial T}{\partial u}(z,1) = S(z)$ we obtain that $S(z)$ is the solution of

$$\begin{aligned} S'(z) &= \sum_{r\geq 0} \varphi_r f_r (T(z))^r + \sum_{r\geq 1} r\varphi_r (T(z))^{r-1} S(z) \\ &= (\varphi \odot F)(T(z)) + S(z)\varphi'(T(z)) \end{aligned}$$

with initial condition $S(0) = 0$. The general solution to the homogeneous equation is given by $CT'(z)$, and the usual variation of constants method gives the claim. □

Theorem 3.20. *[2] Let $\mathcal{T}$ be a very simple family of increasing trees and let s be the parameter of total path length. The EGF S of s and the expected total path length $\mathbb{E}X_n$ satisfy:*

(i) *Recursive trees:*

$$\begin{aligned} S(z) &= \frac{1}{1-z}\left(\log\left(\frac{1}{1-z}\right) - z\right) \\ \mathbb{E}X_n &\sim n\log(n) \end{aligned}$$

(ii) *α-PORT:*

$$\begin{aligned} S(z) &= \frac{\alpha}{(\alpha+1)^2}(1-(\alpha+1)z)^{-\frac{\alpha}{\alpha+1}}\left(\log\left(\frac{1}{1-(\alpha+1)z}\right) - (\alpha+1)z\right) \\ \mathbb{E}X_n &\sim \frac{\alpha}{\alpha+1} n\log(n) \end{aligned}$$

(iii) *d-ary increasing trees:*

$$\begin{aligned} S(z) &= \frac{d}{(d-1)^2}(1-(d-1)z)^{-\frac{d}{d-1}}\left(\log\left(\frac{1}{1-(d-1)z}\right) - (d-1)z\right) \\ \mathbb{E}X_n &\sim \frac{d}{d-1} n\log(n) \end{aligned}$$

Proof sketch. The expectation of X_n satisfies

$$\mathbb{E}X_n = \frac{[z^n]S(z)}{[z^n]T(z)} = n!\frac{[z^n]S(z)}{T_n},$$

therefore we will at first calculate the EGF S. From Theorem 3.18 we obtain

$$S(z) = T'(z)\int_0^z \frac{yT''(y)}{T'(y)}\,\mathrm{d}y,$$

where we used $f_n = n-1$ to simplify the expression in the numerator of the integrand. Plugging in the explicit solution for T from Theorem 3.10, one obtains from elementary calculations the claimed expressions for S. Applying singularity analysis to S and exploiting the asymptotic counts of T_n from the same theorem as used before finally leads to the asymptotic formulas for the expectation $\mathbb{E}X_n$. □

Definition 3.21. An additive parameter s with toll function $(f_n)_{n\in\mathbb{N}}$ is called **elementary** if there are $C, \alpha, r \in \mathbb{R}$ with $r \geq 0$ such that $f_n = Cn^\alpha \log^r(n)$. The triple (C, α, r) is called **type** of the parameter s.

Theorem 3.22. *[2] Let $\mathcal{T}$ be a polynomial family of increasing trees of degree d with EGF $T(z)$ and let s be an elementary size-dependent additive parameter of type (C, α, r). Let $F(z)$ be the GF of the toll function $(f_n)_{n\in\mathbb{N}}$ and define $\rho = \int_0^\infty \frac{1}{\varphi(t)}\,\mathrm{d}t$ and $\delta = \frac{1}{d-1}$. Denote by X_n the random variable induced by s on $\mathcal{T}_n$.*

(i) *If $\alpha < 1$, then*

$$\mathbb{E}X_n \sim \lambda n \quad \textit{with} \quad \lambda = \frac{1}{\rho}\int_0^\rho \frac{(F\odot T)'(t)}{T'(t)}\,\mathrm{d}t.$$

(ii) *If $\alpha = 1$, then*

$$\mathbb{E}X_n \sim \lambda n \log^{r+1}(n) \quad \textit{with} \quad \lambda = \frac{C(1+\delta)}{r+1}.$$

(iii) *If $\alpha > 1$, then*

$$\mathbb{E}X_n \sim \lambda n^\alpha \log^r(n) \quad \textit{with} \quad \lambda = \frac{C(\alpha+\delta)}{\alpha-1}.$$

Proof. The proof relying on Theorem 3.18 and singularity analysis can be found in [2]. □

Theorem 3.23. *[9, Theorems 6.8–6.10] Let $\mathcal{T}$ be a very simple family of increasing trees and $X_{n,k}$ be the random variable counting the number of nodes of out-degree $k \geq 0$ in a random tree $\tau \in \mathcal{T}_n$. In the case of d-ary trees, additionally assume $k \leq d$. Then $X_{n,k}$ is asymptotically normal with mean and variance proportional to n. Furthermore, the mean $\mathbb{E}X_{n,k}$ satisfies:*

(i) *Recursive trees:*

$$\mathbb{E}X_{n,k} = \frac{n}{2^{k+1}} + O((\log(n))^{k+1})$$

(ii) *α-PORT:*

$$\mathbb{E}X_{n,k} = \frac{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(2\alpha+k+2)}n + O(n^{1/(\alpha+1)})$$

(iii) *d-ary increasing trees:*

$$\mathbb{E}X_{n,k} = \frac{(d-1)d^{\underline{k}}}{(2d-1)^{\underline{k}}}n + O(n^{1-1/(d-1)} + n^{(k-1)/(d-1)})$$

Proof. The proof relying on Theorem 3.19 and bivariate singularity analysis can be found in [9, Theorems 6.8–6.10]. □

3.3 Grown simple families of increasing trees

By Theorem 3.25, grown simple families of increasing trees and very simple families of increasing trees admit the same natural probability distributions.[6] These families are of particular interest for two reasons. Firstly, the description via a probabilistic growth process allows for new methods, which are illustrated by the proofs of Theorem 3.27 and Theorem 3.35. Secondly, such a growth process also indicates why these increasing tree models naturally appear in many applications.

Definition 3.24. Let $\mathcal{T}$ be a simple family of increasing trees. The family $\mathcal{T}$ is called **grown simple family of increasing trees** if it can be obtained by a probabilistic growth process: For any tree $\tau' \in \mathcal{T}$ with nodes $v_1, \dots, v_n$, there exist probabilities $p_{\tau'}(v_1), \dots, p_{\tau'}(v_n)$, such that the following holds: If $\tau' \in \mathcal{T}_n$ is chosen at random, then the tree τ obtained by attaching a node with label $n+1$ to any node v_j with probability $p_{\tau'}(v_j)$ is a random tree of size $n+1$.

Equivalent definitions for grown simple families of increasing trees and a complete characterization via their degree-weight generating function were given in [25], and we will state these results here.

Theorem 3.25. *[25] Let $\mathcal{T}$ be a simple family of increasing trees. The following are equivalent:*

(i) *There is a very simple family of increasing trees which induces the same natural probability distributions as $\mathcal{T}$.*

(ii) *The family $\mathcal{T}$ is a grown simple family of increasing trees.*

(iii) *The family $\mathcal{T}$ satisfies for any $n \geq j$: If τ is a random tree of size n and τ' is the tree obtained from τ by deleting all nodes with label greater than j, then τ' is a random tree of size j.*

(iv) *There are c_1, c_2 such that $\frac{T_{n+1}}{T_n} = c_1 n + c_2$ for all $n \geq 1$.*

(v) *There are c_1, c_2, such that the degree-weight generating function $\varphi(t)$ of $\mathcal{T}$ is given by one of the following formulas:*

[6] Even though the theorems in this section are formulated for very simple families of increasing trees, the title of this section shall be a hint towards the use of the stochastic evolution process for these tree families.

(a) $\varphi(t) = \varphi_0 e^{c_1 t/\varphi_0}$, *where* $\varphi_0, c_1 > 0$ *(and* $c_2 = 0$*).*

(b) $\varphi(t) = \varphi_0(1 + c_2 t/\varphi_0)^d$, *where* $\varphi_0, c_1 > 0, c_2 > 0$ *such that* $d := \frac{c_1}{c_2} + 1 \in \mathbb{N}, d \geq 2$.

(c) $\varphi(t) = \varphi_0/(1 + c_2 t/\varphi_0)^{-c_1/c_2 - 1}$, *where* $\varphi_0, c_1 > 0$ *and* $0 < -c_2 < c_1$.

Moreover, if these conditions hold, then the constants c_1 *and* c_2 *in* (iv) *and* (v) *coincide.*

Proof. The proof can be found in [25]. □

Remark 3.26. For very simple families of increasing trees, the probabilities for the attachment of the node with label $n+1$ to a tree τ' of size n follow simple descriptions, which can be explained from Remark 3.7: For recursive trees, any node is equally likely to be chosen as the parent of the new node. For the case of α-PORT, a node v is chosen with probability proportional to $d^+(v) + \alpha$ and there are $d^+(v) + 1$ possibilities where the new node is placed relative to the other children of v. For PORT, this results in a probability of $\frac{1}{2n-1}$ for any possible placement of the node with label $n+1$. In the case of d-ary increasing trees, any node v is chosen as a parent with probability proportional to $d - d^+(v)$. Since for any node v there are $d - d^+(v)$ possible placements for a new node as a child, again any possible placement of the node with label $n+1$ will be chosen with equal probability $\frac{1}{(d-1)n+1}$. The positions where a new node can be placed are often also illustrated via the use of external nodes which do not contribute towards the size of the tree.

3.3.1 The asymptotic joint distribution of degree counts

Theorem 3.27. *[11] Let* $\mathcal{T}$ *be a very simple family of increasing trees and* $X_{n,k}$ *be the random variable counting the number of nodes of out-degree* $k \geq 0$ *in a random tree* $\tau \in \mathcal{T}_n$*. In the case of* d*-ary trees, additionally assume* $k \leq d$*. Then*

$$\frac{1}{n} X_{n,k} \xrightarrow{a.s.} \mu_k \quad \textit{and} \quad \frac{1}{\sqrt{n}}(X_{n,k} - \mu_k n) \xrightarrow{D} V_k,$$

where the V_k *are joint normal distributed random variables with mean* 0*. The value of* μ_k *is given by:*

(i) *Recursive trees:*

$$\mu_k = \frac{1}{2^{k+1}}$$

(ii) α*-PORT:*

$$\mu_k = \frac{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(2\alpha+k+2)}$$

(iii) d*-ary increasing trees:*

$$\mu_k = \frac{(d-1)d^{\underline{k}}}{(2d-1)^{\underline{k}}}$$

Proof. [7] We will only deal with the case of α-PORT; the other cases are treated in detail in [11]. Let $M \in \mathbb{N}_{\geq 1}$ be arbitrary. We reduce the problem to a finite number of random variables by grouping all nodes with out-degree at least M into one variable. We use an urn model with types $0, \ldots, M$, initial state of α balls of type 0 and replacement matrix $A = (A_{i,j})_{i,j=0}^{M}$ given by

$$A_{i,j} = \begin{cases} -(\alpha+i)\delta_{i,j} + (\alpha+i+1)\delta_{i+1,j} + \alpha\delta_{0,j} & \text{if } 0 \leq i < M, 0 \leq j \leq M, \\ \delta_{M,j} + \alpha\delta_{0,j} & \text{if } i = M, 0 \leq j \leq M. \end{cases}$$

This describes a standard generalized Pólya–Eggenberger urn model, where in each step $\alpha+1$ balls are added. Let $Q_n = (Q_{n,0}, \ldots, Q_{n,M})$, where $Q_{n,k}$ denotes the (random) number of balls of type k in the urn after n steps. Then the random variables $Q_{n,k}$ and $X_{n,k}$ are related via the equation

$$X_{n,k} = \frac{1}{\alpha+k} Q_{n,k} \quad \text{for} \quad 0 \leq k < M \quad \text{and} \quad \sum_{k \geq M} (\alpha+k) X_{n,k} = Q_{n,M}.$$

In order to use Theorem 2.12, we now study the eigenvalues of the matrix A. Let $u^{(\lambda)}$ be a right eigenvector of A for some eigenvalue λ. If for $u = (u_0, \ldots, u_M)$ we set $u_{M+1} := u_M$, then equivalently

$$-(\alpha+i)u_i^{(\lambda)} + (\alpha+i+1)\alpha u_{i+1}^{(\lambda)} + \alpha u_0^{(\lambda)} = \lambda u_i^{(\lambda)} \quad \text{for } 0 \leq i \leq M.$$

From this form, we can observe that $u_0 = 0$ would imply $u = 0$, therefore we have $u_0 \neq 0$. Setting $\Delta u_i^{(\lambda)} := u_{i+1}^{(\lambda)} - u_i^{(\lambda)}$ then gives the equivalent formulation

$$(\alpha+i+1)\Delta u_i^{(\lambda)} = (\lambda-1)u_i^{(\lambda)} - \alpha u_0^{(\lambda)} \quad \text{for } 0 \leq i \leq M \quad \text{and} \quad \Delta u_M^{(\lambda)} = 0.$$

By subtracting the i-th equation from the $(i+1)$-th equation, we obtain

$$(\alpha+i+2)\Delta u_{i+1}^{(\lambda)} - (\alpha+i+1)\Delta u_i^{(\lambda)} = (\lambda-1)\Delta u_i^{(\lambda)}$$

or equivalently

$$(\alpha+i+2)\Delta u_{i+1}^{(\lambda)} = (\alpha+i+\lambda)\Delta u_i^{(\lambda)}$$

for $0 \leq i < M$ with $\Delta u_0^{(\lambda)} = \frac{\lambda-(\alpha+1)}{\alpha+1} u_0^{(\lambda)}$. The explicit solution to the recursion is given by

$$\Delta u_i^{(\lambda)} = \left(\prod_{j=1}^{i} \frac{(\alpha+j-1+\lambda)}{(\alpha+j+1)} \right) \frac{\lambda-(\alpha+1)}{\alpha+1} u_0^{(\lambda)}.$$

Since $u_0^{(\lambda)} \neq 0$, the condition $\Delta u_M^{(\lambda)} = 0$ is satisfied iff

$$\lambda = \alpha+1 \quad \text{or} \quad \lambda = -\alpha - j + 1 \text{ for some } j \in \{1, \ldots, M\}.$$

[7] The statement in the case of α-PORT is only given for $\alpha = 1$ in [11]. Their proof allows for an immediate generalization, which is presented here.

Therefore the eigenvalues of A are given by $\lambda_1 = \alpha + 1 > 0$, $\lambda_j = -\alpha - j + 2 < 0$ for $j = 2, \dots, M+1$. Let $v = (v_0, \dots, v_M)$ be the left eigenvector for the eigenvalue λ_1 with the normalization $\sum_{j=0}^{M} v_j = 1$. Then

$$\begin{aligned} \alpha \sum_{j=1}^{M} v_j &= (\alpha+1)v_0, \\ (\alpha+i)v_{i-1} - (\alpha+i)v_i &= (\alpha+1)v_i \quad \text{ for } 0 < i < M, \\ (M+\alpha)v_{M-1} + v_M &= (\alpha+1)v_M. \end{aligned}$$

From the first property and the normalization of v, one obtains $\alpha = (2\alpha+1)v_0$, therefore $v_0 = \frac{\alpha}{2\alpha+1}$. Combining this with the second property, we obtain

$$v_k = (\prod_{j=1}^{k-1} \frac{\alpha+j}{2\alpha+j+1})v_0 = \prod_{j=0}^{k} \frac{\alpha+j}{2\alpha+j+1} = \frac{\Gamma(\alpha+k+1)\Gamma(2\alpha+1)}{\Gamma(\alpha)\Gamma(2\alpha+k+2)}$$

for $0 \leq k < M$. Applying Theorem 2.12 gives $\frac{1}{n}(Q_n \overset{a.s.}{\to} n\lambda_1 v)$ and since clearly $\lambda_2 < \frac{1}{2}\lambda_1$ also

$$\frac{1}{\sqrt{n}}(Q_n - n\lambda_1 v) \overset{D}{\to} \mathcal{N}(0, \Sigma)$$

for some covariance matrix $\Sigma \in \mathbb{R}^{(M+1)\times(M+1)}$. Observe

$$\lambda_1 v_k = (\alpha+1)\frac{\Gamma(\alpha+k+1)\Gamma(2\alpha+1)}{\Gamma(\alpha)\Gamma(2\alpha+k+2)}$$

for $k < M$ and dividing by $\alpha + k$ allows for translation to the corresponding result for $X_{n,k}$. Since M was arbitrary, we have joint asymptotic normality of the infinite random vector $X_n = (X_{n,k})_{k\in\mathbb{N}}$ with the claimed parameters μ_k. □

3.3.2 The height of recursive trees

For the study of the height of recursive trees, we will encode the heights of all nodes of a recursive tree via an incremental string. This correspondence will be used throughout the following analysis.

Definition 3.28. A **(discrete-time) incremental string** $(X_n)_{n\geq1}$ with **transition distribution** $(P_n)_{n\geq1}$ is defined as follows: For any $n \geq 1$ a probability distribution on $\{1, \dots, n\}$ is defined via $P_n = (P_{n,1}, \dots, P_{n,n}) \in \Delta_{n-1}$, such that the probability for $i \in \{1, \dots, n\}$ is given by $P_{n,i}$. The sequence X_n is a random process with initial state $X_1 = 0$ and for any $n \geq 1$ the random variable $X_{n+1} = (X_{n+1,1}, \dots, X_{n+1,n+1}) = (X_{n,1}, \dots, X_{n,n}, X_{n+1,n+1})$ is a sequence of length $n+1$ and a continuation of the sequence X_n. For $i \leq n$, we will denote by $X_n(i)$ the i-th smallest entry of X_n, where values occurring more than once will be counted with their corresponding multiplicity. The entry $X_{n+1,n+1}$ is defined by $X_{n+1,n+1} = 1 + X_n(i)$, where $i \in \{1, \dots, n\}$ is chosen at random with probability $P_{n,i}$.

A **(continuous-time) incremental string** $(X_t)_{t\geq0}$ with **transition rate** $(P_n)_{n\geq1}$ is defined as follows: For any $n \geq 1$ we have a finite sequence $P_n = (P_{n,1}, \dots, P_{n,n})$ with non-negative entries and at least one positive entry. The process $(X_t)_{t\geq 0}$ is a random process with initial state $X_0 = 0$ at time $t_1 := 0$. For any $n \geq 1$ there is a time t_{n+1} such that X_t extends at time $t = t_{n+1}$ to a sequence of length $n+1$ and the value for its last entry is determined as follows: The i-th smallest entry $X_{t_n}(i)$ of X_{t_n} gets assigned a random value $t^{(i)} \in [t_n, \infty)$, where the $t^{(i)}$ are independent shifted exponentially distributed random variables with mean $t_n + \frac{1}{P_{n,i}}$ (if $P_{n,i} = 0$ we set $t^{(i)} = \infty$). We then set $t_{n+1} = \min_{i=1,\ldots,n} t^{(i)}$ and if $j = \operatorname{argmin}_{i=1\ldots,n}(t^{(i)})$ we define $X_{t_{n+1},n+1} = X_{t_n}(j) + 1$. This construction is well-defined almost surely. The time t_n can be described as the moment when the n-th component of X_t is decided, and we will call the sequence $(t_n)_{n\geq 1}$ the **decision times**.

Remark 3.29. We will often require the entries of an incremental string to carry additional information, other than only its value. This is the case because an entry v shall "remember" its parent w, i.e., the position of the entry w used to define the value of v by $v := w + 1$. Formally, this could be done via an increasing tree or simply an encoding in a triple $X_{n,i} = (v, i, j)$, where v is the value, i is its position, and $j < i$ is the position of its parent w. However, for practical uses, we will write $X_{n,i} = v$ and the values of (i, j) are implicit properties of v.

Definition 3.30. The incremental strings $(R_n)_{n\geq 1}$ and $(R_t)_{t\geq 0}$ with transition distribution $P_{n,i} = \frac{1}{n}$ and transition rate $Q_{n,i} = 1$ for all $i = 1, \ldots, n$ and $n \in \mathbb{N}_{\geq 1}$ are both called **incremental string of recursive trees**.

The following study of the height of recursive trees breaks down into three separate parts, which are then combined to prove Theorem 3.35: Firstly, we embed the corresponding discrete-time incremental string into a continuous-time incremental string via Lemma 3.31. Secondly, we analyze the asymptotic behavior of the decision times for such a continuous-time incremental string in Lemma 3.32. Thirdly, in Lemma 3.34, we show that the maximum of the continuous-time incremental string at the decision times grows asymptotically proportional to the decision times.

Lemma 3.31. *[31] Let $(X_n)_{n\geq 1}$ be a discrete-time incremental string with transition distribution $(P_n)_{n\geq 1}$ and let $(Y_t)_{t\geq 0}$ be a continuous-time incremental string with transition rate $(Q_n)_{n\geq 1}$ and decision times $(t_n)_{n\geq 1}$. If Q_n is proportional to P_n for any $n \geq 1$, i.e., there is a constant $c_n > 0$ such that $Q_{n,i} = c_n P_{n,i}$ for all $i = 1, \ldots, n$, then $(X_n)_{n\geq 1} \stackrel{D}{=} (Y_{t_n})_{n\geq 1}$.*

Proof. We prove $(X_k)_{k=1}^n \stackrel{D}{=} (Y_{t_k})_{k=1}^n$ by induction on $n \in \mathbb{N}_{\geq 1}$. The induction start is trivial due to $X_1 = 0 = Y_{t_1}$, so assume that $(X_k)_{k=1}^n \stackrel{D}{=} (Y_{t_k})_{k=1}^n$ for some $n \in \mathbb{N}_{\geq 1}$. Let $t^* > t_n$ be arbitrary and τ_i be independent exponentially distributed random variables with mean $\frac{1}{Q_{n,i}}$ for $i = 1, \ldots, n$. If f_{τ_j} denotes the probability density of τ_j, then the probability that the value of $Y_{t_{n+1},n+1}$ is defined via $Y_{t_n}(j)+1$ is equal to

$$\begin{aligned}\mathbb{P}(\tau_j < \min_{i\neq j}\tau_i) &= \int_0^\infty f_{\tau_j}(t)\mathbb{P}(t<\min_{i\neq j}(\tau_i))\,\mathrm{d}t = \int_0^\infty f_{\tau_j}(t)\prod_{i\neq j}\mathbb{P}(t<\tau_i)\,\mathrm{d}t \\ &= \int_0^\infty Q_{n,j}e^{Q_{n,j}t}\prod_{i\neq j}e^{Q_{n,i}t}\,\mathrm{d}t = Q_{n,j}\int_0^\infty e^{t\sum_{i=1}^n Q_{n,i}}\,\mathrm{d}t \\ &= \frac{Q_{n,j}}{\sum_{i=1}^n Q_{n,i}} = \frac{c_n P_{n,j}}{c_n\sum_{i=1}^n P_{n,i}} = P_{n,j}.\end{aligned}$$

Thus, the transition probability of how $Y_{t_{n+1}}$ is obtained from Y_{t_n} is the same as how X_{n+1} is obtained from X_n, and therefore the claim follows. □

Lemma 3.32. *[26] Let $(X_t)_{t\geq 0}$ be a continuous-time incremental string with transition rate $(P_n)_{n\geq 1}$ and decision times $(t_n)_{n\geq 1}$. If there is some $\beta>0$ such that $\sum_{i=1}^n P_{n,i} = \beta n + O(1)$ for $n\in\mathbb{N}_{\geq 1}$, then*

$$\frac{t_n}{\log(n)} \xrightarrow{a.s.} \frac{1}{\beta}.$$

Proof. Firstly, note that the decisions times $(t_n)_{n\geq 1}$ satisfy $t_{n+1} = t_n + \tau_n$, where the τ_n are independent exponentially distributed with mean $\mu_n = \frac{1}{\sum_{i=1}^n P_{n,i}} = \frac{1}{\beta n + O(1)}$. Therefore $t_n = \sum_{i=1}^{n-1}\tau_i$ and the mean and variance of t_n satisfy

$$\begin{aligned}\mathbb{E}t_n &= \sum_{k=1}^{n-1}\mathbb{E}\tau_k = \sum_{k=1}^{n-1}\mu_k = \sum_{k=1}^{n-1}\frac{1}{\beta k + O(1)} = \frac{1}{\beta}\sum_{k=1}^{n-1}\frac{1}{k}\frac{1}{1+O(\frac{1}{k})} \\ &= \frac{1}{\beta}\sum_{k=1}^{n-1}\frac{1}{k} + \sum_{k=1}^{n-1}O(k^{-2}) = \frac{\log(n)}{\beta} + O(1)\end{aligned}$$

and

$$\mathbb{V}t_n = \sum_{k=1}^{n-1}\mathbb{V}\tau_k = \sum_{k=1}^{n-1}\mu_k^2 = \sum_{k=1}^{n-1}\frac{1}{(\beta k + O(1))^2} = O(1),$$

due to the fact that the last series is a convergent series. Using $\mathbb{V}t_n \leq C$ for some $C>0$ and applying Chebyshev's inequality gives

$$\mathbb{P}\left(|\frac{t_n}{\mathbb{E}t_n} - 1| \geq \frac{1}{\sqrt{\log(n)}}\right) \leq \mathbb{P}\left(|t_n - \mathbb{E}t_n| \geq \frac{\mathbb{E}t_n\sqrt{\mathbb{V}(t_n)}}{\sqrt{C\log(n)}}\right) \leq \frac{C\log(n)}{(\mathbb{E}t_n)^2} = O\left(\frac{1}{\log(n)}\right).$$

Take the subsequence $(t_{2^{m^2}})_{m\geq 1}$ and let $\varepsilon>0$ be arbitrary. Choose $M > \frac{1}{\varepsilon\sqrt{\log(2)}}$ and observe

$$\sum_{m=1}^{\infty}\mathbb{P}\left(|\frac{t_{2^{m^2}}}{\mathbb{E}t_{2^{m^2}}}-1|\geq\varepsilon\right)=\sum_{m=1}^{M-1}\mathbb{P}\left(|\frac{t_{2^{m^2}}}{\mathbb{E}t_{2^{m^2}}}-1|\geq\varepsilon\right)+\sum_{m=M}^{\infty}\mathbb{P}\left(|\frac{t_{2^{m^2}}}{\mathbb{E}t_{2^{m^2}}}-1|\geq\varepsilon\right)$$
$$\leq\sum_{m=1}^{M-1}\mathbb{P}\left(|\frac{t_{2^{m^2}}}{\mathbb{E}t_{2^{m^2}}}-1|\geq\varepsilon\right)+\sum_{m=M}^{\infty}\mathbb{P}\left(|\frac{t_{2^{m^2}}}{\mathbb{E}t_{2^{m^2}}}-1|\geq\frac{1}{m\sqrt{\log(2)}}\right)$$
$$=\sum_{m=1}^{M-1}\mathbb{P}\left(|\frac{t_{2^{m^2}}}{\mathbb{E}t_{2^{m^2}}}-1|\geq\varepsilon\right)+\sum_{m=M}^{\infty}O(m^{-2})<\infty.$$

From the Borel–Cantelli lemma we can therefore conclude that

$$\frac{t_{2^{m^2}}}{\mathbb{E}t_{2^{m^2}}}\stackrel{a.s.}{\to}1.$$

For arbitrary $n\geq 2$ define $m(n)$ as the unique m such that

$$2^{m^2}\leq n<2^{(m+1)^2}.$$

We use $t_{2^{m(n)^2}}\leq t_n\leq t_{2^{(m(n)+1)^2}}$ and

$$\log(2)m(n)^2\leq\log(n)\leq\log(2^{(m(n)+1)^2})=\log(2)(m(n)+1)^2=\log(2)m(n)^2+O(m(n))$$

to obtain

$$\lim_{n\to\infty}\frac{\mathbb{E}t_n}{\mathbb{E}t_{2^{m(n)^2}}}=\lim_{n\to\infty}\frac{\mathbb{E}t_n}{\mathbb{E}t_{2^{(m(n)+1)^2}}}=1.$$

Combining the above results in the almost sure inequality chain

$$1=\lim_{n\to\infty}\frac{t_{2^{m(n)^2}}}{\mathbb{E}t_{2^{m(n)^2}}}\leq\liminf_{n\to\infty}\frac{t_n}{\mathbb{E}t_n}\leq\limsup_{n\to\infty}\frac{t_n}{\mathbb{E}t_n}\leq\lim_{n\to\infty}\frac{t_{2^{(m(n)+1)^2}}}{\mathbb{E}t_{2^{(m(n)+1)^2}}}=1.$$

Therefore, $\frac{t_n}{\mathbb{E}t_n}\stackrel{a.s.}{\to}1$ and by using $\mathbb{E}t_n\sim\frac{\log(n)}{\beta}$ we can conclude $\frac{t_n}{\log(n)}\stackrel{a.s.}{\to}\frac{1}{\beta}$. □

Lemma 3.33. *[26] Let $(R_t)_{t\geq 0}$ be an incremental string of recursive trees. If $T(k)$ denotes the smallest time t, such that $\max(R_t)=k$, then*

$$\frac{T(k)}{k}\stackrel{a.s.}{\to}\frac{1}{e}.$$

Proof. It was shown in [26] that the statement is a consequence of the more general theorem about branching processes given in [15]. We refer the interested reader to these sources for a proof. □

Lemma 3.34. *[26] Let $(R_t)_{t\geq 0}$ be an incremental string of recursive trees with decision times $(t_n)_{n\geq 1}$. Then*

$$\frac{\max(R_{t_n})}{t_n}\stackrel{a.s.}{\to}e.$$

Proof. If $T(k)$ is defined as in Lemma 3.33, we have by definition of T

$$T(\max(R_{t_n})) \leq t_n < T(\max(R_{t_n}) + 1).$$

Since $t_n \overset{a.s.}{\to} \infty$, the second inequality in the above implies that also $\max(R_{t_n}) \overset{a.s.}{\to} \infty$. Therefore, by inverting the inequality chain and multiplying with $\max(R_{t_n})$ we obtain from Lemma 3.33

$$\begin{aligned} e = \lim_{n\to\infty} \frac{\max(R_{t_n})}{T(\max(R_{t_n}) + 1)} &\leq \liminf_{n\to\infty} \frac{\max(R_{t_n})}{t_n} \\ &\leq \limsup_{n\to\infty} \frac{\max(R_{t_n})}{t_n} \leq \lim_{n\to\infty} \frac{\max(R_{t_n})}{T(\max(R_{t_n}))} = e \end{aligned}$$

almost surely, such that also $\frac{\max(R_{t_n})}{t_n} \overset{a.s.}{\to} e$. □

Theorem 3.35. *[26] Let $\mathcal{T}$ be the family of recursive trees and H_n be the random variable denoting the height of a random tree $\tau \in \mathcal{T}_n$. Then*

$$\frac{H_n}{\log(n)} \overset{a.s.}{\to} e.$$

Proof. The heights $(H_n)_{n\geq 1}$ of random recursive trees satisfy $(H_n)_{n\geq 1} \overset{D}{=} \max(X_n)_{n\geq 1}$, where $(X_n)_{n\geq 1}$ is an incremental string of recursive trees. By Lemma 3.31 this incremental string can be embedded into a continuous-time incremental string $(R_t)_{t\geq 0}$ with transition rate $P_{n,i} = 1$ for $i = 1, \dots, n$ and $n \in \mathbb{N}_{\geq 1}$ via the identity $(R_{t_n})_{n\geq 1} \overset{D}{=} (X_n)_{n\geq 1}$ at the decision times $(t_n)_{n\geq 1}$ of $(R_t)_{t\geq 0}$. Applying Lemma 3.32 with $\beta = 1$ and Lemma 3.34 yields $\frac{t_n}{\log(n)} \overset{a.s.}{\to} 1$ and $\frac{\max(R_{t_n})}{t_n} \overset{a.s.}{\to} e$, respectively. Combining these two statements we obtain $\frac{\max(R_{t_n})}{\log(n)} \overset{a.s.}{\to} e$. Finally, the claim then follows from $\max(R_{t_n})_{n\geq 1} \overset{D}{=} (H_n)_{n\geq 1}$. □

Theorem 3.36. *[28, 31] Let $\mathcal{T}$ be the family of recursive trees and let D_n be the random variable denoting the height of the vertex n in a random tree $\tau \in \mathcal{T}_n$. Then for $1 < k < n$ we have*

$$\mathbb{P}(D_n = k) = \frac{\left[{n-1 \atop k}\right]}{(n-1)!},$$

where $\left[{n \atop k}\right]$ denotes the unsigned Stirling numbers of the first kind. Consequently, for any $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that

$$\mathbb{P}(D_n \geq (1 - \eta(\varepsilon))e \log(n)) = O(n^{-1+\varepsilon}).$$

Proof. By calculating a recurrence relation for $\mathbb{P}(D_n = k)$, the first part can be shown. Using well-known asymptotics, one can derive the second claim from the first result. For details, we refer to [28] for the first part and to [31] for the second part. □

4 Bucket increasing trees

Bucket increasing trees provide an interesting generalization of increasing trees, and, for bucket size $b = 1$, the class of increasing trees is fully recovered as a special case. In Section 4.1, we will first give some general enumerative results for simple families of bucket increasing trees as in [16, 19, 20]. Then we will restrict attention to the special case of bucket size $b = 2$, which is based on [3, 19]. In Section 4.2, we will study bucket increasing trees emerging from a stochastic growth process. This part is based on [19, 23], and Theorem 4.19 gives a generalization for one of the main results of [23], which was already conjectured to be possible in [19].

Definition 4.1. Let $b \in \mathbb{N}_{\geq 1}$. A **bucket tree** is defined in the following way: It consists of a rooted tree τ, where any node $v \in V(\tau)$ is assigned a **capacity** $1 \leq c(v) \leq b$. A node with $c(v) = b$ is called **saturated** and the value b is the **maximal capacity** or **bucket size**. Additionally, any interior node of a bucket tree must be saturated. The **size** of the bucket tree is given by $n = \sum_{v \in V(\tau)} c(v)$, and a bucket tree is called **plane** if τ is plane. A **bucket increasing tree** is a bucket tree, where the labels $1, \ldots, n$ are distributed among the nodes of t such that any node $v \in V(\tau)$ gets assigned[1] $c(v)$ many of the labels and the labels of any node are smaller than all of the labels of its children.

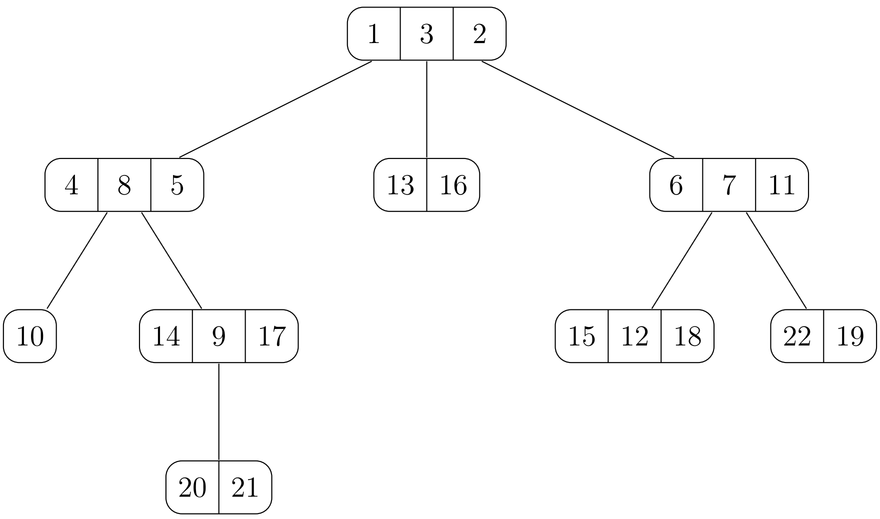

Figure 3: A bucket increasing tree of size 22 and with bucket size $b = 3$.

[1]We assume that the order of the labels within a node is relevant.

4.1 Simple families of bucket increasing trees

In this section, we generalize most of the contents of Section 3.1 to simple families of bucket increasing trees. However, an analogous implicit definition of the generating function and the asymptotic enumeration result for polynomial families is only shown for the case of bucket size $b = 2$.

Definition 4.2. Let $b \in \mathbb{N}_{\geq 1}$, $\psi_1 \ldots, \psi_{b-1} \in [0, \infty)$, $(\varphi_k)_{k\geq 0} \in [0, \infty)^{\mathbb{N}}$ with $\varphi_k > 0$ for some $k \geq 2$. Further assume $\varphi_0 > 0$ or $\psi_j > 0$ for some $j \in \{1, \ldots, b-1\}$. The sequence $\psi_1, \ldots, \psi_{b-1}$ is called **bucket-weight sequence**. To any plane bucket increasing tree τ with bucket size b we assign its **weight** $w(\tau) := \prod_{v\in V(\tau)} w(v)$, where

$$w(v) = \begin{cases} \varphi_{d^+(v)} & \text{if } c(v) = b, \\ \psi_{c(v)} & \text{if } c(v) < b. \end{cases}$$

The family $\mathcal{T}$ then consists of all plane bucket increasing trees of bucket size b and positive weight which satisfy that the labels of any node are increasingly ordered together with their weights. A family $\mathcal{T}$ which can be constructed in this way is called **simple family of bucket increasing trees** and b is its **bucket size**. The sequences $\psi_1, \ldots, \psi_{b-1}$ and $(\varphi_k)_{k\geq 0}$ and the generating function $\varphi(t) := \sum_{k\geq 0} \varphi_k t^k$ are a **bucket-weight sequence**, **degree-weight sequence**, and **degree-weight generating function** of $\mathcal{T}$, respectively. A simple family of bucket increasing trees is a **polynomial family of bucket increasing trees** if its degree-weight generating function is a polynomial. The **degree** of a polynomial family of bucket increasing trees is given by the degree of $\varphi(t)$.

Remark 4.3. The terms "simple families of bucket increasing trees", "very simple families of bucket increasing trees", and "grown simple families of bucket increasing trees" are not used in [16, 19, 20]. However, it is very natural to adopt these names from their special case of increasing tree families. We also altered the condition imposed on the weight sequences by relaxing the constraint of $\varphi_0 > 0$ to a more general setting.

Remark 4.4. If $\mathcal{T}$ is a simple family of bucket increasing trees with bucket size b, degree-weight sequence φ_k and bucket-weight sequence ψ_k, then Remark 3.4 can be generalized as follows: If $a, d > 0$ and the weights φ_k, ψ_k are replaced by $a^b s^{k-1}\varphi_k$ and $a^k s^{-1}\psi_k$, respectively, then the weight of any tree $\tau \in \mathcal{T}_n$ will be scaled by a factor of $a^n s^{-1}$. Therefore, the natural probability distribution for the simple family of increasing trees with the altered weights induces the same natural probability distributions as $\mathcal{T}$. This fact is also discussed in [20].

Lemma 4.5. *[20] A class of bucket increasing trees $\mathcal{T}$ is a simple family of bucket increasing trees iff it can be defined by the formal recursive equation*

$$\mathcal{T} = \sum_{k=1}^{b-1} \psi_k (\mathcal{Z}^{\square})^k + (\mathcal{Z}^{\square})^b \star \varphi(\mathcal{T}) \tag{4.1}$$

for some bucket size b, degree-weight generating function φ and bucket-weight sequence $\psi_1, \ldots, \psi_{b-1}$. The expression $(\mathcal{Z}^{\square})^k \star \mathcal{B}$ with $\mathcal{B} = \mathcal{E}$ or $\mathcal{B} = \varphi(\mathcal{T})$ is an abbreviation for $\mathcal{Z}^{\square} \star (\mathcal{Z}^{\square} \star (\ldots \star (\mathcal{Z}^{\square} \star \mathcal{B})))$, where $\mathcal{Z}^{\square}$ appears k times.

Proof sketch. The proof is similar to that of Lemma 3.5 and can be found in [20]. □

Definition 4.6. The simple family of bucket increasing trees associated with the bucket size b, degree-weight generating function $\varphi(t)$, and bucket-weight sequence $\psi_1, \dots, \psi_{b-1}$ is called the family of

(i) **bucket recursive trees** if

$$\begin{aligned}\varphi(t) &= (b-1)! \exp(bt), \\ \psi_k &= (k-1)!\end{aligned}$$

for $1 \leq k \leq b-1$,

(ii) **(b, α)-plane oriented recursive trees**[2] if

$$\begin{aligned}\varphi(t) &= (b-1)!(\alpha+1)^{b-1}\binom{b-1-\frac{1}{\alpha+1}}{b-1}(1-t)^{-(\alpha+1)b+1}, \\ \psi_k &= (k-1)!(\alpha+1)^{k-1}\binom{k-1-\frac{1}{\alpha+1}}{k-1}\end{aligned}$$

for $1 \leq k \leq b-1$ and some $\alpha > 0$,

(iii) **(b, d)-ary increasing trees** if

$$\begin{aligned}\varphi(t) &= (b-1)!(d-1)^{b-1}\binom{b-1+\frac{1}{d-1}}{k-1}(1+t)^{b(d-1)+1}, \\ \psi_k &= (k-1)!(d-1)^{k-1}\binom{k-1+\frac{1}{d-1}}{k-1}\end{aligned}$$

for $1 \leq k \leq b-1$ and some $d \in \mathbb{Q}$ with $(d-1)b \in \mathbb{N}_{\geq 1}$.

A **very simple family of bucket increasing trees** is a simple family of bucket increasing trees that falls into one of the above three categories.

Remark 4.7. A similar interpretation of these models as in Remark 3.7 can be given. The bucket recursive trees model non-plane bucket trees, where any node of capacity k is substituted by some recursive tree of size k, and the labels are distributed such that we obtain a bucket increasing tree. Likewise, the family of $(b, 1)$-PORT models plane bucket trees, where the nodes are substituted using the family of PORT instead of recursive trees. Finally, in a similar way, the family of (b, d)-ary increasing trees can be interpreted as the substitution of d-ary increasing trees for the nodes of $(b(d-1)+1)$-ary trees.

Lemma 4.8. *[16] Let $\mathcal{T}$ be a simple family of bucket increasing trees with bucket size b, degree-weight generating function $\varphi(t)$, and bucket-weight sequence $\psi_1, \dots, \psi_{b-1}$. Then T satisfies the autonomous ordinary differential equation $T^{(b)}(z) = \varphi(T(z))$ with $T(0) = 0$ and $T^{(k)}(0) = \psi_k$ for $1 \leq k \leq b-1$.*

[2] We will abbreviate this term with (b, α)-PORT.

Proof. The claim is verified immediately by translating the formal recursive equation in Lemma 4.5. □

Theorem 4.9. *[16, 19] Let $\mathcal{T}$ be a very simple family of bucket increasing trees with bucket size b. Then the following formulas hold:*

(i) *Bucket recursive trees:*

$$T(z) = \log\left(\frac{1}{1-z}\right)$$
$$T_n = (n-1)!$$

(ii) *(b, α)-PORT:*

$$T(z) = 1 - (1 - (\alpha+1)z)^{1/(\alpha+1)}$$
$$T_n = -\binom{n - 1/(\alpha+1) - 1}{n}(\alpha+1)^n n!$$
$$T_n \sim -\frac{n^{-1-1/(\alpha+1)}}{\Gamma(-1/(\alpha+1))}(\alpha+1)^n n!$$

(iii) *(b, d)-ary increasing trees:*

$$T(z) = -1 + (1 - (d-1)z)^{-\frac{1}{d-1}}$$
$$T_n = \binom{n + 1/(d-1) - 1}{n}(d-1)^n n!$$
$$T_n \sim \frac{n^{-1+1/(d-1)}}{\Gamma(1/(d-1))}(d-1)^n n!$$

Proof sketch. In [19, 20], a detailed discussion of these tree models is given, leading to the conjectured generating functions as reasonable candidates. It can then easily be verified that the differential equation in Lemma 4.8 is satisfied. The coefficient extraction and asymptotics are standard procedures. □

The special case of bucket size $b = 2$

Due to the absence of a generalization of Lemma 3.9 for arbitrary bucket size b, there is also no immediate generalization for the asymptotic enumeration of polynomial families of bucket increasing trees available. However, for the case of $b = 2$, an analogous treatment as in the case of increasing trees is possible:

Theorem 4.10. *[3, 19] Let $\mathcal{T}$ be a simple family of bucket increasing trees with bucket size $b = 2$, degree-weight generating function $\varphi(t)$, and bucket-weight sequence[3] ψ_1. Then T is the implicit solution of*

$$\int_0^{T(z)} \frac{1}{\sqrt{\psi_1^2 + 2\int_0^t \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}t = z.$$

[3]Since we consider the special case of $b = 2$, the bucket-weight sequence $\psi_1, \ldots, \psi_{b-1}$ is given by a single weight $\psi_1 \in [0, \infty)$.

Proof. By Lemma 4.8 the function T is the solution of

$$T''(z) = \varphi(T(z))$$

with $T(0) = 0$ and $T'(0) = \psi_1$. We multiply by $T'(z)$ and integrate with respect to z to obtain

$$\frac{1}{2}((T'(z))^2 - (T'(0)^2) = \int_0^{T(z)} \varphi(s)\,\mathrm{d}s.$$

Solving for $T'(z)$ and substituting $T'(0) = \psi_1$ then yields

$$T'(z) = \sqrt{\psi_1^2 + 2\int_0^{T(z)} \varphi(s)\,\mathrm{d}s}.$$

Dividing by the right-hand side, integrating, and substituting for T shows

$$z = \int_0^z 1\,\mathrm{d}w = \int_0^z \frac{T'(w)}{\sqrt{\psi_1^2 + 2\int_0^{T(w)} \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}w = \int_0^{T(z)} \frac{1}{\sqrt{\psi_1^2 + 2\int_0^{t} \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}t. \qquad \square$$

Lemma 4.11. *Let $\mathcal{T}$ be a simple family of bucket increasing trees with bucket size $b = 2$, degree-weight generating function $\varphi(t)$, and bucket-weight sequence ψ_1. Let $q_1, q_2 \in \mathbb{N}_{\geq 1}$ be maximal such that $\varphi(t) = \tilde{\varphi}(t^{q_1})$ and $\int_0^t \varphi(s)\,\mathrm{d}s = \tilde{\Phi}(t^{q_2})$ for some $\tilde{\varphi}, \tilde{\Phi} \in \mathbb{C}[[t]]$. The maximum period p of T is given by*

$$p = \begin{cases} 2q_1 & \text{if } \psi_1 = 0, \\ q_2 & \text{if } \psi_1 \neq 0. \end{cases}$$

Furthermore, T is of the form

$$T(z) = \begin{cases} z^2 T^*(z^p) & \text{if } \psi_1 = 0, \\ zT^*(z^p) & \text{if } \psi_1 \neq 0, \end{cases}$$

for some $T^* \in \mathbb{C}[[z]]$.

Proof. The case of $\psi_1 = 0$ is easy to see, since any node then must have capacity 2. The underlying unlabeled tree family is clearly periodic with maximum period q_1, and for any ordinary tree τ, we know $|\tau| \equiv 1 \mod q_1$. This is equivalent to $2|\tau| \equiv 2 \mod 2q_1$, therefore the claim holds in the first case.

For $\psi_1 \neq 0$, the family has a bucket tree of size 1 and any other tree $\tau \in \mathcal{T}$ consists of a node v of capacity $c(v) = 2$ and branches $\tau_1, \ldots, \tau_d \in \mathcal{T}$ for $[t^d]\varphi(t) \neq 0$. By induction, one thus obtains $|\tau| \equiv 1 \mod d+1$ from

$$2 + \sum_{j=1}^{d} |\tau_j| \equiv 2 + \sum_{j=1}^{d} 1 = 2 + d \equiv 1 \mod d+1.$$

This shows that p is a multiple of q_2 and since $\psi_1 \neq 0$, clearly for any d with $[t^d]\varphi(t) \neq 0$ there is some $\tau \in \mathcal{T}$ with $|\tau| = d+2$. It follows that p must be a divisor of q_2 and therefore, $p = q_2$, and the claim also holds in the second case. $\square$

Theorem 4.12. *[3] Let $\mathcal{T}$ be a polynomial family of bucket increasing trees of degree $d \geq 2$ with bucket size $b = 2$, degree-weight generating function $\varphi(t) = \varphi_d t^d + \ldots + \varphi_1 t + \varphi_0$, and bucket-weight sequence ψ_1. Let p be the maximum period of $T(z)$ and define*

$$\rho := \int_0^\infty \frac{1}{\sqrt{\psi_1^2 + 2\int_0^t \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}t, \quad \delta := \frac{2}{d-1}, \quad \text{and} \quad \eta := \left(\frac{\sqrt{\varphi_d}(d-1)\rho}{\sqrt{2(d+1)}}\right)^\delta.$$

Then T_n satisfies

$$\frac{T_n}{n!} = \begin{cases} \frac{p}{\eta\Gamma(\delta)} n^{\delta-1}\rho^{-n} + O((n^{\delta-2} + n^{-\delta-1})\rho^{-n}) & \text{if } n \equiv 2 \mod p \text{ and } \psi_1 = 0 \\ & \text{or } n \equiv 1 \mod p \text{ and } \psi_1 \neq 0, \\ 0 & \text{else.} \end{cases}$$

Proof. From Theorem 4.10 we obtain

$$\rho - z = \int_{T(z)}^\infty \frac{1}{\sqrt{\psi^2 + 2\int_0^t \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}t.$$

Calculate

$$\int_{1/t}^\infty \frac{1}{\sqrt{\psi^2 + 2\int_0^y \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}y = \int_0^t \frac{1}{y^2}\frac{1}{\sqrt{\psi^2 + 2\int_0^{1/y} \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}y$$

and observe, that the integrand is an algebraic function of degree 2, where the smallest exponent is at least of size $-\frac{1}{2}$. Therefore the integral defines a Puiseux expansion, where the smallest exponent is at least of size $\frac{1}{2}$, implying that $\rho - z = H(\frac{1}{\sqrt{T(z)}})$ for some function $H(t)$, which is analytic at 0 with $H(0) = 0$. Extracting the leading order $\frac{1}{t^{(d+1)/2}}$ of the integrand for $t \to \infty$ shows

$$\begin{aligned}
\int_{T(z)}^\infty \frac{1}{\sqrt{\psi^2 + 2\int_0^t \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}t &= \int_{T(z)}^\infty \frac{1}{\sqrt{\frac{2\varphi_d}{d+1}t^{d+1} + \ldots + \varphi_1 t^2 + 2\varphi_0 t + \psi^2}}\,\mathrm{d}t \\
&= \int_{T(z)}^\infty \frac{1}{\sqrt{\frac{2\varphi_d}{d+1}t^{d+1}}} + O(t^{1-(d+1)/2})\,\mathrm{d}t \\
&= -\frac{\sqrt{d+1}}{\sqrt{2\varphi_d}(1-(d+1)/2)} T(z)^{1-(d+1)/2} + O(T(z)^{2-(d+1)/2}) \\
&= \frac{\sqrt{2(d+1)}}{\sqrt{\varphi_d}(d-1)} T(z)^{(1-d)/2} + O(T(z)^{(3-d)/2}).
\end{aligned}$$

Therefore H has a root of order $d-1$ at 0 with $[t^{d-1}]H(t) = \frac{\sqrt{2(d+1)}}{\sqrt{\varphi_d}(d-1)}$. Extracting further orders of $H(t)$ shows that $t^{1-d}H(t)$ is an even function, therefore $H(t)$ is of the form $H(t) = \frac{\sqrt{2(d+1)}}{\sqrt{\varphi_d}(d-1)} t^{d-1} G(t^2)$ for a function G, which is analytic at 0 with $G(0) = 1$. Therefore we obtain

$$\rho - z = \frac{\sqrt{2(d+1)}}{\sqrt{\varphi_d}(d-1)} T(z)^{(1-d)/2} G\left(\frac{1}{T(z)}\right).$$

Due to $G(0) = 1 \neq 0$ also $G^\delta(t) := (G(t))^{2/(d-1)}$ is analytic at 0. Rearranging terms, substituting δ and η and then raising to the power of δ yields

$$\eta\left(1-\frac{z}{\rho}\right)^\delta = \left(\frac{\sqrt{\varphi_d}(d-1)\rho}{\sqrt{2(d+1)}}\left(1-\frac{z}{\rho}\right)\right)^\delta = \frac{1}{T(z)}G^\delta\left(\frac{1}{T(z)}\right).$$

Now we are in the same setting as in the proof of Theorem 3.13, where the only difference stems from Lemma 4.11: In the case of $\psi_1 = 1$ the equality $T(z) = zT^*(z^p)$ must be replaced by $T(z) = z^2T^*(z^p)$ resulting in the altered condition of $n \equiv 2 \bmod p$ in this case. As it was already stated in Remark 3.14, the remaining proof also applies in this case. □

4.2 Grown simple families of bucket increasing trees

Analogous to the case of increasing trees, grown simple families of bucket increasing trees and very simple families of bucket increasing trees induce the same natural probability distributions, which is summarized in Theorem 4.14. This allows for the description of the distribution of the capacities in a random bucket tree via an urn model. Asymptotic probability distributions are then obtained, which are given in Theorem 4.19.

Definition 4.13. Let $\mathcal{T}$ be a simple family of bucket increasing trees. The family $\mathcal{T}$ is called **grown simple family of bucket increasing trees** if it can be obtained by a probabilistic growth process: For any bucket tree $\tau' \in \mathcal{T}$ with nodes $v_1, \dots, v_k$, there exist probabilities $p_{\tau'}(v_1), \dots, p_{\tau'}(v_k)$, such that the following holds: If $\tau' \in \mathcal{T}_n$ is chosen at random, then the bucket tree τ obtained by adding the label $n+1$ to v_j, if $c(v_j) < b$, or attaching a node with label $n+1$ to v_j, if $c(v_j) = b$, with probability $p_{\tau'}(v_j)$, is a random bucket tree of size $n+1$.

Theorem 4.14. *[20] Let $\mathcal{T}$ be a simple family of bucket increasing trees. The following are equivalent:*

(i) *There is a very simple family of bucket increasing trees which induces the same natural probability distributions as $\mathcal{T}$.*

(ii) *The family $\mathcal{T}$ is a grown simple family of bucket increasing trees.*

(iii) *The family $\mathcal{T}$ satisfies for any $n \geq j$: If τ is a random bucket tree of size n and τ' is the bucket tree obtained from τ by deleting all nodes with label greater than j, then τ' is a random bucket tree of size j.*

(iv) *There are c_1, c_2 such that $\frac{T_{n+1}}{T_n} = c_1 n + c_2$ for all $n \geq 1$.*

Proof. Note that any of these conditions implies $\varphi_0 > 0$ for the degree-weight generating sequence. Therefore, the proof in [20] applies. □

Remark 4.15. Similarly to Remark 3.26, the insertion probabilities of the label $n+1$ according to the described growth process for a node v of a random bucket tree τ' of size n are given by

$$p(v) = \begin{cases} \frac{c(v)}{n} & \text{for bucket recursive trees,} \\ \frac{(\alpha+1)c(v)+d^+(v)-1}{(\alpha+1)n-1} & \text{for } (b,\alpha)\text{-PORT,} \\ \frac{(d-1)c(v)+1-d^+(v)}{(d-1)n+1} & \text{for } (b,d)\text{-ary increasing trees.} \end{cases}$$

A detailed proof for this claim is given in [20].

4.2.1 The asymptotic joint distribution of bucket capacities

We summarize here the results about the urn models from [19], which we will use in the proof of Theorem 4.19.

Let $\mathcal{T}$ be a very simple family of bucket increasing trees and let $N_n = (N_{n,1}, \dots, N_{n,b})$ be the random vector where $N_{n,k}$ counts the number of nodes of capacity k in a random bucket tree of size n. Let

$$(\beta, \gamma) = \begin{cases} (1,0) & \text{for bucket recursive trees,} \\ (\alpha+1, -1) & \text{for } (b,\alpha)\text{-PORT,} \\ (d-1, 1) & \text{for } (b,d)\text{-ary increasing trees.} \end{cases}$$

Consider the standard generalized Pólya–Eggenberger urn model with b types of balls and replacement matrix $A = (A_{i,j})_{i,j=1}^b$ given by

$$A_{i,j} = \begin{cases} -(j\beta+\gamma)\delta_{i,j} + ((j+1)\beta+\gamma)\delta_{i,j+1}, & \text{for } 1 \le i \le b,\ 1 \le j < b, \\ (\beta+\gamma)\delta_{i,1} - \gamma\delta_{i,b}, & \text{for } 1 \le i \le b,\ j = b. \end{cases}$$

The replacement matrix A is thus of the form

$$A = \begin{pmatrix} -(\beta+\gamma) & 0 & \cdots & 0 & \beta+\gamma \\ 2\beta+\gamma & -(2\beta+\gamma) & \ddots & \vdots & 0 \\ 0 & 3\beta+\gamma & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & -((b-1)\beta+\gamma) & 0 \\ 0 & 0 & \cdots & b\beta+\gamma & -\gamma \end{pmatrix}.$$

In the initial state, there are $\beta+\gamma \in \{1, \alpha, d\}$ many balls of type 1. Let $Q_n = (Q_{n,1}, \dots, Q_{n,b})$ be the random vector where $Q_{n,k}$ counts the number of balls of type k in the urn after n replacement steps. Then the random vectors Q_n and N_n are related by the equality

$$N_{n,k} = \frac{Q_{n,k}}{n\beta+\gamma}, \quad 1 \le k \le b. \tag{4.2}$$

Furthermore, the characteristic polynomial of A is given by the formula

$$\chi_A(\lambda) = (-\beta)^b \left(\left(\frac{\lambda+\gamma}{\beta}\right)^{\overline{b}} - \left(\frac{\beta+\gamma}{\beta}\right)^{\overline{b}} \right).$$

By letting $\kappa := \frac{\gamma}{\beta} \in (-1, \infty)$, omitting the factor $(-\beta)^b$, and rescaling the argument by a factor of $\frac{1}{\beta}$, we can restrict to studying the roots $\lambda_{b,\kappa}^{(i)}$ of the equation $(\lambda + \kappa)^{\overline{b}} = (1 + \kappa)^{\overline{b}}$ for $i = 1, \ldots, b$. We will also assume that these roots are ordered such that

$$\Re\lambda_{b,\kappa}^{(1)} \geq \Re\lambda_{b,\kappa}^{(2)} \geq \ldots \geq \Re\lambda_{b,\kappa}^{(b)}.$$

Lemma 4.16. *[23] Let $b \in \mathbb{N}_{\geq 1}, \kappa \in [-\frac{1}{2}, \infty)$ and define $q(z) := (z + \kappa)^{\overline{b}}$ and $p(z) := q(z) - q(1)$. Then the following holds for the roots of $p(z)$:*

(i) 1 *is a root, and no root has a real part bigger than 1.*

(ii) $-b - 2\kappa$ *is a root iff b is even.*

(iii) *No other roots are real, and they appear in complex conjugate pairs.*

(iv) *Except for conjugate pairs, no two distinct roots share the same real part.*

(v) *All roots are simple.*

Proof. [4] Clearly 1 is a root of $p(z)$ and for z with $\Re(z) > 1$ we have

$$|q(z)| = \prod_{j=0}^{b-1} |(z + \kappa + j)| \geq \prod_{j=0}^{b-1} (\Re z + \kappa + j) > \prod_{j=0}^{b-1} (1 + \kappa + j) = q(1),$$

therefore there cannot be a root with bigger real part and (i) holds.

The function $\tilde{q}(z) := q(z - \frac{b-1}{2} - \kappa)$ satisfies

$$\begin{aligned}
\tilde{q}(-z) &= \prod_{j=0}^{b-1} \left(-z - \frac{b-1}{2} + j \right) \\
&= \prod_{j=0}^{b-1} \left(-z - \frac{b-1}{2} + (b-1) - j \right) \\
&= \prod_{j=0}^{b-1} \left(-z + \frac{b-1}{2} - j \right) = (-1)^b \tilde{q}(z).
\end{aligned}$$

For even b we obtain

$$q(-b - 2\kappa) = \tilde{q}\left(-\frac{b+1}{2} - \kappa \right) = \tilde{q}\left(\frac{b+1}{2} + \kappa \right) = q(1)$$

and $q(-b - 2\kappa) = -q(1) \neq q(1)$ for odd b, therefore (ii) holds.

The equality

$$\tilde{q}(z) = \frac{z - (b-1)/2}{z + (b+1)/2} \tilde{q}(z + 1)$$

[4]The lemma is stated for the case of $\kappa = 0$ in [23]. Only the proof of (iii) introduces the need for better bounds in the regime of $\kappa \in [-1/2, 0)$.

shows $|\tilde{q}(z)| \leq |\tilde{q}(z+1)|$ for $z \geq 0$. Since q is strictly increasing in $[-\kappa, \infty)$ we have $|q(z)| < q(1)$ for $z \in [\frac{1}{2}, 1) \subseteq [-\kappa, 1)$ and $|q(z)| > q(1)$ for $z \in (1, \infty)$. Next observe

$$|q(-\kappa - z)| = \prod_{j=0}^{b-1} |(-z+j)| \leq \prod_{j=0}^{b-1} |z+j| = q(-\kappa + z) \leq q(1)$$

for $z \in [0, \frac{1}{2}] \subseteq [0, 1+\kappa]$ with equality iff $z = \frac{1}{2}, b = 1$, and $\kappa = -\frac{1}{2}$. Since the previous step already covers the case of $z = \frac{1}{2}$, we therefore get $|q(z)| < q(1)$ for all $z \in [0, 1)$. Combining this result together with iterating the inequality $|\tilde{q}(z)| \leq |\tilde{q}(z+1)|$ we obtain

$$|q(z)| < q(1)$$

for all $z \in [-\frac{b-1}{2} - \kappa, 1)$. Using this, the fact that $q(z) > q(1)$ for $z > 1$, and the symmetry property of $\tilde{q}(z)$, we obtain that $p(z)$ has no other real roots. Since $p(z)$ is a polynomial with real coefficients, the other roots must appear in complex conjugate pairs, which finishes the proof of (iii).

For $x, y_1, y_2 \in \mathbb{R}$ with $|y_1| > |y_2|$, we observe

$$|q(x+iy_1)| = \prod_{j=0}^{b-1} |x+\kappa+j+iy_1| > \prod_{j=0}^{b-1} |x+\kappa+j+iy_2| = |q(x+iy_2)|,$$

which directly implies (iv).

If λ is a root of $p(z)$, then

$$p'(\lambda) = \sum_{\ell=0}^{b-1} \prod_{\substack{j=0 \\ j\neq\ell}}^{b-1} (\lambda+\kappa+j) = q(1) \sum_{\ell=0}^{b-1} \frac{1}{\lambda+\kappa+\ell}.$$

Clearly $p'(1) > 0$ and $p'(-b-2\kappa) < 0$ and thus these two roots are simple. If λ is any other root, then

$$\begin{aligned}\Im(p'(\lambda)) &= q(1) \sum_{\ell=0}^{b-1} \Im\left(\frac{1}{\Re\lambda+\kappa+\ell+i\Im\lambda}\right) \\ &= -q(1)\Im(\lambda) \sum_{\ell=0}^{b-1} \frac{1}{(\Re\lambda+\kappa+\ell)^2+(\Im\lambda)^2}.\end{aligned}$$

Since $\Im(\lambda) \neq 0$, this shows that also the other roots are simple and (v) holds. □

Lemma 4.17. *Let $b \in \mathbb{N}_{\geq 2}$ and define $q_b(z) := z^{\overline{b}}$. Let $\kappa \in [-\frac{1}{2}, \infty)$ and $\lambda_{b,\kappa}^{(1)}, \lambda_{b,\kappa}^{(2)}, \ldots, \lambda_{b,\kappa}^{(b)}$ be the roots of $q_b(z+\kappa) - q_b(1+\kappa) = 0$ with $1 = \lambda_{b,\kappa}^{(1)} > \Re\lambda_{b,\kappa}^{(2)} \geq \ldots \geq \Re\lambda_{b,\kappa}^{(b)}$. If κ is fixed and there is some $b_0 \in \mathbb{N}_{\geq 2}$ such that $\Re\lambda_{b_0,\kappa}^{(2)} + \kappa \geq 0$, then the value of $\Re\lambda_{b,\kappa}^{(2)}$ is increasing with respect to b for $b \in \mathbb{N}_{\geq b_0}$.*

Proof. We assume without loss of generality, that $\Im(\lambda_{b,\kappa}^{(2)}) \geq 0$. For fixed κ, the equation $|q_b(x+iy)|^2 = q_b(1+\kappa)^2$ describes an algebraic curve for $(x,y) \in \mathbb{R}^2$. The proof of Lemma 4.16 shows that for fixed $x \in [-b-\kappa, 1+\kappa]$ the value of $|q_b(x+iy)|$ is strictly increasing with respect to $|y|$, tending to infinity for $|y| \to \infty$, and $|q_b(x)| \leq q_b(1+\kappa)$ with equality iff x is on the boundary of the interval. Therefore, we can define $y : [-b-\kappa, 1+\kappa] \to [0,\infty)$ as the unique $y_b(x) \geq 0$ such that $|q_b(x+iy_b(x))| = q_b(1+\kappa)$ or equivalently $\log(|q_b(x+iy_b(x))| = \log(q_b(1+\kappa))$. Using any analytic continuation of the logarithm, implicit differentiation then gives

$$\begin{aligned} 0 = \frac{d}{dx}\log(q_b(1+\kappa)) &= \frac{d}{dx}\Re\log(q_b(x+iy_b(x)) = \Re\frac{d}{dx}\sum_{j=0}^{b-1}\log(x+j+iy_b(x)) \\ &= \Re\sum_{j=0}^{b-1}\frac{1+iy_b'(x)}{x+j+iy_b(x)} = \sum_{j=0}^{b-1}\frac{x+j+y_b(x)y_b'(x)}{(x+j)^2+y_b(x)^2}. \end{aligned}$$

Define $S_b : \mathbb{C}\setminus\{0,-1,\ldots,-b+1\} \to \mathbb{C}, S_b(z) := \frac{q_b'(z)}{q_b(z)} = \sum_{j=0}^{b-1}\frac{1}{z+j}$ and notice that S_b maps the open upper half-plane into the open lower half-plane. By isolating $y_b'(x)$ in the above calculation, one obtains

$$y_b'(x) = \frac{\Re(S_b(x+iy_b(x))}{\Im(S_b(x+iy_b(x))} \tag{4.3}$$

for $x \in (-b-\kappa, 1+\kappa)$. Since $\Im(S_b(x+iy_b(x)) < 0$ for $x \in (-b-\kappa, 1+\kappa)$, the implicit function theorem justifies the above implicit differentiation and shows that y is even analytic in the interior of its domain. Observe that for $x \geq 0$ clearly $\Re S_b(x+iy_b(x)) \geq 0$ (since $y_b(x) \geq 0$) and thus y_b is decreasing in $[0, 1+\kappa]$.

If we analytically continue the logarithm along the path of $q_b(x+iy_b(x))$ for $x \in [-b-\kappa, 1+\kappa]$, such that for $x \to 1+\kappa$ we obtain the usual real-valued logarithm, then the function $\theta_b : [-b-\kappa, 1+\kappa]$, $\theta_b(x) := \Im(\log(q_b(x+iy_b(x))))$ is a continuous function with $\theta_b(1+\kappa) = 0$. Since y_b is decreasing for $x \in [0, 1+\kappa]$, the complex argument of any factor in $q_b(x+iy_b(x))$ is also decreasing in $[0, 1+\kappa]$. This implies that also θ_b is decreasing in $[0, 1+\kappa]$. Since $\lambda_{b,\kappa}^{(2)} + \kappa$ lies on the algebraic curve $(x, y(x))$, we have

$$\Re\lambda_{b,\kappa}^{(2)} + \kappa \geq 0 \Leftrightarrow 2\pi \in \theta([0, 1+\kappa]). \tag{4.4}$$

In this case, $\Re\lambda_{b,\kappa}^{(2)}$ can be characterized via

$$\Re\lambda_{b,\kappa}^{(2)} + \kappa = \max\{x \in [0, 1+\kappa] \mid \theta_b(x) = 2\pi\}. \tag{4.5}$$

We will now show $y_b(x) \leq y_{b+1}(x)$ for all $x \in [0, 1+\kappa]$. Let $x \in [0, 1+\kappa]$ and observe

$$\begin{aligned} |q_{b+1}(x+iy_b(x))| &= |x+b+iy_b(x)| \cdot |q_b(x+iy_b(x))| \\ &= |x+b+iy_b(x)| q_b(1+\kappa) \\ &= \frac{|x+b+iy_b(x)|}{1+\kappa+b} q_{b+1}(1+\kappa). \end{aligned}$$

We show that the prefactor in the last line can be at most 1. Assume towards a contradiction $|x + b + iy_b(x)| > 1 + b + \kappa$ and let $h : [0, b] \to \mathbb{R}$, $h(t) := (1 + \kappa + t)^2 - |x + t + iy_b(x)|^2$. Then $h'(t) = 2(1 + \kappa + t) - 2(x + t) = 2(1 + \kappa - x) \geq 0$ and thus h is an increasing function. Our assumption can be reformulated as $h(b) < 0$ implying $h(j) < 0$ for all $j = 0, \dots, b - 1$. Rearranging terms, one then obtains

$$|x + \kappa + j + y_b(x)| > |1 + \kappa + j|$$

for $j = 0, \dots, b - 1$, implying the contradiction $|q_b(x + iy_b(x))| > q_b(1 + \kappa)$. Therefore we have shown $|q_{b+1}(x + iy_b(x))| \leq q_{b+1}(1 + \kappa)$ and by the monotonicity of $q_{b+1}(x + iy)$ in $|y|$, we get $y_{b+1}(x) \geq y_b(x)$ as claimed. This also directly implies

$$\begin{aligned}\theta_{b+1}(x) &= \sum_{j=0}^{b} \arg(x + j + iy_{b+1}(x)) \\ &\geq \sum_{j=0}^{b-1} \arg(x + j + iy_{b+1}(x)) \\ &\geq \sum_{j=0}^{b-1} \arg(x + j + iy_b(x)) = \theta_b(x).\end{aligned}$$

By (4.4), we know that $\Re\lambda_{b,\kappa}^{(2)} + \kappa \geq 0$ implies $\Re\lambda_{b+1,\kappa}^{(2)} + \kappa \geq 0$ and by (4.5) if the premise is true, also $\Re\lambda_{b+1,\kappa}^{(2)} \geq \Re\lambda_{b,\kappa}^{(2)}$ holds. □

Remark 4.18. The statement of Lemma 4.17 is a new result. Monotonicity of $\Re\lambda_{b,0}^{(2)}$ with respect to b was already observed numerically in [23] for $b \leq 10^5$. Since $\Re\lambda_{12,0}^{(2)} \approx 0.0422 > 0$, Lemma 4.17 together with numerical verification for the first values implies that $\mathbb{N}_{\geq 2} \to \mathbb{R}, b \mapsto \Re\lambda_{b,0}^{(2)}$ is an increasing sequence. The monotonicity result yields a structural reason for the existence of a threshold value b^*, as stated in the following theorem.

Theorem 4.19. *[23] Let $\mathcal{T}$ be the family of bucket recursive trees, $(b, 1)$-PORT, or (b, d)-ary increasing trees with $d \in \{2, 3, 4, 5, 6\}$ and let $N_{n,k}$ be the random variable counting the number of nodes of capacity $1 \leq k \leq b$ in a random bucket tree τ of size n. Let the parameters b^* and μ_k be given by the following formulas:*

(i) *Bucket recursive trees: $b^* = 26$ and*

$$\mu_k = \frac{1}{\sum_{j=1}^{b} \frac{1}{j}} \cdot \begin{cases} \frac{1}{k(k+1)} & \text{if } 1 \leq k < b, \\ \frac{1}{b} & \text{if } k = b. \end{cases}$$

(ii) *$(b, 1)$-PORT: $b^* = 33$ and*

$$\mu_k = \frac{1}{\sum_{j=0}^{b-1} \frac{1}{2j+1}} \cdot \begin{cases} \frac{1}{(2k+1)(2k-1)} & \text{if } 1 \leq k < b, \\ \frac{1}{2b-1} & \text{if } k = b. \end{cases}$$

(iii) *(b,d)-ary increasing trees with $d \in \{2,3,4,5,6\}$:*

$$b^* = \begin{cases} 25 & \text{if } 2 \le d \le 5, \\ 26 & \text{if } d = 6 \end{cases} \qquad \text{and}$$

$$\mu_k = \frac{1}{\sum_{j=0}^{b-1} \frac{1}{(d-1)(j+1)+1}} \cdot \begin{cases} \frac{1}{((d-1)k+1)((d-1)(k+1)+1)} & \text{if } 1 \le k < b, \\ \frac{1}{((d-1)b+1)d} & \text{if } k = b. \end{cases}$$

Then $\frac{1}{n} N_{n,k} \xrightarrow{a.s.} \mu_k$ and if $b \le b^*$ then

$$\frac{1}{\sqrt{n}} (N_{n,k} - \mu_k n) \xrightarrow{D} V_k,$$

where the V_k are joint normal distributed random variables with mean 0. If $b > b^*$, then there is no joint distributional limit under the same normalization.

Proof. [5] With the notation as in the beginning of this part, the corresponding urn models with transition matrix A is a standard Pólya–Eggenberger urn model. By Lemma 4.16, the largest eigenvalue of A is given by $\lambda_1 = \beta$, which is also the number of balls added in each step. The left eigenvector v corresponding to λ_1 with the correct normalization is given by

$$v_k = \frac{1}{\sum_{j=0}^{b-1} \frac{1}{\beta(j+1)+\gamma}} \cdot \begin{cases} \frac{1}{\beta(k+1)+\gamma} & \text{if } 1 \le k \le b-1, \\ \frac{1}{\beta+\gamma} & \text{if } k = b. \end{cases}$$

Therefore, the general assumptions for Theorem 2.12 are satisfied, and note that, by Lemma 4.16, also all eigenvalues are simple where only conjugate pairs share the same real part. Therefore, only the comparison of $\Re\lambda_{b,\kappa}^{(2)}$ to $\frac{1}{2}$ is left in order to apply Theorem 2.12. The value of $\Re\lambda_{b,\kappa}^{(2)}$ was approximated using Python. The following table contains the rounded results for the relevant values of b and κ:

$b \backslash \kappa$	$-\frac{1}{2}$	0	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1
25	0.4038	0.4719	0.4836	0.4858	0.4889	0.4936	0.4991
26	0.4178	0.4872	0.4994	0.5017	0.5051	0.5102	0.5170
27	0.4309	0.5015	0.5141	0.5165	0.5200	0.5255	0.5334
33	0.4945	0.5692	0.5835	0.5864	0.5907	0.5976	0.6099
34	0.5031	0.5782	0.5927	0.5956	0.6000	0.6071	0.6199

Table 3: Approximation of $\Re\lambda_{b,\kappa}^{(2)}$ for some values of b and κ.

In any case, we obtain $\Re\lambda_{b,\kappa}^{(2)} < 0.5$ for $b \le b^*$ and $\Re\lambda_{b,\kappa}^{(2)} > 0.5$ for $b > b^*$ from Lemma 4.17. By Theorem 2.12, we have $\frac{1}{n} Q_n \xrightarrow{a.s.} \beta v$ and if $b \le b^*$, then

$$\frac{1}{\sqrt{n}} (Q_n - n\beta v) \xrightarrow{D} \mathcal{N}(0, \Sigma)$$

[5] The theorem was previously only known for the case of recursive trees, which was given in [23]. The possibility of generalizing this result was mentioned in [19], and we worked out the details of their ideas to generalize the statement and its proof.

for some covariance matrix Σ. Translating the results into the corresponding results for $N_{n,k}$ by using the relation (4.2) between $N_{n,k}$ and $Q_{n,k}$, we obtain the joint asymptotic normality for $b \leq b^*$. For $b > b^*$, again by Theorem 2.12 we have

$$n^{-\alpha}(Q_{n,k} - n\beta v) - \Re(n^{i\eta}\hat{W}_k) \xrightarrow{a.s.} 0$$

for some $\alpha > 0.5$, $\eta \neq 0$ and random vector $\hat{W}$ in span(v_2), where v_2 is a left eigenvector of A associated to λ_2. Furthermore, with $v_2 = (v_{2,1}, \ldots, v_{2,b})$ we have $\mathbb{E}(\hat{W}_k) \neq 0$ iff $v_{2,1} \neq 0$ and $v_{2,k} \neq 0$. Using the definition of the left eigenvalue, one can easily show that $v_{2,k} = 0$ would imply $v_{2,k+1} = 0$ for $k = 1, \ldots, b-1$. This shows, that $v_{2,1} = 0$ would imply the contradiction $v_2 = 0$, resulting in $v_{2,1} \neq 0$ and further $\mathbb{E}(\hat{W}_1) \neq 0$. This shows that $\hat{W}_1$ must be non-trivial and implies that

$$n^{-1/2}(Q_{n,k} - n\beta v) = n^{\alpha - 1/2} \cdot n^{-\alpha}(Q_{n,k} - n\beta v)$$

cannot have a distributional limit. Again, the use of the relationship between $N_{n,1}$ and $Q_{n,1}$ implies the claim. $\square$

Remark 4.20. The numerical values of $\lambda^{(2)}_{b,\kappa}$ suggest that there might be a monotonicity result of $\Re\lambda^{(2)}_{b,\kappa}$ with respect to κ for suitable values of b and some domain for κ. It would be sufficient to prove that $\lambda^{(2)}_{26,\kappa}$ and $\lambda^{(2)}_{27,\kappa}$ are increasing with respect to κ for $\kappa \in [0, \frac{1}{5}]$, in order to extend Theorem 4.19 to all (b,d)-ary increasing trees with $d \in \mathbb{N}_{\geq 2}$. This is due to the inequality

$$\Re\lambda^{(2)}_{b,0} \leq \Re\lambda^{(2)}_{b,1/(d-1)} \leq \Re\lambda^{(2)}_{b,1/5}$$

for $d \geq 6$ and the fact that bucket recursive trees and $(b,6)$-increasing trees share the same value of $b^* = 26$. The value of b^* for $d \in \mathbb{N}_{\geq 6}$ would then also be given by $b^* = 26$. Further numerical experiments strongly suggest that this claim holds.

5 Increasing circuits

Increasing circuits are a direct generalization of grown simple increasing trees. Their probabilistic definition is inspired by the underlying growth process of these tree families. We present a new framework for increasing circuits, covering the most prominent examples, which we refer to as the standard families of increasing circuits. In Section 5.1, we show an asymptotic normality law for the number of outputs in the standard families of increasing circuits, following [24], and state further results regarding the asymptotic joint distribution of degree counts. Section 5.2 is based on [31] and provides insight into the asymptotics of the depth of a model of random recursive circuits. We then conclude this chapter with a brief outlook on other studied parameters in Section 5.3.

Definition 5.1. An **increasing circuit** is a labeled directed acyclic graph[1], such that the labels along any path are increasing and any node has in-degree 0 or f for some fixed $f \in \mathbb{N}_{\geq 1}$. The integer f is called the **fan-in** of the circuit. A node with in-degree 0 is called **input node** and a node with out-degree 0 is called **terminal node** or **output node**.

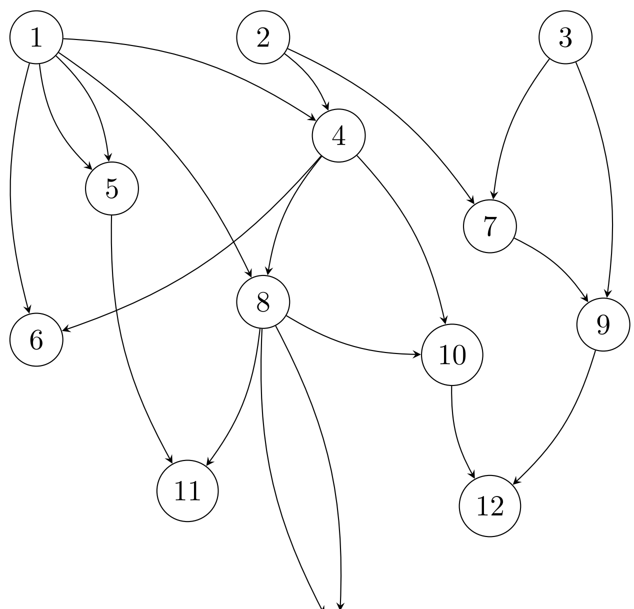

Figure 4: An increasing circuit of size 13 with fan-in $f = 2$, $a = 3$ input nodes and 4 output nodes.

[1] Note that we do not require these graphs to be simple.

We will only study the distributional properties of some parameters for a few interesting classes of increasing circuits. For this reason, we will also only define them via their probability distributions as follows:

Definition 5.2. Let $a, f \in \mathbb{N}_{\geq 1}$ and $(\varphi_k)_{k\in\mathbb{N}} \in [0,\infty)^{\mathbb{N}}$ with $\varphi_k > 0$ for $0 \leq k < f$. Then a class of increasing circuits $\mathcal{C}$ is induced in the following way:

- There is no element in $\mathcal{C}$ of size less than a.
- There is a unique element of size a in $\mathcal{C}$ which is the labeled graph consisting of a many isolated nodes.
- Let $n \geq a$ and assume $\mathcal{C}_n$ is defined. Then for any $c \in \mathcal{C}_n$ and any possible choice[2] for $v_1, \ldots, v_f \in V(c)$, the circuit $\tilde{c}$, consisting of the circuit c together with a node w labeled $n+1$ and an edge from v_k to w for every k, is an element of $\mathcal{C}_{n+1}$ and its probability is given by the following procedure:

  The circuit c is chosen at random and any node v of c is assigned a value $\delta_1(v) := d^+(v)$. For $1 \leq k \leq f$, the node v_k is chosen with probability
  $$\frac{\varphi_{\delta_k(v_k)}}{\sum_{v\in V(c)} \varphi_{\delta_k(v)}},$$
  where δ_{k+1} is defined inductively by $\delta_{k+1}(v_k) := \delta_k(v_k) + 1$ and $\delta_{k+1}(v) := \delta_k(v)$ for $v \in V(c) \setminus \{v_k\}$.

We say that the class $\mathcal{C}$ has **fan-in** f and a many **input nodes**. The class $\mathcal{C}$ is called **family of**

(i) **recursive circuits of first kind (RCFK)** if $\varphi_k = 1$ and the nodes $v_1, \ldots, v_f$ are drawn with replacement,

(ii) **recursive circuits of second kind (RCSK)** if $\varphi_k = 1$, $a \geq f$, and the nodes $v_1, \ldots, v_f$ are drawn without replacement,

(iii) **preferential dynamic attachment circuits (PDAC)** if $\varphi_k = k + 1$ and the nodes $v_1, \ldots, v_f$ are drawn with replacement,

(iv) ***s*-restricted fan-out circuits (*s*-RFC)** if $\varphi_k = \max(s - k, 0)$, $s \geq f$, and the nodes $v_1, \ldots, v_f$ are drawn with replacement.

These four families are called the **standard families of increasing circuits**.

Remark 5.3. These terms and abbreviations are not used within the literature. Most sources call any of these families "recursive circuits" and then specify that the model used is the uniform model (RCFK or RCSK), preferential model (PDAC), or fan-out restricted model (s-RFC). Additionally, in some sources, the input nodes do not contribute towards the size of a recursive circuit. However, we have opted for these changes to emphasize the parallelism to very simple increasing trees and to fit these families into a more general framework.

[2]The nodes may be chosen either with or without repetition and the admissible choice will be specified for each class given.

Remark 5.4. It is possible to obtain combinatorial structures for the standard families of increasing circuits, by reverse-engineering an analogue of Remark 3.26 together with the interpretation of the weights for the very simple families of increasing trees:

Start with a single circuit consisting of a many input nodes, which is the only element of $\mathcal{C}_a$. Assume at first that the parents of any non-input node in all circuits are ordered (i.e., there is a distinguished first parent, second parent,..., f-th parent). If $\mathcal{C}_n$ is already defined, then $\mathcal{C}_{n+1}$ contains any increasing circuit, that can be constructed from a circuit $c \in \mathcal{C}_n$ by choosing parents $v_1, \dots, v_f \in V(c)$ together with "insertion positions" for the edge to the node with label $n+1$. These insertion positions are completely analogous to the case of very simple trees and can also be visualized via external nodes, which do not contribute to the size of a circuit: For recursive circuits, the children of the nodes are not ordered and there is always only one insertion position, while for PDAC the children are ordered and thus a node of out-degree k has $k+1$ insertion positions. Note, however, that the out-degrees of the parent nodes are updated after each step during the construction, which is accounted for by the use of $\delta_1, \dots, \delta_f$. For s-RFC, any node has s positional insertion slots, and any position can only be taken once.

Following the above description, the natural probability distributions of the resulting combinatorial structures agree with our probabilistic definition if circuits are identified via forgetting the order of children and parents.

Contrary to what is stated about PDAC in [32], it is even true for any of the four standard classes that the probability of choosing the nodes $v_1, \dots, v_f$ in this specific order as the parents for a new node w is equiprobable to choosing them in the order $v_{\pi(1)}, \dots, v_{\pi(f)}$ for any permutation $\pi \in S_f$. This follows from the above description of external nodes and from the fact that, in each of the four cases, the total number of external nodes depends affinely on the number of previously chosen parents in a growth step. For RCSK and s-RFC, if one identifies circuits to be equal if they only differ by the ordering of the parent nodes, then any equivalence class contains exactly $f!$ elements. Therefore, in these two cases, the assumption that the parents of any node are ordered is not necessary and can thus also be dropped. In either case, the counts $|\mathcal{C}_n|$ of all standard combinatorial structures satisfy homogeneous linear first-order recurrence relations, which can be solved explicitly.

5.1 The asymptotic distribution of degree counts

The primary result of this section is the asymptotic normality for the number of output nodes stated in Theorem 5.5 for all standard families of increasing circuits. The proof relies on the interpretation of the parameter via an adequate Pólya–Eggenberger urn model.

5.1.1 The number of output nodes

Theorem 5.5. *[24] Let $\mathcal{C}$ be a standard family of increasing circuits with fan-in f and a many input nodes. Let L_n be the random variable counting the number of output nodes in a random circuit in $\mathcal{C}_n$. Then $\frac{1}{n} L_n \xrightarrow{a.s.} \mu$ and*

$$\frac{1}{\sqrt{n}} \left(L_n - \mu n \right) \xrightarrow{D} \mathcal{N}(0, \sigma),$$

where μ and σ^2 are given below.

(i) *RCFK or RCSK:*

$$\mu = \frac{1}{f+1} \quad \text{and} \quad \sigma^2 = \frac{f^2}{(f+1)^2(2f+1)}.$$

(ii) *PDAC:*

$$\mu = \frac{f+1}{2f+1} \quad \text{and} \quad \sigma^2 = \frac{2f^2(f+1)}{(3f+1)(2f+1)^2}.$$

(iii) *s-RFC:*

$$\mu = \frac{s-f}{s(f+1)-f} \quad \text{and} \quad \sigma^2 = \frac{sf^2(s-1)(s-f)}{(s(f+1)-f)^2(s(2f+1)-f)}.$$

Proof. [3] The proof of this theorem relies on the application of Theorem 2.13, and we will start with the easiest case, namely the family of RCSK. Consider the urn model with two types of colors, say white and black balls, representing output nodes and non-output nodes. In the initial state $X_0 = (a, 0)$, there are a white balls and no black balls. In each step, a total of f distinct balls are chosen, and the replacement policy only depends on the number $0 \leq i \leq f$ of chosen white balls. Then, independently of the number i, a total of f black balls and a single white ball are returned to the urn. This process models the probabilistic growth process for a random circuit in the family of RCSK with $L_n = X_{n-a,1}$ for $n \geq a$: If a circuit $c \in \mathcal{C}_n$ grows according to the stochastic growth rule, where the nodes $v_1, \ldots, v_f$ are chosen as the parents for the new node w with label $n+1$, then the nodes $v_1, \ldots, v_f$ are no output nodes in the new circuit, independently of their previous state, and the node w is an output node. The replacement matrix $R \in \mathbb{R}^{(f+1)\times 2}$ is thus given by

$$R = \begin{pmatrix} 1 & 0 & -1 & \cdots & 1-f \\ 0 & 1 & 2 & \cdots & f \end{pmatrix}^T,$$

where the $(i+1)$-th column represents the action of drawing i white balls and the total of balls added to the urn in each step is $r = 1$. In the n-th step, there is a total of $T_n = X_{n,1} + X_{n,2} = n + a$ balls in the urn. Since we are in the case of RCSK, where the parents of a new node are drawn uniformly and without replacement, the decision functions $G_n = (G_n^1, \ldots, G_n^{f+1}) : \Delta_1 \to \Delta_f$ are given by hypergeometric distributions, such that $G_n^{i+1}(x) = \binom{f}{i} \frac{(T_n x_1)^{\underline{i}} (T_n x_2)^{\underline{f-i}}}{T_n^{\underline{f}}}$ for $i = 0, \ldots, f$. These functions converge uniformly in any bounded set with a rate of $O(n^{-1})$ to a binomial distribution $G^{i+1}(x) = \binom{f}{i} x_1^i x_2^{f-i}$. For $x \in \Delta_1$, we calculate

$$\begin{aligned} R^T G(x) &= \begin{pmatrix} \sum_{i=0}^f (1-i)\binom{f}{i} x_1^i x_2^{f-i} \\ \sum_{i=0}^f i \binom{f}{i} x_1^i x_2^{f-i} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^f \binom{f}{i} x_1^i x_2^f - x_1 f \sum_{i=1}^f \binom{f-1}{i-1} x_1^{i-1} x_2^{f-i} \\ x_1 f \sum_{i=1}^f \binom{f-1}{i-1} x_1^{i-1} x_2^{f-i} \end{pmatrix} \\ &= \begin{pmatrix} (x_1+x_2)^f - x_1 f (x_1+x_2)^{f-1} \\ x_1 f (x_1+x_2)^{f-1} \end{pmatrix} = \begin{pmatrix} 1 - x_1 f \\ x_1 f \end{pmatrix}. \end{aligned}$$

[3]The statement is not given for PDAC in [24], and we adapted their proof to fit this case. However, the statement is not new and is instead due to Theorem 5.6.

To apply Theorem 2.13, we require $G \in C^2$ on an open set containing Δ_1. By taking the continuation $\tilde{G} : \mathbb{R}^2 \to \mathbb{R}^{f+1}$, $\tilde{G}(x_1, x_2) := G(x_1, 1 - x_1)$ of G, we can see that G is a polynomial, thus infinitely often differentiable, and we define $F : \mathbb{R}^2 \to \mathbb{R}^2, F(x) := (1-(f+1)x_1, fx_1 - x_2) = R^T\tilde{G}(x) - x$. The only solution u of the equation $F(x) = 0$ in Δ_1 is easily seen to be given by $u = (\frac{1}{f+1}, \frac{f}{f+1})$, and for any $x \in \Delta_1$, we have $F(x) = -(f+1)(x-u)$ such that $(x-u)^T F(x) = -(f+1)\|x-u\|^2$. We are interested in the Jacobian $\mathrm{D}F(u)$ of F at u, which is given by

$$\mathrm{D}F(u) = \begin{pmatrix} -(f+1) & 0 \\ f & -1 \end{pmatrix}.$$

The left eigenvectors of $\mathrm{D}F(u)$ are given by $(1,1)^T$ and $v = (1,0)^T$, which are associated to the eigenvalues $m = -1$ and $\lambda = -(f+1)$, respectively. Since $\lambda = -(f+1) < -\frac{1}{2} = -\frac{r}{2}$, the assumptions for Theorem 2.13 are satisfied and we obtain $\frac{X_n}{n} \overset{a.s.}{\to} \frac{1}{f+1}$. The matrix $C = R^T(\mathrm{diag}(G(u)) - G(u)G(u)^T)R$ can be simplified by applying the binomial theorem multiple times and using $u \in \Delta_1$ to obtain

$$\begin{aligned}
C &= R^T\mathrm{diag}(G(u))R - R^T G(u)(R^T G(u))^T \\
&= \begin{pmatrix} \sum_{i=0}^{f}(1-i)^2 G^{i+1}(u) & \sum_{i=0}^{f}(1-i)iG^{i+1}(u) \\ \sum_{i=0}^{f}(1-i)iG^{i+1}(u) & \sum_{i=0}^{f} i^2 G^{i+1}(u) \end{pmatrix} - \begin{pmatrix} (1-u_1 f)^2 & (1-u_1 f)u_1 f \\ (1-u_1 f)u_1 f & (u_1 f)^2 \end{pmatrix} \\
&= \begin{pmatrix} f(f-1)u_1^2 - fu_1 + 1 & -f(f-1)u_1^2 \\ -f(f-1)u_1^2 & f(f-1)u_1^2 + fu_1 \end{pmatrix} - \begin{pmatrix} (1-u_1 f)^2 & (1-u_1 f)u_1 f \\ (1-u_1 f)u_1 f & (u_1 f)^2 \end{pmatrix} \\
&= \frac{1}{(f+1)^2}\begin{pmatrix} f^2+1 & -f^2+f \\ -f^2+f & 2f^2 \end{pmatrix} - \frac{1}{(f+1)^2}\begin{pmatrix} 1 & f \\ f & f^2 \end{pmatrix} \\
&= \frac{1}{(f+1)^2}\begin{pmatrix} f^2 & -f^2 \\ -f^2 & f^2 \end{pmatrix} = \left(\frac{f}{f+1}\right)^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\end{aligned}$$

Now we can calculate

$$\sigma^2 = \frac{-rv^T Cv}{(2\lambda + r)(v_1 - v_2)^2} = -\frac{f^2}{(f+1)^2(2(-(f+1))+1)(1-0)^2} = \frac{f^2}{(f+1)^2(2f+1)}$$

and obtain from Theorem 2.13 the asymptotic normality

$$\frac{1}{\sqrt{n}}\left(X_{n,1} - \frac{1}{f+1}n\right) \overset{D}{\to} \mathcal{N}\left(0, \frac{f^2}{(f+1)^2(2f+1)}\right).$$

By using $L_n = X_{n-a,1}$ for $n \geq a$, we also obtain

$$\frac{1}{n}L_n \overset{a.s.}{\to} \frac{1}{f+1} \quad \text{and} \quad \frac{1}{\sqrt{n}}\left(L_n - \frac{1}{f+1}n\right) \overset{D}{\to} \mathcal{N}\left(0, \frac{f^2}{(f+1)^2(2f+1)}\right).$$

The treatment for the family of RCFK is similar and only the decision functions G_n have to be replaced, while the limit function G remains the same. We will omit this part and refer the reader to [24] for more details.

We will also deal with the family of PDAC, where the approach via external nodes is similar to the case of s-RFC, which can be found in [24]. We again use an urn model with two types of balls, say white and black balls. These balls now represent the external nodes of the nodes in a circuit, where external nodes are given as described in Remark 5.4. White balls represent the external nodes of output nodes, while black balls represent the external nodes of non-output nodes. Observe that any output node has exactly one external node and thus contributes exactly one white ball to the urn. In this urn model, we again start with an initial state $X_0 = (a, 0)$ with a white balls and no black balls. The replacement policy is again only dependent on the number of white balls drawn and is similar to the previous one. The only difference is that any added edge also introduces another black ball, resulting in the replacement matrix

$$R = \begin{pmatrix} 1 & 0 & -1 & \cdots & 1-f \\ f & f+1 & f+2 & \cdots & 2f \end{pmatrix}^T,$$

where again the $(i+1)$-th column represents the action of drawing i white balls. The total number of balls added to the urn in each step is $r = f + 1$, and in the n-th step, there is a total of $T_n = X_{n,1} + X_{n,2} = n + a + fn = (f + 1)n + a$ balls. The decision functions $G_n : \Delta_1 \to \Delta_f$ are given by the formula

$$G_n^{i+1}(x) = \frac{(T_n x_1)^{\underline{i}}}{T_n^{\overline{f}}} \sum_{0=t_0<t_1<\ldots<t_i<t_{i+1}=f+1} \prod_{j=0}^{i} (T_n x_2 + j + t_j)^{\overline{t_{j+1}-t_j-1}}$$

for $i = 0, \ldots, f$. This formula can be justified in the following way: The balls are drawn successively, where any drawn ball is removed from the urn and replaced by two black balls. This results in the fact that if k balls were already drawn, then there is a total of $T_n + k$ balls left to choose from, which is represented by the denominator in the first factor. When there were already j white balls drawn, then there are $X_{n,1} - j$ balls left, which is represented by the numerator in the first factor. If $t_1, \ldots, t_i$ denote the positions of the i draws in the total f draws, where the white balls are drawn, then the last factor accounts for the possible choices for the black balls drawn in the remaining places. We observe

$$\begin{aligned} G_n^{i+1}(x) &= \left(\frac{(f(n+1)x_1)^i}{(f(n+1))^f} + O(n^{-1}) \right) \left(\sum_{1 \le t_1 < \ldots < t_i \le f} (f(n+1)x_2)^{f-i} + O(n^{-1}) \right) \\ &= x_1^i x_2^{f-i} \sum_{1 \le t_1 < \ldots < t_i \le f} 1 + O(n^{-1}) = \binom{f}{i} x_1^i x_2^{f-i} + O(n^{-1}), \end{aligned}$$

where the $O(n^{-1})$ error can be taken uniformly for any bounded subset of $\mathbb{R}^2$. Thus the functions G_n converge with rate $O(n^{-1})$ uniformly in Δ_1 to $G : \Delta_1 \to \Delta_f$ with $G^i(x) = \binom{f}{i} x_1^i x_2^{f-i}$. For $x \in \Delta_1$, via a similar calculation as before, we obtain

$$R^T G(x) = \begin{pmatrix} 1 - x_1 f \\ x_1 f + f \end{pmatrix}.$$

We take the continuation $\tilde{G} : \mathbb{R}^2 \to \mathbb{R}^{f+1}$, $\tilde{G}(x) := G(x_1, 1 - x_1)$ and define $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x) := R^T \tilde{G}(x) - (f + 1)x$. The unique solution u for $F(x) = 0$ in Δ_1 is given by

$u = (\frac{1}{2f+1}, \frac{2f}{2f+1})$, and for any $x \in \Delta_1$, we have $F(x) = -(2f+1)(x-u)$ such that $(x-u)^T F(x) = -(2f+1)\|x-u\|^2$. The Jacobian $\mathrm{D}F(u)$ at u given by

$$\mathrm{D}F(u) = \begin{pmatrix} -(2f+1) & 0 \\ f & -(f+1) \end{pmatrix}$$

has left eigenvectors $(1,1)^T$ and $v = (1,0)^T$ with corresponding eigenvalues $m = -(f+1)$ and $\lambda = -(2f+1)$, respectively. Due to $\lambda = -(2f+1) < -\frac{f+1}{2} = -\frac{r}{2}$, the conditions for Theorem 2.13 are satisfied and we obtain $\frac{X_{n,1}}{n} \xrightarrow{a.s.} \frac{f+1}{2f+1}$. By defining C in the same way as in the first case and performing a similar calculation, one obtains

$$C = \frac{2f^2}{(2f+1)^2} \begin{pmatrix} 1 & -1 \\ -1 & f \end{pmatrix}.$$

We then get

$$\sigma^2 = \frac{-rv^T Cv}{(2\lambda + r)(v_1 - v_2)^2} = \frac{2f^2(f+1)}{(3f+1)(2f+1)^2}$$

and again by using Theorem 2.13 the asymptotic normality

$$\frac{1}{\sqrt{n}} \left(X_{n,1} - \frac{f+1}{2f+1} n \right) \xrightarrow{D} \mathcal{N}\left(0, \frac{2f^2(f+1)}{(3f+1)(2f+1)^2}\right)$$

follows. Since any output node has exactly one external node, we have the relationship $L_n = X_{n-a,1}$ and thus obtain

$$\frac{L_n}{n} \xrightarrow{a.s.} \frac{f+1}{2f+1} \quad \text{and} \quad \frac{1}{\sqrt{n}} \left(L_n - \frac{f+1}{2f+1} n \right) \xrightarrow{D} \mathcal{N}\left(0, \frac{2f^2(f+1)}{(3f+1)(2f+1)^2}\right).$$

□

5.1.2 Asymptotic joint distributional results

In this part, we provide an overview of existing extensions of the results for the number of output nodes. The two theorems given state joint asymptotic normality laws for multiple degree counts. The first theorem concerns only the numbers of vertices of out-degree 0 and 1, while the second is restricted to the case of fan-in $f = 2$. The corresponding results are due to [32] and [21], respectively, and the reader is referred to these sources for the proofs.

Theorem 5.6. *[32] Let $\mathcal{C}$ be the family of PDAC with fan-in $f \geq 2$ and a input nodes. Let $L_n = (L_{n,0}, L_{n,1})$, where $L_{n,i}$ denotes the number of nodes of out-degree i in a random circuit $c \in \mathcal{C}_n$ for $i \in \{0,1\}$. Let*

$$\mu = \begin{pmatrix} \frac{f+1}{2f+1} \\ \frac{f(f+1)}{(3f+1)(2f+1)} \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \frac{2f^2(f+1)}{(3f+1)(2f+1)^2} & -\frac{4(f+1)^2 f^2}{(4f+1)(3f+1)(2f+1)^2} \\ -\frac{4(f+1)^2 f^2}{(4f+1)(3f+1)(2f+1)^2} & \frac{2f^2(f+1)(48f^3+59f^2+27f+4)}{(5f+1)(4f+1)(3f+1)^2(2f+1)^2} \end{pmatrix}.$$

Then

$$\frac{1}{\sqrt{n}} (L_n - n\mu) \xrightarrow{D} \mathcal{N}(0, \Sigma).$$

Proof. The proof relies on the multivariate martingale convergence theorem and can be found in [32]. □

Remark 5.7. The authors of [32] claimed that their proof for Theorem 5.6 could theoretically be generalized to a central limit theorem for the joint distribution of all out-degrees.

Theorem 5.8. *[21] Let $\mathcal{C}$ be the family of RCSK with fan-in $f = 2$ and $L_{n,k}$ be the random variable counting the number of nodes of out-degree $k \in \mathbb{N}$ in a random circuit $c \in \mathcal{C}_n$. Then*

$$\frac{1}{\sqrt{n}}(L_{n,k} - \mu_k n) \xrightarrow{D} V_k,$$

where $\mu_k = \frac{1}{3}\left(\frac{2}{3}\right)^k$ and the V_k are joint normal distributed random variables with mean 0. Furthermore, for $k_1, k_2 \in \mathbb{N}$, the covariance $\mathrm{Cov}(V_{k_1}, V_{k_2})$ satisfies the recurrence

$$\mathrm{Cov}(V_{k_1}, V_{k_2}) = C_{k_1,k_2} - \sum_{\substack{0\le \ell_1 \le k_1 \\ 0 \le \ell_2 \le k_2 \\ (\ell_1,\ell_2)\neq(k_1,k_2)}} \mathrm{Cov}(V_{\ell_1}, V_{\ell_2}),$$

where $C_{k_1,k_2} = [x^{k_1}y^{k_2}]F(x,y)$ with

$$F(x,y) = \frac{2}{15(1-\frac{2}{3}xy)(1-\frac{2}{5}(x+y))} - \frac{2}{45(1-\frac{2}{3}x)(1-\frac{2}{3}y)(1-\frac{2}{5}(x+y))}.$$

Proof. See [21]. □

5.2 The depth of a random recursive circuit

The depth of a circuit has an important interpretation in computer science. A circuit can be used to model a computer program. The depth then corresponds to the least number of required steps if the program's subtasks are executed in parallel. The main goal of this section is the proof[4] of Theorem 5.20, following the approach presented in [31]. Firstly, the depth of recursive circuits is related to the height of recursive trees using incremental strings, as discussed in Section 3.3.2. The model is then suitably modified to derive asymptotic upper and lower bounds for the depth of recursive circuits.

Definition 5.9. Let c be an increasing circuit. The **depth** of a node $v \in V(c)$, denoted by $\mathrm{depth}(v)$, is defined recursively in the following way: If v is an input node, then set $\mathrm{depth}(v) := 0$. Otherwise, assume that v has parents $v_1, \ldots, v_f$ and that their depth is already defined. Then the depth of v is defined by $\mathrm{depth}(v) := \max\{\mathrm{depth}(v_1), \ldots, \mathrm{depth}(v_f)\} + 1$. The **depth** of the circuit c is defined via $\mathrm{depth}(c) := \max_{v \in V(c)} \{\mathrm{depth}(v)\}$.

[4]Although the statement was already formulated in this form, the proof was originally only done for the case of fan-in $f = 2$ and $a = 1$ input node. The proof of Theorem 5.20 and all prior results leading to the proof were generalized here.

Definition 5.10. Let $(X_t)_{t\geq 0}$ be a continuous-time incremental string with transition rate $(P_n)_{n\geq 1}$, such that $P_{n,i} \leq 1$ for all $i = 1,\ldots,n$ and $n \in \mathbb{N}$. Let $(R_t)_{t\geq 1}$ denote an incremental string of recursive trees with decision times $(t_n)_{n\geq 1}$, then $((X,R,m)_t)_{t\geq 0}$ for fixed $m \geq 0$ denotes the substring of $(R_t)_{t\geq 0}$ defined in the following way: For $t \in [0, t_{m+1})$, let $(X,R,m)_t = R_t$. For $n \geq m$, suppose that $(X,R,m)_{t_n}$ is already defined and let k be the length of $(X,R,m)_{t_n}$. Then $(X,R,m)_t$ for $t \in [t_{n+1}, t_{n+2})$ is defined according to the following case distinction: If $R_{t_{n+1},n+1}$ is decided from an entry which is part[5] of the substring $(X,R,m)_{t_n}$ and is the j-th smallest entry ($j \leq k$), then with probability $P_{k,j}$ we let $(X,R,m)_t$ for $t \in [t_{n+1}, t_{n+2})$ be the continuation of X_{t_n} of length $k+1$ with $(X,R,m)_{t,k+1} := (X,R,m)_{t_n}(j) + 1$. In all other cases, i.e., either with probability $1 - P_{k,j}$ in the case described above, or if the newly selected entry is not contained in the substring $(X,R,m)_{t_n}$, the process remains unchanged. That is, $(X,R,m)_t := (X,R,m)_{t_n}$ for $t \in [t_{n+1}, t_{n+2})$. We call $((X,R,m)_t)_{t\geq 0}$ the **embedding of $(X_t)_{t\geq 0}$ into $(R_t)_{t\geq 0}$ with delay m**.

Remark 5.11. Let $(X_t)_{t\geq 0}$ be a continuous-time incremental string with transition rate $(P_n)_{n\geq 1}$, such that $P_{n,i} \leq 1$ for all $i = 1,\ldots,n$. Then, by construction, the embedding of $(X_t)_{t\geq 0}$ into $(R_t)_{t\geq 0}$ with delay 0 satisfies $(X_t)_{t\geq 0} \overset{D}{=} ((X,R,0)_t)_{t\geq 0}$.

Definition 5.12. Let X and Y be random variables with values in $\mathbb{N}$. We say that Y **stochastically dominates** X, denoted by $X \overset{ST}{\leq} Y$, if for all $i \in \mathbb{N}$ we have

$$\mathbb{P}(X \geq i) \leq \mathbb{P}(Y \geq i).$$

Definition 5.13. Let $(P_n)_{n\geq 1}, (Q_n)_{n\geq 1}$ be transition distributions of some discrete-time incremental strings. We say that Q **dominates** P if for all $i = 1,\ldots,n$ and $n \in \mathbb{N}_{\geq 1}$ we have

$$\sum_{j=i}^{n} P_{n,j} \leq \sum_{j=i}^{n} Q_{n,j}.$$

Lemma 5.14. *[31] Let $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ be two disrete-time incremental strings with transition distributions $(P_n)_{n\geq 1}$ and $(Q_n)_{n\geq 1}$, respectively. If Q dominates P, then $\max(X_n) \overset{ST}{\leq} \max(Y_n)$ for each $n \in \mathbb{N}_{\geq 1}$.*

Proof. The statement of the lemma is independent of the choice of the underlying probability space for the transition distributions P_n and Q_n. We will realize them in a way using a sequence $(U_n)_{n\geq 1}$ of i.i.d uniform random variables on $[0,1]$, such that we obtain $X_{n,i} \leq Y_{n,i}$ for all $i = 1,\ldots,n$ and $n \in \mathbb{N}_{\geq 1}$ by induction. Firstly, $X_{1,1} = 0 \leq 0 = Y_{1,1}$ is trivially true, so suppose that the claim holds for n. Since $X_{n+1,i} = X_{n,i}$ and $Y_{n+1,i} = Y_{n,i}$ for $i = 1,\ldots,n$, we only need to show $X_{n+1,n+1} \leq Y_{n+1,n+1}$ in the induction step. The induction hypothesis also clearly implies $X_n(i) \leq Y_n(i)$ for all $i = 1,\ldots,n$. Let $F_{P_n}(x) := \sum_{i=0}^{\lfloor x \rfloor} P_{n,i}$ and $F_{Q_n}(x) := \sum_{i=0}^{\lfloor x \rfloor} Q_{n,i}$ for $x \in [0,\infty)$ and let $F^{-}_{P_n}(u)$ and $F^{-}_{Q_n}(u)$ for $u \in [0,1]$ be their generalized inverse functions. Then it is well known from the method of inverse transform

[5]Note that we consider the entries as distinguishable objects. We require the exact entry to be part of the substring, not just its value. For clarification see also Remark 3.29

sampling, that $\tilde{P}_n := F^{-}_{P_n}(U_n)$ and $\tilde{Q}_n := F^{-}_{Q_n}(U_n)$ are random variables realizing the transition distributions. Due to the domination of P_n by Q_n, we have $F_{P_n} \geq F_{Q_n}$ resulting in $F^{-}_{P_n} \leq F^{-}_{Q_n}$ and therefore $\tilde{P}_n \leq \tilde{Q}_n$. Using the Iverson bracket

$$[E] := \begin{cases} 1 & \text{if } E \text{ is true} \\ 0 & \text{if } E \text{ if false} \end{cases}$$

for an event E, we obtain

$$\begin{aligned} X_{n+1,n+1} &= 1 + \sum_{j=1}^{n} [\tilde{P}_n = j] X_n(j) \\ &\leq 1 + \sum_{j=1}^{n} [\tilde{P}_n = j] Y_n(j) \\ &\leq 1 + \sum_{j=1}^{n} [\tilde{Q}_n = j] Y_n(j) = Y_{n+1,n+1}. \end{aligned}$$

The last inequality is justified by the fact that $\tilde{P}_n \leq \tilde{Q}_n$ and $Y_n(j)$ is increasing in j. This finishes the induction step and therefore we can conclude $X_{n,i} \leq Y_{n,i}$ for all $i = 1, \ldots, n$ and $n \in \mathbb{N}_{\geq 1}$. This directly implies $\max(X_n) \leq \max(Y_n)$ for all $n \in \mathbb{N}_{\geq 1}$ for this realization of $(X_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$. Clearly this implies $\max(X_n) \overset{ST}{\leq} \max(Y_n)$ for all $n \in \mathbb{N}_{\geq 1}$ and any realization of $(X_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$. □

Lemma 5.15. *Let $(Q_1, \ldots, Q_n) \in \Delta_{n-1}$ be such that $Q_i \leq Q_j$ for $i \leq j$. Let $m \in \mathbb{N}$ and $l_1, \ldots, l_m \in \{2, \ldots, n\}$ with $l_1 < l_2 < \ldots < l_m$ be given. Define $l_0 := 1, l_{m+1} := n + 1$ and*

$$P_i := \frac{1}{l_{k+1} - l_k} \sum_{j=l_k}^{l_{k+1}-1} Q_j$$

for $i \in \{1, \ldots, n\}$ with $l_k \leq i < l_{k+1}$. Then $(P_1, \ldots, P_n) \in \Delta_{n-1}$ and

$$\sum_{j=i}^{n} P_j \leq \sum_{j=i}^{n} Q_j$$

for all $i = 1, \ldots, n$.

Proof. It is immediate from the definition that $P_i \geq 0$ for all $i = 1, \ldots, n$. The intervals $[l_k, l_{k+1})$ induce a partition of $\{1, \ldots, n\}$ with $l_{k+1} - l_k$ elements in the k-th component and such that P_i is constant on any component and equal to the average value of Q on this component. Thus we calculate

$$\sum_{i=1}^{n} P_i = \sum_{k=0}^{m} \sum_{j=l_k}^{l_{k+1}-1} Q_j = \sum_{j=1}^{n} Q_j = 1$$

and obtain $(P_1,\dots,P_n)\in\Delta_{n-1}$. For the second claim, let $i\in\{1,\dots,n\}$ be arbitrary and choose k such that $l_k\le i<l_{k+1}$. By a similar reasoning as before, we obtain

$$\begin{aligned}\sum_{j=i}^{n}P_j-\sum_{j=i}^{n}Q_j&=\sum_{j=i}^{l_{k+1}-1}P_j+\sum_{s=k+1}^{m}\sum_{j=l_s}^{l_{s+1}-1}P_j-\sum_{j=i}^{n}Q_j\\&=\sum_{j=i}^{l_{k+1}-1}P_j+\sum_{s=k+1}\sum_{j=l_s}^{l_{s+1}-1}Q_j-\sum_{j=i}^{n}Q_j=\sum_{j=i}^{l_{k+1}-1}P_j-\sum_{j=i}^{l_{k+1}-1}Q_j.\end{aligned}$$

For $i=l_k$, the above difference is equal to 0, and the claim holds. Suppose $i>l_k$ and assume towards a contradiction that the above difference is greater than 0. By the monotonicity of Q_j and definition of P_j we would have $P_i>Q_i$ and $P_j>Q_j$ for all $j=l_k,\dots,i-1$ resulting in the contradiction

$$\sum_{j=l_k}^{l_{k+1}-1}P_j>\sum_{j=l_k}^{l_{k+1}-1}Q_j.$$

Therefore we must have in any case $\sum_{j=i}^{n}P_j\le\sum_{j=i}^{n}Q_j$ and the second claim holds. □

Lemma 5.16. *[31] Let $f\in\mathbb{N}_{\ge1}$ and B_n,C_n, and $D_n^{(k)}$ be discrete-time incremental string with transition distributions $(L_n)_{n\ge1},(M_n)_{n\ge1}$, and $(N_n^{(k)})_{n\ge1}$ for $k\in\mathbb{N}_{\ge1}$, respectively. Suppose that the transition distributions are given by the following formulas:*

$$L_{n,i}=\begin{cases}1 & \text{if } i=n \text{ and } n\le f,\\ \frac{1}{n-\lceil n(1-1/f)\rceil} & \text{if } \lceil n(1-\frac{1}{f})\rceil<i\le n \text{ and } n>f,\\ 0 & \text{else.}\end{cases}$$

$$M_{n,i}=\frac{i^f-(i-1)^f}{n^f}$$

$$N_{n,i}^{(k)}=\begin{cases}\frac{(n-(s-1)\lfloor n/k\rfloor)^f-(n-s\lfloor n/k\rfloor)^f}{\lfloor n/k\rfloor n^f} & \text{if } n-s\lfloor\frac{n}{k}\rfloor<i\le n-(s-1)\lfloor\frac{n}{k}\rfloor \text{ and } 1\le s\le k,\\ r^{f-1} & \text{if } 1\le i\le r<k \text{ and } n\equiv r \mod k.\end{cases}$$

Then for any $n,k\in\mathbb{N}_{\ge1}$ the following stochastic dominations hold:

$$\max(D_n^{(k)})\overset{ST}{\le}\max(C_n)\overset{ST}{\le}\max(B_n)\ .$$

Proof. We will first show that L dominates M, i.e.,

$$\sum_{j=i}^{n}M_{n,j}\le\sum_{j=i}^{n}L_{n,j}$$

holds for all $i=1,\dots,n$ and $n\in\mathbb{N}_{\ge1}$. For $i\le n(1-\frac{1}{f})$ or if $n\le f$, the sum on the right-hand side equals 1 and the inequality is clearly satisfied. For $f>n$ and $i>n(1-\frac{1}{f})$ calculate

$$M_{n,i}=\frac{1}{n^f}\sum_{l=0}^{f-1}i^l(i-1)^{f-l-1}\le\frac{fi^{f-1}}{n^f}\le\frac{f}{n}=\frac{1}{n-n(1-\frac{1}{f})}\le\frac{1}{n-\lceil n(1-\frac{1}{f})\rceil}=L_{n,i}.$$

This directly implies

$$\sum_{j=i}^{n} M_{n,j} \leq \sum_{j=i}^{n} L_{n,j}.$$

Applying Lemma 5.15 for each $n \in \mathbb{N}_{\geq 1}$ with $Q = M_n$ and $P = N_n^{(k)}$ shows that M dominates $N^{(k)}$ for any $k \in \mathbb{N}_{\geq 1}$. Apply Lemma 5.14 to conclude the proof. □

Lemma 5.17. *[31] Let $f \in \mathbb{N}_{\geq 1}$ and $(B_n)_{n\geq 1}$ be a discrete-time incremental string with transition distribution $(L_n)_{n\geq 1}$ given by*

$$L_{n,i} = \begin{cases} 1 & \text{if } i = n \text{ and } n \leq f, \\ \frac{1}{n - \lceil n(1-1/f) \rceil} & \text{if } \lceil n(1-\frac{1}{f}) \rceil < i \leq n \text{ and } n > f, \\ 0 & \text{else.} \end{cases}$$

Then almost surely

$$\limsup_{n\to\infty} \frac{\max(B_n)}{\log(n)} \leq ef.$$

Proof. By Lemma 3.31 the incremental string B_n can be embedded into a continuous-time incremental string $(\tilde{B}_t)_{t\geq 0}$ with transition rate

$$J_{n,i} = \begin{cases} 1 & \text{if } i = n \text{ and } n \leq f, \\ 1 & \text{if } \lceil n(1-\frac{1}{f}) \rceil < i \leq n \text{ and } n > f, \\ 0 & \text{else.} \end{cases}$$

Combining with Remark 5.11 we further have $(B_n)_{n\geq 1} \overset{D}{=} ((\tilde{B}, R, 0)_{t_n})_{n\geq 1}$, where $(t_n)_{n\geq 1}$ denote the decision times of $((\tilde{B}, R, 0)_t)_{t\geq 0}$. The fact that $(\tilde{B}, R, 0)_{t_n}$ is a substring of R_{t_n} implies that $\max(B_n) \overset{ST}{\leq} \max(R_{t_n})$. Therefore, it is sufficient to prove the upper bound for $H_n := \max(R_{t_n})$ instead of $\max(B_n)$. From $\sum_{i=1}^{n} J_{n,i} = \frac{n}{f} + O(1)$ and Lemma 3.32 we obtain $\frac{t_n}{\log(n)} \overset{a.s.}{\to} f$, because $(t_n)_{n\geq 1}$ is a subsequence of the decision times of $(R_t)_{t\geq 0}$. The same argument shows that Lemma 3.34 implies $\frac{H_n}{t_n} \overset{a.s.}{\to} e$. By combining these two statements, we obtain $\frac{H_n}{\log(n)} \overset{a.s.}{\to} ef$, which concludes the proof. □

Lemma 5.18. *[31] Let $f, k, l \in \mathbb{N}_{\geq 1}$ and $(D_t^{(k)})_{t\geq 0}$ be a discrete-time incremental string with transition rate*

$$K_{n,i}^{(k)} = \begin{cases} \frac{(n-(s-1)\lfloor n/k \rfloor)^f - (n - s\lfloor n/k \rfloor)^f}{n^f - (n - \lfloor n/k \rfloor)^f} & \text{if } n - s\lfloor \frac{n}{k} \rfloor < i \leq n - (s-1)\lfloor \frac{n}{k} \rfloor \text{ and } 1 \leq s \leq k, \\ \frac{\lfloor n/k \rfloor r^{f-1}}{n^f - (n - \lfloor n/k \rfloor)^f} & \text{if } 1 \leq i \leq r < k \text{ and } n \equiv r \mod k. \end{cases}$$

Let $(R_t)_{t\geq 0}$ be an incremental string of recursive trees with decision times $(t_n)_{n\geq 0}$. Further, let $\gamma, \delta > 0$ be arbitrary with $\gamma < \min(\frac{1}{2}, 1 - \sqrt{1 - \eta(\frac{1}{2})})$, where $\eta(\frac{1}{2}) > 0$ is given by Theorem 3.36. Define $\delta_1 := 0$, $\delta_i := \frac{1}{(1-\gamma)^{i-2}}\delta$ for $i = 2, \ldots, l$ and for $n \in \mathbb{N}_{\geq 1}$ the values

$n_i := n^{f\delta_i}, \tilde{t}_i := t_{\lfloor n_i \rfloor}$ for $i = 1, \dots, l$. Choose for any n and i an entry $v_{n,i}$ of $(D, R, \lfloor n^{f\delta} \rfloor)_{\tilde{t}_i}$ with value equal to $H_{n,i} := \max((D, R, \lfloor n^{f\delta} \rfloor)_{\tilde{t}_i})$. Then the events

$$\begin{aligned} A_{n,i} :=& \{H_{n,i} \geq (1-\gamma) f\delta_i e \log(n)\} \\ B_{n,i} :=& \Big\{ \text{All descendants}^6 \text{ of } v_{n,i} \text{ in } (R_t)_{t\geq 0} \text{ which are decided in } [\tilde{t}_i, \tilde{t}_{i+1}) \\ & \text{are entries of } ((D, R, \lfloor n^{f\delta} \rfloor)_t)_{t\geq 0} \Big\} \end{aligned}$$

satisfy $\lim_{n\to\infty} \mathbb{P}(A_{n,i} \cap B_{n,i}) = 1$ *for all* $i = 1, \dots, l$.

Proof. We will do the proof by induction on i. For the induction start, notice that $A_{n,1} = [H_{n,1} \geq 0]$, $v_{n,1} = R_{0,1}$, $\tilde{t}_1 = 0$ and $\tilde{t}_2 = \lfloor n^{f\delta} \rfloor$. Thus, $A_{n,1}$ trivially holds almost surely and $B_{n,1}$ holds almost surely due to the choice of the delay $\lfloor n^{f\delta} \rfloor$. For the induction step, it suffices to show $\mathbb{P}(A_{n,i+1} \cap B_{n,i+1} \mid A_{n,i} \cap B_{n,i}) \to 1$ for $n \to \infty$. Therefore, for the rest of the proof, we condition on $A_{n,i} \cap B_{n,i}$ and all probabilities occurring are to be interpreted in this way.

Suppose towards a contradiction, that $\mathbb{P}(B_{n,i+1}) < 1$. Observe that by the construction of $K_m^{(k)}$, we have $K_{m,i}^{(k)} = 1$ for $m - \lfloor \frac{m}{k} \rfloor < i \leq m$ and $m \in \mathbb{N}_{\geq 1}$. This implies that if $(D, R, \lfloor n^{f\delta} \rfloor)_t$ is of size $m \geq k$ and the next entry in (R_t) is decided from one of the $\lfloor \frac{m}{k} \rfloor$-biggest entries of $(D, R, \lfloor n^{f\delta} \rfloor)_t$, then this new entry must also appear in $(D, R, \lfloor n^{f\delta} \rfloor)_t$. Using this observation with $m = \lfloor n^{f\delta_i} \rfloor$ together with the assumption of $\mathbb{P}(B_{n,i+1}) < 1$ shows that there must be more than $\lfloor \frac{m}{k} \rfloor$ entries at depth at least $H_{n,i}$ in $(D, R, \lfloor n^{f\delta} \rfloor)_t$ and thus also in R_t, for some $t \in [\tilde{t}_i, \tilde{t}_{i+1})$ with positive probability. If $d_{n_{i+1}}$ denotes the height of the last entry of $R_{\tilde{t}_{i+1}}$, we then obtain from our choice of γ and the assumption of $A_{n,i}$ the calculation

$$\begin{aligned} &\mathbb{P}(d_{n_{i+1}} \geq (1-\eta(1/2)) e \log(\lfloor n_{i+1} \rfloor)) \geq \mathbb{P}(d_{n_{i+1}} \geq (1-\gamma)^2 e \log(n_{i+1})) \\ &= \mathbb{P}(d_{n_{i+1}} \geq f\delta_{i+1} e \log(n)) \mathbb{P}(d_{n_{i+1}} \geq (1-\gamma) f\delta_i e \log(n)) \geq \mathbb{P}(d_{n_{i+1}} \geq H_{n,i}) \\ &= \Omega\left(\frac{\frac{m}{\lfloor k \rfloor}}{n_{i+1}}\right) = \Omega\left(\frac{n_i}{k n_{i+1}}\right) = \Omega\left(\frac{n^{f(\delta_i - \delta_{i+1})}}{k}\right) \\ &= \Omega\left(\frac{n^{-f\gamma\delta_{i+1}}}{k}\right) = \Omega\left(\frac{1}{k n_{i+1}^{\gamma}}\right) = \omega\left(\frac{1}{k \lfloor n_{i+1} \rfloor^{1/2}}\right). \end{aligned}$$

By Theorem 3.36, the left-most term is bounded by $O(1/\lfloor n_{i+1} \rfloor^{\frac{1}{2}})$, implying a contradiction. Therefore, we must have $\mathbb{P}(B_{n,i}) = 1$ for almost all $n \in \mathbb{N}_{\geq 1}$ and this concludes the first half of the induction step.

We now proof $\mathbb{P}(A_{n,i+1}) \to 1$ for $n \to \infty$. By the above, we can use $\mathbb{P}(B_{n,i+1}) = 1$. This implies that the descendants of $v_{n,i}$ at time $\tilde{t}_i + t$ which are decided during $[\tilde{t}_i, \tilde{t}_{i+1})$ are distributed as $v_{n,i} + \tilde{R}_t$, where $\tilde{R}_t$ for $t \in [0, \tilde{t}_{i+1} - \tilde{t}_i)$ is an incremental string of recursive trees. Using Lemma 3.32 yields $\frac{\tilde{t}_{i+1}}{\log(n_{i+1})} \xrightarrow{a.s.} 1$ and $\frac{\tilde{t}_i}{\log(n_i)} \xrightarrow{a.s.} 1$ and by substituting for the

[6] An incremental string of recursive trees can be interpreted as a growing sequence of random recursive trees. The term "descendant" of an entry of the incremental string is then defined via this canonical bijection.

definition of n_{i+1} and n_i we further obtain

$$\frac{\tilde{t}_{i+1}-\tilde{t}_i}{\log(n)} \overset{a.s.}{\to} f(\delta_{i+1}-\delta_i).$$

Applying Lemma 3.34 gives

$$\frac{\max(\tilde{R}_{\tilde{t}_{i+1}-\tilde{t}_i})}{\tilde{t}_{i+1}-\tilde{t}_i} \overset{a.s.}{\to} e$$

and by combining both results we have

$$\frac{\max(\tilde{R}_{\tilde{t}_{i+1}-\tilde{t}_i})}{\log(n)} \overset{a.s.}{\to} ef(\delta_{i+1}-\delta_i).$$

Since almost sure convergence implies convergence in probability, we know that

$$\max(\tilde{R}_{\tilde{t}_{i+1}-\tilde{t}_i}) \geq (1-\gamma)f(\delta_{i+1}-\delta_i)e\log(n)$$

holds with probability tending to 1 as $n \to \infty$. Under the assumption of $A_{n,i}$, this implies that also

$$H_{i+1} \geq H_i + \max(\tilde{R}_{\tilde{t}_{i+1}-\tilde{t}_i}) \geq H_i + (1-\gamma)f(\delta_{i+1}-\delta_i)e\log(n) \geq (1-\gamma)f\delta_{i+1}e\log(n),$$

i.e. $A_{n,i+1}$ holds with probability tending to 1 as $n \to \infty$. □

Lemma 5.19. *[31] Let $f \in \mathbb{N}_{\geq 1}$ and $(D_n^{(k)})_{n\geq 1}$ be a discrete-time incremental string with transition distribution $(N_n^{(k)})_{n\geq 1}$*

$$N_{n,i}^{(k)} = \begin{cases} \frac{(n-(s-1)\lfloor n/k\rfloor)^f-(n-s\lfloor n/k\rfloor)^f)}{\lfloor n/k\rfloor} & \text{if } n-s\lfloor \frac{n}{k}\rfloor < i \leq n-(s-1)\lfloor\frac{n}{k}\rfloor \text{ and } 1 \leq s \leq k, \\ r^{f-1} & \text{if } 1 \leq i \leq r < k \text{ and } n \equiv r \mod k \end{cases}$$

for $k \in \mathbb{N}_{\geq 1}$. Then for any $\varepsilon > 0$ there is $k \in \mathbb{N}_{\geq 1}$ such that

$$\lim_{n\to\infty} \mathbb{P}(\max(D_n^{(k)}) \geq (ef-\varepsilon)\log(n)) = 1.$$

Proof. Let $\varepsilon > 0$ be arbitrary and define

$$\beta_k := \frac{1}{k(1-(1-\frac{1}{k})^f)} \quad \text{and} \quad a(k,\delta) := f\delta + \frac{1-f\delta}{\beta_k}$$

for $k \in \mathbb{N}_{\geq 1}$ and $\delta > 0$. Choose small $\gamma, \delta > 0$ and large $k \in \mathbb{N}_{\geq 1}$ such that

(i) $\gamma < \min(\frac{1}{2}, 1-\sqrt{1-\eta(\frac{1}{2})})$, where $\eta(\frac{1}{2}) > 0$ is given by Theorem 3.36,

(ii) $f\delta < 1$, and

(iii) $(1-\gamma)^2 a(k,\delta) \geq f - \frac{\varepsilon}{e}$.

The first and second conditions do not introduce any difficulties into showing the existence of such parameters, and we will only reason why the last inequality is attainable: We have

$$\lim_{k\to\infty}\frac{1}{\beta_k}=\lim_{k\to\infty}\frac{1-(1-\frac{1}{k})^f}{\frac{1}{k}}=\lim_{x\to 0^+}\frac{1-(1-x)^f}{x}=\lim_{x\to 0^+}\frac{f(1-x)^{f-1}}{1}=f.$$

Furthermore, both limits

$$\lim_{\delta\to 0^+}a(k,\delta)=\frac{1}{\beta_k}\quad\text{and}\quad\lim_{\gamma\to 0^+}(1-\gamma)^2a(k,\delta)=a(k,\delta)$$

are uniform with respect to the remaining parameters, if they stay bounded. This implies

$$\lim_{k\to\infty}\lim_{\delta\to 0^+}\lim_{\gamma\to 0^+}(1-\gamma)^2a(k,\delta)=f,$$

and further shows the existence of our choice of parameters.

Let $(\tilde{D}^{(k)}_t)_{t\geq 0}$ be a continuous-time incremental string with transition rate $(K^{(k)}_n)_{n\geq 1}$ as in Lemma 5.18. Let $(R_t)_{t\geq 0}$ be a continuous-time incremental string of recursive trees with decision times $(t_n)_{n\geq 1}$ and let $(t'_{n,m})_{n\geq 1}$ be the decision times of $((\tilde{D}^{(k)},R,\lfloor m^{f\delta}\rfloor)_t)_{t\geq 0}$ for $m\in\mathbb{N}$. Further, let t'_n denote the diagonal sequence of decision times $t'_n:=t'_{n,n}$. It is a consequence of Lemma 3.31 and Lemma 5.15 (with $m=0$) that $\max((\tilde{D}^{(k)},R,\lfloor n^{f\delta}\rfloor)_{t'_n})\overset{ST}{\leq}\max(D^{(k)}_n)$ for $n\in\mathbb{N}_{\geq 1}$. Therefore, it suffices to prove the lower bound for $(\tilde{D}^{(k)},R,\lfloor n^{f\delta}\rfloor)_{t'_n}$.

To determine the asymptotic behavior of t'_n, we observe that t'_n can be written as the sum of two exponential waiting times $t'_n=(t'_{n,n}-t'_{\lfloor n^{f\delta}\rfloor,n})+t'_{\lfloor n^{f\delta}\rfloor,n}$. Clearly, for $n\in\mathbb{N}$, the decision time $t'_{\lfloor n^{f\delta}\rfloor,n}$ is distributed like the $\lfloor n^{f\delta}\rfloor$-th decision time of an incremental string of recursive trees. By Lemma 3.32 and the fact that almost sure convergence implies convergence in probability, we have $\frac{t'_{\lfloor n^{f\delta}\rfloor,n}}{\log(\lfloor n^{f\delta}\rfloor)}\xrightarrow{P}1$ or equivalently

$$\frac{t'_{\lfloor n^{f\delta}\rfloor,n}}{\log(n)}\xrightarrow{P}f\delta.$$

Now observe that $t'_{n,n}-t'_{\lfloor n^{f\delta}\rfloor,n}$ is distributed like the difference of the n-th and $\lfloor n^{f\delta}\rfloor$-th decision times of an incremental string with transition rate given by $(K^{(k)}_{m,i})_{m\geq 1}$. For $m\in\mathbb{N}_{\geq k}$ we can calculate

$$\begin{aligned}\sum_{i=1}^{m}K_{m,i}&=\frac{\lfloor\frac{m}{k}\rfloor m^f}{m^f-(m-\lfloor\frac{m}{k}\rfloor)^f}=\lfloor\frac{m}{k}\rfloor\frac{1}{1-(1-\frac{1}{m}\lfloor\frac{m}{k}\rfloor)^f}=\frac{\frac{m}{k}+O(1)}{1-(1-\frac{1}{k}+O(m^{-1}))^f}\\&=\frac{\frac{m}{k}+O(1)}{1-(1-\frac{1}{k})^f+O(m^{-1})}=\frac{(\frac{m}{k}+O(1))(1+O(m^{-1}))}{1-(1-\frac{1}{k})^f}\\&=\frac{m}{k}\frac{1}{1-(1-\frac{1}{k})^f}+O(1)=m\beta_k+O(1).\end{aligned}$$

With a similar reasoning as for $t'_{\lfloor n^{f\delta}\rfloor,n}$, we can deduce from Lemma 3.32 that

$$\frac{t'_{n,n}-t'_{\lfloor n^{f\delta}\rfloor,n}}{\log(n)}\xrightarrow{P}\frac{1-f\delta}{\beta_k}.$$

Combining the results for both waiting times implies

$$\frac{t'_n}{\log(n)} \xrightarrow{P} a(k,\delta).$$

For $l \in \mathbb{N}_{\geq 2}$, let $\tilde{t}_i, \delta_i$, and H_i be defined as in Lemma 5.18 for $i = 1, \ldots, l$. If we choose $l \in \mathbb{N}_{\geq 2}$, such that $\tilde{t}_{l-1} \leq t'_n < \tilde{t}_l$ holds (note that also l is a random variable), then dividing the inequality chain by $\log(n)$ and applying Lemma 3.32 gives

$$f\delta_{l-1} \leq a(k,\delta) \leq f\delta_l$$

with probability 1. The first inequality gives an upper bound on l depending on f, k, γ, and δ. Thus, there is a finite number of choices of l, to which we apply Lemma 5.18 with $i = l - 1$. This gives

$$\max((\tilde{D}^{(k)}, R, \lfloor n^{f\delta} \rfloor)_{t'_n}) \geq H_{l-1} \geq f(1-\gamma)\delta_{l-1} e \log(n) \geq f(1-\gamma)^2 \delta_l e \log(n)$$

with probability tending to 1 as $n \to \infty$. Taking also into account the lower bound for $f\delta_l$ gives by our choice of the parameters γ, δ, k

$$\max((\tilde{D}^{(k)}, R, \lfloor n^{f\delta} \rfloor)_{t'_n}) \geq (1-\gamma)^2 a(k,\delta) e \log(n) \geq (ef - \varepsilon)\log(n)$$

with probability tending to 1 as $n \to \infty$. □

Theorem 5.20. *[31] Let $\mathcal{C}$ be the family of RCFK with fan-in f and a many input nodes. Let H_n be the random variable denoting the depth of a random circuit in $\mathcal{C}_n$. Then*

$$\frac{H_n}{\log(n)} \xrightarrow{P} ef.$$

Proof. We will first do the proof for the case of $a = 1$. Let B_n, C_n and $D_n^{(k)}$ be defined as in Lemma 5.16. Observe $(H_n)_{n\geq 1} \stackrel{D}{=} (\max(C_n))_{n \geq 1}$ and use Lemma 5.16 to obtain

$$\max(D_n^{(k)}) \stackrel{ST}{\leq} H_n \stackrel{ST}{\leq} \max(B_n)$$

for all $n, k \in \mathbb{N}_{\geq 1}$. Let $\varepsilon > 0$ be arbitrary and use the inequality

$$\mathbb{P}\left(\left|\frac{H_n}{\log(n)} - ef\right| \geq \varepsilon\right) \leq \mathbb{P}\left(\frac{B_n(n)}{\log(n)} \geq ef + \varepsilon\right) + \inf_{k \in \mathbb{N}_{\geq 1}} \mathbb{P}\left(\frac{D_n^{(k)}(n)}{\log(n)} \leq ef - \varepsilon\right)$$

and Lemmata 5.17 and 5.19 to obtain the claim

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{H_n}{\log(n)} - ef\right| \geq \varepsilon\right) = 0.$$

Now consider $a > 1$ and let $\tilde{\mathcal{C}}$ be the family of RCFK with fan-in f and 1 input-node. We assign to a random recursive circuit $\tilde{c} \in \tilde{\mathcal{C}}_n$ a circuit $c \in \mathcal{C}_n$ by deleting all edges ending in the nodes with label in $\{2, \ldots, a\}$. It is immediate from the recursive description that $c \in \mathcal{C}_n$ is also a random recursive circuit under the natural distribution of $\mathcal{C}_n$). If we let $\tilde{H}_n$ denote the height of $\tilde{c}$, then we can immediately observe $\tilde{H}_n - (a-1) \leq H_n \leq \tilde{H}_n$. Dividing by $\log(n)$ and applying the result for $\tilde{H}_n$ shows that the claim also holds for H_n in the case of $a > 1$. □

The following two theorems contain related results about the depth of a random increasing circuit in the family of RFCK.

Theorem 5.21. *[1] Let $\mathcal{C}$ be the family of RCFK with fan-in f and a many input nodes. Let H_n be the random variable denoting the depth of a random circuit in $\mathcal{C}_n$. Then*

$$2.047f \le \liminf_{n\to\infty} \frac{\mathbb{E}H_n}{\log(n)} \le \limsup_{n\to\infty} \frac{\mathbb{E}H_n}{\log(n)} \le ef.$$

Proof. See [1]. $\square$

Theorem 5.22. *[8] Let $\mathcal{C}$ be the family of RCFK[7] with fan-in $f = 2$ and $a = 1$ input node. Let D_n denote the depth of the node with label n in a random circuit $c \in \mathcal{C}_n$. Then*

$$\frac{D_n}{\log(n)} \xrightarrow{P} \lambda \quad \text{and} \quad \frac{\min_{n/2\le x\le n} D_x}{\log(n)} \xrightarrow{P} \frac{\lambda}{2},$$

where $\lambda \approx 4.31107$ is the unique solution of $\lambda \log(2e/\lambda) = 1$ in $(1, \infty)$.

Proof. See [8]. $\square$

5.3 Further parameters

Apart from global parameters such as the depth and the node degree counts, local properties of random circuits are also considered. Here, by local parameters we mean quantities associated with individual nodes, such as their degree or depth, where the node label is either fixed or allowed to depend on the circuit size. The following theorems provide asymptotic results for these parameters. The proofs are omitted here and can be found in the corresponding literature.

Theorem 5.23. *[13] Let $\mathcal{C}$ be the family of RCFK or RCSK with fan-in $f \ge 2$ and a many input nodes. Let X_n be the random variable counting the number of ancestors[8] of the node with label n in a random circuit in $\mathcal{C}_n$. Then with $\nu = \frac{f-1}{f}$ we have*

$$n^{-\nu} X_n \xrightarrow{D} \frac{\pi}{f\sin(\pi/f)}((f-1)X)^{1-\nu},$$

where $X \sim \Gamma(\frac{f}{f-1}, 1)$. Furthermore, for every fixed $r > 0$, convergence of the r-th moments also holds.

Proof. See [13]. $\square$

[7] In [8] also a generalization of this theorem was given for more classes and without the constraint of $f = 2$. However, the description of λ becomes more difficult and the value of $\frac{\lambda}{2}$ must be replaced by $\beta\lambda$ for a suitable $\beta \ge 0$.

[8] An ancestor of a vertex v in a directed graph is any vertex u such that there exists a directed path from u to v. Conversely, in this case, v is a descendant of u. The theorem was formulated for the number of descendants, since the orientation of the edges was flipped in their model.

Theorem 5.24. *[14] Let $\mathcal{C}$ be the family of PDAC[9] with fan-in $f \geq 2$ and a many input nodes. Let X_n be the random variable counting the number of ancestors[8] of the node with label n in a random circuit in $\mathcal{C}_n$. Then with $\nu = \frac{(f-1)(f+1)}{f(f+2)}$ we have*

$$n^{-\nu} X_n \xrightarrow{D} \frac{\Gamma(\nu)\Gamma(\frac{f+1}{f(f+2)} + 1)}{\Gamma(\frac{f+1}{f+2})} \left(\frac{(f-1)(f+2)}{2f+1} X \right)^{1-\nu},$$

where $X \sim \Gamma(\frac{f}{f-1}, 1)$. Furthermore, for every fixed $r > 0$, convergence of the r-th moments also holds.

Proof. See [14]. □

Theorem 5.25. *[22] Let $\mathcal{C}$ be the family of RFCK with fan-in f and a many input nodes. For $j \leq n$, let $D_{j,n}$ denote the degree of the node with label j in a random circuit in $\mathcal{C}_n$. Further, let $j_n \in \mathbb{N}$ with $1 \leq j_n \leq n$ for $n \in \mathbb{N}_{\geq 1}$.*

(i) *If $j_n = o(n)$, then*

$$\frac{D_{j_n,n} - f \log(n/j_n)}{\sqrt{\log(n/j_n)}} \xrightarrow{D} \mathcal{N}(0, f).$$

(ii) *If $j_n \sim cn$ for some $c \in (0,1)$, then*

$$D_{j_n,n} - f \xrightarrow{D} \mathrm{Poi}(f \log(1/c)).$$

Proof. See [22]. □

Theorem 5.26. *[32] Let $\mathcal{C}$ be the family of PDAC with fan-in f and a many input nodes. For $j \leq n$, let $D_{j,n}$ denote the degree of the node with label j in a random circuit in $\mathcal{C}_n$. Then for fixed $j \in \mathbb{N}_{\geq 1}$ the expectation and variance of $D_{j,n}$ satisfy*

$$\begin{aligned}
\mathbb{E} D_{j,n} &\sim \frac{\Gamma(j + \frac{1}{f+1})}{j!} n^{\frac{f}{f+1}}, \\
\mathbb{V}(D_{j,n}) &\sim \left(\frac{2(f+1)\Gamma(j + \frac{2}{f+1})}{((f+1)j)!j!} - \left(\frac{\Gamma(j + \frac{1}{f+1})}{j!} \right)^2 \right) n^{\frac{2f}{f+1}}.
\end{aligned}$$

Proof. See [32]. □

[9] In fact, they have shown a generalization of this theorem for more families than just our model of PDAC.

6 Concurrent processes

In this chapter, we present asymptotic enumeration results for three different classes of increasingly labeled directed graphs. The main reason for the increase in difficulty in comparison to the previous chapter is the lack of a growth description via adding one node at a time. As an application in computer science, we formulate these results in terms of the number of executions in concurrent processes and provide some examples on the average count of admissible executions in some cases. This is also related to the problem of counting the number of linear extensions of a given partial order, which is known to be a hard problem [7]. The three sections in this chapter are dedicated to the study of diamond processes, fork–join processes, and arch processes. These parts are based on [3], [4], and [5], respectively.

Definition 6.1. A **concurrent process** is a simple directed acyclic graph. A **run** or an **execution** of a concurrent process is an increasing labeling of the concurrent process.

Remark 6.2. Some sources use partial orders as the definition of a concurrent process. An execution then describes a linear extension of the partial order. While our definition is more general, the examples given can be interpreted as the transitive reduction of partial orders and are then covered by the more restrictive definition.

6.1 Diamond processes

The study of increasing diamonds is partially a byproduct of our treatment of bucket increasing trees, which is captured by Remark 6.4.

Definition 6.3. Let $(\varphi_k)_{k\in\mathbb{N}} \in [0,\infty)^{\mathbb{N}}$ with $\varphi_k > 0$ for some $k \geq 2$. The labeled family $\mathcal{D}$ specified by the formal recursive equation

$$\mathcal{D} = \mathcal{Z} + \mathcal{Z}^{\square} \star \varphi(\mathcal{D}) \star \mathcal{Z}^{\blacksquare}$$

is called **family of increasing diamonds**. The sequence $(\varphi_k)_{k\in\mathbb{N}}$ and its generating function $\varphi(t) := \sum_{k=0}^{\infty} \varphi_k t^k$ are a **degree-weight sequence** and **degree-weight generating function** of $\mathcal{D}$, respectively. Any element of $\mathcal{D}$ is called **increasing diamond**, and its underlying unlabeled object is called **diamond** or **diamond process**. If $\varphi(t)$ is a polynomial, then $\mathcal{D}$ is a **polynomial family of increasing diamonds.**

Remark 6.4. Families of increasing diamonds can be seen as a special case of simple families of bucket increasing trees of bucket size $b = 2$, namely when the bucket weight sequence $\psi_1 = 1$ is taken. From Lemma 4.5, one can obtain a direct isomorphism of these structures, which is discussed in [19]. This also justifies the use of the term "degree-weight sequence".

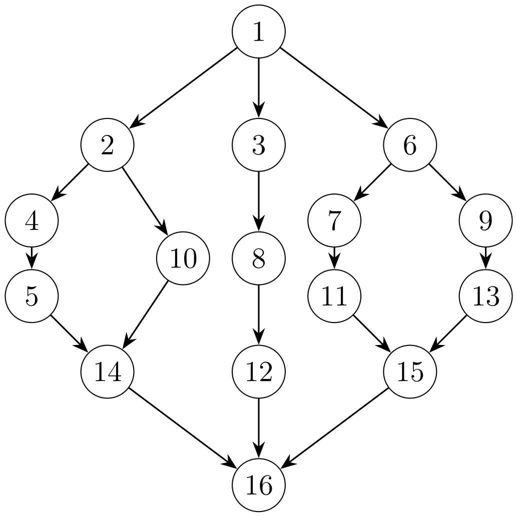

Figure 5: An example of an increasing diamond of size 16.

Theorem 6.5. *[3] Let $\mathcal{D}$ be a family of increasing diamonds with degree-weight generating function $\varphi(t)$. Then $D(z)$ is the implicit solution of*

$$\int_0^{D(z)} \frac{1}{\sqrt{1+2\int_0^t \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}t = z.$$

Proof. The claim follows immediately from the description via bucket increasing trees in Remark 6.4 and then applying Theorem 4.10. □

Definition 6.6. The family of increasing diamonds associated with the degree-weight generating function $\varphi(t)$ is called the family of

(i) **non-plane increasing diamonds** if $\varphi(t) = \exp(t)$,

(ii) **plane increasing diamonds** if $\varphi(t) = \frac{1}{1-t}$,

(iii) ***d*-ary[1] increasing diamonds** if $\varphi(t) = 1 + t^d$ for some $d \in \mathbb{N}$ with $d \geq 2$.

Theorem 6.7. *[3] Let $\mathcal{D}$ be the family of non-plane or plane increasing diamonds. Then the following formulas hold:*

(i) *non-plane increasing diamonds:*

$$D(z) = \log\left(\frac{1}{1-\sin(z)}\right)$$
$$D_n \sim 2n^{-1}\left(\frac{2}{\pi}\right)^n n!$$

[1] The special cases of 2-ary and 3-ary diamonds will be referred to as **binary** and **ternary** diamonds, respectively.

(ii) *plane increasing diamonds:*

$$D(z) = 1 - \sqrt{e} \cdot e^{-(\mathrm{erf}^{-1}(z\sqrt{2/(e\pi)}+\mathrm{erf}(\sqrt{1/2})))^2}$$

$$D_n \sim \frac{\rho}{\sqrt{2}} \log(n)^{-1/2} n^{-2} \rho^{-n} n!, \quad \textit{with} \quad \rho :=\,^{2} \int_0^1 \frac{1}{\sqrt{1-2\log(1-t)}}\,\mathrm{d}t \approx 0.6557.$$

Proof. The proof consists of the use of Theorem 6.5 and singularity analysis. We will only deal with the case of non-plane increasing diamonds. The proof of the other case is similar. Using the substitution $u = e^{-t} - 1$ we obtain

$$\int_0^y \frac{1}{\sqrt{1+2\int_0^t e^s\,\mathrm{d}s}}\,\mathrm{d}t = \int_0^y \frac{1}{\sqrt{2e^t - 1}}\,\mathrm{d}t = \int_0^y \frac{e^{-t}}{\sqrt{2e^{-t} - e^{-2t}}}\,\mathrm{d}t$$
$$= -\int_0^y \frac{-e^{-t}}{\sqrt{1-(e^{-t}-1)^2}}\,\mathrm{d}t = -\int_0^{e^{-y}-1} \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u = -\arcsin(e^{-y}-1).$$

Inverting the implicit equation gives the claimed expression for $D(z)$:

$$z = -\arcsin(e^{-D(z)} - 1) \Leftrightarrow e^{-D(z)} - 1 = -\sin(z) \Leftrightarrow D(z) = \log\left(\frac{1}{1-\sin(z)}\right).$$

The only dominant singularity of $D(z)$ is given by $\rho = \frac{\pi}{2}$ and $D(z)$ satisfies locally

$$\begin{aligned} D(z) &= -\log\left(1 - \cos(\pi/2 - z)\right) \\ &= -\log\left(\frac{(\pi/2-z)^2}{2}(1 + O((\pi/2-z)^2))\right) \\ &= -2\log(\pi/2 - z) + \log(2) + O((\pi/2-z)^2) \end{aligned}$$

for $z \to \frac{\pi}{2}$. Applying the Theorem 2.6 then leads to

$$\frac{D_n}{n!} \sim 2[z^n]\log\left(\frac{2/\pi}{1-2z/\pi}\right) = \frac{2}{n}\left(\frac{2}{\pi}\right)^n. \qquad \square$$

Remark 6.8. In [3], the authors also gave an explicit formula for the generating function of binary increasing diamonds using Weierstrass's $\wp$-function. They further claimed that the generating function for ternary increasing diamonds can also be explicitly described in terms of elliptic integrals. We will not pursue this any further and note in passing that asymptotic enumeration in these cases is also obtained from Theorem 6.9.

Theorem 6.9. *[3] Let $\mathcal{D}$ be a polynomial family of increasing diamonds of degree $d \geq 2$ and degree-weight generating function $\varphi(t) = \varphi_d t^d + \ldots + \varphi_1 t + \varphi_0$. Let $p \in \mathbb{N}_{\geq 1}$ be maximal such that $\int_0^t \varphi(s)\,\mathrm{d}s = \tilde{\Phi}(t^p)$ for some $\tilde{\Phi} \in \mathbb{C}[[t]]$ and define*

$$\rho := \int_0^\infty \frac{1}{\sqrt{1+2\int_0^t \varphi(s)\,\mathrm{d}s}}\,\mathrm{d}t, \quad \delta := \frac{2}{d-1}, \quad \text{and} \quad \eta := \left(\frac{\sqrt{\varphi_d}(d-1)\rho}{\sqrt{2(d+1)}}\right)^\delta.$$

[2] In [3] the authors have used the wrong upper bound of ∞ for the integral.

Then D_n satisfies

$$\frac{D_n}{n!} = \begin{cases} \frac{p}{\eta\Gamma(\delta)} n^{\delta-1}\rho^{-n} + O((n^{\delta-2} + n^{-\delta-1})\rho^{-n}) & \text{if } n \equiv 1 \mod p, \\ 0 & \text{else.} \end{cases}$$

Proof. The claim follows from Remark 6.4, Lemma 4.11, and Theorem 4.12. □

Example 6.10. If we denote by $\tilde{\mathcal{D}}$ the unlabeled combinatorial structure specified by the recursive formal equation $\tilde{\mathcal{D}} = \mathcal{Z} + \mathcal{Z}^2 \times (\mathcal{E} + \tilde{\mathcal{D}}^2)$, then $\tilde{\mathcal{D}}$ describes the underlying unlabeled family for binary increasing diamonds $\mathcal{D}$. The quadratic equation for $\tilde{D}(z)$ can be solved to obtain $\tilde{D}(z) = \frac{1-\sqrt{1-4z^2(z+z^2)}}{2z^2}$. The only dominant singularity $\tilde{\rho} \approx 0.5449$ of $\tilde{D}(z)$ is the unique positive solution of $z^4 + z^3 = 1/4$. One then obtains the local expansion

$$\tilde{D}(z) = \frac{1}{2\tilde{\rho}} - \sqrt{4 + 3/\tilde{\rho}}\sqrt{1 - z/\tilde{\rho}} + O(1 - z/\tilde{\rho})$$

for $z \to \tilde{\rho}$ and applying the transfer theorem gives

$$\tilde{D}_n \sim \sqrt{\frac{4 + 3/\tilde{\rho}}{2\pi n^3}}\tilde{\rho}^{-n}.$$

For the asymptotics of D_n, we use Theorem 6.9 and obtain after simplifying

$$\frac{D_n}{n!} \sim 6n\rho^{-n-2} \quad \text{with} \quad \rho = \int_0^\infty \frac{1}{\sqrt{1 + 2t + \frac{2}{3}t^3}} \,\mathrm{d}t \approx 2.7368.$$

Therefore, we can conclude that the average number of possible executions of a binary diamond process of size n satisfies

$$\frac{D_n}{\tilde{D}_n} \sim \frac{6}{\rho^2}\sqrt{\frac{2\pi}{4 + 3/\tilde{\rho}}} \cdot n^{5/2} \left(\frac{\tilde{\rho}}{\rho}\right)^n n! = Cn^{5/2}\alpha^n n!,$$

with $C \approx 0.6513$ and $\alpha \approx 0.1991$.

Remark 6.11. A similar calculation is possible for the case of plane increasing diamonds, where the generating function for the underlying unlabeled class is again the solution of a quadratic equation. Using Theorem 6.7, one can then show that the average number of executions of a plane diamond process of size n is asymptotically given by

$$\rho\sqrt{3\pi} \cdot \log(n)^{-1/2} n^{-1/2} n!(3\rho)^{-n} = C \log(n)^{-1/2} n^{-1/2} \alpha^n n!,$$

with $\rho = \int_0^1 \frac{1}{\sqrt{1-2\log(1-t)}}\,\mathrm{d}t$, $C \approx 2.0129$ and $\alpha \approx 0.5084$.

For non-plane increasing diamonds and d-ary diamonds with $d \geq 3$, the asymptotic enumeration of the underlying classes would require more work, which we will not pursue further.

6.2 Fork–join processes

The asymptotic enumeration of increasing fork–join processes comes with a major spike in difficulty. We focus on estimating the class by introducing simpler models related to fork–join processes.

Definition 6.12. The labeled family $\mathcal{F}$ specified by the formal recursive equation

$$\mathcal{F} = \mathcal{Z} + \mathcal{Z}^{\square} \star \mathcal{F} + \mathcal{Z}^{\square} \star ((\mathcal{F} \star \mathcal{F}) \boxed{\star} \mathcal{F})$$

is called family of **increasing fork–join processes (increasing FJ processes)**. The underlying unlabeled family $\tilde{\mathcal{F}}$, specified by the formal recursive equation

$$\tilde{\mathcal{F}} = \mathcal{Z} + \mathcal{Z} \times \tilde{\mathcal{F}} + \mathcal{Z} \times \tilde{\mathcal{F}} \times \tilde{\mathcal{F}} \times \tilde{\mathcal{F}},$$

is called family of **fork–join processes (FJ processes)**.

Remark 6.13. In [4], the authors explain how the term "fork–join processes" is inspired by Unix processes: A process may "fork" a subprocess which can then be executed in parallel. A fork can be represented by a node of outdegree 2. The termination of the subprocess is called "join", which is indicated by a node of indegree 2.

There are multiple ways in which one could try to approximate the class of increasing FJ processes with simpler combinatorial structures. Our first attempt at trying to understand increasing FJ processes consists of the study of very simplistic models. In these models, two subprocesses may only be put in parallel if one of them is a (possibly empty) string of nodes, and the maximum number of subprocesses in parallel is restricted.

Definition 6.14. The labeled families $(\mathcal{W}_\ell)_{\ell\in\mathbb{N}}$ specified by the formal system of equations

$$\begin{aligned}
\mathcal{W}_0 &= \mathrm{SET}_{\geq 1}^{\boxed{\star}}(\mathcal{Z}) \\
\mathcal{W}_\ell &= \mathcal{Z} + \mathcal{Z}^{\square} \star \mathcal{W}_\ell + \mathcal{Z}^{\square} \star ((\mathcal{W}_{\ell-1} \star \mathrm{SEQ}(\mathcal{Z}) + \mathrm{SEQ}(\mathcal{Z}) \star \mathcal{W}_{\ell-1} - \mathcal{W}_0 \star \mathcal{W}_0) \boxed{\star}\, \mathcal{W}_\ell)
\end{aligned}$$

are called families of **increasing parallel-constrained fork–join processes**.

Lemma 6.15. *[4] Let $P_1(z) = z(1-2z)$, $Q_1(z) = 1-3z$, and*

$$\begin{aligned}
P_\ell(z) &= z(1-2z)(1-z)^{3\ell-3} Q_{\ell-1}\left(\frac{z}{1-z}\right) \\
Q_\ell(z) &= (1-4z+5z^2)(1-z)^{3\ell-4} Q_{\ell-1}\left(\frac{z}{1-z}\right) - 2z(1-2z)(1-z)^{3\ell-4} P_{\ell-1}\left(\frac{z}{1-z}\right)
\end{aligned}$$

for $\ell \in \mathbb{N}_{\geq 2}$. Then for any $\ell \in \mathbb{N}_{\geq 1}$, the functions P_ℓ and Q_ℓ are polynomials with $\deg(P_\ell) = 3\ell - 1$ *and* $\deg(Q_\ell) = 3\ell - 2$. *Furthermore, for any $\ell \in \mathbb{N}_{\geq 1}$ the two polynomials $P_\ell(z)$ and $Q_\ell(z)$ are coprime.*

Proof. [3] First, observe that for any $k \in \mathbb{N}, a \in \mathbb{C}$, and polynomial $R(z)$ with $\deg(R) \leq k$ the function $(1-az)^k R(\frac{z}{1-az})$ is a polynomial with degree at most k. For $a \neq 0$, by expanding one shows

$$[z^k](1-az)^k R\left(\frac{z}{1-az}\right) = (-a)^k R\left(-\frac{1}{a}\right),$$

[3] This proof is new, since there was no proof provided in [4].

therefore $\deg((1-az)^k R(\frac{z}{1-az})) = k$ iff $R(-\frac{1}{a}) \neq 0$. From this observation with $a = 1, k = 3\ell-1$, and $\ell \in \mathbb{N}_{\geq 1}$, we immediately obtain by induction that $P_\ell(z)$ and $Q_\ell(z)$ are polynomials of degree at most $3\ell-1$ and $3\ell-2$, respectively. To show that these bounds are in fact always attained, we will introduce two auxiliary sequences of polynomials, namely $A_\ell(z) := Q_\ell(-z)$ for $\ell \in \mathbb{N}_{\geq 1}$ and

$$B_\ell(z) := (1+z)^{3\ell-2} A_\ell\left(\frac{z}{1+z}\right) - (1+3z)(1+2z)^{3\ell-3} A_{\ell-1}\left(\frac{z}{1+2z}\right)$$

for $\ell \in \mathbb{N}_{\geq 2}$. Combining both recursions for Q_ℓ and P_ℓ shows

$$A_\ell(z) = (1+4z+5z^2)(1+z)^{3\ell-4} A_{\ell-1}\left(\frac{z}{1+z}\right) - 2z^2(1+z)(1+3z)(1+2z)^{3\ell-6} A_{\ell-2}\left(\frac{z}{1+2z}\right)$$

for $\ell \in \mathbb{N}_{\geq 3}$. Using $(1+4z+5z^2) = (1+3z)(1+z) + 2z^2$, the recursion simplifies to

$$A_\ell(z) = (1+3z)(1+z)^{3\ell-3} A_{\ell-1}\left(\frac{z}{1+z}\right) + 2z^2(1+z) B_{\ell-1}(z)$$

for $\ell \in \mathbb{N}_{\geq 3}$. Applying this recursion for $A_\ell(\frac{z}{1+z})$ in the definition of $B_\ell(z)$ and then simplifying gives

$$B_\ell(z) = z(1+2z)^{3\ell-3} A_{\ell-1}\left(\frac{z}{1+2z}\right) + 2z^2(1+2z)(1+z)^{3\ell-5} B_{\ell-1}\left(\frac{z}{1+z}\right)$$

for $\ell \in \mathbb{N}_{\geq 2}$. Since

$$\begin{aligned} A_2(z) &= 1 + 9z + 27z^2 + 31z^3 + 8z^4 \\ B_2(z) &= z + 9z^2 + 24z^3 + 16z^4, \end{aligned}$$

an easy proof by induction for $\ell \in \mathbb{N}_{\geq 2}$ involving the first order recursions for A_ℓ and B_ℓ shows that $A_\ell(z)$ and $B_\ell(z)$ are polynomials of degree $3\ell - 2$ such that $A_\ell(z)$ has only positive coefficients and $B_\ell(z)$ has only non-negative coefficients. This directly implies $\deg(Q_\ell) = 3\ell - 2$ and $Q_\ell(-1) > 0$, which also implies $\deg(P_\ell) = 3\ell - 1$ for $\ell \in \mathbb{N}_{\geq 1}$.

For the proof of $\gcd(P_\ell, Q_\ell) = 1$, we will show by induction on $\ell \in \mathbb{N}_{\geq 1}$ that P_ℓ and Q_ℓ do not share a common root. For $\ell = 1$, the roots of P_ℓ and Q_ℓ are given by $0, \frac{1}{2}$ for P_ℓ and $\frac{1}{3}$ for Q_ℓ, therefore the base case is true. Let $\ell \in \mathbb{N}_{\geq 2}$ and assume $\gcd(P_{\ell-1}, Q_{\ell-1}) = 1$. Suppose towards a contradiction that P_ℓ, Q_ℓ share a common root $z_0 \in \mathbb{C}$. From $P_\ell(z_0) = 0$, we conclude that either $z_0 \in \{0, \frac{1}{2}, 1\}$ or $\zeta_0 := \frac{z_0}{1-z_0}$ is a root of $Q_{\ell-1}(z)$. Since also $P_{\ell-1}(\frac{1}{2}) = 0$, by the induction hypothesis, we have $Q_\ell(\frac{1}{2}) \neq 0$, and thus we can conclude

$$Q_\ell(0) > 0, \quad Q_\ell(1/2) = Q_{\ell-1}(1/2) \neq 0, \quad Q_\ell(1) = 2[z^{3\ell-4}]P_{\ell-1}(z) \neq 0$$

by the properties already shown for the polynomials. Therefore, we are in the second case, and ζ_0 must be a root of $Q_{\ell-1}$. But then by

$$0 = Q_\ell(z_0) = z_0(1-2z_0)(1-z_0)^{3\ell-4} P_{\ell-1}(\zeta_0),$$

it follows that ζ_0 is also a root of $P_{\ell-1}(z)$, which contradicts the induction hypothesis. Therefore, $\gcd(P_\ell, Q_\ell) = 1$ must hold, which finishes the induction step. □

Theorem 6.16. *[4] Let $(\mathcal{W}_\ell)_{\ell\geq 0}$ be the families of increasing parallel-constrained FJ processes. Let*[4]

$$P_0(z) = z, \quad Q_0(z) = 1 - z, \quad P_1(z) = z(1-2z), \quad Q_1(z) = 1 - 3z,$$
$$P_\ell(z) = z(1-2z)(1-z)^{3\ell-3} Q_{\ell-1}\left(\frac{z}{1-z}\right), \quad \text{and}$$
$$Q_\ell(z) = (1-4z+5z^2)(1-z)^{3\ell-4} Q_{\ell-1}\left(\frac{z}{1-z}\right) - 2z(1-2z)(1-z)^{3\ell-4} P_{\ell-1}\left(\frac{z}{1-z}\right)$$

for $\ell \in \mathbb{N}_{\geq 2}$. Then for all $\ell \in \mathbb{N}$, the function[5] *$\mathcal{L}_c W_\ell(z)$ is a rational function with $\mathcal{L}_c W_\ell(z) = \frac{P_\ell(z)}{Q_\ell(z)}$, where $P_\ell(z)$ and $Q_\ell(z)$ are coprime polynomials. If $\ell \geq 1$, then the degrees of P_ℓ and Q_ℓ are given by $3\ell - 1$ and $3\ell - 2$, respectively.*

Proof. It is immediate from the definition that $W_0(z) = e^z - 1$, resulting in $\mathcal{L}_c W_0(z) = \frac{z}{1-z}$. Using the identities

$$\mathcal{L}_c(A \cdot \exp)(z) = \frac{1}{1-z}(\mathcal{L}_c A)\left(\frac{z}{1-z}\right) \qquad \text{and} \qquad (\mathcal{L}_c B)'(z) = \frac{1}{z}\mathcal{L}_c B(z),$$

with $A = W_{\ell-1}$ and $B = W_\ell$, the defining equation for the class $\mathcal{W}_\ell$ translates into the functional equation

$$\mathcal{L}_c W_\ell(z) = \frac{z(1-z)}{(1-z)^2 - 2z(\mathcal{L}_c W_{\ell-1}(\frac{z}{1-z}) - \frac{z^2}{1-2z})}.$$

We refer to [4] for the details of this step. The claim is then shown by induction. Plugging in $\ell = 1$ gives

$$\mathcal{L}_c W_1(z) = \frac{z(1-2z)}{1-3z},$$

and using $\mathcal{L}_c W_{\ell-1}(z) = \frac{P_{\ell-1}(z)}{Q_{\ell-1}(z)}$ in the above functional equation leads to (after simplification)

$$\mathcal{L}_c W_\ell(z) = \frac{z(1-z)(1-2z)Q_{\ell-1}(\frac{z}{1-z})}{(1-4z+5z^2)Q_{\ell-1}(\frac{z}{1-z}) - 2z(1-2z)P_{\ell-1}(\frac{z}{1-z})}.$$

Multiplying the numerator and denominator with $(1-z)^{3\ell-4}$ shows $\mathcal{L}_c W_\ell(z) = \frac{P_\ell(z)}{Q_\ell(z)}$. The remaining parts are due to Lemma 6.15. $\square$

Remark 6.17. One can use Theorem 6.16 and singularity analysis to obtain formulas for the average number of executions for the presented model of parallel-constrained fork–join processes. These models are heavily constrained, which limits their ability to capture the behavior of general FJ processes. However, the linear growth of the degree of Q_ℓ is a major advantage for the numerical computation of the dominant singularity. In [4], two other variants of parallel-constrained FJ processes are given. These classes increase expressivity at the cost of a significant spike in computational complexity.

[4]Note the additional factor of $(1-z)$ in both recursive formulas compared to the formulas in [4], due to a mistake in the paper.

[5]See page 3 for the definition of the combinatorial Laplace transform.

Our next approximation for the class of increasing FJ processes will be via increasing diamonds. The diamonds used are essentially FJ processes, where the process at first may fork without any restrictions, then a string of nodes in series is executed, and then the subprocesses are joined back together. In order to increase expressivity, we will put an arbitrary number of diamonds in series. In our model, the definition differs slightly from the approach in [4] in order to eliminate the issue of counting the same processes multiple times due to multiple representations for strings. See also Remark 6.20 for more details on this issue.

Definition 6.18. Let $\mathcal{D}$ be the family of increasing diamonds with degree-generating function $\varphi(t) = 1 + t + t^2$ and let $\mathcal{D}^{\langle 1\rangle}$ be defined by $\mathcal{D}^{\langle 1\rangle} + \mathrm{SET}^{\boxed{\star}}_{\geq 1}(\mathcal{Z}) = \mathcal{D}$. The labeled family $\mathcal{S}$ specified by the formal equation

$$\mathcal{S} + \mathcal{E} = \mathrm{SET}^{\boxed{\star}}(\mathcal{Z}) \boxed{\star}\, \mathrm{SET}^{\boxed{\star}}(\mathrm{SET}^{\boxed{\star}}_{\geq 1}(\mathcal{D}^{\langle 1\rangle}) \boxed{\star}\, \mathrm{SET}^{\boxed{\star}}_{\geq 1}(\mathcal{Z})) \boxed{\star}\, \mathrm{SET}^{\boxed{\star}}(\mathcal{D}^{\langle 1\rangle})$$

is called family of **increasing series-constrained fork–join processes**.

Theorem 6.19. *[4] Let $\mathcal{S}$ be the family of increasing series-constrained FJ processes and*[6]

$$\rho := \int_0^\infty \frac{1}{\sqrt{1 + 2t + t^2 + \frac{2}{3}t^3}}\,\mathrm{d}t \approx 2.5278.$$

Then

$$S_n \sim \frac{6}{\rho^2} n \rho^{-n} n!.$$

Let $G(z) := \frac{1-z^2-\sqrt{1-(3z^4+4z^3+2z^2)}}{2z^2} - \frac{z}{1-z} + z$ and $\sigma > 0$ be the unique positive real solution of $G(\sigma) = 1$. Then σ is the only positive real root of the quartic polynomial $z^4 - 3z^3 + 3z^2 + 2z - 1$, and the average number of executions for the underlying series-constrained FJ processes of size n is asymptotically given by

$$\frac{6\sigma G'(\sigma)}{\rho^2} n \left(\frac{\sigma}{\rho}\right)^n n! = C n \alpha^n n!$$

with $C \approx 0.7137$ and $\alpha \approx 0.1442$.

Proof. Let $\mathcal{D}$ be the family of increasing diamonds with degree-weight generating function $\varphi(t) = 1 + t + t^2$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}^{\langle 1\rangle}$ be defined by

$$\begin{aligned}
\mathcal{D}^{\langle 1\rangle} + \mathrm{SET}^{\boxed{\star}}_{\geq 1}(\mathcal{Z}) &= \mathcal{D},\\
\mathcal{A} &= \mathrm{SET}^{\boxed{\star}}_{\geq 1}(\mathcal{Z}),\\
\mathcal{B} &= \mathrm{SET}^{\boxed{\star}}_{\geq 1}(\mathcal{D}^{\langle 1\rangle}),\\
\mathcal{C} &= \mathrm{SET}^{\boxed{\star}}(\mathcal{B} \boxed{\star}\, \mathcal{A}),
\end{aligned}$$

such that

$$\mathcal{S} = (\mathcal{A} + \mathcal{E}) \boxed{\star}\, \mathcal{C} \boxed{\star}\, (\mathcal{B} + \mathcal{E}).$$

[6]Note that in [4] the 1 inside the square root is missing.

Similarly to Example 6.10 we obtain $D_n \sim \frac{6}{\rho^2} n\rho^{-n} n!$ and clearly $D_n^{\langle 1\rangle} \sim D_n$ holds. Also observe $A(z) = e^z - 1$. This shows that the conditions for [4, Theorem 2.2 (i)] are satisfied and we thus obtain $B_n \sim D_n^{\langle 1\rangle}$. Using [4, Theorem 2.1], which is justified by the remark following their theorem, we obtain

$$[z^n](B(z) \boxast A(z)) \sim B_{n-1} + 2A_{n-4} \sim D_{n-1}^{\langle 1\rangle}.$$

We then apply [4, Theorem 2.2 (i)] again and obtain

$$C_n \sim D_{n-1}^{\langle 1\rangle}.$$

Finally, apply [4, Theorem 2.1] twice in order to obtain

$$[z^n]((A(z)+1) \boxast C(z)) \sim A_n + C_n \sim D_{n-1}^{\langle 1\rangle}$$

and

$$S_n + 1 \sim [z^n]((A(z)+1) \boxast C(z)) + B_n \sim D_n^{\langle 1\rangle} \sim \frac{6}{\rho^2} n\rho^{-n} n!,$$

which concludes the proof for the first claim. For the second claim, let $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}^{\langle 1\rangle}$ denote the underlying unlabeled families for $\mathcal{D}$ and $\mathcal{D}^{\langle 1\rangle}$, respectively. The underlying unlabeled family $\tilde{\mathcal{S}}$ for $\mathcal{S}$ is given by

$$\tilde{\mathcal{S}} + \mathcal{E} = \mathrm{SEQ}(\mathcal{Z}) \times \mathrm{SEQ}(\mathrm{SEQ}_{\geq 1}(\tilde{\mathcal{D}}^{\langle 1\rangle}) \times \mathrm{SEQ}_{\geq 1}(Z)) \times \mathrm{SEQ}(\tilde{\mathcal{D}}^{\langle 1\rangle}).$$

Translating the formal equation into a functional equation gives after simplifying

$$\tilde{S}(z) = \frac{1}{1 - (\tilde{D}^{\langle 1\rangle}(z) + z)} - 1.$$

Solving the quadratic equation $\tilde{D}(z) = z(\tilde{D}(z)^2 + \tilde{D}(z) + 1)$ for $\tilde{D}(z)$ and using $\tilde{D}^{\langle 1\rangle}(z) = \tilde{D}(z) - \frac{z}{1-z}$ we get

$$\tilde{D}^{\langle 1\rangle}(z) = \frac{1 - z^2 - \sqrt{1 - (3z^4 + 4z^3 + 2z^2)}}{2z^2} - \frac{z}{1-z},$$

leading to $\tilde{S}(z) = \frac{1}{1-G(z)} - 1$ The unique dominant singularity $\tilde{\rho}$ of $G(z)$ is the only positive real solution of $3z^4 + 4z^3 + 2z^2 = 1$, which is approximately $\tilde{\rho} \approx 0.469$. From this, one can show that $G(\tilde{\rho}) > 1$ and therefore the coefficient asymptotics of $\tilde{S}(z)$ fall into the supercritical sequence scheme, [10, Theorem V.1]. We obtain

$$\tilde{S}_n \sim \frac{1}{\sigma G'(\sigma)} \sigma^{-n},$$

where $\sigma \in (0, \tilde{\rho})$ is the positive real root of the equation $G(z) = 1$. By simplifying this condition, one obtains that σ is the only positive real root of the quartic equation $z^4 - 3z^3 + 3z^2 + 2z - 1 = 0$. The average number of executions for a series-constrained FJ process, given by the quotient $S_n/\tilde{S}_n$, then satisfies

$$\frac{S_n}{\tilde{S}_n} \sim \frac{6\sigma G'(\sigma)}{\rho^2} n \left(\frac{\sigma}{\rho}\right)^n n!. \qquad \square$$

Remark 6.20. Although the authors of [4] have recognized the issue of multiple counts of strings, their claim for the irrelevance of this multiple counting in the asymptotics is wrong. This is due to the fact that not only strings will be counted multiple times, but any process with at least one string of length at least 2 between two consecutive diamonds will be counted multiple times. While the unambiguous definition in [4] truly does not contribute towards the asymptotics for increasing series-constrained FJ processes, this is not the case for the underlying unlabeled combinatorial structure. These observations are noticed by comparing the statement of Theorem 6.19 with its counterpart in [4].

For the rest of the section, we will summarize further claims about the asymptotics of fork–join processes from [4].

Theorem 6.21. *[4] Let $\tilde{\mathcal{F}}$ be the family of fork–join processes. Then the dominant singularity $\tilde{\rho}$ of $\tilde{F}(z)$ is given by $\tilde{\rho} = 1/(1 + 3 \cdot 2^{-2/3})$ and $\tilde{F}_n$ satisfies asymptotically*

$$\tilde{F}_n \sim \frac{1}{2\sqrt{3\tilde{\rho}\pi}} n^{-3/2} \tilde{\rho}^{-n}.$$

Proof sketch. The formal recursive definition shows that the class $\tilde{\mathcal{F}}$ also describes a simple variety of trees, i.e., the underlying unlabeled combinatorial structure of a simple family of increasing trees. Locating the singularity and using the method of singularity analysis then leads to the claim. For more details, see [4].

The formal recursive definition can be translated into a cubic equation for $\tilde{\mathcal{F}}$. Solving for $\tilde{F}$ and using the method of singularity analysis then leads to the claim. For more details, see [4]. □

Theorem 6.22. *[4] Let $\mathcal{F}$ be the family of increasing fork–join processes, $r = 2.31197$, and $\varepsilon = 2 \cdot 10^{-5}$. Then for all $n \geq 127$ we have*

$$\frac{6}{(r+\varepsilon)^2} n (r+\varepsilon)^{-n} n! \leq F_n \leq \frac{6}{r^2} n r^{-n} n!.$$

Remark 6.23. The statement of Theorem 6.22 differs from the one in [4] in three aspects. Firstly, their statement contains a typo resulting in the prefactors $6(r+\varepsilon)^2$ and $6r^2$ instead of $\frac{6}{(r+\varepsilon)^2}$ and $\frac{6}{r^2}$. Secondly, they have replaced $n \cdot n!$ by the asymptotically equivalent term $(n+1)!$. Thirdly, they have divided F_n by the count $\tilde{F}_n$ of the underlying unlabeled family, while dividing the upper and lower bound by the asymptotic formula of $\tilde{F}_n$ given in Theorem 6.21.

A proof for the upper bound of our version of Theorem 6.22 is given in [4], and they claim that the proof for the lower bound can be done in an even simpler way. Their proof relies on numerical verification of these bounds, similar bounds for different regimes of small n, and an inductive proof using the recurrence relation of F_n. Due to their typo in the prefactors, we have verified the validity of the numerical claims in their proof for $n \leq 1000$.

We also checked numerically for $n \leq 1000$ the validity of

$$\frac{12\sqrt{3\pi\tilde{\rho}}}{(r+\varepsilon)^2} n^{3/2} \left(\frac{\tilde{\rho}}{r+\varepsilon}\right)^n (n+1)! \leq \frac{F_n}{\tilde{F}_n} \leq \frac{12\sqrt{3\pi\tilde{\rho}}}{r^2} n^{3/2} \left(\frac{\tilde{\rho}}{r}\right)^n (n+1)!.$$

While the upper bound is true for any $n \leq 1000$, the lower bound does not hold for $n \leq 407$, but it is true for $408 \leq n \leq 1000$. This shows that the version of Theorem 6.22 in [4] is still flawed, even after correcting the prefactors as mentioned before.

Theorem 6.24. *[4] Let $\mathcal{F}$ be the family of increasing fork–join processes and $\tilde{\mathcal{F}}$ be the family of fork–join processes. If F has a unique dominant singularity ρ and there exists a Δ-domain Δ_0 such that F is analytic in $\rho\Delta_0$, then*

$$F_n \sim \frac{6}{\rho^2} n \rho^{-n} n!.$$

If $\tilde{\rho} = 1/(1 + 3 \cdot 2^{-2/3})$ denotes the unique dominant singularity of $\tilde{F}$, then the average number of executions of a fork–join process of size n satisfies

$$\frac{F_n}{\tilde{F}_n} \sim \frac{12\sqrt{3\pi\tilde{\rho}}}{\rho^2} n^{5/2} \left(\frac{\tilde{\rho}}{\rho}\right)^n n! = C n^{5/2} \alpha^n n!,$$

with $C \approx 4.0542$ and $\alpha \approx 0.1497$.

Remark 6.25. The theorem was given in [4] without proof. Comparison with the result of Theorem 6.22 suggests that the roles of the singularities in [4] were mixed up, which was corrected here.

6.3 Arch processes

This section is dedicated to the study of arch processes, with the main result being a recursion for the number of increasing labelings of a given arch process.

Definition 6.26. Let $n, k \in \mathbb{N}$ with $k \leq n + 1$. The **(n, k)-arch process** is a directed graph G defined in the following way: There are a total of $n + 2k$ nodes $v_1, \ldots, v_{n+2k}$, and the set of edges $E(G)$ is given by

$$E(G) = \{v_i v_{i+1} \mid i = 1, \ldots, n + k - 1\} \cup \{v_i v_{n+k+i},\ v_{n+k+i} v_{n+i} \mid i = 1, \ldots, k\}.$$

Remark 6.27. We will mainly use the alternative variable names for the nodes $v_1, \ldots, v_{n+2k}$ of the (n, k)-arch process given by

$$(a_1, \ldots, a_k, x_1, \ldots, x_{n-k}, c_1, \ldots, c_k, b_1, \ldots, b_k) = (v_1, \ldots, v_{n+2k}).$$

If $k = n$, then there is no x_i variable, and if $k = n + 1$, this renaming shall be understood in the sense that there is no x_i variable and additionally $a_k = c_1$.

Example 6.28. Figure 6 illustrates the previous remark and serves as an example for an arbitrary arch process. Additionally, a possible increasing labeling is depicted, and also a planar representation of this graph is given. Using this planar representation, counting the number of executions becomes straightforward. The number of executions is equal to the number of ways b_1, b_2 can be inserted into the string $a_1 a_2 x_1 c_1 c_2$, where b_i must be placed between a_i and c_i for $i = 1, 2$. There are 3 gaps for b_1 and b_2 each, and if both are placed in the same gap, i.e., between a_2 and x_1 or between x_1 and c_1, then either b_1b_2 or b_2b_1 is placed in this gap. This corresponds to a total of $3 \cdot 3 + 2 = 11$ number of executions of the $(3, 2)$-arch process.

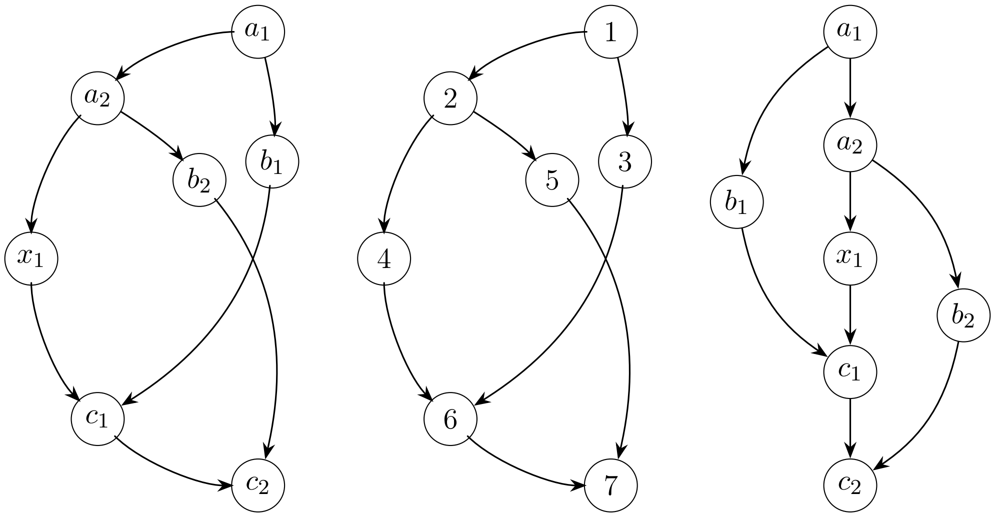

Figure 6: A $(3, 2)$-arch process, an example execution of the process, and a plane representation.

Remark 6.29. Note that the $(0, 1)$ arch process is a cycle of size 2 and thus not a concurrent process. The number of executions of this process is 0, as there is no increasing labeling of this graph. We also note here that the $(0, 0)$-arch process has exactly one increasing labeling. We do not exclude these cases, which become practical in Theorem 6.30.

Theorem 6.30. *[5] Let $t_{n,k}$, for $n, k \in \mathbb{N}$, be recursively defined by*

$$t_{n,k} = \frac{n + 2k - 1}{2} t_{n,k-1} + \frac{n - k}{2} t_{n+1,k-1}$$

for $k \geq 1$ and $t_{n,0} = 1$. If $n, k \in \mathbb{N}$ with $k \leq n + 1$, then the number of executions of the (n, k)-arch processes is given by $t_{n,k}$.

Proof. We prove the claim by induction on k. For any $n \in \mathbb{N}$, the $(n, 0)$-arch process is a string of length n and thus has a single execution. Therefore, the base case for the inductive proof holds. Let $k \in \mathbb{N}_{\geq 1}$ and assume that for any $n \in \mathbb{N}_{\geq k-2}$, the number of executions of the $(n, k-1)$-arch process is given by $t_{n,k-1}$. For the inductive step, let $n \in \mathbb{N}_{\geq k-1}$ be arbitrary. Let $G^{(0)}_{n,k}, G^{(1)}_{n,k}, G^{(2)}_{n,k}, G^{(3)}_{n,k}$ be the graphs defined in the following way:

- $G^{(0)}_{n,k}$ is the (n, k)-arch process.
- $G^{(1)}_{n,k}$ is $G^{(0)}_{n,k}$ without the edge b_1c_1.
- $G^{(2)}_{n,k}$ is $G^{(1)}_{n,k}$ with the edge x_1b_1 added and without the edge a_1b_1.

- $G^{(3)}_{n,k}$ is $G^{(2)}_{n,k}$ with the edge[7] b_1c_1 added.

Let $L^{(i)}_{n,k}$ denote the set of increasing labelings of $G^{(i)}_{n,k}$ for $i \in \{0,1,2,3\}$. Assume at first $n \geq k$. Any increasing labeling of $G^{(1)}_{n,k}$ is either an increasing labeling of $G^{(0)}_{n,k}$, or the label of c_1 is smaller than the label of b_1. Since the label of c_1 is at least as large as the label of x_1 (with equality iff $n = k$ and thus $x_1 = c_1$), any labeling satisfies the second case iff it is an increasing labeling of $G^{(2)}_{n,k}$ but not an increasing labeling of $G^{(3)}_{n,k}$. Since $L^{(3)}_{n,k} \subseteq L^{(2)}_{n,k} \subseteq L^{(1)}_{n,k}$, we obtain

$$|L^{(0)}_{n,k}| = |L^{(1)}_{n,k} \setminus (L^{(2)}_{n,k} \setminus L^{(3)}_{n,k})| = |L^{(1)}_{n,k}| - |L^{(2)}_{n,k} \setminus L^{(3)}_{n,k}| = |L^{(1)}_{n,k}| - |L^{(2)}_{n,k}| + |L^{(3)}_{n,k}|. \tag{6.1}$$

We want to find a recurrence formula for $L^{(0)}_{n,k}$ by analyzing the other terms.

Starting from an element of $L^{(1)}_{n,k}$, we obtain an element in $L^{(0)}_{n,k-1}$ by removing a_1 and b_1 and renaming $c_1 \to x_{n-k+1}$, $a_i \to a_{i-1}$, $c_i \to c_{i-1}$ for $i = 2, \dots, k$. To invert the construction, it is also required to choose a relative position for the execution of the node b_1 under the constraint that b_1 is executed after a_1. There are $n + 2k - 1$ nodes other than b_1, and the label of b_1 can be the successor of any node. Therefore, there are $n + 2k - 1$ possibilities leading to $|L^{(1)}_{n,k}| = (n + 2k - 1)|L^{(0)}_{n,k-1}|$.

Starting from an element of $L^{(2)}_{n,k}$, we obtain an element in $L^{(0)}_{n,k}$ by removing a_1, adding a node c_{k+1} together with an edge c_kc_{k+1}, and then renaming $x_1 \mapsto a_{k+1}$, $b_1 \mapsto b_{k+1}$ $c_1 \mapsto x_{n-k+1}$, $a_i \mapsto a_{i-1}, b_i \mapsto b_{i-1}, c_i \mapsto c_{i-1}$ for $i = 2, \dots k+1$ and $x_i \mapsto x_{i-1}$ for $i = 2, \dots, n-k+1$. This describes a bijective correspondence and thus $|L^{(2)}_{n,k}| = |L^{(0)}_{n,k}|$.

Let $j \in \{2, \dots, n-k+1\}$ be fixed. Starting from an element of $L^{(3)}_{n,k}$, we obtain an element in $L^{(0)}_{n+1,k-1}$ by removing the node a_1, renaming[8] $x_i \mapsto x_{i+1}$ for $i = j, \dots, n-k$, $b_1 \mapsto x_j$ and $a_i \mapsto a_{i-1}, b_i \mapsto b_{i-1}, c_i \mapsto c_{i-1}$ for $i = 1, \dots, k$, and adding/removing edges accordingly. For any choice of j, this map describes a different injection and the images form a partition of $L^{(0)}_{n+1,k-1}$. This yields $|L^{(3)}_{n,k}| = (n-k)|L^{(0)}_{n+1,k-1}|$.

Substituting in (6.1) gives

$$|L^{(0)}_{n,k}| = (n + 2k - 1)|L^{(0)}_{n,k-1}| - |L^{(0)}_{n,k}| + (n-k)|L^{(0)}_{n+1,k-1}|.$$

Solving for $|L^{(0)}_{n,k}|$ and using the induction hypothesis leads to

$$\begin{aligned} |L^{(0)}_{n,k}| &= \frac{n+2k-1}{2}|L^{(0)}_{n,k-1}| + \frac{n-k}{2}|L^{(0)}_{n+1,k-1}| \\ &= \frac{n+2k-1}{2}t_{n,k-1} + \frac{n-k}{2}t_{n+1,k-1} = t_{n,k}. \end{aligned}$$

A similar proof shows[9]

$$|L^{(0)}_{k-1,k}| = (3k-2)|L^{(0)}_{k-1,k-1}| - |L^{(0)}_{k-1,k}| - |L^{(0)}_{k,k-1}|.$$

[7]If $n = k$ let $x_1 = c_1$.

[8]In the original proof they claim that removing also b_1 leads to an $(n+1, k-1)$-arch, which is not correct. The $(n+1, k-1)$-arch is only attained after inserting b_1 as a new x_j node.

[9]Note that the prefactor $3k$ instead of $(3k-2)$ in [5] is an error.

After rearranging terms, we obtain that also in the case $n = k - 1$ we get $|L_{n,k}^{(0)}| = t_{n,k}$, finishing the proof. □

Remark 6.31. It is possible to iterate the recurrence relation in Theorem 6.30 to obtain an explicit formula for the number of executions of the (n, k)-arch process. The explicit formula is given in [5] and is rather complicated, thus it provides little insight into the growth behavior of the sequence. An attempt in the asymptotic analysis of the double sequence via singularity analysis is therefore made in [5].

While there is no immediate combinatorial interpretation of $t_{n,k}$ for $k > n+1$, the previous theorem justifies the following definition.

Definition 6.32. Let $t_{n,k}$ for $n, k \in \mathbb{N}$ be recursively defined by

$$t_{n,k} = \frac{n+2k-1}{2} t_{n,k-1} + \frac{n-k}{2} t_{n+1,k-1}$$

for $k \geq 1$ and $t_{n,0} = 1$. The bivariate function

$$A(z,u) := \sum_{n,k\geq 0} \frac{t_{n,k}}{k!} z^n u^k$$

is called the **generating function of increasing arch processes.**

Lemma 6.33. *[5] The generating function of increasing arch processes $A(z, u)$ satisfies the following partial differential equation*

$$(2zu-2z-u)\frac{\partial}{\partial u}A(z,u)+(z-2)A(z,u)+z(z+1)\frac{\partial}{\partial z}A(z,u)+u\frac{\partial}{\partial u}A(0,u)+2A(0,u)=0.$$

Proof sketch. Translating the recurrence relation for $t_{n,k}$ in Theorem 6.30 into a functional equation for $A(z, u)$ leads, after simplification, to the claim. □

Theorem 6.34. *[5] The generating function of increasing arch processes $A(z, u)$ satisfies the following algebraic equation*

$$\begin{aligned}
&(8u^3 - 15u^2 + 12u - 4)(z^3 + 6zu + 3z^2 - 3z - 1)A(z,u)^3 \\
&+\ 6z^2(8u^3 - 15u^2 + 12u - 4)A(z,u)^2 \\
&+\ 6(12zu^3 - 18zu^2 - 2u^2 + 13zu + 2u - 3z - 1)A(z,u) + 2 = 0.
\end{aligned}$$

Remark 6.35. A proof for the theorem is given in [5], which relies on guessing, simplifying, and then verifying the polynomial solution. We have found multiple contradictions in the claims about algebraic equations in [5]: Firstly, specifying $z = 0$ in Theorem 6.34 contradicts their claim for the algebraic equation satisfied by $A(0, u)$ in the constant term. Secondly, in [5], they claimed that the function $B(z, u) := A(z/u, u)$ satisfies the equation

$$\begin{aligned}
&(9u^2 + 12u - 4)(z^3 + 3z^2 + 6u - 3z - 1)B(z,u)^3 \\
&+\ 6z^2(9u^2 + 12u - 4)B(z,u)^2 \\
&+\ 6(18zu^2 - 18u^2 + 6zu + 9u - 3z - 1)B(z,u) + 6(u-1)^2 = 0.
\end{aligned}$$

Since $B(0,u) = A(0,u)$, specifying $z = 0$ in this equation should give an algebraic equation satisfied by $A(0,u)$, but this would contradict Theorem 6.34. We also verified Theorem 6.34 symbolically using the sympy package in Python for the orders $n, k \leq 15$, justifying why we believe in the validity of the proof in [5]. By substituting z/u for z in the algebraic equation of $A(z,u)$ and then multiplying by u^3, it is possible to obtain a different algebraic equation satisfied by $B(z,u)$. We were unable to recreate the original ideas envisioned by the authors of [5] and therefore cannot provide any asymptotics for the number of executions of a given arch process.

7 Outlook and some open problems

We have presented increasing trees as the most basic objects in a hierarchy of increasingly labeled structures. However, we have only given a glimpse of the broad topics covered in the literature, focusing only on the easiest families of increasing trees and their most important results.

Bucket increasing trees can be seen as a way of generalizing upon the concept of increasing trees without dismissing the underlying tree structure. In [16, 19, 20], more parameters of the very simple families of bucket increasing trees have been studied. In [18], some alternative tree models with multiple labels per node, such that an increasing structure is present, were discussed. Another possibility of studying labeled tree structures, where the labels obey an increasing constraint, arises by allowing labels to be repeated. For an example of such an approach, we refer the reader to [6].

For increasing circuits, we have already given an outlook on further results at the end of Sections 5.1 and 5.2 and also in Section 5.3. There are some improvements on the distribution of degree counts beyond the number of outputs for some special cases, as illustrated by Theorems 5.6 and 5.8. However, to the best of our knowledge, a general result for the asymptotic joint distribution of degree counts in the standard families of increasing circuits, generalizing Theorem 3.27, is currently not known. While Theorem 5.20 gives a detailed description of the behavior of the depth of random circuits at least for one model of the recursive circuits, an analogous treatment of the other families considered is not available. In [31], they have also stated an interest in studying parameters somehow related to the "width" of a recursive circuit. These parameters are of great interest from a computer scientist's point of view, as it gives an insight into the number of processors needed in order to execute a program in parallel. The authors of [14] have revealed their interest in another open problem, which is related to Theorems 5.23 and 5.24. If X_n denotes the number of ancestors in a random circuit of size n for $n \in \mathbb{N}$, then their interest lies in the study of the asymptotics of $\max\{X_{n+1}, \ldots, X_{n+i}\}$ for fixed $i \geq 2$ or potentially for $i = i_n$, where i_n lies in some growth regime.

Finally, the chapter on concurrent processes also leaves room for further exploration. In [17], different models of so-called series–parallel processes are studied by drawing a connection to bucket recursive trees and other increasing tree-like structures, similarly to what is done in Section 6.1. Regarding Theorem 6.24, an unconditional result of the asymptotics of increasing fork–join processes is left open. Lastly, we mention that this chapter paves the way for another branch of combinatorics, namely the related problem of counting the number of linear extensions of a given partial order.

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