# Dissertation

# Studies on several parameters in lattice paths

ausgeführt zum Zwecke der Erlangung des akademischen Grades eines Doktors der Naturwissenschaften

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# DECLARATION

I declare in lieu of oath, that I wrote this thesis and performed the associated research myself, using only literature cited in this volume. If text passages from sources are used literally, they are marked as such. I confirm that this work is original and has not been submitted elsewhere for any examination, nor is it currently under consideration for a thesis elsewhere. This thesis however draws on previous publications of the author. For a complete list of my relevant publications, I refer to page IV.

place, date

Valerie Roitner

# ABSTRACT

This thesis deals with enumerative as well as asymptotic aspects of directed lattice paths. Several parameters appearing in lattice paths will be analyzed, e.g. the area enclosed by or the number of contacts between two paths or the number of occurrences of certain patterns in a path.

The first chapter gives an overview over the history of lattice path theory as well as basic definitions and an overview of the applications of lattice paths in mathematical models arising in natural sciences or computer science. In the second chapter the methods used in enumerative and asymptotic combinatorics will be introduced: combinatorial classes and their generating functions for exact enumeration as well as singularity analysis for asymptotic results.

There are two lattice path configurations on which this thesis mostly focuses: non-intersecting pairs (or tuples) of paths and paths which avoid patterns, i.e., fixed sequences of consecutive steps. Chapter 3 deals with non-intersecting pairs of lattice paths. We will derive results about their average number of contacts as well as the average area between them.

Chapter 4 deals with pattern avoidance in lattice paths. First, the vectorial kernel method developed by Andrei Asinowski, Axel Bacher, Cyril Banderier, and Bernhard Gittenberger will be introduced, since it is a very powerful tool for enumerating lattice paths avoiding a fixed pattern as well as enumerating the occurrences of a fixed pattern in a lattice path. Then it will be generalized in two directions: for enumerating lattice paths with longer steps and for enumerating lattice paths which avoid several patterns at once. The tools developed in this section have also been used to prove a conjecture by David Callan about the asymptotic behavior of the expected number of ascents in Schröder paths.

In Chapter 5 we will combine the methods from Chapter 3 and 4 for studying pattern avoidance as well as the lower height in pairs of non-intersecting lattice paths.

Some of the results in this thesis have already been published in scientific papers. For a complete list of the publications this thesis is based on, see page IV.

# ZUSAMMENFASSUNG

Diese Dissertation behandelt abzählende und asymptotische Aspekte von gerichteten Gitterwegen. Es werden zahlreiche in Gitterwegen auftauchende Parameter untersucht, wie beispielsweise die Fläche oder die Anzahl der Kontakte zwischen zwei nicht-schneidenden Gitterwegen oder die Anzahl des Auftretens eines gewissen Musters in einem Pfad.

Das erste Kapitel liefert sowohl einen geschichtlichen Überblick über Gitterweg-Theorie als auch einen Überblick über die Anwendungsgebiete von Gitterwegs-Modellen in den Naturwissenschaften und der Informatik. Des weiteren werden auch die Begriffe Gitter und Gitterweg präzise definiert. Im zweiten Kapitel werden die in der abzählenden und asymptotischen Kombinatorik verwendeten Methoden vorgestellt: kombinatorische Klassen und ihre erzeugenden Funktionen für exakte Abzählung sowie Singularitätenanalyse für asymptotische Resultate.

Der Schwerpunkt dieser Dissertation liegt auf den folgenden beiden Gitterwegskonfigurationen: einerseits nicht-schneidende Paare (oder Tupel) von Gitterwegen, andererseits Gitterwege, die ein oder mehrere Muster, d. h. eine vorgegebene Folge von aufeinanderfolgenden Schritten, vermeiden. In Kapitel drei werden nicht-schneidende Paare von Gitterwegen behandelt und die Anzahl der Kontakte zwischen ihnen sowie die Fläche zwischen ihnen analysiert.

Kapitel vier handelt von Mustervermeidung in Gitterwegen. Zunächst wird die von Andrei Asinowski, Axel Bacher, Cyril Banderier und Bernhard Gittenberger entwickelte vektorielle Kernel-Methode vorgestellt, da sie sich sowohl bei der Aufzählung von Gitterwegen, die ein Muster vermeiden, als auch bei Untersuchungen, wie oft ein gewisses Muster in Gitterwegen vorkommt, als sehr nützlich erweist. Anschließend wird sie in zwei Richtungen verallgemeinert: einerseits zur Aufzählung von Gitterwegen mit längeren Schritten, andererseits zur Aufzählung von Gitterwegen, die mehrere Muster zugleich vermeiden. Die in diesem Abschnitt entwickelten Methoden wurden auch dazu verwendet, um eine Vermutung von David Callan über das asymptotische Verhalten über die erwartete Anzahl von Aufstiegen in Schröder-Pfaden zu beweisen.

In Kapitel fünf werden die Methoden aus Kapitel drei und vier kombiniert, um Mustervermeidung sowie die untere Höhe in Paaren von nicht-schneidenden Gitterwegen zu untersuchen.

Manche der Resultate aus dieser Dissertation wurden bereits in Form von wissenschaftlichen Artikeln veröffentlicht. Für eine vollständige Liste der Publikationen, die Teil dieser Dissertation sind, siehe Seite IV.

## PUBLICATIONS

This thesis is based on the following publications and preprints:

[4] A. Asinowski, C. Banderier, and V. Roitner. *Generating functions for lattice paths with several forbidden patterns*. Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics Article #95, 12 pp., 2020.

[67] V. Roitner. *Contacts and returns in 2-watermelons without wall*. Bulletin of the ICA (89), pp. 75–92, 2020.

[68] V. Roitner. *The vectorial kernel method for walks with longer steps*. preprint, 2020. ArXiv: 2008.02240

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Last, but not least, I want to thank my partner Patrick as well as my family and friends for their continuous encouragement and support.

# CONTENTS

# 1

# INTRODUCTION

In this chapter we are going to introduce lattice paths. We will first present a brief historical introduction before giving precise definitions. We will finish the chapter by giving an overview of some applications of lattice path theory as well as several special classes of lattice paths.

## 1.1 THE BEGINNINGS OF LATTICE PATH THEORY – A HISTORICAL INTRODUCTION

The so-called ballot problem was one of the earliest occurrences of lattice path combinatorics. It is formulated as follows:

> *"Suppose that two candidates $A$ and $B$ are in an election. The number of the voters is $\mu$. $A$ obtains $m$ votes and is elected, $B$ obtains $\mu - m$. Find the probability that, during the counting of the votes, the number of votes for $A$ always exceeds those of his competitor."*
>
> J. Bertrand. *Solution d'un problème.* (Translated from French by M. Renault [65])

In 1887 Joseph Bertrand published an answer to this question in [17], where he sketches a proof using recursion relations and deduces that the probability is

$$\frac{2m - \mu}{\mu}.$$

He also remarks that it might be possible that such a simple result could be proved by a more direct method. Such a proof was later given by Désiré André in [2] with a bijection between certain permutations. This later became known as André's reflection method, although André himself did not use any reflections in his proof. The reflection method is a variation of André's proof, however it is not accurate to say that André employed the reflection method in his proof (more details on this can be seen in [65]).

The sequence of such votes can be visualized by paths in the Euclidean plane. We start at the origin $(0, 0)$ and move one step for each vote: an $(1, 1)$-step ("up-step") for each vote for candidate $A$ and an $(1, -1)$-step ("down-step") for each vote for candidate $B$. If there are $\mu$ votes in total and candidate $A$ receives $m$ votes, we end in the point $(\mu, 2m - \mu)$, since we do $m$ up-steps and $\mu - m$ down-steps, thus ending up at altitude $m - (\mu - m) = 2m - \mu$. The condition that candidate $A$ is always ahead of candidate $B$ then corresponds to the condition that the lattice path never touches the $x$-axis (except at $(0, 0)$ in the beginning).

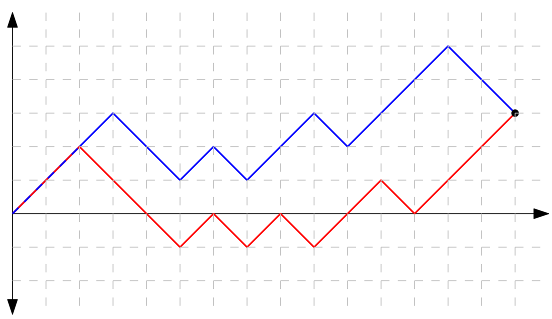


Figure 1: Two sequences of votes represented as lattice paths. Only the blue path has the property that candidate $A$ is always ahead of candidate $B$.

In Figure 1, the blue lattice path corresponds to the sequence of votes $AAABBABAABAAABB$ and the red path corresponds to the sequence $AABBBABABAABAAA$ (for the example in the figure the values for $m$ and $\mu$ are 9 and 15 respectively). In both cases, candidate $A$ wins the election with $m$ out of $\mu$ votes. However, only the first (blue) sequence has the desired property that candidate $A$ is always ahead of candidate $B$.

If we want to compute the probability of this event, we have to count all lattice paths from $(0,0)$ to $(\mu, 2m-\mu)$ with the desired property that the lattice path only touches the $x$-axis at the origin as well as all lattice paths from $(0,0)$ to $(\mu, 2m-\mu)$ (without any constraints) and then compute their quotient.

The latter one is easily achieved: We take $\mu$ steps in total, $m$ of them being up-steps and these up-steps can be anywhere among the $\mu$ steps. Thus, we have to choose $m$ out of $\mu$ steps and end up with $\binom{\mu}{m}$ paths in total.

Counting all lattice paths that obey the constraint that they only touch the $x$-axis at the origin turns out to be a bit trickier. First of all, let us note that each of these paths has to start with an up-step. Otherwise, candidate $B$ receives the first vote, thus violating the condition from the very beginning (also, the lattice path would have to touch the $x$-axis somewhere between $(1,-1)$ and $(\mu, 2m-\mu)$, which we do not want). Thus, what remains is a path from $(1,1)$ to $(\mu, 2m-\mu)$ that never goes below $x=1$.

To count these objects we use the approach: "The good guys are everyone minus the bad guys". The "good guys" being all lattice paths from $(1,1)$ to $(\mu, 2m-\mu)$ that never go below $x=1$, "everyone" being all lattice paths from $(1,1)$ to $(\mu, 2m-\mu)$ without any further constraints and the "bad guys" being all lattice paths from $(1,1)$ to $(\mu, 2m-\mu)$ which do go below $x=1$. The paths we labeled "everyone" are easy to count. By a similar argument as before, we obtain that their number is $\binom{\mu-1}{m-1}$.

Counting the "bad guys" turns out to be a bit trickier, but can be achieved via the following reflection: Each "bad" path touches the $x$-axis at least once. Let us consider the first intersection of the path and the $x$-axis and reflect the remaining path along the $x$-axis. We end up with a path from $(1,1)$ to $(\mu, -2m+\mu)$, as can be seen in Figure 2 (the initial segment of the red path continued by the dashed red path).

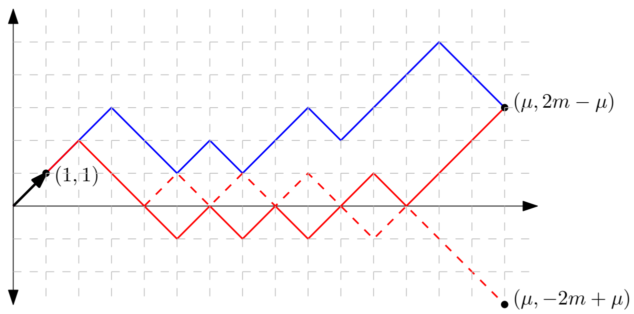



Figure 2: The reflection argument.

Conversely, each path from $(1,1)$ to $(\mu, -2m+\mu)$ can be mapped to a "bad" path via the same reflection argument. Thus the number of "bad" paths is the same as the number of all paths from $(1,1)$ to $(\mu, -2m+\mu)$. But these paths can be easily counted by a similar computation as above (choosing up-steps among all steps). We obtain that their number is $\binom{\mu-1}{\mu-m-1} = \binom{\mu-1}{m}$.

Thus, we now can compute the number of the "good guys", the paths we are actually interested in, and obtain that it is

$$\binom{\mu-1}{m-1} - \binom{\mu-1}{m}.$$

After some computations with binomial coefficients, this becomes

$$\binom{\mu-1}{m-1} - \binom{\mu-1}{m} = \frac{2m-\mu}{\mu}\binom{\mu}{m}.$$

The probability of candidate $A$ always being ahead of candidate $B$ is then given by

$$\frac{\frac{2m-\mu}{\mu}\binom{\mu}{m}}{\binom{\mu}{m}} = \frac{2m-\mu}{\mu}.$$

*Variants and special cases of the ballot problem*

In the original ballot problem we were interested in the probability that candidate $A$ is strictly ahead of candidate $B$ when counting the votes. However, we could also ask how likely it is that candidate $A$ is ahead (but not necessarily strictly, i.e., ties are allowed) of candidate $B$ throughout the counting of the votes. In this case the probability is $\frac{2m-\mu+1}{m+1}$, as can be shown by similar methods. Visualizing these vote sequences as lattice paths, we obtain lattice paths that never go below the $x$-axis.

If we now consider the special case that the election as such also ends in a tie between candidate $A$ and $B$, we obtain lattice paths with steps $(1,1)$ and $(1,-1)$ from $(0,0)$ to $(2n,0)$ that never go below the $x$-axis. These class of lattice paths is called *Dyck paths*. These paths will reappear at many places throughout this thesis.

This introductory example of the ballot problem already shows some of the reasons why lattice paths are useful and interesting. They turned out to be very helpful in various mathematical models (e.g., for vote counting in an election, as in the ballot problem example). In physics, lattice paths appear in models of wetting and melting processes [35] or Brownian motion [57]. In computer science lattice paths are used in the analysis of algorithms, e.g. shellsort [49] or dual-pivot-quicksort [5]. In chemistry lattice paths are used to describe certain polymers or DNA-denaturation as well as DNA-unzipping, as can be seen in [58, 66, 45]. Furthermore, lattice paths can also be used to describe birth-death-processes, see for example [36].

Lattice paths also stand in bijection with numerous other combinatorial objects (for example, the permutations from André's proof). Dyck paths in particular stand in bijection with numerous combinatorial objects. In his book [71] Stanley lists over 200 bijections between objects counted by the Catalan numbers (listed as A000108 in the OEIS [63]), Dyck paths of length $2n$ being one of them. Other examples include:

- Expressions containing $n$ pairs of correctly matched parentheses.
- Plane rooted trees with $n+1$ nodes.
- Different ways a convex $(n+2)$-gon can be triangulated.
- Permutations of the set $\{1,2,\ldots,n\}$ that avoid 321. We say a permutation $\pi$ avoids 321 if there is no subsequence $xyz$ of $\pi$ such that $x > y > z$ (i.e., $x, y$ and $z$ are ordered in the same way as 321).
- Schröder paths (lattice paths on $\mathbb{Z}^2$ with step set $\{(1,1),(1,-1),(2,0)\}$ that never go below the $x$-axis) of length $2n$ where no up-step is immediately followed by a down step. This is probably not the most famous example of objects counted by the Catalan numbers, but since this thesis also deals with pattern avoidance in lattice paths in Chapter 4, it is listed here as some kind of preview.

## 1.2 WHAT IS A LATTICE PATH?

In the historical introduction we have already dealt with lattice paths without properly defining them. This will be made up for in this section where we will give precise definitions. This section is mostly based on [54].

In order to define a lattice path, we first need to define a lattice.

**Definition 1.2.1.** *A* lattice $\Lambda = (V, E)$ *is a mathematical model of a discrete space. It consists of a set* $V \subseteq \mathbb{R}^d$ *of vertices and a set* $E \subseteq V \times V$ *of edges. If two vertices are connected via an edge, we call them* nearest neighbors.

*An important subclass of lattices are* periodic *lattices. A lattice is called periodic if the there are vectors* $v_1, \dots, v_k$ *such that the lattice is mapped to itself under any translation of the form* $\sum_{j=1}^{k} \alpha_j v_j$ *where* $\alpha_j \in \mathbb{Z}$ *for* $j = 1, \dots, k$.

Some examples of lattices can be seen in Figure 3.

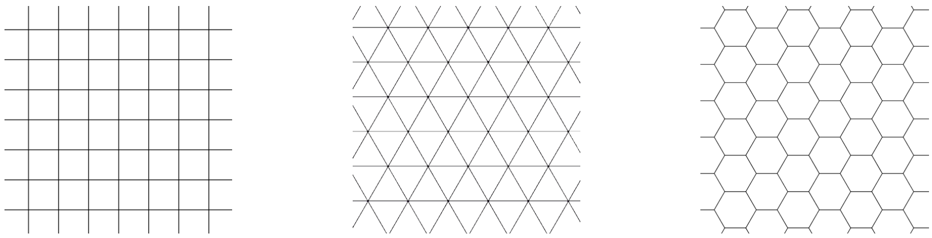


Figure 3: Three examples of periodic lattices. From left to right: the Euclidean (or square) lattice $\mathbb{Z}^2$, the triangular lattice and the hexagonal lattice.

The term "lattice" comes from physics, where lattices are studied in crystallography, solid-state physics or surface physics. In mathematics and computer science lattices are also called graphs.

A *lattice path* is exactly what the name suggests: a path on some lattice, taking steps from a vertex to one of its neighbors. More precisely,

**Definition 1.2.2.** *A n-step lattice path or lattice walk on a lattice* $\Lambda = (V, E)$ *from* $s \in V$ *to* $x \in V$ *is a sequence* $w = (w_0, w_1, \dots, w_n)$ *of vertices such that*

1. $w_0 = s$ *and* $w_n = x$
2. $(w_i, w_i + 1) \in E$ *for* $i = 0, \dots, n-1$

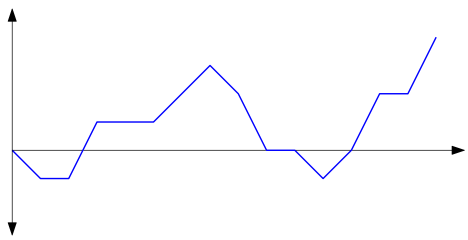


Figure 4: A lattice path

In this thesis we are going to work on the *Euclidean lattice* $\mathbb{Z}^d$. On this lattice, an alternative definition of lattice paths via the so-called step set can be used.

**Definition 1.2.3.** *An n-step lattice path from* $s \in \mathbb{Z}^d$ *to* $x \in \mathbb{Z}^d$ *relative to a step set* $\mathcal{S}$ *is a sequence* $w = (w_0, w_1, \dots, w_n)$ *of points in* $\mathbb{Z}^d$ *such that*

1. $w_0 = s$ *and* $w_n = x$
2. $(w_i, w_i + 1) \in \mathcal{S}$ *for* $i = 0, \dots, n-1$

In this definition the set of possible edges is implicitly defined over the set of allowed steps. Note that the step set is defined globally, thus the lattice has the same structure at each vertex. The advantage of this definition is its compact form. Unless stated otherwise, we will be using this definition in this thesis and the underlying lattice will always be $\mathbb{Z}^2$.

The step set can be both finite or infinite. Unless stated otherwise, we will only consider finite step sets in this thesis.

**Definition 1.2.4.** *A lattice path in $\mathbb{Z}^2$ is called* directed *if all its steps have positive first coordinate. A lattice path is in $\mathbb{Z}^2$ called* simple *if all of its steps are of the form $(1, b)$. These objects are essentially one-dimensional objects.*

**Definition 1.2.5.** *Let $w$ be a directed lattice path in $\mathbb{Z}$. Then we define the following parameters:*

- *The first entry $u_i$ of a step is called its* length*. The length of a walk $w$, denoted by $|w|$ is the sum of the lengths of all its steps, i.e. $|w| = u_1 + \cdots + u_m$. If the length of a certain class of walks is always even, we defined the* semilength *of a walk to be half of its length.*

- *The* size *of a walk is defined to be the number of its steps. Length and size of a walk do not always coincide. Only for simple lattice paths length and size are the same.*

- *The* final altitude *of a walk $w$, denoted by $alt(w)$ is the sum of the altitudes of all steps, i.e. $alt(w) = v_1 + \cdots + v_m$. Thus, a walk starting in $(0, 0)$ terminates in $(|w|, alt(w))$.*

**Definition 1.2.6.** *For some application it is useful to associate* weights *to the steps of a lattice path. In this case, the step set $\mathcal{S}$ is coupled with a system of weights $\mathcal{G} = \{w_j\}_j$ where $w_j > 0$ is the weight associated to the step $s_j = (a_j, b_j) \in \mathcal{S}$. The weight of a path is defined to be the product of the weight of its steps.*

Often used choices of weights are:

- Combinatorial paths in the standard sense, where $w_j = 1$ for all steps

- Paths with colored steps, where $w_j \in \mathbb{Z}^+$. The weight $w_j = k$ means that there are $k$ possible ways to color the corresponding step $s_j$.

- Probabilistic models, where $\sum_j w_j = 1$. In this model the step $s_j$ is chosen with probability $w_j \in (0, 1]$ among all steps in the step set.

**Example:** A famous historical example of a lattice path with weighted steps is the visualization *gambler's ruin problem*. This problem can be stated in the following way:

Two players, Alice and Bob make a sequence of bets. For each bet, Alice wins with probability $p$ and Bob wins with probability $q := 1 - p$. The loser of a bet has to pay 1 dollar to the winner. In the beginning, Alice has $a$ dollars and Bob has $b$ dollars. The game ends if one of the players is broke and the other player has $a + b$ dollars.

This can be visualized as lattice path, illustrating Alice's finances over time. The path starts at $(0, a)$, never going below the boundary line $y = 0$ (Alice runs out of money) and never going above $y = a + b$ (Alice has won all the money). At each point in time, an up-step is taken with probability $p$ and a down-step is taken with probability $1 - p$. The lattice path ends if it hits one of the boundaries.

In the symmetric case, i.e., if $p = q = \frac{1}{2}$, the probability that Alice wins all the money is $\frac{a}{a+b}$. In the asymmetric case, i.e., if $p \neq q$, the probability that Alice wins all the money is $\frac{1-(q/p)^a}{1-(q/p)^{a+b}}$.

**Definition 1.2.7.** *Let $w$ be a simple lattice path with step set $\mathcal{S} = \{s_1, \ldots, s_m\}$ with associated weights $\mathcal{G} = \{w_1, \ldots, w_m\}$. The* step polynomial *of $w$ is then defined as*

$$P(u) := \sum_{j=1}^{m} w_j u^{s_j}.$$

*For most step sets $P$ is actually a Laurent polynomial in $u$ and not a polynomial in the classical sense.*

As we will see in later chapters, when working with generating functions, the step polynomial is a convenient way to encode the step set. In Chapter 4, we will also see a generalization of the step polynomial to walks that are not simple.

## 1.3 IMPORTANT CLASSES OF LATTICE PATHS

There are several important subclasses of lattice paths. One way to categorize them is by their step set. This gives rise to the following definitions:

- We have already encountered *Dyck paths*, which are paths from $(0,0)$ to $(2n,0)$ that never go below the $x$-axis with step set $\mathcal{S} = \{\mathsf{U}, \mathsf{D}\} = \{(1,1),(1,1)\}$. Note that the endpoint $(2n,0)$ is always even, because we have to take the same number of up- and down-steps in order to return to altitude zero. In this case, $n$ is called the *semilength* of the path. If we drop the condition on the endpoint or the region the path is not allowed to leave, but still have the same step set, we say the path has *Dyck step set* (and analogously for other kinds of paths defined here).
- If we also allow horizontal steps, we obtain *Motzkin paths*. More precisely, Motzkin paths are lattice paths from $(0,0)$ to $(n,0)$ that never go below the $x$-axis and have the step set $\mathcal{S} = \{\mathsf{U}, \mathsf{H}, \mathsf{D}\} = \{(1,1),(1,0),(1,-1)\}$.
- A similar concept as Motzkin-paths are Schröder paths, however, their horizontal step is longer: These are paths from $(0,0)$ to $(2n,0)$ that never go below the $x$-axis and have the step set $\mathcal{S} = \{\mathsf{U}, \mathsf{F}, \mathsf{D}\} = \{(1,1),(2,0),(1,-1)\}$.
- *Łukasiewicz paths* are paths where all step vectors lie in $\{1\} \times (\mathbb{N} \cup \{-1\})$. Dyck and Motzkin paths are a special case of Łukasiewicz paths. However, Schröder paths are not Łukasiewicz paths.
- Walks, where the step set is a subset of the eight cardinal directions

  $$\mathcal{S} \subseteq \{(0,1),(1,1),(1,0),(1,-1),(0,-1),(-1,-1),(-1,0),(-1,1)\},$$

  or more intuitively,

  $$\mathcal{S} \subseteq \{\text{N, NE, E, SE, S, SW, W, NW}\}$$

  are called *walks with small steps*. There are $2^8 = 256$ such step sets. All quarter-plane-walks with these step sets have been classified according to the algebraic nature of their generating function in [23].

The above list is, of course, far from complete, but gives a brief overview about important and often used step sets.

Many applications of lattice paths in models also call for restrictions on either the endpoints of the lattice path or the region the path is not allowed to leave. This leads us to the following concepts for directed lattice paths:

- A *walk* is an unconstrained lattice path.
- A *bridge* is a lattice path whose endpoint lies on the $x$-axis.
- A *meander* is a lattice path that lies in the quarter-plane $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. For directed lattice paths, this is equivalent to lattice paths that never attain negative altitude.
- An *excursion* is a lattice path that is both a bridge and a meander, i.e., a lattice path that ends on the $x$-axis, but never crosses the $x$-axis.
- An *arch* is an excursion that stays at strictly positive altitude except at the start and the end point.
- Lattice paths confined to a *strip* are lattice paths that never go below altitude $g$ and never exceed altitude $h$, i.e., all the lattice points $(x,y)$ along the path fulfill $g \leq y \leq h$. An example of lattice paths confined to a strip is the gamblers ruin problem, where $g = 0$ and $h = a + b$.

There are several identities among these objects, for example "An excursion is a sequence of arches." Such identities and decompositions will become of interest and importance when calculating generating functions for combinatorial objects, as can be seen in Chapter 2. More such identities as well as generalizations of some of the objects defined above (e.g. generalized arches) can be found in [12].

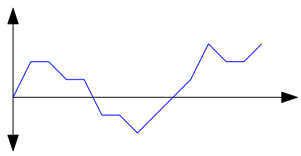
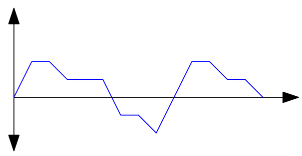
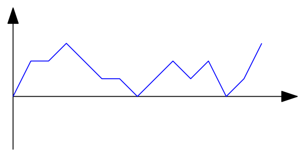
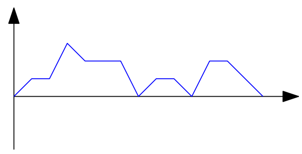
| | ending anywhere | ending at 0 |
|---|---|---|
| unconstrained on $\mathbb{Z}$ | walks  $W(t,1) = \frac{1}{1-tP(1)}$ | bridges  $B(t) = W_0(t) = t\sum_{i=1}^{c} \frac{u_i'(t)}{u_i(t)}$ |
| constrained on $\mathbb{Z}_{\geq 0}$ | meanders  $M(t,1) = \frac{1}{1-tP(1)} \prod_{i=1}^{c}(1-u_i(t))$ | excursions  $E(t) = M_0(t) = \frac{(-1)^{c-1}}{p_{-c}t} \prod_{i=1}^{c} u_i(t)$ |

Figure 5: Walks, bridges, meanders, and excursions. The functions listed below the graphics are the generating functions of the counting sequences of these objects which were derived in [9].

Other conditions that sometimes will be imposed on lattice paths are self-avoiding lattice paths, mutually avoiding (non-touching or non-crossing) pairs or tuples of lattice paths or avoidance of certain patterns (by pattern we mean a consecutive sequence of steps). We will not be dealing with self-avoiding lattice paths in this thesis, more information on them can be found in [42, 46]. Mutually avoiding pairs of walks will be the main focus of Chapter 3, while Chapter 4 will be dealing with walks that avoid one or more patterns.

For a good overview about several other conditions that can be imposed on lattice paths, see [52].

# 2 METHODS – GENERATING FUNCTIONS AND ANALYTIC COMBINATORICS

The main focus of this thesis is the enumeration of certain combinatorial objects, namely lattice paths. A useful tool for handling such enumeration problems are generating functions. The idea behind generating functions is to encode combinatorial objects with formal power series. Let $x$ be a variable. Then an object of size $n$ is encoded by $x^n$. Thus, the number of all objects of size $n$ corresponds to the coefficient of $x^n$ in the power series. Or, as Herbert Wilf very vividly puts it in his textbook on generating functions [76]:

> *"A generating function is a clothesline on which we hang up a sequence of numbers for display."*
>
> H. Wilf. *generatingfunctionology*, p. 1.

The usage of generating functions turned out to be very fruitful and gave rise to many new solution strategies, e.g. deriving a functional equation for the generating function instead of dealing with recurrences which are often hard to solve (this is called the kernel method and will be explained in more detail in Chapter 4). The rich structure of the ring of generating functions is also very useful, especially since operations like addition, multiplication or formal derivative of formal power series also have a combinatorial interpretation.

Furthermore, generating functions also give rise to many interdisciplinary approaches. One field that turned out to be particularly useful is complex analysis, which can be used for obtaining asymptotic results for the growth of the coefficients of a series. By interpreting a formal power series as an analytic function on its disk of convergence one also gains access to the tools and theorems of complex analysis. This particular branch of combinatorics is called *Analytic Combinatorics*. A detailed overview over this field can be found in the book of Flajolet and Sedgewick [37].

## 2.1 FORMAL POWER SERIES AND GENERATING FUNCTIONS

In this section we will introduce formal power series and discuss their relation with combinatorial objects via their generating function. This section is based on [76] as well as [37].

### 2.1.1 *Formal power series*

**Definition 2.1.1.** *Let $R$ be a ring with unity. The ring of* formal power series *over $R$, denoted by $R[[z]]$ consists of all formal sums of the form*

$$\sum_{n\geq 0} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots,$$

*where the coefficients $a_n$ lie in $R$. The* sum *of two formal power series $A(z) := \sum_{n\geq 0} a_n z^n$ and $B(z) := \sum_{n\geq 0} b_n z^n$ is given by*

$$A(z) + B(z) := \sum_{n\geq 0} (a_n + b_n) z^n.$$

*The* product *of two formal power series $A(z)$ and $B(z)$ is given by*

$$A(z) \cdot B(z) = \left(\sum_{n\geq 0} a_n z^n\right) \cdot \left(\sum_{n\geq 0} a_n z^n\right) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n.$$

*This type of product is also called* Cauchy product *or* convolution.

A series

$$A(z) = \sum_{n\geq 0} a_n z^n \in R[[z]]$$

is (multiplicatively) invertible if and only if its constant coefficient $a_0$ is invertible in $R$. Suppose $B(z) = \sum_{n\geq 0} b_n z^n$ is the inverse of $A$, i.e., $A \cdot B = 1$, then by comparing coefficients we obtain

$$b_0 = \frac{1}{a_0}$$

and

$$b_n = -\frac{1}{a_0}\left(\sum_{k=1}^{n} a_k b_{n-k}\right) \qquad \text{for } n \geq 1.$$

These two equations explain why $a_0 \in R^*$ is both a necessary and sufficient condition for $A$ to be invertible.

An important special case is the geometric series

$$\frac{1}{1-z} = \sum_{n\geq 0} z^n,$$

which is the inverse of $1-z$. The geometric series can be seen as a special case of another important series, the binomial series

$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n$$

where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$. Taking $\alpha = -1$ and $x = -z$ yields the above geometric series.

**Definition 2.1.2.** *Given the power series $A(z)$ and $B(z)$, where $a_0 = 0$, the* composition *of $A$ and $B$ is given by*

$$B(A(z)) = \sum_{n\geq 0} b_n A(z)^n = \sum_{n\geq 0} c_n z^n$$

*where the coefficients $c_n$ can be computed by expanding the powers of $A$:*

$$c_n = \sum_{0\leq k\leq n, j_1+\cdots+j_k=n} b_k a_{j_1} \ldots a_{j_k}.$$

*The condition $a_0 = 0$ is important to make sure that the coefficients $c_n$ are well defined.*

The series $I(z) = z$ is the neutral element of the composition. If $R$ is a field, then $A(z)$ is invertible under composition if $a_0 = 0$ and $a_1$ is an invertible element in $R$.

**Example 2.1.3.** The series

$$A(z) = e^z - 1 = \sum_{n\geq 1} \frac{z^n}{n!}$$

and

$$B(z) = \log(1+z) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} z^n$$

are inverse under composition.

**Definition 2.1.4.** *Let $A(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series. Then*

$$A'(z) := \sum_{n\geq 0} (n+1) a_{n+1} z^n$$

*is the* formal derivative *of $A$. Sometimes the formal derivative is also denoted by $DA(z)$.*

**Theorem 2.1.5.** *The formal derivative satisfies the following properties:*

- *Linearity:* $(A(z) + B(z))' = A'(z) + B'(z)$ *and* $(c \cdot A(z))' = c \cdot A'(z)$.
- *Product rule:* $(A \cdot B)' = A'B + AB'$.
- *If* $R$ *is commutative, the formal derivative also satisfies the chain rule:* $A(B(z))' = A'(B(z)) \cdot B'(z)$.

**Definition 2.1.6.** *Let* $A(z) = \sum_{n \geq 0} a_n z^n$ *be a formal power series. Then the operator* $[z^n] : R[[z]] \to R$ *defined as*

$$[z^n]A(z) := a_n$$

*is called the* coefficient extractor operator.

**Theorem 2.1.7.** *The coefficient extractor operator fulfills the following properties:*

- *Linearity:* $[z^n](A(z) + B(z)) = [z^n]A(z) + [z^n]B(z)$ *and* $[z^n](cA(z)) = c[z^n]A(z)$
- $[z^n]A(cz) = c^n[z^n]A(z)$
- $[z^n]z^\ell A(z) = [z^{n-\ell}]A(z)$

The following formula often turns out to be helpful for coefficient extraction.

**Theorem 2.1.8.** Lagrange inversion formula. *The coefficients of an inverse function and its powers are determined by the coefficients of powers of the direct function: let* $\phi(T)$ *be a formal power series in* $T$ *with* $\phi(0) = 1$. *Then there is a unique formal power series* $T = T(z)$ *that satisfies* $z = \frac{T}{\phi(T)}$. *Then one has for any* $k \in \mathbb{N}$:

$$[z^n]T(z) = \frac{1}{n}[T^{n-1}]\phi(T)^n, \quad [z^n]T(z)^k = \frac{k}{n}[T^{n-k}]\phi(T)^n. \tag{1}$$

*Furthermore, if* $F$ *is any formal power series in* $T$ *the value* $F(T(z))$ *when expanded into a power series in* $z$ *satisfies*

$$[z^n]F(T(z)) = \frac{1}{n}[T^{n-1}]F'(T)\phi(T)^n. \tag{2}$$

*Proof.* A proof for 2 can be found in [76]. Note that 1 is just a special case of 2 with $F(T) = T$ or $F(T) = T^k$ respectively. □

If $R$ is a field of characteristic zero, we furthermore have the following relation between the formal derivative and the coefficient extractor operator:

$$a_n = \frac{1}{n!}(D^n A(z))|_{z=0}$$

where the notation $f(x)|_{x=a}$ stands for the evaluation of the function $f$ at $x = a$.

When viewing formal power series as a complex-valued function (as will be done in Section 2.2 when dealing with coefficient asymptotics and singularity analysis), the following definition is of importance:

**Definition 2.1.9.** *Let* $A(z) = \sum_{n \geq 0} a_n z^n$ *be a power series. The* radius of convergence *is defined as*

$$R := \sup\{z : \sum_{n \geq 0} a_n z^n \text{ converges}\}.$$

*For* $|z| < R$ *we have that* $a_n z^n \to 0$ *(as* $n \to \infty$*) and the series converges. Similarly, for* $|z| > R$ *we have that* $a_n z^n \to \infty$ *and the series diverges. For* $|z| = R$ *both convergence and divergence is possible, however, there has to be at least one point on the circle of convergence such that the series diverges.*

Sometimes it is also useful to allow (finitely many) negative powers in a formal power series. This leads to the following

**Definition 2.1.10.** *A* formal Laurent series *is a series of the form*

$$A(z) = \sum_{n \geq n_0} a_n z^n$$

*for some* $n_0 \in \mathbb{Z}$

The condition that there are only finitely many negative powers is important for the multiplication of two formal Laurent series to be well defined. It can be defined in a similar fashion to the Cauchy-Product. More precisely, if $A(z) = \sum_{n \geq n_0} a_n z^n$ and $B(z) = \sum_{n \geq m_0} b_n z^n$ then the coefficient $z^n$ of the product of these series is

$$c_n = \sum_{i \in \mathbb{Z}} a_i b_{n-i}.$$

In particular, $c_n = 0$ if $n < n_0 + m_0$.

**Definition 2.1.11.** *For a nonzero Laurent series $A(z)$, the* order *of $A$, denoted by* $\operatorname{ord}(A)$ *is the smallest integer $n$ such that $a_n \neq 0$.*

**Definition 2.1.12.** *If a Laurent series only has finitely many nonzero coefficients it is called a* Laurent polynomial*. The largest integer $n$ such that $a_n \neq 0$ is called its* degree*, denoted by* $\deg(A)$*.*

**Definition 2.1.13.** *Let $A(z) := \sum_{n \geq n_0} a_n z^n$ be a formal Laurent series. Then the negative part operator $\{z^{<0}\}A(z)$ is defined as*

$$\{z^{<0}\}A(z) := \sum_{n=n_0}^{-1} a_n z^n.$$

*This sum is zero if* $\operatorname{ord}(A) > -1$*. Similarly we define $\{z^{>0}\}A(z)$, $\{z^{\leq 0}\}A(z)$ and $\{z^{\geq 0}\}A(z)$.*

These operators fulfill similar properties as the coefficient extraction operator, e.g., linearity.

**Definition 2.1.14.** *Let $K$ be a field of characteristic zero. Then formal power series in $K[[z]]$ can be classified in the following way:*

- *A formal power series $F(z) \in K[[z]]$ is called* rational *if there are polynomials $P(z), Q(z) \in K[z]$ (with $Q(z) \neq 0$) such that*
$$F(z) = \frac{P(z)}{Q(z)}.$$
- *A formal power series $F(z) \in K[[z]]$ is called* algebraic *if there are polynomials $P_0(z), \ldots P_d(z) \in K[z]$ (with $P_d(z) \neq 0$) such that $F$ fulfills the equation*
$$P_d(z)F(z)^d + P_{d-1}(z)F(z)^{d-1} + \cdots + P_1(z)F(z) + P_0(z) = 0.$$
*The smallest positive integer $d$ for which such an equation is satisfied by $F$ is called the* degree *of $F$.*
- *A formal power series $F(z) \in K[[z]]$ is called* D-finite *(or* holonomic*) if there are polynomials $P_0(z), \ldots P_d(z) \in K[z]$ (with $P_d(z) \neq 0$) such that $F$ satisfies a linear differential equation with polynomial coefficients*
$$P_d(z)F^{(d)} + P_{d-1}(z)F^{(d-1)} + \cdots + P_1 F'(z) + P_0(z)F = 0,$$
*where $F^{(j)} = \frac{d^j F}{dz^j}$.*

It is easy to see from these definitions that each rational power series is also algebraic and each algebraic power series is also D-finite. We have the following hierarchy (a proof for this can be found in [37] p. 749):

$$\begin{array}{c} \text{D-finite} \\ \uparrow \\ \text{algebraic} \\ \uparrow \\ \text{rational} \end{array}$$

This hierarchy is not exhaustive, there are various other classes that could be added, but these three are the ones most useful in enumerative combinatorics (for several other classes as well as their definition see for example [78]). This classification is of particular interest in computer algebra, since there exist efficient algorithms for problems within these specific classes.

**Example 2.1.15.** Rational, algebraic and holonomic series:

- The geometric series $G(z) = \frac{1}{1-z}$ is rational since it can be written as $\frac{P(z)}{Q(z)}$ where $P(z) = 1$ and $Q(z) = 1 - z$. It is also algebraic since $(1-z)G(z) = 1$ (this already illustrates the idea of the proof why every rational series is also algebraic) and D-finite since it fulfills $(1-z)F'(z) - F(z) = 0$.
- The generating function for Dyck paths $D(z) = \frac{1-\sqrt{1-4z}}{2}$ (which we will encounter several times in this thesis) is algebraic, since it fulfills the polynomial equation $zD^2 - D + 1 = 0$.
- The power series $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \ldots$ is D-finite since it satisfies $\cos'' + \cos = 0$. However, it is neither algebraic nor rational (without proof). Similarly, the power series $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \ldots$ is D-finite.

Another reason why holonomic functions are of interest are their rich closure properties, as can be seen in the following theorem.

**Theorem 2.1.16.** *The class of holonomic functions in one variable is closed under the following operations:*

- *sum*
- *product*
- *differentiation* ($\partial_z$)
- *indefinite integration* ($\int \ldots dz$)
- *algebraic substitution* ($z \to a(z)$ *for some algebraic function* $a(z)$)

*Proof.* See [37] p. 749. □

### 2.1.2 *Combinatorial classes and ordinary generating functions*

In combinatorics, mathematicians are usually interested in counting objects with certain properties of a given size, e.g., all possible ways to write $n$ as a sum of positive integers, all Dyck paths of semilength $n$ or all words of size $n$ over a given alphabet. Formalizing such problems leads us to the notion of combinatorial classes.

**Definition 2.1.17.** *A* combinatorial class, *or simply a* class, *is a finite or denumerable set* $\mathcal{C}$ *and a function* $w : \mathcal{C} \to \mathbb{N}$, *called the* size function, *such that all sets*

$$w^{-1}(\{n\}) = \{c \in \mathcal{C} : w(c) = n\} \quad n \in \mathbb{N}$$

*are finite, i.e., the number of elements of a given size is finite. An element* $c \in \mathcal{C}$ *is called* combinatorial object *and* $w(c)$ *is called the* size *of* $c$. *Sometimes we will also write* $|c|$ *for the size (or* $|c|_{\mathcal{C}}$ *if the underlying class is not clear from context).*

The enumeration problem is to determine the number of objects of a given size, or, phrased differently, to determine the numbers

$$c_n := |w^{-1}(\{n\})|$$

for all $n \in \mathbb{N}$. Writing $\mathcal{C}_n$ for the sets $w^{-1}(\{n\})$ we obtain a decomposition of $\mathcal{C}$ into disjoint sets.

**Definition 2.1.18.** *The* counting sequence *of a combinatorial class $\mathcal{C}$ is defined as the sequence of integers $(c_n)_{n\geq 0}$. This sequence can be written as the coefficients of a formal power series*

$$C(z) = \sum_{n\geq 0} c_n z^n.$$

*This series is called the* generating function *of the class $\mathcal{C}$. We say that the variable z marks (or encodes) the size in the generating function.*

As we will see later, the usage of generating functions has many benefits. Many combinatorial constructions directly translate to generating functions. Furthermore, viewing a formal power series as an analytic function and using theorems from complex analysis turns out to be a powerful tool for the analysis of the asymptotic growth of the counting sequence.

**Definition 2.1.19.** *Two combinatorial structures $\mathcal{A}$ and $\mathcal{B}$ are called (combinatorially) isomorphic, written $\mathcal{A} \cong \mathcal{B}$ if their counting sequences are identical. This is equivalent to the existence of a bijection from $\mathcal{A}$ to $\mathcal{B}$ that preserves size.*

Such a bijection is not always easy to find, despite its need to exist. Some bijections between isomorphic structures can be constructed in a very straightforward manner, others however are very complicated to construct.

**Example 2.1.20.** In Section 1.1 we mentioned that Dyck paths of size $2n$ and planar rooted trees with $n+1$ nodes are counted by the same sequence. Thus, there has to be a bijection between these two objects. This bijection is easy to construct: Start at the root of the tree then traverse the nodes of the tree in the order they are discovered by a depth first search. Whenever a new node is discovered, take an up-step. Whenever we have to backtrace to an already discovered node, take a down-step. Or, phrased more visually, but less precise: draw a line around the tree, passing each inner node twice, whenever we pass a node from below, take an up-step, whenever we pass it from above take a down-step. This construction can be seen in Figure 6.

Since each inner node is passed twice it is clear that the resulting path has length $2n$. It is also clear that it has to end at the $x$-axis, but never goes below the $x$-axis, since we start and finish in the root and the root has no parent.

The inverse mapping is constructed similarly: an up-step means we discover a new node as a child of the current node, a down step means we go back to the current node's parent.

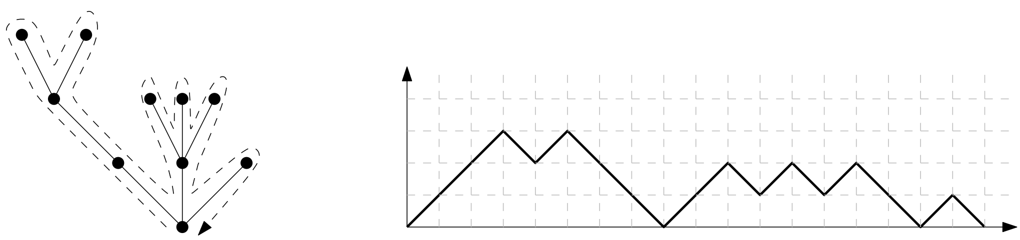


Figure 6: The bijection between planar rooted trees with $n+1$ nodes and Dyck paths of size $2n$

It is obvious from the definition, but important to note that two isomorphic structures have the same generating function. Sometimes, the fact that two structures are isomorphic is only known because they have the same generating function, an explicit construction of the bijection however is unknown.

**Definition 2.1.21.** *There are two special classes which often appear as building blocks for more complicated classes. The first one is the* empty class $\mathcal{E}$*, it consists only of one element of size zero (namely the empty set, hence the name even if the class itself is not empty). Its generating function is given by* $E(z) = 1$*. The second one is the* atomic class $\mathcal{Z}$*, which consists of one element of size one, i.e., an "atom". Its generating function is* $Z(z) = z$*. It can be viewed as the smallest building block which cannot be decomposed into smaller parts, hence the name.*

There are several constructions in which combinatorial structures can be glued together to create a new structure, some of the most important ones listed below. In the following, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are classes, $a, b, c$ their objects, $(a_n), (b_n), (c_n)$ their counting sequences and $A(z), B(z), C(z)$ their generating functions.

- Disjoint union: Let $\mathcal{A}$ and $\mathcal{B}$ be two disjoint classes. Their union (or sum) $\mathcal{C} = \mathcal{A} \dot{\cup} \mathcal{B}$ represents a new class with size defined as $|c|_{\mathcal{C}} = |c|_{\mathcal{A}}$ if $c \in \mathcal{A}$ and $|c|_{\mathcal{C}} = |c|_{\mathcal{B}}$ if $c \in \mathcal{B}$. This translates to $c_n = a_n + b_n$. Thus, the generating function of $\mathcal{C}$ is
$$C(z) = A(z) + B(z) = \sum_{n \geq 0} (a_n + b_n) z^n.$$
- Cartesian product: Let $\mathcal{A}$ and $\mathcal{B}$ be two classes. Then their Cartesian product is defined as
$$\mathcal{C} = \mathcal{A} \times \mathcal{B} = \{c = (a, b) : a \in \mathcal{A}, b \in \mathcal{B}\},$$
i.e., the set of ordered pairs of objects from $\mathcal{A}$ and $\mathcal{B}$. For an object $c = (a, b) \in \mathcal{C}$ the size is defined as
$$|c|_{\mathcal{C}} = |a|_{\mathcal{A}} + |b|_{\mathcal{B}}.$$
If we want to count all objects of size $n$ in $\mathcal{C}$, we have to consider all pairs of objects in $\mathcal{A}$ and $\mathcal{B}$ whose sizes sum up to $n$ in the manner of a Cauchy product. Hence
$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$
which translates to generating functions as
$$C(z) = A(z) \cdot B(z) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) z^n.$$
Note that the empty class is the neutral element with respect to Cartesian product, i.e.,
$$\mathcal{A} \times \mathcal{E} \cong \mathcal{E} \times \mathcal{A} \cong \mathcal{A}.$$
- Sequence: Using sum and product, we can define the *sequence class* $\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$ as
$$\mathrm{SEQ}(\mathcal{A}) := \mathcal{E} + \mathcal{A} + \mathcal{A} \times \mathcal{A} + \mathcal{A} \times \mathcal{A} \times \mathcal{A} + \dots .$$
This construction only makes sense if $\mathcal{A}$ contains no object of size zero. Otherwise the union would contain infinitely many objects of size zero, which contradicts the condition that the number of elements of a given size has to be finite for a combinatorial class. Using the knowledge about the generating functions for sum and product, we obtain for the generating functions of sequences
$$C(z) = 1 + A(z) + A(z)^2 + A(z)^3 + \dots = \frac{1}{1 - A(z)}.$$
- Multiset: If we "forget about the order" in a sequence of objects, we obtain the *multiset class* $\mathcal{C} = \mathrm{SET}(\mathcal{A})$. Again, we have to assume that the class $\mathcal{A}$ contains no object of size zero for this construction to make sense. For the generating function we have that
$$C(z) = \exp\left( A(z) + \frac{A(z^2)}{2} + \frac{A(z^3)}{3} + \dots \right).$$

The usefulness of these constructions are probably best seen in examples.

**Example 2.1.22.** Unrestricted paths with Dyck step set. Let $\mathcal{W}$ be the class of unrestricted lattice paths on $\mathbb{Z}^2$ with step set $\mathcal{S} = \{(1,1),(1,-1)\}$. We want to compute how many such paths of length $n$ there are. A very natural way to solve this problem is to use a step-by-step construction, as illustrated in figure 7.

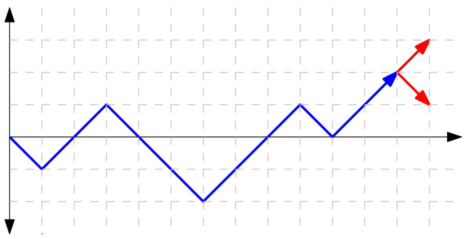


Figure 7: The step-by-step construction: The two steps marked in red are possible extensions of a lattice path of length $n$ (marked in blue) to a lattice path of length $n+1$.

A path of length $n+1$ is obtained by appending a step at the end of a path of length $n$. This is indeed always a path lying in $\mathcal{W}$, since there are no restrictions on the path. This gives us the following recurrence relation

$$w_{n+1} = 2w_n,$$

since there are two possible steps that can be appended in the end, an up-step or a down-step. Using $w_0 = 1$ (the empty path has length 0) we obtain $w_n = 2^n$.

Alternatively, we can solve this problem using generating functions. A walk in $\mathcal{W}$ is either the empty walk or a shorter walk with an up-step (1,1) or a down-step (1,-1) appended in the end. It is easy to check that these three classes of objects (the empty walk, walks ending with an up-step and walks ending with a down step) involved in the construction are disjoint. This gives us

$$\mathcal{W} = \mathcal{E} + \mathcal{W} \times \mathcal{Z}_{\mathrm{U}} + \mathcal{W} \times \mathcal{Z}_{\mathrm{D}}, \tag{3}$$

where $\mathcal{Z}_{\mathrm{U}}$ denotes the class that consists only of one up-step and $\mathcal{Z}_{\mathrm{D}}$ is defined analogously. Both an up-step and a down-step have size one and since we do not distinguish between up- and down-steps, we have $\mathcal{Z}_{\mathrm{U}} \sim \mathcal{Z}_{\mathrm{D}} \sim \mathcal{Z}$. Hence, when translate the step-by-step construction (3) to the world of generating functions we obtain:

$$W(z) = 1 + zW(z) + zW(z) = 1 + 2zW(z). \tag{4}$$

Solving this equation for $W(z)$ we obtain

$$W(z) = \frac{1}{1-2z} = \sum_{k\geq 0} 2^k z^k$$

from which we can easily read off the coefficients $w_n$ and obtain that the number of unrestricted paths with step set $\mathcal{S}$ of length $n$ is

$$w_n = [z^n]W(z) = 2^n.$$

At first glance, the generating function approach might look much more complicated than the direct approach via recurrences. However, the recurrence approach often fails where the generating function approach still turns out to be fruitful.

**Example 2.1.23.** Let $\mathcal{D}$ be the class of Dyck paths. In this case, a step-by-step construction does not work as easily as in the previous example, since if the path is on the $x$-axis, it is not allowed to take a down step. However, we can solve this problem by decomposing a Dyck path at its first return to the $x$-axis, as illustrated in Figure 8 (this is called a *first passage decomposition*, an argument often used in lattice path combinatorics). A Dyck path is either empty or consists of an up-step, a path from $(1,1)$ to $(2m-1,1)$ $(0 \leq m \leq n)$ which never goes below the line $y = 1$, a down step from $(2m-1,1)$ to $(2m,0)$ (the first return to the $x$-axis), and a path from $(2m,0)$ to $(2n,0)$ which never goes below the $x$-axis. By a simple shift argument, we see that both the path from $(1,1)$ to $(2m-1,1)$ as well as the path from $(2m,0)$ to $(2n,0)$ themselves are Dyck-paths.

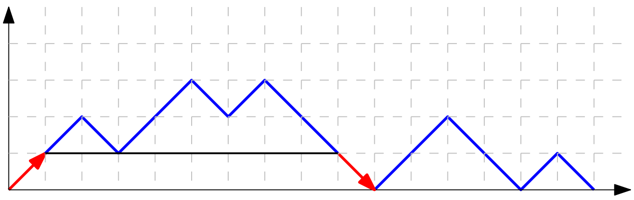


Figure 8: A Dyck path decomposed into two Dyck paths (blue) as well as an up- and a down-step (red) at its first return to the $x$-axis.

This gives us the following decomposition:

$$\mathcal{D} = \mathcal{E} + \mathcal{Z}_{\mathsf{U}} \times \mathcal{D} \times \mathcal{Z}_{\mathsf{D}} \times \mathcal{D}. \tag{5}$$

Since we do not distinguish between up- and down-steps and both these classes consist of one object of size one we have $\mathcal{Z}_{\mathsf{U}} \sim \mathcal{Z}_{\mathsf{D}} \sim \mathcal{Z}$. Translating equation (5) into generating functions we obtain:

$$D(z) = 1 + z^2 D(z)^2. \tag{6}$$

This equation can be solved with the quadratic formula and we obtain

$$D(z) = \frac{1 - \sqrt{1-4z^2}}{2z^2}.$$

The solution with $+\sqrt{1-4z^2}$ can be discarded, since it is not a power series.

From this generating function we already see that there are no Dyck paths of odd length. So, instead of counting Dyck paths of length $2n$ we can count them by their semilength, i.e., half their length. Using the binomial series and property 3 from Theorem 2.1.7 we obtain for the numbers $d_n$ which count Dyck paths of length $2n$ (resp. semilength $n$):

$$d_n := [z^{2n}]D(z) = [z^n]\frac{1-\sqrt{1-4z}}{2z} = -\frac{1}{2} \cdot \binom{\frac{1}{2}}{n+1}(-4)^{n+1}.$$

After some elementary manipulations of the binomial coefficients this becomes

$$d_n = \frac{1}{n+1}\binom{2n}{n} = C_n,$$

the $n$-th Catalan number.

*Remark:* We will later see how a functional equation for the generating function of Dyck paths can still be derived and solved with a step-by-step construction when introducing the kernel method in Chapter 4.

### 2.1.3 *Multivariate generating functions*

Often, combinatorialists are not only interested in counting certain objects of size $n$, but are also in keeping track of some additional parameters. To name some lattice path examples, one might ask how many walks with Dyck step set of length $n$ end at final altitude $k$. Or how many Dyck paths of semilength $n$ there are that return to altitude 0 exactly $k$ times? Such questions can be tackled by generalizing the concept of generating functions to multivariate generating functions.

In the case of (univariate) functions, the coefficients and powers of the variable $z$ kept track of how many objects of size $n$ there are. The idea is now to introduce a second (or third, fourth, ...) variable that keeps track of the parameters we are interested in.

More precisely, consider a sequence of numbers $(f_{n,k})$ depending on two integer-valued indices $n$ and $k$. Usually, $f_{n,k}$ is the number of objects $\phi$ in a combinatorial class $\mathcal{F}$ such that their size $|\phi|$ is equal to $n$ and some parameter $\chi(\phi)$ is equal to $k$. This sequence can be encoded with the help of a bivariate generating function, consisting of a primary variable $z$ corresponding to the size and a secondary variable $u$ corresponding to the value of the parameter. In the case of several parameters, one has several secondary variables.

Furthermore, multivariate generating functions help us answer questions about random combinatorial structures, like *"What does a large random object look like?"*. With the help of multivariate generating functions we gain easy access to mean and variance – or even higher moments – of parameters of combinatorial structures. This allows a precise characterization of large random structures. How this can be done will be the subject of the next subsection.

**Definition 2.1.24.** *The* bivariate generating function *of a double indexed sequence or array $(f_{n,k})$ is the formal power series in two variables defined by*

$$f(z,u) := \sum_{n,k} f_{n,k} z^n u^k.$$

*Furthermore we define the horizontal generating functions $f_n(u)$ to be*

$$f_n(u) := \sum_k f_{n,k} u^k = [z^n] f(z,u)$$

*and the vertical generating functions $f^{\langle k \rangle}(z)$ as*

$$f^{\langle k \rangle}(z) := \sum_n f_{n,k} z^n = [u^k] f(z,u).$$

*Remark:* The terminology of the vertical and horizontal generating function becomes more intuitive if we imagine the elements $(f_{n,k})$ arranged as an infinite matrix, where $f_{n,k}$ is placed in row $n$ and column $k$. Then the horizontal and the vertical generating functions appear as generating functions of the rows or columns respectively. Naturally one has

$$f(z,u) = \sum_k u^k f^{\langle k \rangle}(z) = \sum_n f_n(u) z^n.$$

**Definition 2.1.25.** *Let $\mathcal{A}$ be a combinatorial class. A* parameter *is a function $\chi : \mathcal{A} \to \mathbb{N}$ that associates to any object $a \in \mathcal{A}$ an integer value $\chi(a)$. The sequence*

$$a_{n.k} = |\{a \in \mathcal{A} : |a| = n, \chi(a) = k\}|$$

*is called the* counting sequence *of the pair $(\mathcal{A}, \chi)$. The* bivariate generating function *of $\mathcal{A}$ and $\chi$ is defined as*

$$A(z,u) := \sum_{n,k \geq 0} a_{n,k} z^n u^k.$$

*One says that the variable $z$* marks size *and the variable $u$* marks the parameter *$\chi$.*

*Remark:* A parameter can also be defined as a function $\chi : \mathcal{A} \to \mathbb{Z}$ which gives us a formal Laurent series (in $u$) as generating function. This is useful for parameters that can attain negative values like for example final altitude of an unrestricted walk.

Naturally we have that $A(z,1) = A(z)$ where $A(z)$ is the generating function associated with the structure $\mathcal{A}$. Similarly we have

$$a_n = [z^n] A(z,1) = \sum_k a_{n,k}.$$

A word of caution however: In general, we have to be careful when inserting special values (like $u = 1$ here) into formal power series. These series are not always convergent or well defined, so we have to ensure that all operations are legitimate.

Trivariate or multivariate generating functions are defined in a similar manner as bivariate generating functions (Definition 2.1.24). Combinatorial classes with several parameters are defined analogously to combinatorial classes with one parameter (Definition 2.1.25). The following example illustrates the usage of multivariate generating functions.

**Example 2.1.26.** Unrestricted walks with Dyck step set and final altitude $k$. For counting unrestricted walks with final altitude $k$ we will use a similar approach as in Example 2.1.22 where we counted the same objects without paying any attention to final altitude. This time, we will introduce a new variable that marks altitude in addition to the variable marking size. Using a step-by-step construction we see that a walk is either the empty walk or a shorter walk with an up- or down-step attached in the end. This gives us the following equation for the class $\mathcal{W}$ of walks:

$$\mathcal{W} = \mathcal{E} + \mathcal{W} \times \mathcal{Z}_{\mathsf{U}} + \mathcal{W} \times \mathcal{Z}_{\mathsf{D}}.$$

This time however we have to distinguish between up- and down-steps: while both contribute $z^1$ to size, an up-step contributes $u^1$ to altitude and a down-step contributes $u^{-1}$ to altitude. This gives us the following bivariate generating function for $W(z, u)$:

$$W(z, u) = 1 + W(z, u) \cdot zu + W(z, u) \cdot zu^{-1}.$$

Solving it for $W$ we obtain

$$W(z, u) = \frac{1}{1 - z(u + u^{-1})}. \tag{7}$$

In order to read off the coefficient $[z^n u^k]$ we first expand $W$ into a geometric series and then use the binomial theorem to expand the powers of $u$ and obtain

$$W(z, u) = \sum_{n \geq 0} z^n (u + u^{-1})^n = \sum_{n \geq 0} z^n \sum_{j=0}^{n} \binom{n}{j} u^{n-j} (u^{-1})^j = \sum_{n \geq 0} z^n \sum_{j=0}^{n} \binom{n}{j} u^{n-2j}.$$

Making the substitution $n - 2j =: k$ we obtain

$$W(z, u) = \sum_{n \geq 0} \sum_{k=-n}^{n} \binom{n}{\frac{n-k}{2}} z^n u^k.$$

Now we can easily read off the $[z^n u^k]$-th coefficient and obtain that the number of unrestricted walks of length $n$ and final altitude $k$ is

$$w_{n,k} = [z^n u^k] W(z, u) = \binom{n}{\frac{n-k}{2}}.$$

Plugging in $u = 1$ into 7 we obtain

$$W(z) = W(z, 1) = \frac{1}{1 - 2z}$$

as expected from the result we had earlier in Example 2.1.22.

### 2.1.4 *Probability generating functions*

As already mentioned in the previous subsection, combinatorialists are often interested in the mean, variance or distribution of a certain parameter. This leads us to the notion of probability generating functions.

**Definition 2.1.27.** *Let $X$ be a discrete random variable taking values in $\mathbb{N}$. Then the* probability generating function *of $X$ is defined as*

$$P(u) = \mathbb{E}(u^X) = \sum_{k \geq 0} p(k)u^k \tag{8}$$

*where*

$$p(k) := \mathbb{P}(X = k).$$

Since $\sum_{k=0}^{\infty} p(k) = 1$ we immediately obtain that

$$P(1) = 1$$

for any probability generating function $P$.

Let us recall several important definitions from probability theory:

**Definition 2.1.28.** *Let $X$ be a discrete random variable.*

- *The* expected value *or* mean *of $X$ is the probability-weighted average over all its possible values, i.e.*

$$\mathbb{E}[X] = \sum \mathbb{P}(X = x) \cdot x.$$

- *The* variance *of $X$ is the expected value of the squared deviation from the mean, i.e.*

$$\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

- *The $r$-th* moment *of $X$ is defined as*

$$m_r(X) = \mathbb{E}[X^r].$$

The mean, the variance, and higher moments of a random variable can be easily computed from its associated probability generating function as the following theorem shows.

**Theorem 2.1.29.** *Let $X$ be a discrete random variable taking values in $\mathbb{N}$ and $P$ its associated probability generating function. Then the mean, the variance, and the $r$-th moment of $X$ can be computed from the probability generating function in the following way:*

- $\mathbb{E}[X] = P'(1)$.
- $\mathbb{V}[X] = P''(1) + P'(1) - (P'(1))^2$.
- $m_r(X) = (uD_u)^r P(u)|_{u=1}$.

*Proof.* For $P(u) = \sum_{k\geq 0} p_k u^k$ we have that $P'(x) = \sum_{k\geq 0} k p_k u^k$ thus $P'(1) = \sum_{k\geq 0} k p_k$. On the other hand $\mathbb{E}X = \sum_{k\geq 0} k p_k$. The other two parts of the theorem follow similarly. □

*Probability generating functions from bivariate generating functions:* Given a combinatorial class $\mathcal{A}$ with parameter $\chi$, we have that

$$\mathbb{P}_{\mathcal{A}_n}(\chi = k) = \frac{a_{n,k}}{a_n} = \frac{a_{n,k}}{\sum_k a_{n,k}}. \tag{9}$$

Let $A(z,u)$ be the bivariate generating function of the combinatorial class $\mathcal{A}$ with parameter $\chi$. Then from (9) and the definition of probability generating functions (8) we obtain that the probability generating function of $\chi$ over $\mathcal{A}_n$ is given by

$$\sum_k \mathbb{P}_{\mathcal{A}_n}(\chi = k)u^k = \frac{[z^n]A(z,u)}{[z^n]A(z,1)}. \tag{10}$$

**Corollary 2.1.30.** Mean, variance, and moments from BGFs. *Let $A(z,u)$ be a bivariate generating function of a combinatorial structure $\mathcal{A}$ with parameter $\chi$. Then the expected value, variance, and moments of the parameter in an object of size n are given by*

$$\begin{aligned}
\mathbb{E}_{\mathcal{A}_n}[\chi] &= \frac{[z^n]\partial_u A(z,u)|_{u=1}}{[z^n]A(z,1)},\\
\mathbb{V}_{\mathcal{A}_n}[\chi] &= \frac{[z^n]\partial_u^2 A(z,u)|_{u=1}}{[z^n]A(z,1)} + \frac{[z^n]\partial_u A(z,u)|_{u=1}}{[z^n]A(z,1)} - \left(\frac{[z^n]\partial_u A(z,u)|_{u=1}}{[z^n]A(z,1)}\right)^2,\\
\mathbb{E}_{\mathcal{A}_n}[\chi^r] &= \frac{[z^n](u\partial_u)^r A(z,u)|_{u=1}}{[z^n]A(z,1)}.
\end{aligned}$$

*Proof.* Follows immediately from Equation (10) and Theorem 2.1.29. □

**Example 2.1.31.** Returns to the $x$-axis in Dyck paths. The number of returns to the $x$-axis in Dyck paths as well as several other parameters (like peaks or valleys, i.e. consecutive UD- or DU-steps) related to Dyck paths have been studied in [27]. Let $\mathcal{D}$ be the class of Dyck-paths of semilength $n$ with a parameter $\chi$ that counts how many times the Dyck path touches the $x$-axis. Furthermore, let $D(z,u)$ be its generating function where $z$ marks semilength and $u$ marks returns.

Using the decompositions "a Dyck path is a sequence of arches and at the end of each arch there is a return" and "an arch is a Dyck path only touching the $x$-axis at its very end, it can be decomposed into an up-step, a Dyck path, and a down-step" we obtain the following equation for the generating function

$$D(z,u) = \frac{1}{1 - zuD(z)}.$$

Note that since we are considering semilength, the pair of the up-step and the down-step at the beginning and end of an arch contributes only $z^1$ to semilength. We already know that

$$D(z) = D(z,1) = \frac{1-\sqrt{1-4z}}{2z}$$

from Example 2.1.23 (again, note that $z$ now encodes semilength where it previously encoded length, hence the occurrence of $z$ instead of $z^2$). Plugging in we obtain

$$D(z,u) = \frac{2}{2 - u + u\sqrt{1-4z}}.$$

In order to compute the expected value of returns we need

$$\partial_u D(z,u)|_{u=1} = \frac{1 - 3z + (z-1)\sqrt{1-4z}}{2z^2}.$$

Reading off coefficients we obtain

$$[z^n]\partial_u D(z,u)|_{u=1} = \frac{1}{n+2}\binom{2n+2}{n+1} - \frac{1}{n+1}\binom{2n}{n} = \frac{3n}{(n+1)(n+2)}\binom{2n}{n}.$$

Thus we have that

$$\mathbb{E}_{D_n}[\chi] = \frac{[z^n]\partial_u D(z,u)|_{u=1}}{[z^n]D(z,1)} = \frac{3n}{n+2}.$$

Some important probability distributions are:

- A random variable $X$ which takes value 1 with probability $p$ and value 0 with probability $1-p =: q$ is called *Bernoulli distributed*. We denote this by $X \sim \mathcal{B}(p)$. An experiment which is described by a Bernoulli distributed random variable is called a *Bernoulli trial*. From this fundamental distribution many other discrete probability distributions are derived.

- The *binomial distribution* describes the number of successes in a sequence of $n$ independent and identically distributed Bernoulli trials. Let $X$ be the random variable counting the number of successes in $n$ trials and the probability of success in each trial be denoted by $p$. Then its probability mass function (PMF) is given by

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $0 \le k \le n$. This is denoted by $X \sim \mathbb{B}(n, p)$.

- The geometric distribution describes the number of failures before the first success in a (potentially arbitrarily long) sequence of Bernoulli trials. Let $X$ be a random variable counting this number of failures. Then

$$\mathbb{P}(X = k) = (1-p)^k p$$

and we write $X \sim \text{Geom}(p)$.

- The *negative binomial distribution* describes the number of failures in a sequence of independent and identically distributed Bernoulli trials before a specified number of successes (called $r$) occurs. Let $X$ be the random variable counting the number of failures. Then

$$\mathbb{P}(X = k) = \binom{k+r-1}{k} \cdot (1-p)^k p^r.$$

We denote this by $X \sim NB(r, p)$. The geometric distribution is a special case of the negative binomial distribution where $r = 1$.

If the support of a distribution is not discrete we enter the realm of continuous probability distributions. Instead of being characterized by their probability mass function they are characterized by their cumulative distribution function (i.e. the function $f(x)$ which describes the probability that the random variable $X$ will take a value less than or equal to $x$). If their mass function exists, we call it probability density function (PDF).

- The *Gaussian* or *normal distribution* is the probably most famous and most often appearing probability distribution. It is given by the density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where the parameter $\mu$ is its expected value and $\sigma$ is its standard deviation. A Gaussian distributed normal variable $X$ is denoted by $X \sim \mathcal{N}(\mu, \sigma)$. One reason why the normal distribution appears in many places is the central limit theorem, which states that the normalized sum of independent and identically distributed random variables with finite variance tends to normal distribution (even if the original random variables were not normally distributed).

- The *half-normal distribution* is a probability distribution closely related to the normal distribution. Let $Y$ be a normal distributed variable with mean $\mu = 0$. Then $X := |Y|$ follows a half-normal distribution. We denote this by $X \sim \mathcal{H}(\sigma)$. The density function of a half-normal distribution is given by

$$f(x) = \frac{\sqrt{2}}{\sqrt{\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}.$$

- The *Rayleigh distribution* with parameter $\sigma$ is obtained by considering the Euclidean norm of two independent normally distributed variables $Y_1, Y_2$ with $Y_i \sim \mathcal{N}(0, \sigma^2)$. Then the random variable $X := \sqrt{Y_1^2 + Y_2^2}$ is Rayleigh distributed, denoted by $X \sim \mathcal{R}(\sigma)$. Its density function is given by

$$f(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}.$$

Tables 1 and 2 summarize the results:

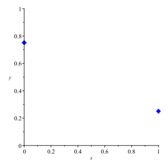
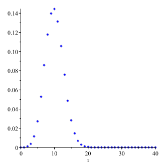
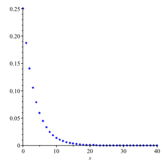
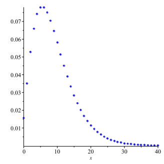
| Distribution | Bernoulli | binomial | geometric | negative binomial |
|---|---|---|---|---|
| Plot |  |  |  |  |
| Support | $k \in \{0, 1\}$ | $k \in \{0, 1, \dots, n\}$ | $k \in \mathbb{N}$ | $k \in \mathbb{N}$ |
| PMF | $\begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = 0 \end{cases}$ | $\binom{n}{k} p^k (1-p)^{n-k}$ | $(1-p)^k p$ | $\binom{k+r-1}{k} \cdot (1-p)^k p^r$ |
| Mean | $p$ | $np$ | $\frac{1-p}{p}$ | $\frac{r(1-p)}{p}$ |
| Variance | $p(1-p)$ | $np(1-p)$ | $\frac{1-p}{p^2}$ | $\frac{r(1-p)}{p^2}$ |

Table 1: An overview over some important discrete distributions: Bernoulli, binomial, geometric, and negative binomial. For the graphics, the values $p = \frac{1}{4}, n = 40$ and $r = 3$ were used.

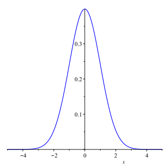
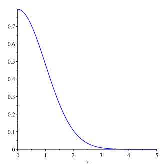
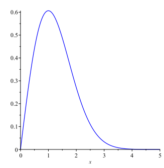
| Distribution | Gaussian | half-normal | Rayleigh |
|---|---|---|---|
| Plot |  |  |  |
| Support | $x \in \mathbb{R}$ | $x \in \mathbb{R}_{\geq 0}$ | $x \in \mathbb{R}_{\geq 0}$ |
| PDF | $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ | $f(x) = \frac{\sqrt{2}}{\sqrt{\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$ | $f(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$ |
| Mean | $\mu$ | $\sigma \frac{\sqrt{2}}{\sqrt{\pi}}$ | $\sigma \frac{\sqrt{\pi}}{\sqrt{2}}$ |
| Variance | $\sigma^2$ | $\sigma^2 \left(1 - \frac{2}{\pi}\right)$ | $\sigma^2 \left(2 - \frac{\pi}{2}\right)$ |

Table 2: An overview over some important continuous distributions: Gaussian, half-normal, and Rayleigh. For the graphics, the values $\mu = 0$ and $\sigma = 1$ were used.

## 2.2 COEFFICIENT ASYMPTOTICS

Often combinatorialists are not only interested in counting the exact numbers of certain combinatorial objects of size $n$, but also in giving an estimate on how fast these numbers grow as $n$ grows large. This leads us to coefficient asymptotics. This section is based on the book [37].

### 2.2.1 *Preliminaries: Asymptotic notation and complex analysis*

**Definition 2.2.1.** Asymptotic notation. *Let $S$ be a set on which the notion of neighborhoods exists and $x_0 \in S$. Let $f$ and $g$ be two functions $f, g : S \setminus \{x_0\} \to \mathbb{R}$ or $\mathbb{C}$.*

- *$\mathcal{O}$-notation: Define*

$$f(x) = \mathcal{O}(g(x)) \quad \text{for } x \to x_0$$

*if the quotient* $\frac{f(x)}{g(x)}$ *stays bounded as* $x \to x_0$*. This is equivalent to the existence of a neighborhood* $U$ *of* $x_0$ *and a constant* $c > 0$ *such that*

$$|f(x)| \leq c|g(x)|, \quad \textit{for all } x \in U \setminus \{x_0\}.$$

- *o-notation: Define*

$$f(x) = o(g(x)) \quad \textit{for } x \to x_0$$

*if the quotient* $\frac{f(x)}{g(x)}$ *tends to zero as* $x \to x_0$*. In other words, for all* $\varepsilon > 0$ *there exists a neighborhood* $U_\varepsilon$ *of* $x_0$ *such that*

$$|f(x)| \leq \varepsilon|g(x)|, \quad \textit{for all } x \in U_\varepsilon \setminus \{x_0\}.$$

- $\sim$*-notation: Define*

$$f(x) \sim (g(x)) \quad \textit{for } x \to x_0$$

*if* $\frac{f(x)}{g(x)} = 1$ *as* $x \to x_0$*. One says that* $f$ is asymptotically equivalent to $g$ *(for* $x$ *tends to* $x_0$*) if this holds. The relation* $\sim$ *is indeed an equivalence relation which can easily be verified.*

In most cases, we will be using these notations for $x_0 = \infty$.

**Example 2.2.2.** In Example 2.1.31 we computed that the average number of returns of a Dyck path of semilength $n$ to the $x$-axis is $\frac{3n}{n+2}$. For $n \to \infty$ this number asymptotically behaves like $\frac{3n}{n+2} \sim 3$.

If the coefficients of a formal power series are explicitly known and consist of expressions involving factorials and binomial coefficients (as the sizes of combinatorial objects often do) the following formula often turns out to be helpful for analyzing their growth.

**Theorem 2.2.3.** Stirling's formula. *For* $x \to \infty$ *one has*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$$

*where*

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt$$

*is the Gamma function, an extension of the factorial to non-integer arguments such that* $\Gamma(n+1) = n!$*. More precisely*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + \mathcal{O}\left(\frac{1}{x^4}\right)\right).$$

*Proof.* Five different proofs can be found in [37] (p. 407, p. 410, p. 555, p. 760, and p. 766). □

**Example 2.2.4.** In Example 2.1.23 we computed that the number of Dyck paths of semilength $n$ is given by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)(n!)^2}.$$

Applying Stirling's formula gives us after some cancellations that the Catalan numbers $C_n$ asymptotically behave like

$$C_n \sim \frac{4^n}{\sqrt{\pi n^3}}.$$

If the coefficients of a generating function are explicitly known, it is easier to analyze the asymptotic growth of its coefficients, as seen in the example about the asymptotic behavior of the Catalan numbers or the number of returns to the $x$-axis in Dyck paths. However, sometimes obtaining explicit expressions for coefficients is very hard or even impossible. In such cases however it is sometimes still possible to obtain statements about the asymptotic growth with the help of singularity analysis. The fact that the coefficients need not be explicitly known to obtain information about their asymptotics is one of the main advantages of singularity analysis.

When doing singularity analysis we are going to examine generating functions from the viewpoint of complex analysis. More specifically, we will be interested in their singularities, since they provide information about the growth of the coefficients. Analyzing singularities also appears in many other fields of mathematics. To name a famous example, Euler [32] recognized the fact that the Riemann zeta function $\zeta(s)$ becomes infinite (and thus has a singularity) at $s = 1$ implies the existence of infinitely many prime numbers.

First, let us recall some important theorems from complex analysis. In the following, we will assume that all paths are piecewise continuously differentiable and loops are oriented positively.

**Theorem 2.2.5** (Null Integral Property). *Let $f$ be analytic in $\Omega$ and let $\gamma$ be a simple loop in $\Omega$. Then*

$$\int_\gamma f(z)dz = 0.$$

For $f$ meromorphic, we have a similar result. Curve integrals along a positively oriented simple loop only depend on the poles and their residues enclosed by the loop. The next theorem makes this more precise.

**Theorem 2.2.6** (Cauchy's residue theorem). *Let $h(z)$ be a meromorphic function in the region $\Omega$. Furthermore, let $\gamma$ pe a positively oriented simple loop in $\Omega$ along which h is analytic. Then*

$$\frac{1}{2i\pi}\int_\gamma h(z)dz = \sum_s \mathrm{Res}_s(h(z))$$

*where the sum runs over all poles s of $h(z)$ in the region enclosed by $\gamma$.*

**Theorem 2.2.7** (Cauchy's integral formula). *If $f$ is analytic in $\Omega$, $z_0 \in \Omega$, and $\gamma$ is a simple loop in $\Omega$ encircling $z_0$ one has*

$$f(z_0) = \frac{1}{2i\pi}\int_\gamma f(\zeta)\frac{d\zeta}{\zeta - z_0}.$$

*Proof.* Follows directly since $\mathrm{Res}_{z_0}\left(\frac{f(\zeta)}{\zeta - z_0}\right) = f(z_0)$. ☐

By differentiation with respect to $z_0$ it follows that

$$\frac{1}{k!}f^{(k)}(z_0) = \frac{1}{2i\pi}\int_\gamma f(\zeta)\frac{d\zeta}{(\zeta - z_0)^{k+1}}. \tag{11}$$

An important application of the Cauchy's integral formula in combinatorics is the following

**Theorem 2.2.8.** *Let $f(z)$ be analytic in a region $\Omega$ with $0 \in \Omega$ and let $\gamma$ be a simple positive oriented loop around 0 in $\Omega$. Then the coefficient $f_n = [z^n]f(z)$ has the following integral representation*

$$f_n = \frac{1}{2i\pi}\int_\gamma f(z)\frac{dz}{z^{n+1}}.$$

*Proof.* Using $[z^n]f(z) = \frac{1}{n!}f^{(n)}(0)$ and Equation (11) with $z_0 = 0$ the theorem immediately follows. ☐

### 2.2.2 *Singularity analysis*

In order to define singularities, we first need the concept of analytic continuation.

**Definition 2.2.9.** *Let $f(z)$ be an analytic function defined over a region $\Omega$ determined by the interior of a simple closed curve $\gamma$. Let $z_0$ be a point on the curve $\gamma$. The function $f(z)$ is called* analytic continuable *at $z_0$ if there exists an analytic function $f^*(z)$ defined over some open set $\Omega^*$ containing $z_0$ such that $f^*(z) = f(z)$ in $\Omega \cap \Omega^*$. The function $f^*$ is then called an* analytic continuation *of $f$.*

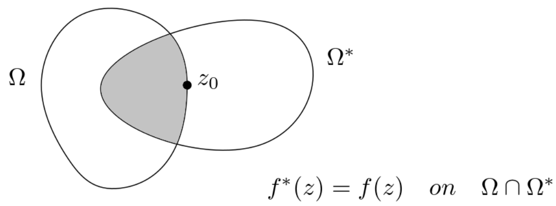



Figure 9: Analytic continuation.

Now we can define singularities as well as dominant singularities, which play an important rôle in coefficient asymptotics.

**Definition 2.2.10.** *Let $f$ be a function defined in a region $\Omega$ determined by the interior of a simple closed curve $\gamma$. A point $z_0$ on the boundary of the region is called a* singular point *or a* singularity *if $f$ is not analytically continuable at $z_0$. A point $z_0$ where $f$ is not singular is also called* regular. *A singularity is called* dominant singularity *of $f$ if there is no other singularity of $f$ with smaller modulus.*

The following two theorems provide information about the location of singularities of certain types of functions.

**Theorem 2.2.11.** *A function $f$ which is analytic at the origin and whose series expansion at the origin has finite radius of convergence $R$ always has a singularity on the boundary of its disk of convergence.*

*Proof.* The main idea of the proof is to construct a contradiction. We know that there can be no singularity of $f$ inside the disk $|z| < R$. Suppose there is also no singularity on the boundary of the disk $|z| = R$. Then it can be shown that the series expansion of $f$ would have a larger convergence radius $r > R$, a contradiction. The details can be found in [37], p. 240. □

**Theorem 2.2.12** (Pringsheim's Theorem)**.** *Let $f(z)$ be an analytic function representable by a series expansion around the origin that has only non-negative coefficients and radius of convergence $R$. Then the point $z = R$ is a singularity of $f$.*

*Proof.* See [37]. □

This theorem is of particular interest in analytic combinatorics, since the series arising as generating functions related to combinatorial problems have non-negative integer coefficients (since the coefficients encode sizes of certain combinatorial objects). Thus, in order to find a dominant singularity one only has to determine the radius of convergence of the generating function. A function can have more than one dominant singularity, each of them contributing to coefficient asymptotics.

Next we will give the asymptotics for some standard functions. As we will see later in Theorem 2.2.16, they also turn out helpful for determining asymptotics of other functions not fitting these standard forms.

**Theorem 2.2.13.** Standard function scale. *Let $\alpha$ be a number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Then the coefficient of $z^n$ in*

$$f(z) = (1 - z)^{-\alpha}$$

*admits for large $n$ the asymptotical expansion*

$$[z^n]f(z^n) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k}\right)$$

*where $e_k$ is a polynomial in $\alpha$ of degree $2k$. In particular, the first few terms are given by*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \frac{\alpha(\alpha-1)}{2n} + \mathcal{O}\left(\frac{1}{n^3}\right)\right).$$

*Proof.* (Sketch.) By the binomial theorem $[z^n](1-z)^{-\alpha} = (-1)^n\binom{-\alpha}{n} = \binom{n+\alpha-1}{n}$. This can be expanded with the help of Stirlings's formula 2.2.3, which gives us the result.

Another way to prove this is with the help of Cauchy's integral formula 2.2.7 along a suitable curve. The details of this can be found in [37], pp. 381-383. □

**Theorem 2.2.14.** Standard function scale for logarithms. *Let $\alpha$ be a number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Then the coefficient of $z^n$ in the function*

$$f(z) = (1-z)^{-\alpha} \left(\frac{1}{z} \log \frac{1}{1-z}\right)$$

*admits for large $n$ the asymtotic expansion*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^{\beta} \left(1 + \frac{c_1}{\log n} + \frac{c_2}{\log^2 n} + \dots\right)$$

*where*

$$c_k = \binom{\beta}{k} \Gamma(\alpha) \frac{d^k}{ds^k} \frac{1}{\Gamma(s)}\bigg|_{s=\alpha}.$$

*Proof.* (Sketch.) Again, Cauchy's integral formula 2.2.7 on a suitable contour can be employed, for details see [37]. □

Note that it is enough to consider only singularities at $z = 1$ since with the virtue of $[z^n]A(cz) = c^n[z^n]A(z)$ (see Theorem 2.1.7) we can rescale the function $A$ its singularity lies at $z = 1$.

These two theorems already give us asymptotics for a large class of functions. However, they turn out for deriving asymptotics for many other functions, too, because of the so-called Transfer Theorem. In essence, it says that if two functions behave similarly, their coefficients also behave similarly. Before we can state the theorem, we first need the definition of $\Delta$-analyticity for technical reasons.

**Definition 2.2.15.** *Let $R$ be a real number greater than one, and $\phi$ be an angle such that $0 < \phi < \frac{\pi}{2}$. An open* $\Delta$-domain *(at 1), denoted $\Delta(\phi, R)$ is then defined as*

$$\Delta(\phi, R) := \{z : |z| < r, z \neq 1, |\arg(z-1)| < \phi\}.$$

*For any complex number $\zeta \neq 0$ a* $\Delta$-domain at $\zeta$ *is the image of a $\Delta$-domain at 1 under the mapping $z \mapsto \zeta z$. A function is called* $\Delta$-analytic *if it is analytic in some $\Delta$-domain.*

**Theorem 2.2.16** (Transfer theorems)**.** *Let $\alpha$ and $\beta$ be arbitrary real numbers and let $f(z)$ be a function that is $\Delta$-analytic.*

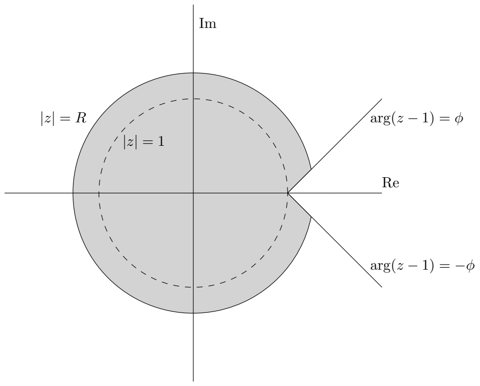



Figure 10: A $\Delta$-domain at 1.

1. *Assume that $f(z)$ satisfies*

$$f(z) = \mathcal{O}\left((1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}\right)$$

*in the intersection of a neighborhood of 1 with its $\Delta$-domain. Then*

$$[z^n]f(z) = \mathcal{O}\left(n^{\alpha-1}(\log n)^{\beta}\right).$$

2. *Assume that $f(z)$ satisfies*

$$f(z) = o\left((1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}\right)$$

*in the intersection of a neighborhood of 1 with its $\Delta$-domain. Then*

$$[z^n]f(z) = o\left(n^{\alpha-1}(\log n)^{\beta}\right).$$

*Proof.* (Sketch.) The main idea is to apply Cauchy's coefficient formula on a curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ which is internal to the $\Delta$-domain of $f$. More precisely, $\gamma$ consists of

- a small inner circle $\gamma_1 = \{z : |z-1| = \frac{1}{n}, |\arg(z-1)| \geq \theta\}$,
- a line segment $\gamma_2 = \{z : \frac{1}{n} \leq |z-1|, |z| \leq r, \arg(z-1) = \theta\}$,
- an outer circle $\gamma_3 = \{z : |z| = r, |\arg(z-1)| \geq \theta\}$
- and another line segment $\gamma_4 = \{z : \frac{1}{n} \leq |z-1|, |z| \leq r, \arg(z-1) = -\theta\}$,

Then one proceeds by finding bounds for the absolute value of each integral. The main contribution comes from the small circle and the line segments and is exactly of the form stated in the theorem, the contribution from the integral over the outer circle is exponentially small. The details can be found in [37], pp. 390–392. □

**Corollary 2.2.17** (sim-transfer)**.** *Assume that $f(z)$ is $\Delta$-analytic and*

$$f(z) \sim (1-z)^{-\alpha}$$

*as $z \to 1$ (for $z \in \Delta$) and $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. Then the coefficients of $f$ fulfill*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}.$$

*Proof.* Observe that $f(z) \sim g(z) \Leftrightarrow f(z) = g(z) + o(g(z))$. Using $g(1-z)^\alpha$ and applying Theorem 2.2.13 to the main term as well as Theorem 2.2.16 to the $o$-term completes the proof. $\square$

These three theorems and the corollary lie at the heart of coefficient asymptotics. They define the so-called *singularity analysis*. This process can be summarized in an almost algorithmic manner:

1. Locate singularities: Determine the dominant singularities of $f$ and check that $f(z)$ as a single singularity $\rho$ on its circle on convergence.

2. Check continuation: Find a $\Delta$-domain at $\rho$ such that $f$ is analytic in this domain.

3. Express with standard functions: Find an expression of the form
$$f(z) = g(z/\rho) + \mathcal{O}(h(z/\rho))$$
for $z \to \rho$ with $h(z) = o(g(z))$ such that $g$ and $h$ belong to the standard scale of functions from Theorems 2.2.13 and 2.2.14.

4. Transfer: Transfer the main term $g$ using Theorem 2.2.13 and the error term $h$ using Theorem 2.2.14. Use the Transfer Theorem 2.2.16 to conclude
$$[z^n]f(z) = \rho^{-n}[z^n]g(z) + \mathcal{O}(\rho^{-n}[z^n]h(z))$$
for $n \to \infty$.

**Example 2.2.18.** Let us have again a look at Dyck paths of semilength $n$, counted by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. Their generating function is $D(z) = \frac{1-\sqrt{1-4z}}{2z}$, as computed in Example 2.1.23. We already computed their asymptotic growth with the help of Stirling's formula in Example 2.2.4. This time we will do it with the help of singularity analysis.

1. First, we have to find the dominant singularities of the generating function. It can easily be seen that $D(z)$ has its only singularity at $z = \frac{1}{4}$.

2. Since $D(z)$ is analytic in $\mathbb{C} \setminus \{\frac{1}{4}\}$ it is in particular analytic in any $\Delta$-domain around $z = \frac{1}{4}$.

3. Making the substitution $x := 4z$ gives us
$$[z^n]\frac{1-\sqrt{1-4z}}{2z} = 4^n \cdot 2[x^{n+1}] - \sqrt{1-x}.$$
This expresses the function we are interested in in terms of standard functions.

4. Applying a sim-transfer (see Corollary 2.2.17) and Theorem 2.2.13 with $\alpha = -\frac{1}{2}$ we obtain
$$[z^n]\frac{1-\sqrt{1-4z}}{2z} \sim -4^n \cdot 2 \cdot \frac{n^{-3/2}}{\Gamma(-\frac{1}{2})} = \frac{4^n}{\sqrt{\pi n^3}},$$
as expected.

# 3 PARAMETERS RELATED TO NONINTERSECTING LATTICE PATHS (WATERMELONS)

## 3.1 INTRODUCTION AND DEFINITIONS

In this chapter we are going to study a certain class of *vicious walkers*, i.e., pairs or tuples of lattice paths where no two paths occupy the same lattice site. This model has been introduced by Fisher [35] to study wetting and melting processes. These objects have since been of much interest because they can also be used to model polymer networks [30] as well as DNA denaturation [58, 66]. They also are of purely mathematical interest because they stand in bijection with other combinatorial objects like Young tableaux [43], certain walks in the quarter plane [23] or random matrices [7].

Friendly walkers form a similar notion, where the paths are allowed to touch but not to cross each other. Watermelons are a special case of vicious (respectively friendly) walkers where the underlying step set is the Dyck step set (i.e. (1,1) and (1,-1)) and there are certain conditions on the start- and endpoints of the paths.

**Definition 3.1.1.** *A $p$-*watermelon *of length $n$ is a family of $p$ nonintersecting paths $P_1, \dots, P_p$ in $\mathbb{Z}^2$ with Dyck step set such that*

1. *$P_i$ starts at $(0, 2i-2)$ and ends at $(n, h+2i-2)$ where $n \equiv h$ mod $2$.*
2. *no two paths occupy the same lattice point.*

The name *watermelon* might seem odd at first glance, but becomes quite descriptive if we color the areas between the paths alternating in light and dark green. The resulting image almost looks like the skin of a watermelon, hence the name.

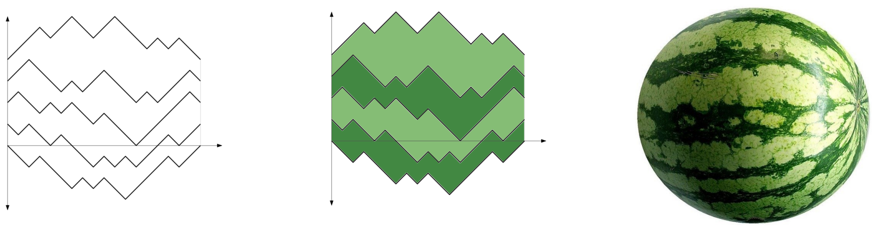


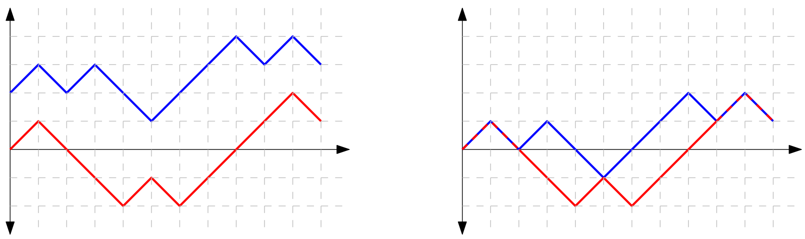
Figure 11: A 5-watermelon with deviation zero ... and the reason why these objects are called watermelons. (Image to the right by user Lebensmittelfotos on Pixabay)

**Definition 3.1.2.** *The y-coordinate of the endpoint $(n, h)$ is called the deviation of the watermelon.*

It is also possible to use the following definition:

**Definition 3.1.3.** *A $p$-*watermelon *of length $n$ is a family of paths with Dyck step set (i.e., $\mathcal{S} = \{(1,1), (1,-1)\}$) such that all paths start at $(0,0)$ and end at $(n,h)$, where these paths may touch but not cross each other. In other words, if $(m, y_i)$ denotes the coordinates of the $i$-th path after $m$ steps, we have that $y_1 \leq y_2 \leq \cdots \leq y_p$ for all $m$.*

By a simple shift argument (moving the $i$-th path $2i-2$ units up/down) we can see that Definition 3.1.3 and 3.1.1 are equivalent.



Note that ”no two paths occupy the same lattice site” is not the same as ”no two paths intersect” for any underlying step set. It is the same for Dyck paths (if all start points are congruent modulo two), but not for Motzkin paths, as can be seen in the following example:

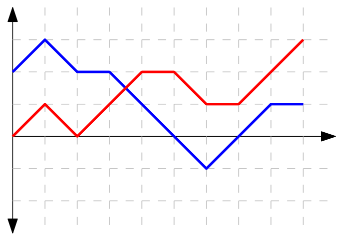


Figure 12: Two walks with Motzkin-step-set. Here ”no two paths occupy the same lattice site” is not the same as ”no two paths intersect”. The above paths never occupy the same lattice site, they do intersect, however.

If we drop the condition on the endpoints of the paths, we obtain the following

**Definition 3.1.4.** *A p-star of length n is a family of p nonintersecting lattice paths $P_1, \dots, P_p$ in $\mathbb{Z}^2$ with Dyck step set such that*

1. *$P_i$ starts at $(0, 2i-2)$ and ends at $(2n, e_i)$ where $e_1 < \cdots < e_p$ (and $e_i \equiv n \bmod 2$).*
2. *no two paths occupy the same lattice point.*

This is equivalent to

**Definition 3.1.5.** *A p-star of length n is a family of p lattice paths $P_1, \dots, P_p$ in $\mathbb{Z}^2$ with Dyck step set such that*

1. *$P_i$ starts at $(0, 0)$ and ends at $(2n, d_i)$ where $d_1 < \cdots < d_p$ (and $d_i \equiv n \bmod 2$).*
2. *no two paths cross each other.*

Often, there are also constraints imposed on the region the paths of the watermelon are allowed to be in.

**Definition 3.1.6.** *Region constraints for stars and watermelons.*

- *A* wall condition *means that none of the paths is allowed to go below the x-axis. It is sufficient to impose this condition only on the lowest path, because the non-touching (or non-crossing) condition immediately implies that all other paths do not go below the x-axis.*
- *Vicious (or non-crossing) walkers confined to a* strip *of* width *h are tuples of non-touching (or non-crossing) lattice paths such that they never go below the x-axis and never go above the horizontal line $y = h$. Again, it is enough to impose the condition of not going below on the lowest path only and the condition of not going above the line $y = h$ on the highest path only.*

*Known results*

Several parameters related to watermelons and stars have already been studied. We will give a short overview of some known results.

Let us start with a lemma that plays an important role in counting pairs of nonintersecting paths. It was first formulated by Lindström [55] in 1973 and later proven by Gessel and Viennot [39] in 1989.

**Lemma 3.1.7.** Lindström–Gessel–Viennot Lemma. *Let $G$ be a locally finite, directed acyclic graph. We assign a weight $w_e$ to each edge $e$, where all weights are elements of some commutative ring. Let $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_p\}$ be the set of start and destination vertices respectively. For each directed path $P$ let $w(P)$ be the product of the weights of the edges contained in this path. For any two vertices $a$ and $b$ let $e(a,b) = \sum_{P:a\to b} w(P)$ the sum of the weights of all paths from $a$ to $b$ (this sum is well defined since for any two vertices there are only finitely many paths between them). If each edge has weight 1, then $e(a,b)$ counts the number of paths between $a$ and $b$.*
*Define*

$$M := (e(a_i, b_j))_{1\le i,j\le n}.$$

*A $p$-tuple of nonintersecting paths from $A$ to $B$ is an $n$-tuple $(P_1, \ldots, P_p)$ of paths in $G$ such that*

- *There exists a permutation $\sigma$ of $\{1, 2, \ldots, p\}$ such that $P_i$ is a path from $a_i$ to $b_{\sigma(i)}$ for all $i$.*
- *For $i \neq j$ the paths $P_i$ and $P_j$ have no vertex in common.*

*Given a $p$-tuple of paths we denote by $\sigma(P)$ the permutation from the first condition. Then we have that*

$$\det(M) = \sum_{(P_1,\ldots,P_n):A\to B} sign(\sigma(P)) \prod_{i=1}^{n} w(P_i).$$

*Remark:* If the only possible permutation is the identity (every tuple of nonintersecting paths takes $a_i$ to $b_i$) and all weights are 1, then $\det(M)$ is exactly the number of non-intersecting $n$-tuples of paths from $A$ to $B$. In our setting of $p$-watermelons this is the case.

*Proof.* First, we introduce some notation. An $p$-path from a $p$-tuple $(a_1, \ldots, a_p)$ of vertices to another $p$-tuple $(b_1, \ldots, b_p)$ of vertices is a $p$-tuple of paths $(P_1, \ldots, P_p)$ in the underlying graph $G$, where each $P_i$ begins in $a_i$ and ends in $b_i$. An $p$-path will be called nonintersecting if for $i \neq j$ the paths $P_i$ and $P_j$ have no two vertices (not even endpoints) in common. The weight $w(P)$ of an $p$-path $P = (P_1, \ldots, P_p)$ is defined as the product of the weights of the paths it consists of, i.e. $w(P) := w(P_1) \ldots w(P_p)$.

A twisted $p$-path from $(a_1, \ldots, a_p)$ to $(b_1, \ldots, b_p)$ is an $p$-path from $(a_1, \ldots, a_p)$ to $(b_{\sigma(1)}, \ldots, b_{\sigma(p)})$ where $\sigma \in S_p$ is some permutation of $p$ elements. This permutation will be called the twist of the twisted $p$-path, denoted by $\sigma(P)$ (where $P$ is the original $p$-path).

Expanding the determinant of $M$ as a signed sum of permutations, we get

$$\begin{aligned}
\det(M) &= \sum_{\sigma\in S_p} \operatorname{sgn}(\sigma) \cdot e(a_1, b_{\sigma(1)}) \ldots e(a_p, b_{\sigma(p)}) \\
&= \sum_{\sigma\in S_p} \cdot \sum_{\substack{P \text{ is } p-\text{path} \\ \text{from } (a_1,\ldots,a_p) \\ \text{to } (b_{\sigma(1)},\ldots,b_{\sigma(p)})}} w(P).
\end{aligned}$$

We can simplify the right hand side using the definition of a twisted $p$-path to

$$\sum_{\substack{P \text{ is twisted } p-\text{path} \\ \text{from } (a_1,\ldots,a_p) \\ \text{to } (b_1,\ldots,b_p)}} \operatorname{sgn}(\sigma(P)) w(P).$$

It remains to prove that this is equal to

$$\sum_{(P_1,\ldots,P_p):A\to B} \operatorname{sgn}(\sigma(P)) \prod_{i=1}^{p} w(P_i) = \sum \operatorname{sgn}(\sigma(P))w(P),$$

where the latter sum runs over all nonintersecting twisted $p$-paths from $(a_1, \ldots, a_p)$ to $(b_1, \ldots, b_p)$. Phrased differently, we have to prove that the sum of the expressions $\operatorname{sgn}(\sigma(P))w(P)$ over all twisted $p$-paths equals the same sum but only over nonintersecting $p$-paths. In order to do this, we will prove that the sum of $\operatorname{sgn}(\sigma(P))w(P)$ over all twisted $p$-paths that are not nonintersecting vanishes.

The idea is to find an involution of the set of all not nonintersecting twisted $p$-paths from $(a_1, \ldots, a_p)$ to $(b_1, \ldots, b_p)$. This involution will flip the sign $\operatorname{sgn}(\sigma(P))$ but leaves the weight $w(P)$ invariant. Thus the sum over all intersecting twisted $p$-paths has to be 0 because the involution splits it into pairs of mutually canceling summands.

It remains to construct this involution, which will be called $f$. The idea is to take two intersecting paths $P_i$ and $P_j$ and switch their tails after the point of intersection. Because there are (in general) several pairs of intersecting paths and two paths can intersect several times, a choice needs to be made. Let $i$ be the smallest index such that the path $P_i$ (the path starting in $a_i$) contains an intersection. Let $q$ be the first point along $P_i$ where $P_i$ intersects another path and let $j$ be the largest index such that $q$ lies on $P_j$. Then we define $f(P)$ to be the same set of paths as $P$ but with the tails (the segments from $q$ to the endpoints) of the two paths $P_i$ and $P_j$ swapped. Clearly, $f(P)$ is a twisted $p$-path and its twist $\sigma(f(P))$ differs from $\sigma(P)$ by a transposition of $\sigma(i)$ and $\sigma(j)$. Thus $\operatorname{sgn}(\sigma(f(P))) = -\operatorname{sgn}(\sigma(P))$.

It is easy to see that $f$ is indeed an involution. In $f(P)$, the smallest index of an intersecting path will again be $i$, the first point of intersection along it will again be $q$ and the largest index of a path containing $q$ will again be $j$. The existence of such an involution with the desired properties completes the proof. □

Both exact and asymptotic results about the number of watermelons and stars with or without various constraints have been given by Guttmann, Krattenthaler, Owczarek and Viennot in the series of articles [43, 50, 51].

**Theorem 3.1.8.** Stars, arbitrary endpoints, no wall. *The number of stars with p branches of length n with no wall is equal to*

$$\prod_{1\le i\le j\le n} \frac{p+i+j-1}{i+j-1} \tag{12}$$

*Proof.* The proof can be found in [43]. □

**Theorem 3.1.9.** Stars, fixed endpoints, no wall. *The number of $p$-stars with starting points $A_i = (0, 2i-2)$ and end points $E_i = (n, e_i)$ (where $n$ and all $e_i$ have the same parity) where no wall condition is imposed is given by*

$$2^{-\binom{p}{2}} \prod_{i=1}^{p} \frac{(n-i-p)!}{\left(\frac{n+e_i}{2}\right)!\left(\frac{n-e_i}{2}+p-1\right)!} \prod_{1\le i<j\le p} (e_j - e_i). \tag{13}$$

*Proof.* The proof can be found in [50]. □

**Corollary 3.1.10.** *The number of watermelons of length n (with deviation k) and no wall is given by*

$$\prod i = 1^p \frac{(n-i-p)!}{(\frac{n+2i-2+k}{2})!(\frac{n-2i+2-k}{2}+p-1)!} \prod_{1\le i<j\le p} (j-i) \tag{14}$$

*Proof.* Follows directly from Equation (13) with $e_i = 2i - 2 + k$ and some cancellations. □

**Theorem 3.1.11.** Stars, arbitrary endpoints, no wall, asymptotics. *The number of stars with p branches of length n is asymptotically*

$$\begin{cases} 2^{np+p^2/4} n^{-p^2/4+p/4} \pi^{-p/4} \left(\prod_{\ell=1}^{p/2} (2\ell-2)!\right) (1+\mathcal{O}(n^{-1}) & \text{for } p \text{ even} \\ 2np+p^2/4-1/4 n^{-p^2/4+p/4} \pi^{-p/4+1/4} \left(\prod_{ell=1}^{(p-1)/2} (2\ell-1)!\right) (1+\mathcal{O}(n^{-1}) & \text{for } p \text{ odd} \end{cases} \tag{15}$$

*as n tends to infinity.*

*Proof.* The proof can be found in [50]. □

**Theorem 3.1.12.** Stars, arbitrary endpoints, wall. *The number of stars of length n with p branches which do not go below the x-axis and whose endpoints hate y-coordinate at least s (with $s \equiv m$ mod 2) is given by*

$$\prod_{i=1}^{p} \prod_{j=1} (m+s)/2 \prod_{k=1}^{(m-s)/2} \frac{i+j+k-1}{i+j+k-2}. \tag{16}$$

*Proof.* The proof can be found in [50]. □

**Theorem 3.1.13.** Stars, fixed endpoints, wall. *Let $0 \le e_1 < e_2 < \cdots < e_p$ with $e_i \equiv n$ mod 2, $i = 1, \ldots, p$. The number of stars with p branches and wall, where the i-th branch goes from $A_i = (0, 2i-2)$ to $E_i = (n, e_i)$, is given by*

$$2^{-p^2+p} \prod_{i=1}^{p} \frac{(e_i+1)(n+2i-2)!}{(\frac{n+e_i}{2}+p)!(\frac{n-e_i}{2}+p-1)!} \prod_{1\le i<j\le p} (e_j-e_i)(e_j+e_i+2). \tag{17}$$

*Proof.* The proof can be found in [50]. □

**Corollary 3.1.14.** *The number of watermelons of length n with p branches and wall and deviation $k \ge 0$ is given by*

$$\prod_{i=0}^{\frac{p}{2}-1} \frac{(k+2p-1-2i)!}{(k+2i)!} \prod_{j=0}^{p-1} \frac{(n+2j)!j!}{(\frac{n-k}{2}+j)!(\frac{n+k}{2}+j+p)!}.$$

*Alternatively, this expression can be written as*

$$\prod_{\ell=1}^{p} \frac{(\ell-1)!(k+2l-1)_{p-\ell+1}(n+2\ell-2)!}{(\frac{n-k}{2}+j)!(\frac{n+k}{2}+j+p)!}.$$

**Theorem 3.1.15.** Stars, arbitrary endpoints, wall, asymptotics. *The number of stars with p branches of length n which do not go below the x-axis and whose endpoints have y-coordinates at least s (with $s \equiv n$ mod 2) is asymptotically*

$$2^{np+p^2-p/2} n^{-p^2/2} \pi^{-p/2} \left(\prod_{ell=1}^{p} (\ell-1)!\right) (1+\mathcal{O}(1/n))$$

*as $n \to \infty$.*

*Proof.* The proof can be found in [50]. □

**Theorem 3.1.16.** Stars, fixed endpoints, strip. *Let $0 \le a_1 < a_2 < \ldots a_p \le h$ such that all $a_i$ have the same parity. Furthermore, let $0 \le e_1 < e_2 < \cdots < e_p \le h$ such that all $e_i$ are of the same parity and $a_i + e_i \equiv n$ mod 2 for $i = 1, 2, \ldots, p$. The number of vicious walkers with p branches of length n, with the i-th branch running from $A_i = (0, a_i)$ to $E_i = (n, e_i)$, which go neither below the x-axis nor above the line $y = h$ is given by*

$$\det_{1\le s,t\le p} \left( \sum_{k=-\infty}^{\infty} \left( \binom{n}{\frac{n+e_t-a_s}{2}+k(h+2)} - \binom{n}{\frac{n+e_t+a_s}{2}+k(h+2)+1} \right) \right) \tag{18}$$

*Proof.* The proof can be found in [51]. The main idea is using formulas for lattice paths with given start- and endpoints in a strip (as given in [59]) and then apply the Lindström–Gessel–Viennot lemma 3.1.7. □

For certain start- or endpoints the above determinant formula can be simplified a bit. There is also a way to rewrite this formula involving trigonometric functions instead of binomial coefficients which is useful for deriving asymptotics. The details on this can also be seen in [51].

**Theorem 3.1.17.** Stars, fixed endpoints, strip, asymptotics. *Let $0 \le a_1 < a_2 < \dots a_p \le h$ such that all $a_i$ have the same parity. Furthermore, let $0 \le e_1 < e_2 < \dots < e_p \le h$ such that all $e_i$ are of the same parity and $a_i + e_i \equiv n \bmod 2$ for $i = 1, 2, \dots, p$. Then the number of vicious walkers with $p$ branches of length $n$ with the $i$-th branch running from $A_i = (0, a_i)$ to $E_i = (n, e_i)$, which go neither below the $x$-axis nor above the line $y = h$ is asymptotically*

$$\frac{4^{p^2}}{(h+2)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2}\right)^n \prod_{1\le s<t\le p}\sin\frac{\pi(a_t-a_s)}{2(h+2)}\cdot\sin\frac{\pi(e_t-e_s)}{2(h+2)}$$
$$\times\prod_{1\le s<t\le p}\sin\frac{\pi(a_t+a_s+2)}{2(h+2)}\cdot\sin\frac{\pi(e_t+e_s+2)}{2(h+2)}$$

*Proof.* The proof can be found in [51]. □

Some parameters related to watermelons have already been studied as well, for example the height as well as the range of watermelons.

**Definition 3.1.18.** *The height of a watermelon is the $y$-coordinate of the highest lattice point along any of the paths of the watermelon (due to the non-touching constraint it will always be along the uppermost path).*
*A similar parameter is the range of a watermelon: The range of a watermelon is the difference of the maximum of its uppermost branch and the minimum of its lowest branch. In the case of watermelons with wall height and range always coincided (the lowest branch is not allowed to go below altitude zero, but it has altitude zero at the beginning).*

In his paper [38] Fulmek proved the following

**Theorem 3.1.19.** *The average height $H(n,p)$ of a $p$-watermelon with deviation zero and wall of length $2n$ is given by*

$$H(n,p) = \frac{1}{C(n,p)}\sum_{h=1}^{n+2p-2} C(n,p) - C(n,p,h-1),$$

*where $C(n,p)$ is the number of all $p$-watermelons with deviation zero and wall of length $2n$ which can be computed as a special case of Corollary 3.1.14 and equals*

$$C(n,p) = \prod_{j=0}^{p-1}\frac{\binom{2n+2j}{n}}{\binom{n+2j+1}{n}},$$

*and $C(n,p,h)$ is the number of all $p$-watermelons of length $2n$ which do not exceed height $h$ and given by the following determinant*

$$C(n,p,h) = \det\left(\sum_{k\in\mathbb{Z}}\left(\binom{2n}{n-i+j-k(h+2)} - \binom{2n}{n+i+j-k(h+2)+1}\right)\right)_{0\le i,j\le p-1}.$$

*Proof.* The proof can be found in [38]. The Lindström–Gessel–Viennot formula again plays a central role in this proof. □

Fulmek also analyzed the asymptotic behavior for the average height of 1-watermelons with wall (i.e. Dyck paths) and 2-watermelons with wall in [38]. These results have later been generalized by Feierl in [33] by computing asymptotics for $p$-watermelons with $p \ge 2$ as well as computing higher moments. In [34] the same was done for the height and range of $p$-watermelons without wall.

## 3.2 CONTACTS AND RETURNS IN 2-WATERMELONS WITHOUT WALL

In this section we are going to study the number of contacts in a 2-watermelon, i.e., the number of times when the two paths meet at a lattice point. We will distinguish between two types of contacts whether the paths were apart before and meet again (returns) or whether they take a common step and also analyze these numbers. We will also derive the complete probability distribution for all these parameters.

This chapter is based on the article [67].

In the following we will be using the friendly walkers model, since in this model the notion of contacts is more visible and intuitive.

**Definition 3.2.1.** *A contact in a 2-watermelon is a point (not counting the starting point) where both paths occupy the same lattice point, i.e. all points $(m, y)$ such that $(m, y)$ lies both on the lower path $P_1$ and the upper path $P_2$.*

Note that for more than two paths there are several possible ways to define contacts – either as points lying on all of the paths or as a point lying on two (or more) of the paths, but not necessarily on all of them. The first version is much more restrictive than the second one, each point that is a contact in the first sense is also a contact in the second sense. One could also count weighted contacts, i.e. if two paths meet it is counted as a simple contact with weight $c$, if three paths meet it is counted as double contact with weight $d$ and so on.

*Average number of contacts*

**Theorem 3.2.2.** *Let $X_n$ be the random variable counting the number of contacts in a 2-watermelon without wall, where the watermelon is chosen uniformly at random among all possible 2-watermelons of length $n$ and arbitrary deviation. Then*

$$\mathbb{E}X_n = \frac{(7n+13)n}{(n+4)(n+3)} = 7 - \frac{36}{n} + \frac{168}{n^2} + O\left(\frac{1}{n^3}\right).$$

Before giving the proof of Theorem 3.2.2, let us observe the following bijection, which turns out to be helpful for this proof, but also for other parameters related to 2-watermelons without wall and arbitrary deviation, since this bijection preserves many parameters.

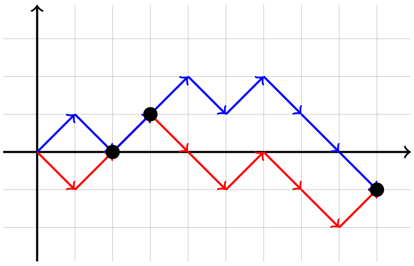
**Lemma 3.2.3.** *2-watermelons without wall and arbitrary deviation stand in bijection with weighted Motzkin paths (excursions) where there are two different kinds of level steps.*

*Proof.* We can construct a bijection between 2-watermelons with arbitrary deviation and weighted Motzkin paths that start and end on the $x$-axis, but never cross the $x$-axis in the following way:

| | | | | |
|---|---|---|---|---|
| step of the upper path | $\nearrow$ | $\nearrow$ | $\searrow$ | $\searrow$ |
| step of the lower path | $\searrow$ | $\nearrow$ | $\searrow$ | $\nearrow$ |
| step of the Motzkin path | $\nearrow$ | $\xrightarrow{u}$ | $\xrightarrow{d}$ | $\searrow$ |

These are weighted Motzkin paths with two different kinds of level steps. The height of the Motzkin path corresponds to (half of) the distance of the paths in the watermelon. Because the paths are not allowed to cross, this distance may not be negative, i.e., the Motzkin path may never cross the $x$-axis. The condition that both paths of the watermelon end on the same altitude corresponds to the condition that the Motzkin path has to end on the $x$-axis. □

A contact between the two paths occurs each time the Motzkin path touches the $x$-axis. Hence we are interested in counting the number of returns of the Motzkin path to the $x$-axis.



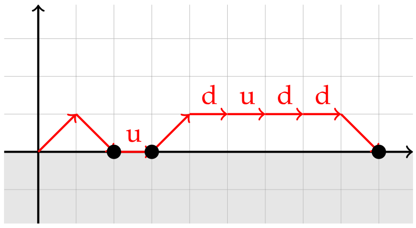



Figure 13: The bijection between 2-watermelons and weighted Motzkin paths

Now, let $F$ denote the generating function. A Motzkin path can be constructed as a sequence of the following objects: a level step with weight $u$, a level step with weight $d$, an up-step and a down step with a Motzkin path in-between.

Figure 13 illustrates the bijection between 2-watermelons with arbitrary deviation and Motzkin paths with wall. Contacts are marked with black dots.

Using this bijection and the decomposition "a Motzkin path is a sequence of arches and level steps" we obtain a functional equation:

$$F(z) = \frac{1}{1 - z^2F(z) - 2z}.$$

Multiplying with the denominator and solving the quadratic equation gives us

$$F(z) = \frac{1 - 2z - \sqrt{1-4z}}{2z^2} \tag{19}$$

Technically we get two solutions, but the solution with $+\sqrt{1-4z}$ does not make sense since it is not a power series.

*Proof of Theorem 3.2.2:* Using the bijection from Lemma 3.2.3 and introducing a new variable $u$, which counts the number of contacts of the Motzkin-path with the $x$-axis, we get

$$F(z,u) = \frac{1}{1 - u(z^2F(z,1) + 2z)}.$$

Using $F(z,1) = F(z)$ and (19) we obtain

$$F(z,u) = \frac{2}{2 - u - 2uz + u\sqrt{1-4z}} \tag{20}$$

Differentiating with respect to $u$ and plugging in $u = 1$ we get

$$\partial_u F(z,u)|_{u=1} = \frac{2(1 + 2z - \sqrt{1-4z}}{(1 - 2z - \sqrt{1-4z})^2}. \tag{21}$$

By rationalizing this fraction we can rewrite this as

$$\partial_u F(z,u)|_{u=1} = \frac{2 - 8z + 2z^2 + 4z^3 + (2z^2 + 4z - 2)\sqrt{1-4z}}{4z^4}.$$

The average number of contacts of a watermelon of length $n$ (with deviation, without wall) is given by

$$\mathbb{E}X_n = \frac{[z^n]\partial_u F(z,u)|_{u=1}}{[z^n]F(z,1)}.$$

By expanding $\sqrt{1-4z}$ with the binomial series, we can read off coefficients from (19) and obtain

$$[z^n]F(z,1) = C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}, \tag{22}$$

where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the $n$-th Catalan number.

To obtain $[z^n]\partial_u F(z,u)|_{u=1}$ we use that $[z^n]\sqrt{1-4z} = -2C_{n-1}$. We have

$$\begin{aligned}[z^n]\partial_u F(z,u)|_{u=1} &= [z^n]\frac{2-8z+2z^2+4z^3}{4z^4} + [z^n]\frac{(2z^2+4z-2)\sqrt{1-4z}}{4z^4} \\ &= \frac{1}{2}[z^{n+2}]\sqrt{1-4z} + [z^{n+3}]\sqrt{1-4z} - [z^{n+4}]\frac{1}{2}\sqrt{1-4z} \\ &= -C_{n+1} - 2C_{n+2} + C_{n+3}.\end{aligned}$$

Now we can compute

$$\mathbb{E}X_n = \frac{C_{n+3} - 2C_{n+2} - C_{n+1}}{C_{n+1}} = \frac{\frac{1}{n+4}\binom{2n+6}{n+3} - \frac{2}{n+3}\binom{2n+4}{n+2} - \frac{1}{n+2}\binom{2n+2}{n+1}}{\frac{1}{n+2}\binom{2n+2}{n+1}}.$$

Pulling out the common factor $\frac{1}{n+2}\binom{2n+2}{n+1}$ this becomes after some simplifications

$$\mathbb{E}X_n = \frac{(7n+13)n}{(n+4)(n+3)}. \tag{23}$$

Expanding (23) as a series, we get the assertion of Theorem 3.2.2. □

*Variance of the number of contacts*

**Theorem 3.2.4.** *Let $X_n$ be defined as in Theorem 3.2.2. Then the variance of the number of contacts in a 2-watermelon is given by*

$$\mathbb{V}X_n = \frac{12n(2n^5+13n^4+17n^3-7n^2-19n-6)}{(n+3)^2(n+4)^2(n+5)(n+6)} = 24 - \frac{444}{n} + \frac{5136}{n^2} + O\left(\frac{1}{n^3}\right).$$

*Proof.* The variance of the number of contacts in a watermelon of length $n$ is given by

$$\mathbb{V}X_n = \frac{[z^n]\partial^2_{uu}F(z,1)}{[z^n]F(z,1)} + \frac{[z^n]\partial_u F(z,1)}{[z^n]F(z,1)} - \left(\frac{[z^n]\partial_u F(z,1)}{[z^n]F(z,1)}\right)^2.$$

Because the last two terms can be computed via (23) it remains to determine

$$\partial^2_{uu}F(z,u)|_{u=1} = \frac{(-z^4-4z^3-z^2+4z-1)\sqrt{1-4z} - 2z^5 - 3z^4 + 6z^3 + 7z^2 - 6z + 1}{z^6}.$$

Reading off coefficients gives us

$$[z^n]\partial^2_{uu}F(z,1) = 2C_{n+1} + 8C_{n+2} + 2C_{n+3} - 8C_{n+4} + 2C_{n+5}.$$

Hence

$$\frac{[z^n]\partial^2_{uu}F(z,1)}{[z^n]F(z,1)} = \frac{66n^4+276n^3+54n^2+396n}{(n+3)(n+4)(n+5)(n+6)}. \tag{24}$$

Combining (23) and (24) we obtain

$$\mathbb{V}X_n = \frac{12n(2n^5+13n^4+17n^3-7n^2-19n-6)}{(n+3)^2(n+4)^2(n+5)(n+6)} = 24 - \frac{444}{n} + \frac{5136}{n^2} + O\left(\frac{1}{n^3}\right),$$

which finishes the proof. □

*Returns and common steps*

**Definition 3.2.5.** *A* return *is a point where the two paths of a watermelon meet, but have been apart one step before. In the Motzkin path setting this corresponds to a step which ends on the x-axis but does not start on the x-axis. Here the only possible return is being at altitude 1 and then taking a down-step. Level steps at altitude 0 do not count as return.*

**Definition 3.2.6.** *A* common step *occurs if both paths of a watermelon are at the same altitude and then take either an up-step or a down-step together. In the Motzkin path setting this corresponds to a level step at height 0.*

Obviously the number of returns plus the number of common steps is the number of contacts. Thus it is sufficient to analyze only one of these numbers. We will consider returns. Their mean and variance can be computed in a similar manner as the mean and variance of contacts.

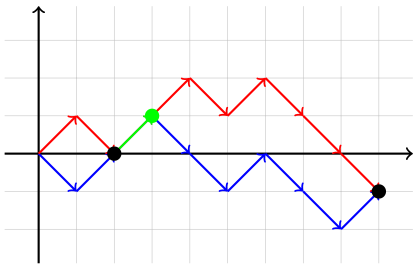


Figure 14: Returns and common steps in a 2-watermelon (with deviation -1). Returns are marked in black, common steps are marked in green.

*Average number of returns*

**Theorem 3.2.7.** *Let $Y_n$ be the random variable counting the number of returns in a 2-watermelon without wall, where the watermelon is chosen uniformly at random among all possible 2-watermelons of length $n$ and arbitrary deviation. Then*

$$\mathbb{E}Y_n = \frac{3n(n-1)}{(n+4)(n+3)} = 3 - \frac{24}{n} + \frac{132}{n^2} + O\left(\frac{1}{n^3}\right).$$

*Proof.* The generating function which counts returns is given by

$$F(z,x) = \frac{1}{1-(xz^2F(z,1)+2z)}$$

where $x$ encodes the number of returns to the $x$-axis. Here we only have to mark contacts occurring from a down step, thus the $+2z$-part remains unmarked. Plugging in the expression of $F(z,1)$ we computed in (19) we obtain

$$F(z,x) = \frac{2}{2-x(1-2z-\sqrt{1-4z})-4z}. \tag{25}$$

Derivating with respect to $x$ and plugging in $x=1$ this becomes

$$\partial_x F(z,x)|_{x=1} = \frac{2(1-2z-\sqrt{1-4z})}{(1-2z+\sqrt{1-4z})^2}. \tag{26}$$

This looks very similar to what we had in Formula (21) when computing contacts. The only difference is the term $(1-2z-\sqrt{1-4z})$ instead of $(1+2z-\sqrt{1-4z})$ in the numerator. Multiplying out and then rationalizing, we get

$$\partial_x F(z,x)|_{x=1} = \frac{(-3z^2+4z-1)\sqrt{1-4z}+1-6z+9z^2-2z^3}{2z^4}.$$

Reading off coefficients, we obtain:

$$[z^n]\partial_x F(z,x)|_{x=1} = [z^n]\frac{(-3z^2+4z-1)\sqrt{1-4z}}{2z^4} = 3C_{n+1} - 4C_{n+2} + C_{n+3}$$
$$= \frac{3}{n+2}\binom{2n+2}{n+1} - \frac{4}{n+3}\binom{2n+4}{n+2} + \frac{1}{n+4}\binom{2n+6}{n+3}.$$

Now we can compute the average number of returns in a watermelon of length $n$ via

$$\mathbb{E}_r = \frac{[z^n]\partial_x F(z,x)|_{x=1}}{[z^n]F(z,1)} = \frac{\frac{3}{n+2}\binom{2n+2}{n+1} - \frac{4}{n+3}\binom{2n+4}{n+2} + \frac{1}{n+4}\binom{2n+6}{n+3}}{\frac{1}{n+2}\binom{2n+2}{n+1}}.$$

Pulling out common factors, we get that the above expression is

$$\mathbb{E}_r = \frac{3n(n-1)}{(n+4)(n+3)} = 3 - \frac{24}{n} + \frac{132}{n^2} + O\left(\frac{1}{n^3}\right), \tag{27}$$

which completes the proof. □

**Corollary 3.2.8.** *A 2-watermelon has asymptotically on average 7 contacts and 3 returns. Thus it has asymptotically on average $7-3=4$ common steps.*

*Variance of the number of returns*

**Theorem 3.2.9.** *Let $Y_n$ be defined as in Theorem 3.2.7. Then the variance of the number of returns in a 2-watermelon is given by*

$$\mathbb{V}Y_n = \frac{4n(n-1)(n^4-4n^3+4n^2+279n+450)}{(n+3)^2(n+4)^2(n+5)(n+6)} = 4 - \frac{120}{n} + \frac{2004}{n^2} + O\left(\frac{1}{n^3}\right)$$

*Proof:* The variance of the number of returns is given by

$$\mathbb{V}_{Y_n} = \frac{[z^n]\partial^2_{xx}F(z,1)}{[z^n]F(z,1)} + \frac{[z^n]\partial_x F(z,1)}{[z^n]F(z,1)} - \left(\frac{[z^n]\partial_x F(z,1)}{[z^n]F(z,1)}\right)^2.$$

Since the last two terms in this expression can be obtained with the help of (27) it remains to compute

$$\partial^2_{xx}F(z,x)|_{x=1} = \frac{1-10z+35z^2-50z^3+25z^4-2z^5+(-5z^4+20z^3-21z^2+8z-1)\sqrt{1-4z}}{z^6}.$$

Thus, we get that

$$\frac{[z^n]\partial^2_{xx}F(z,1)}{[z^n]F(z,1)} = \frac{10n^4-60n^3+110n^2-60n}{(n+3)(n+4)(n+5)(n+6)}. \tag{28}$$

Combining (27) and (28) we get that the variance of the number of returns in a watermelon of size $n$ is given by

$$\mathbb{V}_{Y_n} = \frac{4n(n-1)(n^4-4n^3+4n^2+279n+450)}{(n+3)^2(n+4)^2(n+5)(n+6)}.$$

Asymptotic expansion of this expression finishes the proof. □

*Distributions*

*The number of contacts*

**Theorem 3.2.10.** *Let $X_n$ be the random variable counting the number of contacts in a 2-watermelon without wall, where the watermelon is chosen uniformly at random among all possible 2-watermelons of* *length $n$ and arbitrary deviation. Then the probability that such a watermelon has exactly $k$ contacts is given by*

$$\mathbb{P}(X_n = k) = \frac{\frac{1}{2^k}\sum_{\ell=0}^{n}\sum_{m=0}^{k}\binom{k}{m}\binom{m}{n-\ell}2^{n+\ell}(-1)^{k-m+\ell}\binom{\frac{k-m}{2}}{\ell}}{\frac{1}{n+2}\binom{2n+2}{n+1}}.$$

*Proof.* In order to figure out the distribution of the number of contacts, we need to consider

$$\mathbb{P}(X_n = k) = \frac{[z^n u^k]F(z,u)}{[z^n]F(z,1)}$$

where $X_n$ is the random variable counting the number of contacts in a 2-watermelon of length $n$ (without wall). We rationalize (20) to get rid of the square root in the denominator

$$F(z,u) = \frac{2}{2-u-2uz+u\sqrt{1-4z}} = \frac{1}{2}(2-u-2uz-u\sqrt{1-4z})\frac{1}{u^2z^2+2u^2z-2uz-u+1}.$$

The idea is to decompose

$$R(z,u) := \frac{1}{u^2z^2+2u^2z-2uz-u+1} = \frac{a(z)}{1-\alpha(z)u} + \frac{b(z)}{1-\beta(z)u}$$

by partial fraction decomposition. From this expression we then can read off the coefficient of $u^k$. The zeroes of the denominator (as a quadratic polynomial in $u$) are

$$u_1(z) = \frac{1+2z+\sqrt{1-4z}}{2z(z+2)} \quad \text{and} \quad u_2(z) = \frac{1+2z-\sqrt{1-4z}}{2z(z+2)}.$$

Thus, we have

$$\begin{aligned} R(z,u) &= \frac{1}{\sqrt{1-4z}}\left(\frac{1}{u_2(z)-u} - \frac{1}{u_1(z)-u}\right) \\ &= \frac{1}{\sqrt{1-4z}}\left(\frac{1}{u_2(z)(1-\frac{1}{u_2(z)}u)} - \frac{1}{u_1(z)(1-\frac{1}{u_1(z)}u)}\right). \end{aligned}$$

Now we can read off coefficients using $[u^k]\frac{1}{1-cu} = c^k$ and obtain

$$[u^k]R(z,u) = \frac{1}{\sqrt{1-4z}}\left(\frac{1}{u_2(z)^{k+1}} - \frac{1}{u_1(z)^{k+1}}\right).$$

Plugging in $u_1(z)$ and $u_2(z)$ this becomes

$$\begin{aligned} [u^k]R(z,u) &= \frac{1}{\sqrt{1-4z}}\left(\frac{(2z(z+2))^{k+1}}{(1+2z-\sqrt{1-4z})^{k+1}} - \frac{(2z(z+2))^{k+1}}{(1+2z+\sqrt{1-4z})^{k+1}}\right) \\ &= \frac{(1+2z+\sqrt{1-4z})^{k+1}-(1+2z-\sqrt{1-4z})^{k+1}}{2^{k+1}\sqrt{1-4z}}. \end{aligned}$$

Thus, we get

$$\begin{aligned} [u^k]F(z,u) &= \frac{2}{2}[u^k]R(z,u) - \frac{1+2z+\sqrt{1-4z}}{2}[u^{k-1}]R(z,u) \\ &= \frac{(1+2z+\sqrt{1-4z})^{k+1}-(1+2z-\sqrt{1-4z})^{k+1}}{2^{k+1}\sqrt{1-4z}} \\ &\quad - \frac{(1+2z+\sqrt{1-4z})((1+2z+\sqrt{1-4z})^k-(1+2z-\sqrt{1-4z})^k)}{2^{k+1}\sqrt{1-4z}} \\ &= \frac{(1+2z-\sqrt{1-4z})^k}{2^k}. \end{aligned} \tag{29}$$

For reading off the coefficient of $[z^n]$ of this expression, the expansion into a binomial series is helpful:

$$\begin{aligned}\frac{(1+2z-\sqrt{1-4z})^k}{2^k} &= \frac{1}{2^k}\sum_{m=0}^{k}\binom{k}{m}(1+2z)^m(-1)^{k-m}\sqrt{1-4z}^{k-m}\\ &= \frac{1}{2^k}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\left(\sum_{r=0}^{m}\binom{m}{r}2^r z^r\right)\left(\sum_{\ell\geq 0}\binom{\frac{k-m}{2}}{\ell}(-4z)^\ell\right).\end{aligned}$$

If we want to read off $[z^n]$, the variables $r$ and $\ell$ have to add up to $n$. Thus

$$[z^n u^k]F(z,u) = \frac{1}{2^k}\sum_{\ell=0}^{n}\sum_{m=0}^{k}\binom{k}{m}\binom{m}{n-\ell}2^{n+\ell}(-1)^{k-m+\ell}\binom{\frac{k-m}{2}}{\ell}. \tag{30}$$

Dividing (30) by the number of all watermelons of length $n$ as counted by (22) we obtain the statement of the theorem. □

*The number of returns*

**Theorem 3.2.11.** *Let $Y_n$ be the random variable counting the number of contacts in a 2-watermelon without wall, where the watermelon is chosen uniformly at random among all possible 2-watermelons of length n and arbitrary deviation. The probability that such a watermelon has exactly k contacts is given by*

$$\mathbb{P}(X_n = k) = \frac{\sum_{j=0}^{n} A_j^{(k)} B_{n-j}^{(k)}}{\frac{1}{n+2}\binom{2n+2}{n+1}}.$$

*where*

$$A_n^{(k)} := \frac{1}{2^k}\sum_{\ell=0}^{n}\sum_{m=0}^{k}\binom{k}{m}\binom{m}{n-\ell}\binom{\frac{k-m}{2}}{\ell}2^{n+l}(-1)^{k-m+n}$$

*and*

$$B_n^{(k)} := [z^n]\frac{1}{(1-2z)^{k+1}} = \binom{n+k}{n}2^n.$$

*Proof.* We want to compute

$$\mathbb{P}(X_n = k) = \frac{[z^n x^k]F(z,x)}{[z^n]F(z,1)}.$$

From rationalizing (25) we get

$$F(z,x) = \frac{2-4z-x(1-2z+\sqrt{1-4z})}{2(x^2z^2-4xz^2+4xz+4z^2-x-4z+1)}.$$

Again, we apply a partial fraction decomposition to

$$R(z,x) = \frac{1}{x^2z^2-4xz^2+4xz+4z^2-x-4z+1}$$

and read off coefficients from that. The zeros of the denominator are

$$x_1(z) = \frac{(1-2z)(1-2z-\sqrt{1-4z})}{2z^2} \quad \text{and} \quad x_2(z) = \frac{(1-2z)(1-2z+\sqrt{1-4z})}{2z^2}.$$

We obtain

$$R(z,x) = \frac{1}{(1-2z)\sqrt{1-4z}}\left(\frac{1}{x_1(z)(1-\frac{x}{x_1(z)})} - \frac{1}{x_2(z)(1-\frac{x}{x_2(z)})}\right).$$

Reading off coefficients, we obtain

$$\begin{aligned}
[x^k]R(z,x) &= \frac{1}{(1-2z)\sqrt{1-4z}}\left(\frac{1}{x_1(z)^{k+1}} - \frac{1}{x_2(z)^{k+1}}\right) \\
&= \frac{1}{(1-2z)^{k+2}\sqrt{1-4z}}\left(\frac{2^{k+1}z^{2k+2}}{(1-2z-\sqrt{1-4z})^{k+1}} - \frac{2^{k+1}z^{2k+2}}{(1-2z+\sqrt{1-4z})^{k+1}}\right). \\
&= \frac{1}{2^{k+1}(1-2z)^{k+2}\sqrt{1-4z}}\left((1-2z+\sqrt{1-4z})^{k+1} - (1-2z-\sqrt{1-4z})^{k+1}\right).
\end{aligned}$$

Rewriting (25) as

$$F(z,x) = (1-2z)R(z,x) - \frac{1-2z+\sqrt{1-4z}}{2}xR(z,x),$$

reading off the coefficient $[x^k]$ and simplifying, we obtain

$$[x^k]F(z,x) = \frac{(1-2z-\sqrt{1-4z})^k}{2^k(1-2z)^{k+1}} = A^{(k)}(z)B^{(k)}(z),$$

where

$$A^{(k)}(z) = \frac{(1-2z-\sqrt{1-4z})^k}{2^k} \quad \text{and} \quad B^{(k)}(z) = \frac{1}{(1-2z)^k}.$$

Expanding

$$B_n^{(k)} := [z^n]\frac{1}{(1-2z)^{k+1}} = \binom{n+k}{n}2^n$$

in a binomial series and a similar reasoning as before yields

$$A_n^{(k)} := [z^n]A^{(k)}(z) = \frac{1}{2^k}\sum_{\ell=0}^{n}\sum_{m=0}^{k}\binom{k}{m}\binom{m}{n-\ell}\binom{\frac{k-m}{2}}{\ell}2^{n+\ell}(-1)^{k-m+n}.$$

Note that $A_n^{(k)}$ looks similar to (30), the only difference between these two expressions are the powers of $(-1)$, namely $(-1)^{k-m+n}$ and $(-1)^{k-m+\ell}$ respectively.

Using the Cauchy-Product of $A$ and $B$ we obtain

$$[z^n x^k]F(z,x) = \sum_{j=0}^{n} A_j^{(k)}B_{n-j}^{(k)} = \sum_{j=0}^{n}\sum_{\ell=0}^{j}\sum_{m=0}^{k}\binom{k}{m}\binom{m}{j-\ell}\binom{\frac{k-m}{2}}{\ell}\binom{n-j+k}{n-j}2^{n+\ell-k}(-1)^{k-m+j}.$$

Dividing this by the number of all watermelons of length $n$ given by (22) we get the assertion of the theorem. □

*Asymptotic behavior of the distributions*

In this section we will analyze the asymptotic behavior of the distributions of contacts and returns in 2-watermelons. The theoretical background of this section are the methods for coefficient asymptotics from Flajolet and Sedgewick as introduced in Section 2.2.

*Contacts*

**Theorem 3.2.12.** *Let $X_n$ be the random variable counting the average number of contacts be defined as in Theorem 3.2.2. Then $X_n \to X$ in probability where $X$ is distributed as follows*

$$\mathbb{P}(X=0) = 0 \quad \textit{and} \quad \mathbb{P}(X=k) = \mathbb{P}(B=k-1) \quad \textit{for } k \geq 1$$

*where $B$ is a negative-binomially distributed random variable with parameters $r=2$ and $p=\frac{3}{4}$, i.e.*

$$\mathbb{P}(B=k) = \binom{r+k-1}{k}p^k(1-p)^r.$$

*Proof.* We want to compute $[z^n u^k]F(z,u)$ for $n \to \infty$ and $k$ fixed. The function

$$[u^k]F(z,u) = \frac{(1+2z-\sqrt{1-4z})^k}{2^k}$$

has its dominant singularity at $z = \frac{1}{4}$ (and no other singularities). Using Theorem 2.2.13 we obtain

$$[z^n u^k]F(z,u) \sim -\frac{k}{2^k}\left(\frac{3}{2}\right)^{k-1}\frac{n^{-3/2}4^n}{\Gamma(-\frac{1}{2})} = \frac{k3^{k-1}4^n}{4^k\sqrt{\pi n^3}}$$

for $n \to \infty$. Using $C_n \sim \frac{4^n}{\sqrt{\pi n^3}}\left(1+O\left(\frac{1}{n}\right)\right)$ we obtain than

$$\mathbb{P}(X_n = k) = \frac{[z^n u^k]F(z,u)}{C_{n+1}} \sim \frac{k3^{k-1}}{4^{k+1}}\left(1+O\left(\frac{1}{n}\right)\right)$$

for $n \to \infty$. Observing that $\mathbb{P}(X=0) = 0$ and introducing a new random variable $B$ with $\mathbb{P}(X=k) = \mathbb{P}(B=k-1)$ we see that $B$ is a negative-binomially distributed random variable with parameters $r = 2$ and $p = \frac{3}{4}$, which finishes the proof. $\square$

*Returns*

**Theorem 3.2.13.** *Let the random variable $Y_n$ counting the number of returns be defined as in Theorem 3.2.7. Then $Y_n \to Y$ in probability where $Y$ is distributed as follows*

$$\mathbb{P}(Y=0) = 0 \quad \textit{and} \quad \mathbb{P}(Y=k) = \mathbb{P}(\tilde{B} = k-1) \quad \textit{for } k \geq 1$$

*where $\tilde{B}$ is a negative-binomially distributed random variable with parameters $r = 2$ and $p = \frac{1}{2}$.*

*Proof.* The proof of this theorem resembles the proof of Theorem 3.2.12. Now we look at $[z^n x^k]F(z,x)$ for $n \to \infty$ and $k$ fixed. The function

$$F(z,x) = \frac{(1-2z-\sqrt{1-4z})^k}{2^k(1-2z)^k}$$

has singularities at $z = \frac{1}{4}$ and $z = \frac{1}{2}$. The singularity at $z = \frac{1}{4}$ is dominant, the other singularity at $z = \frac{1}{2}$ lies outside of every $\Delta$-region around $z = \frac{1}{4}$. By expanding the denominator with the help of the binomial series and by using Theorem 2.2.13 we obtain

$$[z^n x^k]F(z,x) \sim -\frac{k}{2^k}\cdot\frac{(\frac{1}{2})^k}{(\frac{1}{2})^{k+1}}\cdot\frac{n^{-3/2}4^n}{\Gamma(-\frac{1}{2})} = \frac{k4^n}{2^{k-1}\sqrt{n^3\pi}}$$

for $n \to \infty$. We obtain

$$\mathbb{P}(X_n = k) = \frac{[z^n x^k]F(z,x)}{C_{n+1}} \sim \frac{k}{2^{k+1}}\left(1+O\left(\frac{1}{n}\right)\right)$$

for $n \to \infty$. Introducing the random variable $\tilde{B}$ as in the theorem and observing that it indeed is negative-binomially distributed with parameters $r = 2$ and $p = \frac{1}{2}$ concludes the proof. $\square$

*Comparison with similar results*

There are some other results which also study contacts in watermelons in slightly different settings:

- In [53] both exact and asymptotic results for the average number of contacts between the lowest path in a $p$-watermelon and the wall were derived.

- In [64] the authors considered the number of contacts in 2-watermelons in a strip. They compute the generating function as well as the free energy $\kappa(c, w) = \lim_{n\to\infty} 1/n \log(Z_n(c, w))$ (where $w$ is the width of the strip and $c$ is the variable that encodes the number of contacts in the generating function) of the system.

- In [73] the authors studied contacts in a 3-watermelon without wall. They obtained the generating function of these objects with the help of the kernel method. They also studied the singularity structure and phase transitions of the model.

There are also some ways in which the results of this section could be expanded: the deviation, i.e. the height of the endpoints of our watermelons, was arbitrary. A natural question would be to ask what happens for watermelons with a fixed deviation. This can be encoded in the following way: let $z$ mark the length of the watermelon (or Motzkin path) and $y$ the deviation. The deviation is then given by the number of level steps marked with $u$ minus the number of level steps marked with $d$. Taking this into account when constructing the functional equation we obtain

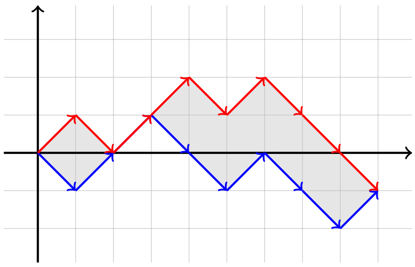
$$F(z, y) = \frac{1}{1 - (z^2 F + (y + y^{-1})z)}.$$

This then could be used to derive similar results for watermelons with a given deviation.

## 3.3 THE AREA BETWEEN 2-WATERMELONS WITHOUT WALL

In this section we are going to study the area enclosed by the two paths of a 2-watermelon without wall and arbitrary deviation. As already hinted earlier the bijection with Motzkin paths presented in Theorem 3.2.3 turns out to be helpful again, since it also preserves the area (up to a scaling factor).

More precisely, the area between the two paths of the 2-watermelon corresponds to twice the area between the Motzkin-path and the $x$-axis. This can easily be verified by considering what appending a new step at the end contributes to the area in each setting. If the two paths are $2k$ units apart (the distance is always even since the underlying steps are Dyck steps) the next step contributes $2k + \varepsilon$ to the area where $\varepsilon$ is 1 if the upper path goes up and the lower path goes down, 0 if both paths go up or down and $-1$ if the lower path goes up and the upper path goes down. In the Motzkin-path-setting, if the path is at altitude $k$ and a new step is appended the contribution to the area is $k + \delta$ where $\delta$ is $\frac{1}{2}$ for an up step, 0 for a level step labeled with $u$ or $d$ and $-\frac{1}{2}$ for a down step.



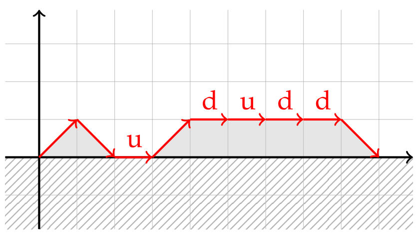



Figure 15: The bijection between 2-watermelons and weighted Motzkin paths. On the left is a 2-watermelon with area 12, on the right a weighted Motzkin path with area 6.

Thus, we are interested in the area under an excursion. These areas have been studied by Banderier and Gittenberger [10]. Using their results, we can derive results for the average area between the two paths of a 2-watermelon as well as the paths of 2-stars (which correspond to meanders in the Motzkin-setting).

**Definition 3.3.1.** *Let $f_{nkm}$ denote the number of stars of length $n$ with final difference of the altitudes of the paths being $k$ and area $m$, which is the same as the number of weighted Motzkin paths of length $n$, final altitude $k$, and area $\frac{m}{2}$. Then the area generating function is*

$$F(z,u,q) = \sum_{n,k,m\geq 0} f_{nkm} z^n u^k q^m = \sum_{k\geq 0} F_k(z,q) u^k.$$

For watermelons, we want the final difference of the paths to be zero, i.e. to consider $F(z,0,q)$. By a step by step construction we obtain a functional equation for the generating function.

**Theorem 3.3.2.** *The area generating function satisfies the following functional equation*

$$F(z,u,q) = 1 + zP(uq)F(z,uq^2,q) - z(uq)^{-1}F_0(z,q)$$

*where $P(u) = u^{-1} + 2 + u$ is the step polynomial of weighted Motzkin paths with two different kinds of level steps.*

*Proof.* A Motzkin path is either an empty path, this corresponds to the 1 on the right-hand side of the functional equation, or a shorter path with a step attached in the end. This corresponds to $zP(uq)F(z,uq^2,q)$ because the step polynomial describes the steps one is allowed to attach and attaching a step at height $k$ adds a rectangle of double area $2k$ and a triangle described by the step that is added, which explains the occurrence of $uq$ instead of $q$ in the step polynomial and the occurrence of $uq^2$ in the generating function. Finally we have to make sure that the path doesn't cross the $x$-axis. This is ensured by $-z(uq)^{-1}F_0(z,q)$, i.e. by subtracting those walks that do cross the $x$-axis. □

This is a special case of Theorem 1 in [10]:

**Theorem 3.3.3.** *For a weighted walk with step polynomial*

$$P(u) = \sum_{i=-c}^{d} p_i u^i$$

*where $c$ is the largest downward jump and $d$ is the largest upward jump, where the walk may not cross the $x$-axis, the area enclosed by the walk and the $x$-axis is described by the generating function $F(z,u,q)$ which satisfies*

$$F(z,u,q) = 1 + zP(uq)F(z,uq^2,q) - z\sum_{k=0}^{c-1} r_k(uq)F_k(z,q)q^k$$

*where $r_k$ are Laurent polynomials defined by*

$$r_k(u) := [u^{<0}](P(u)u^k) = \sum_{j=-c}^{-k-1} p_j u^{j+k}.$$

*Again $z$ encodes length, $u$ encodes final altitude and $q$ encodes doubled[1] area.*

*Proof.* See [10], Theorem 1. □

**Definition 3.3.4.** *A Łukasiewicz walk is a lattice path with $(1,-1) \in S$ and all other steps lie in $\{1\} \times \mathbb{N}$, i.e., there is only one downwards jump of length 1.*

For these kinds of paths the area between excursions and the $x$-axis respectively meanders and the $x$-axis has been well studied.

[1] The reason why doubled area is considered is mainly convenience: with doubled area the generating function has integer exponents. Using just the area leads to Puiseux series, i.e. power series that also allow negative and fractional exponents of the indeterminate.

**Theorem 3.3.5.** *The generating function for the average area below a (weighted) Łukasiewicz excursion is given by*

$$\partial_q F_0(z,1) = \frac{2}{zp_{-1}} u_1 + \frac{u_1 u_1''}{p_{-1} u_1'} - \frac{2}{p_{-1}} u_1'. \tag{31}$$

*where $u_1$ is the (unique) small root of the kernel $1 - zP(u) = 0$ and $p_{-1}$ is the coefficient of $z^{-1}$ in the step polynomial $P(u)$. The average area below an excursion behaves asymptotically like*

$$\frac{\tau\sqrt{\pi P(\tau)}}{\rho p_{-1}\sqrt{2P''(\tau)}} n^{3/2} - \frac{3}{\rho p_{-1}} n - \frac{3\tau\sqrt{\pi P(\tau)}}{8p_{-1}\rho\sqrt{2P''(\tau)}}\sqrt{n} + \frac{7}{2\rho p_{-1}} + O\left(\frac{1}{\sqrt{n}}\right). \tag{32}$$

*Proof.* The proof can be found in [10], Theorem 5. □

To make sure we can apply these results in this weighted Motzkin-path setting we need to check that the following technical properties (which were needed in the proof of the above theorem) are fulfilled:

- There are $c$ distinct roots $u_1(z), \dots, u_c(z)$ of the kernel $1 - zP(u) = 0$ which are analytic at zero.
- There is a unique positive real number $\tau$ such that $P'(\tau) = 0$. The radius of convergence of the generating function is given by $\rho := \frac{1}{P(\tau)}$.
- The root $u_1$ which is singular at $\rho$ has square root behavior near its singularity, i.e. $u_1(z) \sim \tau + K\sqrt{1 - z/\rho}$, the other roots are analytical at $\rho$.

In our case $P(u) = u^{-1} + 2u + u$, thus $c = 1$ and $d = 1$. The kernel $1 - zP(u)$ has the roots

$$u_- = \frac{1 - 2z - \sqrt{1-4z}}{2z} \quad \text{and} \quad u_+ = \frac{1 - 2z + \sqrt{1-4z}}{2z}.$$

The root $u_1 := u_-$ is analytical at zero, the other root is not. Thus, we have indeed $c = 1$ distinct roots of the kernel which are analytical at zero. The derivative of the step polynomial is

$$P'(u) = -u^2 + 1$$

and its zeros are $\pm 1$. Thus $\tau = 1$ and $\rho = \frac{1}{P(\tau)} = \frac{1}{4}$. The root $u_1$ has indeed square root behavior at its singularity at $\frac{1}{4}$.

Plugging in these values in (31) we obtain after some simplifications that the generating function of the average area below a Motzkin path is

$$\partial_q F_0^{(MP)}(z,1) = \frac{(2z^2 - 4z + 1)\sqrt{1-4z} - 8z^2 + 6z - 1}{z^2(1-4z)^{3/2}}. \tag{33}$$

Asymptotically the average area below such a Motzkin path is

$$\frac{[z^n]\partial_q F_0^{(MP)}(z,1)}{[z^n]F_0^{(MP)}(z,1)} \sim 4\sqrt{\pi} n^{3/2} - 12n - \frac{3\sqrt{\pi}}{2}\sqrt{n} + 14 + O\left(\frac{1}{\sqrt{n}}\right). \tag{34}$$

Since the generating function (33) already describes the double area, it is also the generating function of the average area below a 2-watermelon without wall.

For 2-stars with arbitrary endpoints and no wall, similar results can be obtained.

**Theorem 3.3.6.** *The generating function of the average area below Łukasiewicz meanders is given by*

$$(\partial_q F)(z,1,1) = \frac{\delta z(1 + zP(1))(1 - u_1)}{(1 - zP(1))^3} + \frac{2zP(1)u_1}{(1 - zP(1))^2} - \frac{u_1 + \frac{zu_1u_1''}{u_1'} - zu_1'}{1 - zP(1)} \tag{35}$$

*where $\delta := P'(1)$ is the drift of the walk. The asymptotics of the area also depend on the drift:*

- *If $\delta > 0$ then the area asymptotically behaves like*

$$\tau\sqrt{\frac{P''(\tau)}{P(\tau)}}\sqrt{2\pi}n^{3/2} - 2n + O(\sqrt{n}).$$

- *If $\delta = 0$ the area asymptotically behaves like*

$$\frac{3}{4}\sqrt{\frac{P''(\tau)}{P(1)}}\sqrt{2\pi}n^{3/2} - 2\sqrt{\pi}n + O(\sqrt{n}).$$

- *If $\delta > 0$ then the area asymptotically behaves like*

$$\frac{\delta}{P(1)}n^2 + O(n^{3/2}).$$

*Proof.* The proof can be found in [10], Theorem 6. □

In the case of weighted Motzkin-paths, the drift $\delta$ is zero. Plugging in and carrying out some simplifications the generating function for the average area under a meander becomes

$$\partial_q F^{(MP)}(z,1,1) = \frac{(-6z+3)\sqrt{1-4z}+12z-3}{(1-4z)^{5/2}}. \tag{36}$$

The area under such a meander behaves asymptotically like

$$\frac{[z^n]\partial_q F^{(MP)}(z,1,1)}{[z^n]F^{(MP)}(z,1,1)} = \frac{3\sqrt{\pi}}{4}n^{3/2} - 2\sqrt{\pi}n + O(\sqrt{n}). \tag{37}$$

The functional equation (36) also describes the average area enclosed by the two paths of a 2-star without wall.

A natural question arising in this context would be "What is the average area between the two paths of a 2-watermelon with wall?" (or variations thereof, like "What is the average area enclosed by the upper path of a 2-watermelon with wall and the $x$-axis?"). However, since the bijection used here does not encode the wall condition, this approach will probably fail for answering these questions.

We however will encounter this bijection between Motzkin paths and watermelons again in Chapter 5 when dealing with pattern avoidance in watermelons.

# 4 THE VECTORIAL KERNEL METHOD AND PATTERN AVOIDANCE IN LATTICE PATHS

## 4.1 DEFINITIONS AND NOTATIONS

In this chapter we are going to analyze lattice paths that avoid certain patterns.

Let $w$ be a lattice path with a simple step set, i.e., where all steps have length one. A *pattern* is a fixed path (or word)

$$p = [a_1, \ldots, a_\ell]$$

where $a_i \in \mathcal{S}$. The length of a pattern is the number of its steps. An *occurrence* of a pattern $p$ in a lattice path $w$ is a contiguous sub-string of $w$, which coincides with $p$. We say a lattice path $w$ *avoids* the pattern $p$ if there is no occurrence of $p$ in $w$. For example, the path $w = [1, 3, 3, 1, -2, 3, 1]$ (where $i$ stands for the step $(i, 1)$) has two occurrences of the pattern $p = [3, 1]$ but avoids the pattern $\tilde{p} = [-2, -2]$.

A *prefix* of length $k$ of a string is a contiguous non-empty sub-string that matches the first $k$ letters (or steps, to phrase it with words more familiar for a lattice path setting). Similarly, a *suffix* of length $k$ of a string is a contiguous non-empty sub-string that matches the last $k$ letters. For example, $[1, 3, 3]$ is a prefix (of length 3) of the path $w$ from the previous example and $[-2, 3, 1]$ is a suffix. A *presuffix* of a pattern is a non-empty string that is both prefix and suffix. In our above example, $[1]$ is the only presuffix of this given path.

Several patterns in lattice paths have been studied in the past, both in the context of counting paths avoiding a pattern as well as counting the number of occurrences of a fixed pattern in a path, as can be seen in [6, 15, 27, 60, 61, 72, 69]. The methods used are many different techniques like bijective proofs, decomposition arguments as well as finite operator calculus and have to be adapted and reinvented for each different pattern (or certain classes of patterns). The vectorial method, however, is a unified approach to solve the enumeration problem for lattice paths avoiding any kind of pattern.

Some authors use a different definition of a pattern, namely when the pattern is contained in the path as non-contiguous substring, see for example [6]. The path $w$ as defined in the previous example contains $[1, 3, 1, 3]$ in the non-contiguous-sense, but not in the contiguous sense. Lattice paths avoiding patterns in the non-contiguous sense also can be dealt with the vectorial kernel method. In this chapter however we will only consider consecutive patterns, unless mentioned otherwise.

In order to describe pattern avoidance we will need the concept of finite automata.

**Definition 4.1.1.** *A* finite automaton *is a quadruple* $(\Sigma, \mathcal{M}, s_0, \delta)$ *where*

- $\Sigma$ *is the input alphabet (in our case,* $\Sigma$ *will usually be associated to the step set)*
- $\mathcal{M}$ *is a finite, nonempty set of states*
- $s_0 \in \mathcal{M}$ *is the initial state*
- $\delta : \mathcal{M} \times \Sigma \to \mathcal{M}$ *is the state transition function. In many cases, it is useful to allow* $\delta$ *to be a partial function as well, i.e., not every image* $\delta(S_i, x)$ *has to be defined. Especially for pattern avoidance the usage of partial functions is very helpful.*

*Sometimes there is also a set* $F \subseteq \mathcal{M}$ *of final states given in the definition of a finite automaton. Here, however, we will not have any final states (i.e.* $F = \emptyset$*).*

A finite automaton can be described as a weighted directed graph (the states being the vertices, the edges and their weights given by the transition function) or by an adjacency matrix $A$, where the entry $A_{ij}$ consists of the formal sum of all letters $x$ that, when being in state $S_i$ and reading the letter $x$, transition to state $S_j$. Phrased differently,

$$A_{ij} = \sum_{x:\delta(S_i,x)=S_j} x.$$

**Example 4.1.2.** Let $\mathcal{S} = \{\mathsf{U}, \mathsf{H}, \mathsf{D}\}$ where $\mathsf{U} = (1,1), \mathsf{H} = (1,0)$ and $\mathsf{D} = (1,-1)$ be the step set and $p = [\mathsf{U}, \mathsf{H}, \mathsf{U}, \mathsf{D}]$ the forbidden pattern. We will build an automaton with $s = 4$ states, where $s$ is the number of steps in the pattern. Each state corresponds to a proper prefix of $p$ collected so far by walking along the lattice path. Let us label these states $X_0, \ldots, X_{s-1}$ (in our case $X_0, \ldots, X_3$). The first state $X_0$ is labeled by the empty word. The next states are labeled by proper prefixes of $p$, more precisely $X_i$ is labeled by $X_i = [a_1, \ldots, a_i]$ where $a_j$ are the letters of the forbidden pattern. For $i, j \in \{1, \ldots, s\}$ we have $\delta(X_i, \lambda) = X_j$ (or, in the graph setting, an arrow labeled $\lambda$) if $j$ is the maximal number such that $X_j$ is a suffix of $X_i\lambda$.

When the automaton reads a path $w$, it ends in the state labeled with the longest prefix of $p$ that coincides with a suffix of $w$. The automaton is completely determined by the step set and the pattern.

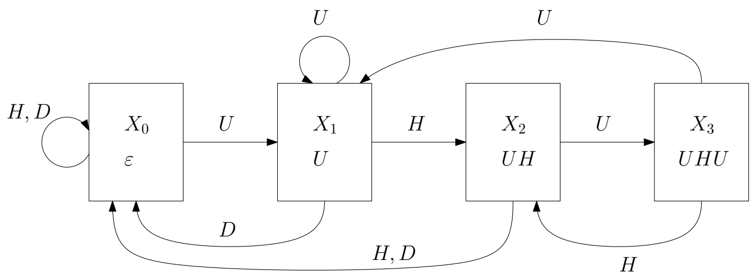



Figure 16: The automaton for $\mathcal{S} = \{\mathsf{U}, \mathsf{H}, \mathsf{D}\}$ and $p = [\mathsf{U}, \mathsf{H}, \mathsf{U}, \mathsf{D}]$

The adjacency matrix is then given by the adjacency matrix of the directed weighted graph described by the automaton (note that we use the final altitudes of the steps encoded as powers of the variable $u$ instead of the steps themselves, i.e., $\sigma = \{u, 1, \frac{1}{u}\}$ instead of $\sigma = \{\mathsf{U}, \mathsf{H}, \mathsf{D}\}$):

$$A = A(u) = \begin{pmatrix} 1+u^{-1} & u & 0 & 0 \\ u^{-1} & u & 1 & 0 \\ 1+u^{-1} & 0 & 0 & u \\ 0 & u & 1 & 0 \end{pmatrix}.$$

In each row except the last one, all entries sum up to $P(u)$, because at each state except the last one, all possible steps are allowed. The entries in the last row of the matrix sum up to $P(u) - w_s$, where $w_s$ is the weight of the last step in the forbidden pattern $p$. This is because in the last state $X_{s-1}$ all steps except the one that would make $p$ complete are allowed.

Automata can not only be used to describe the avoidance of one pattern, but also for other constraints, e.g. the aforementioned non-contiguous notion of patterns, the avoidance of several patterns at once (see Section 4.5) or height constraints.

**Definition 4.1.3.** *The* kernel *of an automaton is defined to be the determinant of $I - tA$, where $A$ is the adjacency matrix of the automaton, i.e.,*

$$K(t,u) := \det(I - tA).$$

Why this kernel is of importance and what we will use it for will be explained in the next sections after we have introduced the kernel method. For certain kinds of automata, for example the automata that arise when considering walks that avoid a pattern, there are easier expressions for the kernel that avoid the computation of the adjacency matrix and its determinant. For more details on this, see [3].

## 4.2 THE KERNEL METHOD – AN INTRODUCTION

The kernel method[1] is a tool to study generating functions that satisfy functional equations. The main idea behind the kernel method is to introduce a catalytic variable and then bind the variables in a way such that one side of the equation vanishes. In its easiest form, i.e., for solving equations of the type

$$K(z,u)F(z,u) = A(z,u) + B(z,u)G(z), \tag{38}$$

where $K$, $A$ and $B$ are known functions, $F$ and $G$ are unknown functions, and where the kernel $K(z,u) = 0$ has only one small root $u_1(z) = 0$. The kernel method has been folklore in statistics and statistical physics. One identifiable source is Knuth's book [48] from 1968, where he used this idea as a new method for solving the ballot problem. Ever since there have been several extensions and applications of this method, see for example [13, 14, 21, 23, 11]. One very recent extension is the so-called *vectorial kernel method* developed by Asinowski, Bacher, Banderier, and Gittenberger in [3] which deals with the enumeration of simple walks avoiding a pattern. We will examine this particular extension in the next section, then we will expand it even further to walks with longer steps and to walks avoiding more than one pattern.

In the current section we will explain the idea behind the kernel method. This section is mostly based on the talk "What is ... the kernel method?" I gave at the SFB Statusseminar meeting in Strobl, 2019.

The kernel method can be decomposed into the following four steps:

1. Enlarge the class of objects we want to count by add a *catalytic* or also called *auxiliary* variable that contains additional information about them.

2. Establish a functional equation. Rewrite it in kernel form.

3. Eliminate one of the unknowns.

4. Extract the generating function and read off the coefficients.

In Equation (38), the variable $z$ encodes the quantity we are interested in, whereas $u$ is an auxiliary variable. This equation is already in kernel form, where the unknown $G$ depending only on the initial variable $z$ and the unknown $F$ depending both on the initial variable $z$ and the auxiliary variable $u$ are on different sides of the equation.

There are two ways to execute step three. The first possibility is to eliminate the left hand side in

$$K(z,u)F(z,u) = A(z,u) + B(z,u)G(z).$$

In order to do so, we insert values $u(z)$ for $u$ such that the kernel vanishes, reducing the equation to

$$0 = A(z,u(z)) + B(z,u(z))G(z).$$

Solving it for $G(z)$ then gives us the information we are interested in. Then we could also obtain $F(z,u)$, which would provide further information.

The other option is to eliminate the right-hand side of (38). This can be done by letting a group of transformations act on the kernel equation, which leaves the kernel fixed and then combining the equations obtained under the group action in a way such that the right-hand side vanishes.

How these two variants work out is probably best explained by an example. In the following, we will give two examples, one for each variant.

[1] The "kernel method" mentioned here has nothing to do with the "kernel method" or "kernel trick" from statistics or machine learning.

**Example 4.2.1.** The first example comes from the aforementioned book [48], which is one of the earliest identifiable sources of the kernel method:

> *"Consider a word composed of n 'S' symbols and n 'X' symbols, where S stands for 'add an element' to some specific stack and X stands for 'remove an element' from the stack. Such a word is called admissible if it specifies no operations that cannot be performed – i.e. if the number of X's never exceeds the number of S's when read from left to right. Find the number of admissible words as a function of n."*
>
> D.E. Knuth, [48], Exercise 2.2.1.4

We have already encountered this example several times in this thesis, albeit in different clothes. Rephrasing it, it boils down to "Find the number of Dyck paths of semilength $n$" since the symbol $S$ can be interpreted as up step, the symbol $X$ as down step, and the admissibility condition corresponds to the condition that the path never goes below the $x$-axis. We solved this enumeration problem with the help of the reflection principle in section 1.1, with the help of a first passage decomposition and generating functions in Example 2.1.23 , but now "We present here a new method for solving the ballot problem with the use of double generating functions, since this method lends itself to the solution of more difficult problems ...", as Knuth put it in [48].

1. First, we want to introduce an auxiliary variable. Let $z$ be the variable that marks the length of the walk, i.e., the quantity we are interested in. Now, introduce another variable $u$ that encodes the final altitude of the walk. This variable will play the role of the catalytic variable in the following.

2. Using a step-by-step construction "A walk is either empty or a shorter walk with a step attached in the end" we obtain the following functional equation

$$F(z,u) = 1 + z(u+\overline{u})F(z,u) - z\overline{u}F(z,0),$$

   where $\overline{u}$ is shorthand for $\frac{1}{u}$ and the term $-z\overline{u}F(z,0)$ ensures that the walks never go below the $x$-axis. We are interested in the number of Dyck paths that end on the $x$-axis, which are encoded by the generating function $F(z,0)$.

   Rewriting it in kernel form "Bulk on the left, initial and boundary on the right" we obtain

$$\underbrace{(1 - z(u+\overline{u}))}_{\text{kernel}}F(z,u) = 1 - z\overline{u}F(z,0).$$

3. Now, we want to eliminate one of the unknowns $F(z,u)$ or $F(z,0)$. Since we are interested in $F(z,0)$, we are going to eliminate $F(z,u)$. To do so, we will insert certain values for $u$ that will make the left hand side vanish. First, multiply the kernel equation by $(-u)$ to get rid of negative powers

$$(zu^2 - u + z)F(z,u) = zF(z,0) - u.$$

   We have that

$$K(z,u) = zu^2 - u + z = z\left(u - \frac{1-\sqrt{1-4z^2}}{2z}\right)\left(u - \frac{1-\sqrt{1+4z^2}}{2z}\right).$$

   Plugging in $u = u_1(z) = \frac{1-\sqrt{1-4z^2}}{2z}$ or $u = u_2(z) = \frac{1+\sqrt{1-4z^2}}{2z}$ into the kernel equation causes the left hand side to vanish. However, plugging in $u_2$ would lead to a solution which is not a power series. Hence, we plug in $u_1$ and obtain

$$0 = zF(z,0) - u_1(z).$$

4. Thus, extracting the generating function gives us

$$F(z,0) = \frac{u_1(z)}{z} = \frac{1-\sqrt{1-4z^2}}{2z^2}.$$

Reading off coefficients we obtain that the number of Dyck paths of semilength $n$ is $\frac{1}{n+1}\binom{2n}{n}$. If we for some reason are also interested in the final altitude, we can plug the above solution for $F(z,0)$ in the kernel equation and obtain

$$F(z,u) = \frac{u_1(z)-s}{zu^2-u+z} = \frac{1-\sqrt{1-4z^2}-2zu}{2z(zu^2-u+z)}.$$

As we have seen above, not all roots of the kernel lead to solutions that are power series. In order to decide which roots to plug in, we need the notion of small and large roots.

**Definition 4.2.2.** *A* small root *is a root $u_i(z)$ of the kernel $K(z,u)$ which tends to zero as $z$ tends to zero. Similarly, a* large root *is a root $u_i(z)$ of the kernel $K(z,u)$ such that $|u_i(z)|$ tends to infinity as $z$ tends to zero.*

For the kernel method to work we have to use small roots.

**Example 4.2.3.** Consider walks with step set $\mathcal{S} = \{W, N, SE\} = \{\overline{x}, y, x\overline{y}\}$ in the quarter plane. How many such walks of length $n$ are there?

This example is taken from [23], where not only this step set, but all other walks with small steps in the quarter plane have been classified according to the algebraic nature of their generating function (except for Gessel walks, i.e., the step set $\mathcal{S} = \{W, E, NE, SW\}$). The algebraicity of the generating function of these walks has been proved with the aid of the computer by Kauers, Koutschan, and Zeilberger [47] in 2008, seven years later in 2015 Bostan, Kurkova, and Raschel [20] finally gave a proof which was not computer aided.

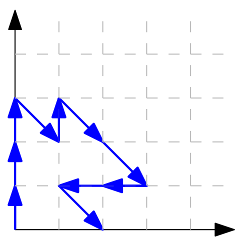


Figure 17: A walk in the quarter plane consisting of steps $W, N$, and $SE$.

Again, we will be using the kernel method to solve this problem.

1. Let $t$ mark the variable we are interested in in the generating function, namely the length of the walk. Furthermore let us introduce two catalytic variables $x$ and $y$, encoding the coordinates of the end point.

2. A step-by-step construction gives us the following functional equation in the three unknowns $F(x,y,t), F(x,0,t)$, and $F(0,y,t)$:

$$F(x,y,t) = 1 + t(\overline{x} + y + x\overline{y})F(x,y,t) - t\overline{x}F(0,y,t) - tx\overline{y}F(x,0,t). \tag{39}$$

Note that now we have to take care not to leave the region the walk is confined to at two instances. We are not allowed to take a step to the west when we are on the $y$-axis because by doing so we would leave the quarter plane, this corresponds to the term $-t\overline{x}F(0,y,t)$. Furthermore, we are not allowed to take a south-east step when we are on the $x$-axis, this corresponds to the term $-tx\overline{y}F(x,0,t)$.

If our walk had a south-west step we would have take extra care. If we are at the origin and then take an south-west-step, leaving the quarter plane, we counted this twice, once by the correction terms coming from the $x$-axis, once by the correction terms coming from the $y$-axis. Hence, we would have to add the term corresponding to taking an south-west step at the origin in order to fix this double counting, in an inclusion-exclusion-like manner. In three or more dimensions the inclusion-exclusion-like nature becomes even more visible, as can be seen in [19].

Rewriting Equation (39) in kernel form by bringing all unknowns depending on both $x$ and $y$ to the left hand side we obtain:

$$(1-t(\bar{x}+y+x\bar{y}))F(x,y,t) = 1 - t\bar{x}F(0,y,t) - tx\bar{y}F(x,0,t).$$

To get rid of negative powers we multiply with $xy$ and obtain:

$$\underbrace{(1-tS(x,y))}_{\text{kernel } K(x,y)}xyF(x,y) = xy + \underbrace{tyF(0,y)}_{\text{depends only on } y} - \underbrace{tx^2F(x,0)}_{\text{depends only on } x}, \tag{40}$$

where $S(x,y) := (\bar{x}+y+x\bar{y})$ is shorthand for the step polynomial.

3. Now, we want the right-hand side to vanish. Since it contains two unknowns whereas the left-hand side contains only one unknown, this seems like a wise choice. To do so, we consider a group of birational transformations that leave the kernel fixed. Furthermore, we want the generators $\Phi$ and $\Psi$ of this group also to leave $x$ and $y$ respectively fixed in order to obtain cancellations of the terms $tx^2F(x,0)$ and $tyF(0,y)$.

   Such birational transformations can be found by writing the step polynomial as

   $$S(x,y) = A_{-1}(x)\bar{y} + A_0(x) + A_1(x)y = B_{-1}(y)\bar{x} + B_0(y) + B_1(y)x,$$

   where $A_1, A_{-1}, B_1, B_{-1} \neq 0$. Then define

   $$\Phi : (x,y) \mapsto \left(\bar{x}\frac{B_{-1}(y)}{B_1(y)}, y\right) \quad \text{and} \quad \Psi : (x,y) \mapsto \left(x, \bar{y}\frac{A_{-1}(x)}{A_1(x)}\right).$$

   Clearly the actions $\Phi$ and $\Psi$ leave $S(x,y)$ fixed. Furthermore, we have that $\Phi^2 = id$ and $\Psi^2 = id$. For the group acting on the Kernel Equation (40), consider the group generated by $\Phi$ and $\Psi$. In our case we have

   $$\begin{array}{lll} A_1(x) = 1 & A_0(x) = \bar{x} & A_{-1}(x) = x \\ B_1(y) = \bar{y} & B_0(y) = y & B_{-1}(x) = 1 \end{array}$$

   Thus we arrive at

   $$\Phi(x,y) = (\bar{x}y, y) \quad \text{and} \quad \Psi(x,y) = (x, x\bar{y}).$$

   The group of the walk is then given by

   $$\begin{aligned} (x,y) &\overset{\Phi}{\leftrightarrow} (\bar{x}y,y) \overset{\Psi}{\leftrightarrow} (\bar{x}y,\bar{x}) \overset{\Phi}{\leftrightarrow} (\bar{y},\bar{x}) \\ &\overset{\Psi}{\leftrightarrow} (\bar{y},x\bar{y}) \overset{\Phi}{\leftrightarrow} (x,x\bar{y}) \overset{\Psi}{\leftrightarrow} (x,y). \end{aligned}$$

   Let this group act on the functional equation and form alternating sums (also called *orbit sums*). Each group element leaves the step polynomial and the kernel fixed. Furthermore, all these actions are indeed well defined, since these objects are all power series in $t$ (the dependency on $t$ will be omitted in the following). Thus we arrive at

$$\begin{aligned}
K(x,y)xyF(x,y) &= xy - tyF(0,y) - tx^2F(x,0) \\
-K(x,y)\overline{x}y^2F(\overline{x}y,y) &= -\overline{x}y^2 + tyF(0,y) + t\overline{x}^2y^2F(\overline{x}y,0) \\
K(x,y)\overline{x}^2yF(\overline{x}y,\overline{x}) &= \overline{x}^2y - t\overline{x}F(0,\overline{x}) - t\overline{x}^2y^2F(\overline{x}y,0) \\
-K(x,y)\overline{xy}F(\overline{y},\overline{x}) &= -\overline{xy} + t\overline{x}F(0,\overline{x}) + t\overline{y}^2F(\overline{y},0) \\
K(x,y)x\overline{y}^2F(\overline{y},x\overline{y}) &= x\overline{y}^2 - tx\overline{y}F(0,x\overline{y}) - t\overline{y}^2F(\overline{y},0) \\
-K(x,y)x^2\overline{y}F(x,x\overline{y}) &= -x^2\overline{y} + tx\overline{y}F(0,x\overline{y}) + tx^2F(x,0)
\end{aligned}$$

Summing up we obtain:

$$\begin{aligned}
&K(x,y)(xyF(x,y) - \overline{x}y^2F(\overline{x}y,y) + \overline{x}^2yF(\overline{x}y,\overline{x}) - \overline{xy}F(\overline{y},\overline{x}) + x\overline{y}^2F(\overline{y},x\overline{y}) - x^2\overline{y}F(x,x\overline{y})) \\
&= xy - \overline{x}y^2 + \overline{x}^2y - \overline{xy} + x\overline{y}^2 - x^2\overline{y}. \qquad (41)
\end{aligned}$$

Note that the right-hand side now does not contain any unknowns anymore. Furthermore, the left hand side contains only one expression that has both positive $x$-powers and positive $y$-powers. This will help us extracting the generating function.

4. Dividing Equation (41) by the kernel and $xy$ and reading off nonnegative parts we obtain

$$F(x,y) = \{x^{\geq 0}y^{\geq 0}\}\frac{1 - \overline{x}^2y + \overline{x}^3 - \overline{x}^2\overline{y}^2 + \overline{y}^3 - x\overline{y}^2}{K(x,y)}.$$

   Let $n = 3m + 2i + j$ where $(i,j)$ are the coordinates of the endpoint and $n$ is the length of the walk. Then we have

$$[x^iy^jt^n]F(x,y,t) = \frac{(i+1)(j+1)(i+j+2)(3m+2i+j)!}{m!(m+i)!(m+i+j)!}.$$

   This follows from $K(x,y) = 1 - t(\overline{x} + y + x\overline{x})$ and $[x^iy^j](\overline{x} + y + x\overline{x}) = \frac{(3m+2i+j)!}{m!(m+i)!(m+i+j)!}$ by the binomial theorem and some simplifications.

A word of caution however: The approach with the orbit sums does not always work. There are two phenomena that can occur:

1. The right-hand side can vanish entirely. This happens for example for the walks with step set $\mathcal{S} = \{N, E, SW\}$. In this case, looking at half-orbit sums turns out to be helpful.
2. The group can be infinite. This happens for example for walks with the step set $\mathcal{S} = \{N, E, NW, SE\}$. In this case the argument fails, because we would encounter infinite sums. Proving that the group of a walk is infinite is not easy, either. How this can be done can be seen in [23].

## 4.3 THE VECTORIAL KERNEL METHOD

Let us now return to the problem of walks with a simple step set avoiding a pattern from [3], where the vectorial kernel method was developed as an unified approach for dealing with such problems.

**Definition 4.3.1.** *For any word $p$, let $\mathcal{Q}$ be the set of its presuffixes. Then the* autocorrelation polynomial *of $p$ is*

$$R(t,u) := \sum_{q\in\mathcal{Q}} t^{|\overline{q}|}u^{\mathrm{alt}(\overline{q})}, \qquad (42)$$

*where $\overline{q}$ denotes the complement of $q$ in $p$, i.e. the suffix of $p$ such that $p = q\overline{q}$ is the concatenation of $q$ and $\overline{q}$.*

**Example 4.3.2.** For example, consider the pattern $p = [1,2,1,-3,1,1,2,1]$. It has the following three presuffixes: $[1], [1,2,1]$ and $[1,2,1,-3,1,1,2,1]$. They contribute to the autocorrelation polynomial as follows:

| Presuffix | $[1]$ | $[1,2,1]$ | $[1,2,1,-3,1,1,2,1]$ |
|---|---|---|---|
| Length of complement | 7 | 5 | 0 |
| Height of complement | 5 | 2 | 0 |
| Contribution | $t^7u^5$ | $t^5u^2$ | 1 |

Thus we have that the autocorrelation polynomial of $p$ is $R(t,u) = 1 + t^5u^2 + t^7u^5$.

If $p$ is a pattern without any autocorrelation, we have $\mathcal{Q} = \{p\}$ and thus $R(t,u) = 1$.

The notion of the autocorrelation polynomial already appeared back in 1985 in a work of Guibas and Odlyzko [41] when they analyzed a string searching algorithm and properties of periodic words. It also stands in relation with the kernel of a walk avoiding a pattern, as we will see in the following theorem and its corollaries.

Equipped with these definitions we can now finally state the theorems from [3] for generating functions for walks, bridges, meanders, and excursions avoiding a pattern:

**Theorem 4.3.3.** *Let $\mathcal{S}$ be a simple set of steps and let $p$ be a pattern with steps from $\mathcal{S}$. Then the bivariate generating function for walks avoiding the pattern $p$ is given by*

$$W(t,u) = \frac{(1,0,\dots,0)\,\mathrm{adj}(I-tA)\vec{\mathbf{1}}}{K(t,u)}.$$

*It can also be written as*

$$W(t,u) = \frac{R(t,u)}{K(t,u)}.$$

*If we do not keep track of the final altitude, we obtain*

$$W(t) = W(t,1) = \frac{1}{1 - tP(1) + \frac{t^\ell}{R(t,1)}}$$

*where $\ell$ is the length of the pattern.*

*Proof.* By a step-by-step construction, we obtain the following functional equation

$$(W_1,\dots,W_\ell) = (1,0,\dots,0) + t(W_1,\dots,W_\ell)A$$

since the adjacency matrix describes into which state we transition after taking a step. This can be rewritten as

$$(W_1,\dots,W_\ell)(I-tA) = (1,0,\dots,0)$$
$$(W_1,\dots,W_\ell) = (1,0,\dots,0)\frac{\mathrm{adj}(I-tA)}{\det(I-tA)}.$$

Note that the generating function $W(t,u)$ is the sum of all the generating functions $W_\alpha(t,u)$ over all states. This gives us

$$W(t,u) = \sum_{\alpha=1}^{\ell} W_\alpha = (W_1,\dots,W_\ell)\vec{\mathbf{1}} = \frac{(1,0,\dots,0)\,\mathrm{adj}(I-tA)\vec{\mathbf{1}}}{\det(I-tA)}.$$

Since $K(t,u)$ was defined as $\det(I-tA)$ we obtain

$$W(t,u) = \frac{(1,0,\dots,0)\,\mathrm{adj}(I-tA)\vec{\mathbf{1}}}{K(t,u)}.$$

On the other hand, the generating function $W(t,u)$ can be constructed by the following combinatorial argument. Denote $W^{\{p\}}(t,u)$ the generating function of walks with step set $\mathcal{S}$ that end with $p$ but have no other occurrence of $p$. If we add a letter from $\mathcal{S}$ to a $p$-avoiding walk, we either obtain another $p$-avoiding walk or a walk with a single occurrence of $p$ at the end. This gives us

$$W + W^{\{p\}} = 1 + tPW. \tag{43}$$

If we append $p$ at the end of a $p$-avoiding walk, we obtain the same result as we would obtain by appending the complement of a presuffix of $p$ to a walk with a single occurrence of $p$ at the end. Phrased differently

$$Wt^{\ell}u^{\mathrm{alt}(p)} = W^{\{p\}}R(t,u). \tag{44}$$

The equations (43) and (44) form a system of two linear equations in the two unknowns $W$ and $W^{\{p\}}$. Solving it for $W$ we obtain

$$W(t,u) = \frac{R(t,u)}{(1-tP(u))R(t,u) + t^{|p|}u^{\mathrm{alt}(p)}}.$$

Thus we have two representations for $W(t,u)$ and obtain

$$W(t,u) = \frac{(1,0,\dots,0)\,\mathrm{adj}(I-tA)\vec{\mathbf{1}}}{\det(I-tA)} = \frac{R(t,u)}{(1-tP(u))R(t,u) + t^{|p|}u^{\mathrm{alt}(p)}}. \tag{45}$$

□

**Corollary 4.3.4.** *Let $\mathcal{S}$ be a simple set of steps and let $p$ be a pattern with steps from $\mathcal{S}$. The the generating function for bridges avoiding the pattern $p$ is given by*

$$B(t) = -\sum_{i=1}^{e} \frac{u_i'(t)}{u_i(t)} \frac{R(t,u_i)}{K_t(t,u_i)},$$

*where $u_1(t),\dots,u_e(t)$ are the small roots of the kernel $K(t,u)$ and $K_t$ is derivative of $K$ with respect to $t$.*

*Proof.* The generating function of bridges is given by $B(t) = [u^0]W(t,u)$. This coefficient can be computed with the help of Cauchy's coefficient formula (2.2.7):

$$B(t) = [u^0]W(t,u) = \frac{1}{2\pi i}\int_{|u|=\varepsilon} \frac{W(t,u)}{u}.$$

By Theorem 4.3.3 we have that

$$W(t,u) = \frac{R(t,u)}{K(t,u)}$$

hence the poles are exactly at the small roots $u_1(t),\dots,u_e(t)$ of the kernel and we obtain

$$B(t) = \sum_{i=1}^{e} \mathrm{Res}_{u=u_i} \frac{W(t,u)}{u}.$$

Computing these residues we obtain

$$\mathrm{Res}_{u=u_i(t)} \frac{W(t,u)}{u} = \left.\frac{R(t,u)}{\frac{d}{du}(u((1-tP(u))R(t,u) + t^{\ell}u^{\mathrm{alt}(p)})}\right|_{u=u_i(t)}.$$

The denominator of this expression is

$$\left.-tuP'(u)R(t,u) + u(1-tP(u))R_u(t,u) + \mathrm{alt}(p)t^{\ell}u^{\mathrm{alt}(p)}\right|_{u=u_i(t)}. \tag{46}$$

Differentiating $K(t,u_i) = 0$ with respect to $t$ we obtain an expression for $P'(u_i(t))$. Substituting it into (46), we obtain the corollary. □

We can state another corollary that allows easier computation of the kernel in the case of the avoidance of a single pattern. This is of interest because the autocorrelation polynomial is usually much easier to compute than the determinant of the adjacency matrix.

**Corollary 4.3.5** (Alternative representation of the kernel)**.** *Let $\mathcal{S}$ be a step set and $p$ a pattern with steps from $\mathcal{S}$. Furthermore, let $A$ be the adjacency matrix of the corresponding automaton. Then we have*

$$K(t,u) := \det(I - tA) = (1 - tP(u))R(t,u) + t^{|p|}u^{\mathrm{alt}(p)}.$$

*Proof.* It remains to show that the two representations from Equation (45) are the same, i.e., that numerators and denominators in both fractions are the same. To do so, notice that $\det(I - tA)$ is a polynomial of degree $\ell$ in $t$ and constant term 1. This is also the case for $(1 - tP(u))R(t,u) + t^{|p|}u^{\mathrm{alt}(p)}$. From this we obtain that the two numerators in Equation (45) are in fact equal. □

Before proving a similar result as Theorem 4.3.3 for meanders we first need the following

**Lemma 4.3.6.** *All roots $u(t)$ of the kernel $K(t,u)$ are either small or large. Let $d_K$ denote the degree of $K(t,u)$ in $u$ and let $\ell_K$ denote the lowest power of $u$ in the monomials of $K$. Then $K$ has exactly $e = \max(0, -\ell_K)$ small roots and $f = \max(0, d_K)$ large roots.*

*Proof.* See [3]. □

**Theorem 4.3.7.** *Let $\mathcal{S}$ be a simple set of steps and let $p$ be a pattern with steps from $\mathcal{S}$. The bivariate generating function of meanders avoiding the pattern $p$ is*

$$M(t,u) = \frac{G(t,u)}{u^e K(t,u)} \prod_{i=1}^{e} (u - u_i(t)), \tag{47}$$

*where $u_1(t), \ldots, u_e(t)$ are the small roots of the kernel $K(t,u)$ and $G(t,u)$ is a polynomial in $u$ which will be characterized in the proof.*

*Proof.* By a similar step-by-step construction as in Theorem 4.3.3 we obtain the vectorial functional equation for the generating function of meanders:

$$(M_1, \ldots, M_\ell) = (1, 0, \ldots, 0) + t(M_1, \ldots, M_\ell)A - t\{u^{<0}\}((M_1, \ldots, M_\ell)A). \tag{48}$$

However, we have to take into account that the meanders may not cross the $x$-axis, which is taken care of by $-t\{u^{<0}\}((M_1, \ldots, M_\ell)A)$ (remember that $\{u^{<0}\}$ denotes all powers of $u$ which are negative). Rewriting 48 we obtain

$$(M_1, \ldots, M_\ell)(I - tA) = (1, 0, \ldots, 0) - t\{u^{<0}\}((M_1, \ldots, M_\ell)A). \tag{49}$$

The right-hand side of (49) is a vector, its components are power series in $t$ and Laurent polynomials in $u$ (their lowest degree is $\geq -c$). Denote this vector by $F$ and its components by $F_\alpha = F_\alpha(t,u)$ for $\alpha = 1, \ldots, \ell$ (the letter $F$ can be interpreted as shorthand for "forbidden", since this vector encodes the forbidden transitions below the $x$-axis). Using this notation we have that

$$(M_1, \ldots, M_\ell)(I - tA) = (F_1, \ldots, F_\ell). \tag{50}$$

We multiply (50) from the right by $(I - tA)^{-1} = \frac{(\mathrm{adj}(I - tA)) \cdot \vec{\mathbf{1}}}{\det(I - tA)}$, where $\vec{\mathbf{1}}$ is the column vector consisting only of ones, i.e., $\vec{\mathbf{1}} = (1, \ldots, 1)^\top$. Furthermore, denote $\vec{\mathbf{v}} := \vec{\mathbf{v}}(t,u) = (\mathrm{adj}(I - tA)) \cdot \vec{\mathbf{1}}$. We obtain

$$M(t,u) = \frac{(F_1, \ldots, F_\ell)\vec{\mathbf{v}}}{K(t,u)}. \tag{51}$$

Write

$$\Phi(t,u) := u^e (F_1(t,u), \ldots, F_\ell(t,u)) \cdot \vec{\mathbf{v}} \tag{52}$$

where $e$ is the number of small roots of $K(t,u)$ and multiply (51) with $u^e K(t,u)$ to get rid of the denominator and negative $u$-powers. We obtain

$$u^e K(t,u)M(t,u) = \Phi(t,u). \tag{53}$$

What we have done so far were steps one and two of the kernel method. The variable $u$ plays the rule of a catalytic variable (we are not actually interested in final height, although the additional information provided by it is nice to have) and after lots of rewriting we arrived at Equation (53) that resembles the kernel form. We now want to make the left-hand side of (53) vanish. This can be done by plugging in $u = u_i(t)$ into (53) where $u_i$ is any small root of the kernel. Thus, the equation

$$\Phi(t,u) = 0$$

is satisfied by every small root of the kernel. Note that $\Phi$ is a Laurent polynomial since $\Phi = u^e F \cdot \vec{\mathbf{v}}$ and $F_i$ as well as $\vec{\mathbf{v}}$ are Laurent polynomials by construction. Furthermore, because of Equation (53) we have that $\Phi = u^e M(t,u)K(t,u)$ and since $M$ is a power series in $u$ and $u^e K(t,u)$ is a polynomial in $u$, the function $\Phi(t,u)$ has no negative powers of $u$ and is thus a polynomial in $u$. Since every small root $u_i(t)$ of the kernel $K(t,u)$ is a root of the polynomial equation

$$\Phi(t,u) = 0$$

it follows that

$$\Phi(t,u) = G(t,u)\prod_{i=1}^{e}(u - u_i(t)) \tag{54}$$

for some $G(t,u)$ which is a power series in $t$ and a polynomial in $u$. Substituting this into (51) we obtain

$$M(t,u) = \frac{G(t,u)}{u^e K(t,u)}\prod_{i=1}^{e}(u - u_i(t)),$$

as stated in the theorem. □

**Corollary 4.3.8.** *Let $\mathcal{S}$ be a simple set of steps and let $p$ be a pattern with steps from $\mathcal{S}$. The generating function of excursions avoiding the pattern $p$ is*

$$E(t) = M(t,0) = \lim_{t\to 0}\frac{G(t,u)}{u^e K(t,u)}\prod_{i=1}^{e}(u - u_i(t)), \tag{55}$$

*where $G$ and $u_i$ are as in Theorem 4.3.7.*

*Proof.* Follows directly from Theorem 4.3.7 and the fact that excursions are meanders ending at altitude 0. □

**Example:** To illustrate Theorem 4.3.7 and its corollary, let us have a look at Dyck paths of semilength $n$ avoiding the pattern DUDU. For the means of obtaining a generating functions for such excursions, let us first consider full length and then transition back to semilength. The pattern is $p = \text{DUDU}$ and it can be described by the automaton given in Figure 18.

The corresponding adjacency matrix is

$$A = \begin{pmatrix} u & u^{-1} & 0 & 0 \\ 0 & u^{-1} & u & 0 \\ u & 0 & 0 & u^{-1} \\ 0 & u^{-1} & 0 & 0 \end{pmatrix}.$$

Thus

$$K(t,u) = \det(I - tA) = \frac{1}{u}(-(t^3+t)u^2 + (1+t^2+t^4)u - (t^3 - t)).$$

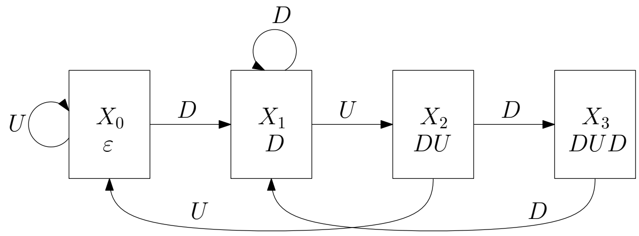



Figure 18: The automaton describing DUDU-avoiding Dyck paths.

Its zeroes are

$$u_{1/2}(t) = \frac{1+t^2+t^4 \pm \sqrt{1-2t^2-5t^4-2t^6+t^8}}{2t(1+t^2)}$$

the solution $u_1(t)$ with minus being a small root, the other one a large root. The number of small roots is $e = 1$. Computing the autocorrelation vector $\vec{\mathbf{v}}$ via $\vec{\mathbf{v}} = \operatorname{adj}(I - tA) \cdot \vec{\mathbf{1}}$ we obtain

$$\vec{\mathbf{v}} = \begin{pmatrix} 1+t^2 \\ 1+t^2-t^3u \\ 1 \\ -(t^2+1)(tu-1) \end{pmatrix}.$$

In order to compute $F$ we first need to compute

$$\{u^{<0}\}(M_0,\dots,M_3)A = \{u^{<0}\}(u(M_0+M_2), u^{-1}(M_0+M_1+M_3), uM_1, u^{-1}M_2) \quad = (0, u^{-1}m, 0, 0),$$

where $m := [u^0]M(t,u) = [u^0]M_0 + M_1 + M_2 + M_3$. The last equation holds because $[u^0]M_2 = 0$, since meanders counted by $M_2$ end in an up step and thus always at altitude greater than zero. We arrive at

$$F = (1,0,0,0) - t\{u^{<0}\}(M_0,\dots,M_3)A = (1, -tu^{-1}m, 0, 0).$$

Thus we have that

$$\Phi(t,u) = u^e F\vec{\mathbf{v}} = (t^4m + t^2 + 1)u - t^3m - tm.$$

Using $\Phi = G(t,u)(u - u_1(t))$ and comparing coefficients we obtain that $\deg_u(G) = 0$ and

$$\begin{aligned} G &= t^4m + t^2 + 1 \\ Gu_1 &= t^3m + tm. \end{aligned}$$

Solving this linear system in the two unknowns $G$ and $m$ we obtain

$$\begin{aligned} G &= -\frac{t^4 - 2t^2 + 1}{t^3u_1 - t^2 - 1} \\ m &= -\frac{u_1(t^2+1)}{t(t^3u_1 - t^2 - 1)} \end{aligned}$$

We could now compute $M(t,u)$ and subsequently $E(t)$ via Theorem 4.3.7 and Corollary 4.3.8, but since we reasoned that $m = [u^0]M(t,u) = E(t)$ earlier there is no need to do that anymore, since we already know $m$. Plugging in the value for $u_1$ we obtain after some simplifications

$$E(t) = \frac{1+t^2-t^4-\sqrt{1-2t^2-5^t4-2t^6+t^8}}{2t^2}.$$

Transitioning to semilength we obtain that the number of Dyck paths of semilength $n$ avoiding DUDU is counted by the generating function

$$E(x) = \frac{1+x-x^2-\sqrt{1-2x-5x^2-2x^3+x^4}}{2x}.$$

For certain types of patterns, the polynomial $G$ from Theorem 4.3.7 has a nice form and we obtain the following special cases.

**Definition 4.3.9.** *A* quasimeander *is a lattice path which does not cross the x-axis, except, possibly, at the last step. A* reversed meander *is a lattice path whose end point has a strictly smaller y-coordinate than all other points along the path.*

**Theorem 4.3.10** (Generating function of meanders, quasimeander pattern subcase)**.** *Let $p$ be a pattern which is a quasimeander. Then the bivariate generating function of meanders avoiding the pattern $p$ is given by*

$$M(t,u) = \frac{R(t,u)}{u^c K(t,u)} \prod_{i=1}^{e} (u - u_i(t)),$$

*where $u_1(t), \dots, u_e(t)$ are the small roots of $K(t,u) = 0$.*

*Proof.* See [3]. □

**Theorem 4.3.11** (Generating function of meanders, reversed meander pattern subcase)**.** *Let $p$ be a reversed meander. The bivariate generating function of meanders avoiding the pattern $p$ is given by*

$$M(t,u) = \frac{1}{u^e K(t,u)} \prod_{i=1}^{e} (u - u_i(t)),$$

*where $u_1(t), \dots, u_e(t)$ are the small roots of $K(t,u) = 0$.*

*Proof.* See [3]. □

As illustrated in [3] the vectorial kernel method can not only be used to count walks avoiding a given pattern, but also for counting the number of occurrences of a given pattern. Instead of prohibiting the step that would complete the pattern in the automaton, one weights this step with a new variable, marking the number of times the pattern occurs. This leads us to the use of trivariate generating functions. We will also use this in Example 4.4.2 when counting ascents in Schröder paths.

**Definition 4.3.12.** *An occurrence of a pattern $p$ in a walk $w$ is any substring of $w$ that coincides with $p$. When counting them, these occurrences need not be disjoint. For example, 121111 has three occurrences of the pattern 11.*

**Theorem 4.3.13.** *The trivariate generating function of the number of occurrences of the pattern $p$ in walks (where $t$ encodes length, $u$ encodes final altitude and $v$ encodes the number of occurrences) is given by*

$$W(t,u,v) = \frac{1}{1 - tP(u) - t^\ell u^{\mathrm{alt}(p)}(v-1)/(1-(v-1)(R(t,u)-1))}. \tag{56}$$

*Proof.* Using a similar approach as for counting walks with a forbidden pattern, let $W \equiv W(t,u,v)$ be the generating function of all walks and $W_p \equiv W_p(t,u,v)$ be the generating function of walks ending with $p$. Then the following two identities hold:

$$1 + WtP = W - W_p + v^{-1} W_p \tag{57}$$

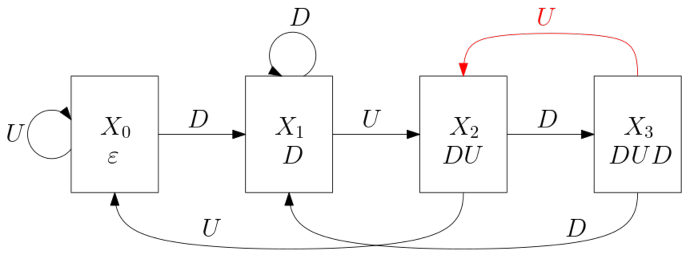
$$W t^\ell u^{\mathrm{alt}(p)} = v^{-1} W_p R - (R-1) W_p. \tag{58}$$

Identity (57) can be shown by taking a walk and appending a step to it. If the resulting walk does not end with $p$ it is counted by $W - W_p$, otherwise it is counted by $v^{-1}W_p$, the factor $v^{-1}$ comes from the fact that the new occurrence of $p$ hasn't been marked on the left-hand side.

To obtain identity (58), take a walk with $i$ occurrences of $p$ and consider what appending $p$ at the end of the walk (denoted by $w.p$) contributes to both sides of the equation. Adding the pattern $p$ at the end of the walk $w$ creates $j \geq 1$ extra occurrences of $p$ (this number may be greater than 1 because of overlaps). Thus, there are $j$ different ways in which $w.p$ can be written as $w'.r$ where $w'$ ends with $p$ and $r$ is an autocorrelation factor, or $j-1$ ways if we impose $r \neq \varepsilon$. Therefore the word $w.p$ contributes a factor $v^i$ to $Wt^\ell u^{\mathrm{alt}(p)}$, a factor $v^{i+1}+\cdots+v^{i+j}$ to $W_pR$ and with a factor $v^{i+1}+\cdots+v^{i+j-1}$ to $W_p(R-1)$.

Solving the system of equations (57) and (58) for $W$ then gives us formula (56). $\square$

In order to get a similar result for meanders, let us again consider the automaton associated with the pattern. Instead of forbidding the transition that completes the pattern we now associate a weight $v$ to the edge whose transition would lead to an occurrence of $p$. The adjacency matrix $A(u,v)$ has now entries which are Laurent polynomials in $u$ and $v$. This is illustrated in the following example where the automaton counts the occurrences of the pattern DUDU in a Dyck path. The arrow completing the pattern and the weight $v$ are marked in red.




$$A = \begin{pmatrix} u & u^{-1} & 0 & 0 \\ 0 & u^{-1} & u & 0 \\ u & 0 & 0 & u^{-1} \\ 0 & u^{-1} & uv & 0 \end{pmatrix}$$

Accordingly we define the trivatiate kernel as

$$K(t,u,v) := \det(I - tA(u,v)).$$

Note that for $v=0$ we get the same kernel as in the avoidance case and for $v=1$ we get $K(t) = 1-tP(u)$, which is the kernel for walks where any patterns are allowed. The formulas for the trivariate generating functions for meanders can now be obtained as in Theorem 4.3.7, where the $u_i$'s now are the small roots of the trivariate kernel. Similarly, the trivariate generating function for bridges or excursions can be obtained as in Corollary 4.3.4 and 4.3.8.

With the notion of the automaton we also obtain the following alternative form for the trivatiate generating function of walks:

$$W(t,u,v) = \frac{(1,0,\dots,0)\,\mathrm{adj}(I-tA(u,v))b\vec{f}1}{\det(I-tA(u,v))}. \tag{59}$$

As a consequence of Theorem 4.3.13 and Equation (59) we obtain

$$K(t,u,v) := \det(I-tA(u,v)) = (1-v)\left((1-tP(u))R(t,u)t^\ell u^{\mathrm{alt}(p)}\right) + v(1-tP(u)).$$

This equation follows because the denominators of the rational functions in Theorem 4.3.13 and Equation (59) are the same irreducible polynomial of degree $\ell$ in t.

These trivatiate generating functions allow us to study the asymptotic behavior of the number of occurrences of a pattern in a walk. Before we do so we first need to define generic walks.

**Definition 4.3.14.** *A constrained walk model is called* generic *if it fulfills the following five properties.*

- *The generating functions for bridges, meanders, and excursions denoted by $B(t)$, $M(t)$ and $E(t)$ are algebraic and not rational.*
- *These generating functions have a unique dominant singularity, which is algebraic and not a pole.*
- *The factor $G(t,u)$ defined in Equation (54) is a polynomial in $t$.*
- *Let $\varrho$ be the smallest positive real number such that a large branch meets a small branch at $t=\varrho$ (the branches refer to the roots of the kernel $K(t,u)=0$). No large negative branch (i.e. a branch such that $\lim_{t\to 0^+} = -\infty$) meets a small negative branch at $t=\varrho$.*

- *The smallest positive root of $K(t, 1)$ is simple.*

These properties are natural and hold in most cases. For a detailed analysis of cases where they are not holding, we refer to [3].

**Theorem 4.3.15.** *Let $X_n$ be the random variable counting the number of occurrences of a pattern in a generic walk, bridge, meander or excursion model. Then $X_n$ has Gaussian limiting distribution with expected vaule $\mathbb{E}[X_n] = \mu n + O(1)$ and variance $\mathbb{V}[X_n] = \sigma^2 n + O(1)$ for some constants $\mu > 0$ and $\sigma^2 \geq 0$, i.e.*

$$\frac{1}{\sqrt{n}}(X_n - \mathbb{E}[X_n]) \to \mathcal{N}(0, \sigma^2).$$

*Proof.* See [3]. □

This is an instance of what Flajolet and Sedgewick [37] called Borges' Theorem: *Any pattern not forbidden by design will appear a linear number of times with Gaussian fluctuations in large enough structures.*

Note that this is more a principle or credo than a mathematical theorem, the claim still needs to be established rigorously in each case. However, this claim is proven to hold true in many combinatorial structures like maps, trees, Markov chains, permutations, context-free grammars, and – as seen above – lattice paths.

The name Borges' Theorem is a tribute to the short story "The Library of Babel" by the Agentinian writer Jorge Luis Borges. This library is described as so huge that it contains:

> "All – *the detailed history of the future, the autobiographies of the archangels, the faithful catalog of the Library, thousands and thousands of false catalogs, the proof of the falsity of those false catalogs, a proof of the falsity of the true catalog, the gnostic gospel of Basilides, the commentary upon that gospel, the commentary on the commentary on that gospel, the true story of your death, the translation of every book into every language, the interpolations of every book into all books, the treatise Bede could have written (but did not) on the mythology of the Saxon people, the lost books of Tacitus."*
>
> J. L. Borges. *The Library of Babel*. (Translated from Spanish by Andrew Hurley)

## 4.4 THE VECTORIAL KERNEL METHOD FOR WALKS WITH LONGER STEPS

In the previous section we studied simple walks avoiding a pattern. This will be generalized to walks with longer steps now. This section is based on the paper [68].

The vectorial kernel method also works for directed walks with longer steps if the right adaptions are made. Instead of the adjacency matrix $A = A(u)$ we now have to consider the adjacency matrix $A(t, u)$ that takes into account the different lengths of the steps by weighting them with the corresponding powers of $t$, i.e. a step of length $i$ is weighted with $t^i$.

The length of a pattern is now defined to be the sum of the length of its steps, which does not necessarily coincide with the number of steps in the pattern anymore.

**Example:** Let $\mathcal{S} = \{\mathsf{U}, \mathsf{F}, \mathsf{D}\}$ where $\mathsf{U} = (1, 1), \mathsf{F} = (2, 0)$ and $\mathsf{D} = (1, -1)$ be the step set and $p = [\mathsf{U}, \mathsf{F}, \mathsf{U}, \mathsf{D}]$ the forbidden pattern. It has length 5 albeit it has only four steps, i.e., size four.

When computing the adjacency matrix of this automaton we have to keep track not only of the altitude but also of the length of the steps. We obtain

$$A = \begin{pmatrix} t^2 + tu^{-1} & tu & 0 & 0 \\ tu^{-1} & tu & t^2 & 0 \\ t^2 + tu^{-1} & 0 & 0 & tu \\ 0 & tu & t^2 & 0 \end{pmatrix}.$$

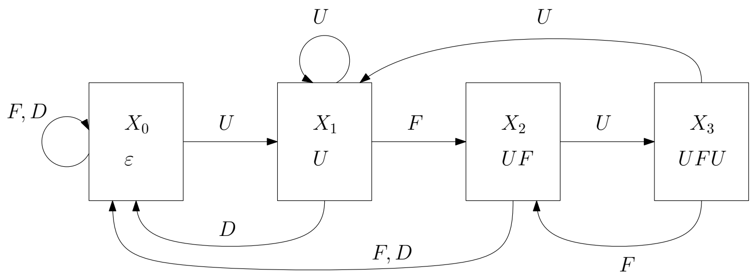



Figure 19: The automaton for $\mathcal{S} = \{\mathsf{U}, \mathsf{F}, \mathsf{D}\}$ and $p = [\mathsf{U}, \mathsf{F}, \mathsf{U}, \mathsf{D}]$

**Definition 4.4.1.** *The* kernel *of an automaton is defined to be the determinant of $I - A(t,u)$, where $A$ is the adjacency matrix of the automaton, i.e.,*

$$K(t,u) := \det(I - A(t,u)).$$

For simple walks we can pull out the factor $t$ from $A(t,u) = tA(u)$ where $A(u)$ is now independent of the variable $t$ and obtain the definition of the kernel we had earlier.

With this adapted notion of the adjacency matrix we obtain the following theorems:

**Theorem 4.4.2.** *The bivariate generating function for walks obeying constraints that can be described by a finite automaton (e.g. pattern avoidance) is given by*

$$W(t,u) = \frac{(1,0,\ldots,0)\operatorname{adj}(I - A(t,u))\vec{\mathbf{1}}}{\det(I - A(t,u))} \tag{60}$$

*where $t$ marks length and $u$ marks final altitude.*

**Theorem 4.4.3.** *The bivariate generating function for meanders obeying constraints that can be described by a finite automaton is given by*

$$M(t,u) = \frac{G(t,u)}{u^e K(t,u)} \prod_{i=1}^{e} (u - u_i(t)) \tag{61}$$

*where $t$ marks length and $u$ marks final altitude, $u_i$ $(i = 1, \ldots, e)$ are the small roots of the kernel $K(t,u)$ and $G(t,u)$ is a polynomial in $u$ which will be characterized in the proof in Equation (65).*

*Proof of Theorem 4.4.2:* The proof follows the same idea as in the case with steps of length one, which was studied in [3]. Writing $W_i := W_i(t,u)$ for the generating function of walks ending in state $X_i$ and using a step-by-step-construction we obtain the functional equation

$$(W_1, \ldots, W_\ell) = (1,0,\ldots,0) + (W_1, \ldots, W_\ell) \cdot A(t,u),$$

or equivalently

$$(W_1, \ldots, W_\ell)(I - A(t,u)) = (1,0,\ldots,0).$$

Multiplying this equation from the right with $(I - A(t,u))^{-1} = \frac{\operatorname{adj}(I-A(t,u))}{\det(I-A(t,u))}$ gives us

$$(W_1, \ldots, W_\ell) = \frac{(1,0,\ldots,0)\operatorname{adj}(I - A(t,u))}{\det(I - A(t,u))}.$$

The generating function $W(t,u)$ is the sum of the generating functions $W_i(t,u)$, thus

$$W(t,u) = (W_1, \ldots, W_\ell)\vec{\mathbf{1}} = \frac{(1,0,\ldots,0)\operatorname{adj}(I - A(t,u))\vec{\mathbf{1}}}{\det(I - A(t,u))}$$

which finishes the proof. $\square$

**Corollary 4.4.4.** *The generating function for bridges is*

$$B(t) = [u^0]W(t,u) = \frac{1}{2\pi i}\int_{|u|=\varepsilon} \frac{W(t,u)}{u} = \sum_{i=1}^{e} \mathrm{Res}_{u=u_i} \frac{W(t,u)}{u}.$$

*Proof of Theorem 4.4.3:* This proof works similarly as the one for walks, but now we also have to take care that the walk is not allowed to attain negative altitude. Writing $M_i = M_i(t,u)$ for the generating function of meanders ending in state $X_i$ of the automaton and using a step-by step construction we obtain the following vectorial functional equation

$$(M_1,\ldots,M_\ell) = (1,0,\ldots,0) + (M_1,\ldots,M_\ell)\cdot A(t,u) - \{u^{<0}\}((M_1,\ldots,M_\ell)\cdot A(t,u)).$$

This is equivalent to

$$(M_1,\ldots,M_\ell)(I - A(t,u)) = (1,0,\ldots,0) - \{u^{<0}\}((M_1,\ldots,M_\ell)\cdot A(t,u)).$$

Writing $F := (F_1,\ldots,F_\ell)$ for the right-hand side of the above equation leads to

$$(M_1,\ldots,M_\ell)(I - A(t,u)) = (F_1,\ldots,F_\ell). \tag{62}$$

Multiplying (62) from the right by $(I - A(t,u))^{-1} = \frac{\mathrm{adj}(I-A(t,u))}{\det(I-A(t,u))}$ gives us

$$(M_1,\ldots,M_\ell) = (F_1,\ldots,F_\ell)\cdot\frac{\mathrm{adj}(I-A(t,u))}{\det(I-A(t,u))}.$$

The generating function $M(t,u)$ is the sum of all the generating functions $M_i$. Using this, defining

$$\vec{v} := \mathrm{adj}(I - A(t,u))\vec{\mathbf{1}}$$

and using

$$\det(I - A(t,u)) = K(t,u)$$

we obtain

$$M(t,u) = \frac{(F_1,\ldots,F_\ell)\vec{v}}{K(t,u)}. \tag{63}$$

Let $u_i = u_i(t)$ be a small root of the kernel $K(t,u)$. We plug $u = u_i$ into (62). The matrix $(I - A(t,u))|_{u=u_i}$ is singular. Furthermore, we observe that $\vec{v}_{u=u_i}$ is an eigenvector of $(I - A(t,u))|_{u=u_i}$ for the eigenvalue $\lambda = 0$.

Hence, multiplying (62) from right with $\vec{v}_{u=u_i}$ causes the left-hand side of the equation to vanish. Said differently, the equation

$$(F_1(t,u),\ldots,F_\ell(t,u))\vec{v}(t,u) = 0$$

is satisfied by all small roots $u_i(t)$ of $K(t,u)$.

Define

$$\Phi(t,u) := u^e(F_1(t,u),\ldots,F_\ell(t,u))\vec{v}(t,u). \tag{64}$$

Note that $\Phi$ is a Laurent polynomial in $u$, because $F_i$ and $\vec{v}$ are Laurent polynomials in $u$ by construction. Because of (63) we have that

$$\Phi(t,u) = u^e M(t,u)K(t,u)$$

and because $M$ is a power series in $u$ and $K$ has exactly $e$ small roots the Laurent-polynomial $\Phi$ contains no negative powers in $u$ and is a polynomial in $u$. Each small root $u_i$ is a root of the polynomial equation $\Phi(t,u) = 0$. This gives us the factorization

$$\Phi(t,u) = G(t,u)\prod_{i=1}^{e}(u - u_i(t)) \tag{65}$$

where $G(t,u)$ is a polynomial in $u$ and formal power series in $t$. Plugging $G$ in (63) we obtain

$$M(t,u) = \frac{G(t,u)}{u^e K(t,u)} \prod_{i=1}^{e} (u - u_i(t))$$

which finishes the proof. □

**Corollary 4.4.5.** *The generating function $E(t)$ for excursions with restrictions described by a finite automaton $A(t,u)$ satisfies*

$$E(t) = M(t,0) = \frac{G(t,u)}{u^e K(t,u)} \prod_{i=1}^{e} (u - u_i(t)) \Bigg|_{u=0} .$$

*Examples*

In the following we will consider some examples illustrating applications of the previous theorems. The first example is more of the simple and introductory kind and deals with Schröder paths avoiding the pattern UF, the second one counts Schröder paths having $k$ ascents and proves a conjecture about the asymptotic behavior of the expected number of ascents.

*Number of Schröder paths of semilength $n$ avoiding* UF

Schröder paths are lattice paths consisting of the steps $\mathsf{U} = (1,1), \mathsf{D} = (1,-1)$ and $\mathsf{F} = (2,0)$ starting at $(0,0)$, ending at $(2n,0)$ and never going below the $x$-axis. Here we consider Schröder paths of length $2n$ avoiding the pattern $p = \mathsf{UF}$. These objects are enumerated by OEIS A007317 and have been studied by Yan in [77], where bijections between Schröder paths avoiding UF and Schröder paths without peaks at even level as well as two pattern avoiding partitions were constructed.

The generating function for Schröder paths avoiding the pattern UF can be obtained by a first passage decomposition – if $S^*$ denotes all Schröder paths avoiding UF, then

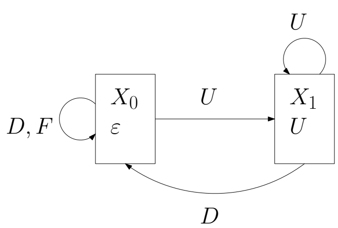
$$S^* = \varepsilon \cup \mathsf{F} \times S^* \cup \mathsf{UD} \times S^* \cup \mathsf{U} \times (S^* \setminus \{\varepsilon \cup \mathsf{F} \times S^*) \times \mathsf{D} \times S^*,$$

i.e. a Schröder path avoiding UF is either empty, or starts with either F followed by another Schröder path avoiding UF, UD and another Schröder path avoiding UF or starts with an up step, followed by an nonempty Schröder path avoiding UF which does not start with F (otherwise we would obtain an occurrence of UF), a down step to altitude zero (the first passage) and another Schröder path avoiding UF. For generating functions, this translates to

$$F(x) = 1 + 2xF(x) + x(F(x) - 1 - xF(x))F(x),$$

where $x$ encodes semilength. From here, the generating function can be obtained by solving a quadratic equation. However, in other cases a first passage decomposition may not be possible whereas the enumeration problem can still be solved by the vectorial kernel method.

The automaton describing Schröder paths avoiding UF is



Its adjacency matrix is

$$A(t,u) = \begin{pmatrix} t^2 + tu^{-1} & tu \\ tu^{-1} & tu \end{pmatrix}.$$

Thus the kernel is

$$K(t,u) = \det(I - A) = \frac{t^3u^2 - t^2u - tu^2 - t + u}{u}.$$

Its roots are

$$u_{1/2} = \frac{1 - t^2 \pm \sqrt{1 - 6t^2 + 5t^4}}{2t(1 - t^2)},$$

the root with minus being the small root.

Denote $M_0$ the generating function of the walks ending in state $X_0$, i.e., with a D or F-step, and $M_1$ the generating function of the walks ending in state $X_1$, i.e., in an U-step. A step-by-step construction gives us the following system of equations for the generating functions:

$$(M_0, M_1) = 1 + (M_0, M_1)A - \{u^{<0}\}(M_0, M_1)A.$$

This can be rewritten as

$$(M_0, M_1)(I - A) = 1 - \{u^{<0}\}(M_0, M_1)A.$$

We have that

$$\{u^{<0}\}(M_0, M_1)A = (tu^{-1}m_0, 0),$$

where $m_0 = [u^0]M_0 + M_1$. Thus the forbidden vector $F$ is

$$F = 1 - \{u^{<0}\}(M_0, M_1)A = (1 - tu^{-1}m_0, 0).$$

Using

$$\mathrm{adj}(I - A) = \begin{pmatrix} 1 - tu & tu \\ tu^{-1} & 1 - tu^{-1} - t^2 \end{pmatrix}$$

we obtain

$$\vec{v} = \mathrm{adj}(I - A) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - t^2 \end{pmatrix}.$$

Thus

$$\Phi(t,u) = u^e F\vec{v} = u - tm_0.$$

Using

$$\Phi(t,u) = G(t,u)(u - u_1)$$

and comparing coefficients we obtain

$$G(t,u) = 1.$$

From

$$M(t,u) = \frac{G(t,u)}{u^e K(t,u)}(u - u_1(t)) = \frac{1}{t^3u - t^2u - tu^2 - t + u}\left(u - \frac{1 - t^2 - \sqrt{1 - 6t^2 + 5t^4}}{2t(1 - t^2)}\right)$$

we obtain for the generating function $M(t)$ of meanders

$$M(t) = M(t,1) = \frac{2\,t^3 - t^2 - 2\,t - \sqrt{5\,t^4 - 6\,t^2 + 1} + 1}{2t\,(t^2 - 1)\,(t^3 - t^2 - 2\,t + 1)}$$

and the generating function $E(t)$ of excursions

$$E(t) = M(t,0) = \frac{1 - t^2 - \sqrt{1 - 6t^2 + 5t^4}}{2t^2(1 - t^2)}.$$

Making a transition to semilength (i.e., the substitution $x := t^2$) we end up exactly the same result for the generating function as in [77].

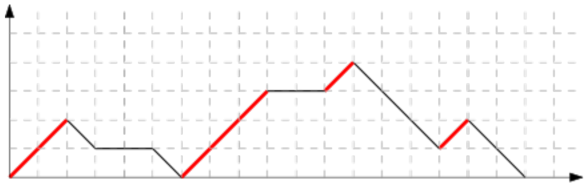


Figure 20: A Schröder path with $k = 4$ ascents (marked in red).

*Schröder paths of semilength n having k ascents*

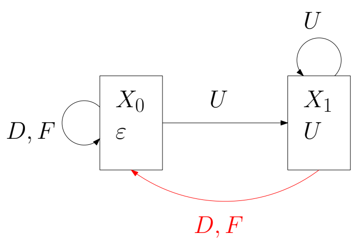
**Definition 4.4.6.** *An* ascent *in a Schröder path is a maximal string of up-steps.*

**Theorem 4.4.7.** *Let $X_n$ be the random variable counting ascents in a Schröder path of length $2n$ which is chosen uniformly at random among all Schröder paths of length $2n$. Then $\mathbb{E}[X_n] \sim (\sqrt{2} - 1)n$ for $n \to \infty$*

*Remark:* This theorem was formulated as conjecture by D. Callan in the OEIS, entry A090981.

*Proof.* The (contiguous) patterns UD and UF mark the end of an ascent. Thus, when counting ascents we want to enumerate how many times these two patterns occur. Problems like this can also be dealt with the vectorial kernel method: Instead of forbidding a transition from one state to another which would complete the pattern, we mark such transitions with a new variable and then read off the corresponding coefficients in the generating function in order to obtain the number of walks where this pattern occurs $k$ times, since it is encoded by the $k$-th power of this new variable.

Our problem can be described by the automaton:




The red arrow marks the ascents we want to count and will be marked by a new variable $v$ in the adjacency matrix. The adjacency matrix of this automaton is

$$A = \begin{pmatrix} tu^{-1} + t^2 & tu \\ (tu^{-1} + t^2)v & tu \end{pmatrix}$$

where $u$ encodes altitude, $t$ encodes length of the path, and $v$ counts the number of ascents. Thus we have

$$I - A = \begin{pmatrix} 1 - tu^{-1} - t^2 & -tu \\ -tu^{-1}v - t^2v & 1 - tu \end{pmatrix}.$$

The kernel is then given by

$$K(t,u) = \det(I - A) = u^{-1}((t^3 - t^3v - t)u^2 + (1 - t^2v)u - t). \tag{66}$$

Its roots are

$$u_{1,2} = \frac{1 - t^2v \pm \sqrt{t^4(v-2)^2 - 2t^2(v+2) + 1}}{2t(1 + t^2(v-1))},$$

the one with minus being the small root. Hence, the number of small roots is $e = 1$.

Writing $M_i = M_i(t,u,v)$ for the walks ending in state $X_i$ we obtain the vectorial functional equation

$$(M_0, M_1)(I - A) = (1,0) - \{u^{<0}\}((M_0, M_1)A). \tag{67}$$

We are interested in $M(t,0,v) = M_0(t,0,v)$, i.e. walks ending at altitude zero (since walks ending in state $X_1$ end in an up-step, they have final altitude at least 1, they will not contribute). In order to compute the forbidden vector $F = (1,0) - \{u^{<0}\}((M_0, M_1)A$ we cosider

$$\{u^{<0}\}((M_0, M_1)A = (tu^{-1}M_0 + t^2M_0 + tu^{-1}vM_1 + t^2vM_1, tu(M_0 + M_1)).$$

Writing $m_0 := [u^0]M_0(t,u)$ and using $[u^0]M_1(t,u) = 0$ we arrive at

$$\{u^{<0}\}((M_0, M_1)A) = (tu^{-1}m_0, 0)$$

and subsequently

$$F = (1 - tu^{-1}m_0, 0).$$

The adjoint of the adjacency matrix is

$$\mathrm{adj}(I - A) = \begin{pmatrix} 1 - tu & tu \\ t^2v + tu^{-1}v & -t^2 - tu^{-1} + 1 \end{pmatrix}.$$

Thus the autocorrelation vector $\vec{v}$ is

$$\vec{v} = \mathrm{adj}(I - A) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ t^2v + tu^{-1}v - t^2 - tu^{-1} + 1 \end{pmatrix}.$$

We obtain

$$\Phi(t,u) = u^eF \cdot \vec{v} = u - tm_0.$$

Using

$$\Phi(t,u) = G(t,u)(u - u_1)$$

where $u_1$ is the small root of the kernel we conclude that $\deg_u G = 0$ and by comparing coefficients we obtain that

$$G = 1 \quad \text{and} \quad Gu_1 = tm_0.$$

Thus

$$M(t,0,v) = E(t,v) = m_0 = \frac{Gu_1}{t} = \frac{1 - t^2v - \sqrt{t^4(v-2)^2 - 2t^2(v+2) + 1}}{2t^2(1 + t^2(v-1))}.$$

Transitioning to semilength $x := t^2$ (and omitting the dependency on $u$) we arrive at

$$E(x,v) = \frac{1 - xv - \sqrt{1 - 2x(v+2) + x^2(v-2)^2}}{2x(1 + x(v-1))}.$$

We are interested in the asymptotic behavior of

$$\mathbb{E}X_n = \frac{[x^n]\partial_v E(x,v)|_{v=1}}{[x^n]E(x,1)}.$$

We have

$$E(x,1) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}, \tag{68}$$

which is the generating function of Schröder paths, and

$$\partial_v E(x,v)|_{v=1} = \frac{x^2 - 5x + 2 + (x+2)\sqrt{1 - 6x + x^2}}{2\sqrt{1 - 6x + x^2}} = \frac{x+2}{2} + \frac{x^2 - 5x + 2}{2\sqrt{1 - 6x + x^2}}. \tag{69}$$

By the rules for computing limits we have

$$\lim_{n\to\infty} \mathbb{E}X_n = \lim_{n\to\infty} \frac{[x^n]\partial_v E(x,v)|_{v=1}}{[x^n]E(x,1)} = \frac{\lim_{n\to\infty}[x^n]\partial_v E(x,v)|_{v=1}}{\lim_{n\to\infty}[x^n]E(x,1)}$$

thus it remains to compute the coefficient asymptotics for (68) and (69). This can be done with the help of the methods introduced in Section 2.2.2.

First, we want to compute

$$[x^n]E(x,1) = [x^{n+1}]\frac{-\sqrt{1-6x+x^2}}{2}$$

for $n$ large. The discriminant $1-6x+x^2$ has the roots $x_{1,2} = 3 \pm \sqrt{8}$, where $\rho = 3-\sqrt{8}$ is the dominant singularity and the other singularity at $3+\sqrt{8}$ lies outside every $\Delta$-domain around $\rho$. First, we want to move the dominant singularity to one in order to fit into the framework from Section 2.2.2. This is achieved by the substitution $z = \frac{x}{3-\sqrt{8}}$. We have

$$\begin{aligned}
\sqrt{1-6x+x^2} &= \sqrt{3-\sqrt{8}-x}\cdot\sqrt{3+\sqrt{8}-x}\\
&= \sqrt{3-\sqrt{8}}\sqrt{1-z}\cdot\sqrt{3+\sqrt{8}-(3-\sqrt{8})z}\\
&\sim (3-\sqrt{8})^{1/2}(2\sqrt{8})^{1/2}\sqrt{1-z}
\end{aligned}$$

locally for $z \to 1$. Thus, by Corollary 2.2.17 with $\alpha = -\frac{1}{2}$ we obtain

$$\begin{aligned}
[x^n]E(x,1) &\sim [x^{n+1}]\frac{1}{2}(2\sqrt{8}(3-\sqrt{8}))^{1/2}\left(-\sqrt{1-\frac{x}{3-\sqrt{8}}}\right)\\
&= -\frac{1}{2}(2\sqrt{8}(3-\sqrt{8}))^{1/2}(3-\sqrt{8})^{-n-1}[z^{n+1}]\sqrt{1-z}\\
&= -\frac{1}{2}(2\sqrt{8})^{1/2}(3-\sqrt{8})^{-n-1/2}\frac{(n+1)^{-3/2}}{\Gamma(\frac{1}{2})}\\
&\sim \frac{1}{2}(3-\sqrt{8})^{-n-1/2}\frac{(2\sqrt{8})^{1/2}}{2\sqrt{\pi}}n^{-3/2}
\end{aligned} \tag{70}$$

for $n \to \infty$. In order to compute $[x^n]\partial_v E(x,v)|_{v=1}$ we first determine $[x^n](1-6x+x^2)^{-1/2}$ because this expression will appear in the computation of $[x^n]\partial_v E(x,v)|_{v=1}$. By the substitution $z = \frac{x}{3-\sqrt{8}}$ and Corollary 2.2.17 with $\alpha = \frac{1}{2}$ we obtain

$$\begin{aligned}
[x^n](1-6x+x^2)^{-1/2} &= [x^n]((3-\sqrt{8})-x)^{-1/2}((3+\sqrt{8})-x)^{-1/2}\\
&= [z^n](3-\sqrt{8})^{-n-1/2}(1-z)^{-1/2}((3+\sqrt{8})-(3-\sqrt{8})z)^{-1/2}\\
&\sim (3-\sqrt{8})^{-n-1/2}(2\sqrt{8})^{-1/2}\frac{n^{-1/2}}{\sqrt{\pi}}
\end{aligned} \tag{71}$$

for $n \to \infty$. For $n$ large we have that

$$\begin{aligned}
[x^n]\partial_v E(x,v)|_{v=1} &= \frac{1}{2}[x^n](x^2-5x+2)(1-6x+x^2)^{-1/2}\\
&= \frac{1}{2}[x^{n-2}](1-6x+x^2)^{-1/2} - \frac{5}{2}[x^{n-1}](1-6x+x^2)^{-1/2}\\
&\quad + [x^n](1-6x+x^2)^{-1/2}.
\end{aligned}$$

Using (71) and the fact that $(n-k)^{-1/2} \sim n^{-1/2}$ for $k$ constant and $n \to \infty$, after some simplifications we arrive at

$$[x^n]\partial_v E(x,v)|_{v=1} \sim \frac{(2\sqrt{8})^{1/2}}{\sqrt{\pi}}n^{-1/2}(3-\sqrt{8})^{-n-1/2}(2-\sqrt{2}) \tag{72}$$

Using the expressions for (70) and (72) we obtain that for $n \to \infty$ the expected value of ascents behaves like

$$\mathbb{E}X_n \sim \frac{(3-\sqrt{8})^{-n-1/2}(2-\sqrt{2})}{\sqrt{\pi}n^{1/2}(2\sqrt{8})^{1/2}} \cdot \frac{2\cdot 2\sqrt{\pi}n^{3/2}}{(3-\sqrt{8})^{-n-1/2}(2\sqrt{8})^{1/2}}$$

which, after some simplifications, becomes

$$\mathbb{E}X_n \sim (\sqrt{2}-1)n. \tag{73}$$

This proves Callan's conjecture. □

**Theorem 4.4.8.** *Let $X_n$ be the random variable counting ascents in a Schröder path of length $n$ which is chosen uniformly at random among all Schröder paths of length $n$. Then*

$$\mathbb{V}X_n \sim \frac{188-133\sqrt{2}}{8\sqrt{2}-12}n \approx 0.1317\, n \tag{74}$$

*for $n \to \infty$.*

*Proof.* The variance can be computed using similar means as the expected value. We have that

$$\mathbb{V}(X_n) = \frac{[x^n]\partial_v^2 E(x,v)|_{v=1}}{[x^n]E(x,1)} + \frac{[x^n]\partial_v E(x,v)|_{v=1}}{[x^n]A(x,1)} - \left(\frac{[x^n]\partial_v E(x,v)|_{v=1}}{[x^n]A(x,1)}\right)^2. \tag{75}$$

The second derivative of $E$ with respect to $v$ is given by

$$\partial_v^2 E(x,v)|_{v=1} = (-x^5+11x^4-33x^2+21x^2+2x)(x^2-6x+1)^{-3/2} - \frac{x^4-8x^3+13x^2-2x}{x^2-6x+1}.$$

Using the substitution $z = \frac{x}{3-\sqrt{8}}$ and the tables for the asymptotics of standard functions from [37], p. 388, we obtain

$$\begin{aligned}
[z^n](1-z)^{1/2} &\sim -\frac{1}{\sqrt{\pi n^3}}\left(\frac{1}{2}+\frac{3}{16n}+\frac{25}{256n^2}+\mathcal{O}\left(\frac{1}{n^3}\right)\right),\\
[z^n](1-z)^{-1/2} &\sim \frac{1}{\sqrt{\pi n}}\left(1-\frac{1}{8n}+\frac{1}{128n^2}+\mathcal{O}\left(\frac{1}{n^3}\right)\right),\\
[z^n](1-n)^{-1} &\sim 1\\
[z^n](1-z) &\sim \sqrt{\frac{n}{\pi}}\left(2+\frac{3}{4n}-\frac{7}{64n^2}\mathcal{O}\left(\frac{1}{n^3}\right)\right)
\end{aligned}$$

(we need the additional terms because there will be a cancellation of the leading terms of order $n^2$, just the previously computed terms will not do the trick).

Inserting these as well as the correct asymptotic growth rates into the formula for the variance (75) we arrive at the claim of the theorem after some cancellations and computing limits. □

We can obtain even more information about the limiting distribution of the number of ascents with the help of the Drmota-Lalley-Woods theorem.

**Theorem 4.4.9** (Drmota-Lalley-Woods theorem, limiting distribution version from [8])**.** *Suppose that $\mathbf{y} = \mathbf{P}(z,\mathbf{y},u)$ is a strongly connected and analytically well defined entire or polynomial system of equations that depends on $u$ and has a solution $\mathbf{f}$ that exists in a neighborhood of $u=1$. Furthermore, let $h(z,u)$ be given by*

$$h(z,u) = \sum_{n\geq 0} h_n(u)z^n = H(z,\mathbf{f}(z,u),u),$$

*where $H(z,y,u)$ is entire or a polynomial function with non-negative coefficients that depends on $\mathbf{y}$ and suppose that $h_n(u) \neq 0$ for all $n \geq n_0$ (for some $n_0 \geq 0$).*

*Let $X_n$ be a random variable whose distribution is defined by*

$$\mathbb{E}\left[u^{X_n}\right] = \frac{h_n(u)}{h_n(1)}.$$

*Then $X_n$ has a Gaussian limiting distribution. More precisely, we have $\mathbb{E}[X_n] = \mu n + O(1)$ and $\mathbb{V}[X_n] = \sigma^2 n + O(1)$ for constants $\mu > 0$ and $\sigma^2 \geq 0$ and*

$$\frac{1}{\sqrt{n}}(X_n - \mathbb{E}[X_n]) \to N(0,\sigma^2).$$

*Proof.* See [8] or [29]. □

**Corollary 4.4.10.** *The number of ascents in Schröder paths has a Gaussian limiting distribution with parameters* $\mu = \sqrt{2} - 1$ *and* $\sigma^2 = \frac{188-133\sqrt{2}}{8\sqrt{2}-12}$.

*Proof.* Let

$$P(z, y, u) = z(1 + z(u - 1))y^2 + zuy + 1.$$

Solving the system $y = P(z, y, u)$ gives us

$$f(z, u) = \frac{1 - zu - \sqrt{1 - 2z(u + 2) + z^2(u - 2)^2}}{2z(1 - z(u - 1))}$$

which is a formal power series in a neighborhood of $u = 1$ (the other solution with plus is not and can be disregarded). The function $f$ coincides with $E(x, v)$ (after a substitution $z = x$ and $u = v$). The system is strongly connected since it consists of only one equation in one unknown. Let $H(z, y, u) = y$ such that $H(z, f, u) = f(z, u)$. From the combinatorial interpretation we see that $h_n(u) \neq 0$ for $n \geq n_0$ (remember, $h_n(u)$ counts ascents in Schröder paths of length $n$, thus being a power series of the form $1 + c_1 u + O(u^2)$ for any $n > 0$, the 1 comes from the Schröder path consisting only of flat steps, thus having no ascent). The random variable $X_n$ counting ascents has distribution defined by

$$\mathbb{E}\left[u^{X_n}\right] = \frac{h_n(u)}{h_n(1)}.$$

Thus, we can apply the Drmota-Lalley-Woods theorem and obtain that $X_n$ has Gaussian limiting distribution. We already computed the constants $\mu = \sqrt{2} - 1$ and $\sigma^2 = \frac{188-133\sqrt{2}}{8\sqrt{2}-12}$ earlier in Equations (73) and (74). □

In the light of Borges' Theorem the Gaussian limiting distribution probably does not come as surprise (however, here we were counting the cumulative appearances of two patterns, not one as in Theorem 4.3.15 where only one pattern was considered, thus we could not apply the theorem directly).

This example also serves as some kind of preview for the next section where we will study lattice paths avoiding several patterns at once.

## 4.5 LATTICE PATHS AVOIDING SEVERAL PATTERNS

In this section, the vectorial kernel method will be further generalized to the case where the path avoids several patterns. While it was expected that the generating functions counting such paths would be algebraic, it is a nice surprise that they have a nice closed-form, involving some combinatorial determinants, and directly generalize previous results. This section is based on joint work with Andrei Asinowski and Cyril Banderier [4].

Again, let $t$ mark the length of a walk, $u$ its final altitude, and let $P(u)$ be the step polynomial. Furthermore, let us assume that none of the forbidden patterns $p_i$ is a substring of another pattern $p_j$. There is no loss of generality in this assumption, since otherwise we can restrict the set of patterns to the set of its minimal elements.

In addition to autocorrelation of one pattern, we now have to take care of mutual correlation between two patterns. This leads us to the notion of mutual correlation polynomials and the mutual correlation matrix.

**Definition 4.5.1.** *Let $p_i$ and $p_j$ be two patterns. An* overlap *of $p_j$ and $p_i$ is a non-empty suffix of $p_j$ that is also a prefix of $p_i$. Denote by $\mathcal{C}_{ij}$ the set of all complements $q$ (in $p_i$) of overlaps between $p_j$ and $p_i$. Using these sets, we define the* mutual correlation polynomials

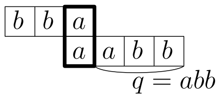
$$C_{ij}(t, u) := \sum_{q \in \mathcal{C}_{ij}} t^{|q|} u^{\mathrm{alt}(q)}. \tag{76}$$

*Furthermore, we define the* mutual correlation matrix *of $m$ patterns to be*

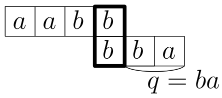
$$C(t,u) := \begin{pmatrix} C_{11} & \dots & C_{1m} \\ \vdots & \ddots & \vdots \\ C_{m1} & \dots & C_{mm} \end{pmatrix}. \tag{77}$$

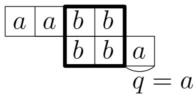
*Remark:* The polynomial $C_{ii}(t,u)$ is the classical autocorrelation polynomial of $p_i$, as defined in (42).

**Example 4.5.2.** Let $p_1 = aabb$ and $p_2 = bba$. Then we have the following overlap (marked with fat black border) between $p_2$ and $p_1$:




Thus, $\mathcal{C}_{12} = \{abb\}$.
Furthermore, we have the following overlaps between $p_1$ and $p_2$:




We obtain that $\mathcal{C}_{21} = \{a, ba\}$.

**Theorem 4.5.3.** *The generating function of walks with steps encoded by the step polynomial $P(u)$ and avoiding the patterns $p_1, \dots, p_m$ is given by*

$$W(t,u) = \frac{\Delta(t,u))}{(1-tP(u))\Delta(t,u) + \sum_{i=1}^{m} t^{|p_i|} u^{\mathrm{alt}(p_i)} \Delta_i(t,u)}, \tag{78}$$

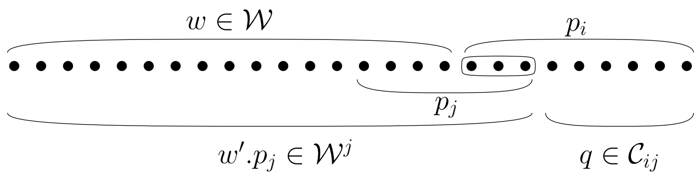
*where $\Delta(t,u) := \det(C(t,u))$ is the determinant of the mutual correlation matrix $C$ defined in (77) and $\Delta_i$ is the determinant of the mutual correlation matrix where the $i$-th row of the matrix is replaced with ones, i.e., $C_{ji} = 1$ for $j = 1, \dots, m$.*

*Proof.* Let $\mathcal{W}$ be the set of walks avoiding all of the patterns $p_1, \dots, p_m$. Let $W(t,u)$ be the generating function of $\mathcal{W}$. Furthermore, let $W^{(i)}(t,u)$ be the generating function of all walks that have exactly one occurrence of $p_i$ at the very end, but no occurrence of $p_i$ earlier, as well as no occurrence of any of the other patterns $p_j$.

If we append one step from $\mathcal{S}$ to a walk from $\mathcal{W}$ we either obtain another walk in $\mathcal{W}$ or a walk with a single occurrence of a pattern $p_i$ at the end. This pattern is uniquely determined, thus these walks are counted by $W^{(i)}$. For the generating functions this means

$$1 + W(t,u)tP(u) = W + \sum_{i=1}^{m} W^{(i)}. \tag{79}$$

Now take a walk $w \in \mathcal{W}$ and append a pattern $p_i$ to it. Write $w.p_i$ for such constructions. We end up with a walk that ends in $p_i$, but might have occurrences of other patterns earlier. More precisely, let $q$ be the maximal (possible empty) suffix of $w.p_i$ such that $w.p_i = w'.p_j.q$ where $w'$ is another walk in $\mathcal{W}$ and $p_j$ is one of the forbidden patterns ($p_j = p_i$ is possible). Then $q$ is the complement of an overlap of $p_j$ and $p_i$.




Recall that these complements of overlaps are described by the mutual correlation polynomials $C_{ij}$ defined in (76). Using these notations we have

$$\begin{cases} Wt^{|p_1|}u^{\mathrm{alt}(p_1)} = \sum_{j=1}^{m} W^{(j)} C_{1j}(t,u) \\ \vdots \\ Wt^{|p_m|}u^{\mathrm{alt}(p_m)} = \sum_{j=1}^{m} W^{(j)} C_{mj}(t,u). \end{cases} \tag{80}$$

The equations (79) and (80) form a linear system in $m+1$ equations and $m+1$ unknowns, namely $W, W^{(1)}, \ldots, W^{(m)}$. We want to solve it for $W$. Because of Equation (79) we have

$$W = \frac{1 - \sum W^{(i)}}{1 - tP(u)}.$$

The sum $\sum W^{(i)}$ can be determined from (80) with the help of Cramer's rule:

$$\sum W^{(i)} = \frac{W}{\Delta} \cdot \left( \sum_{k=1}^{m} t^{|p_i|} u^{\mathrm{alt}(p_i)} \Delta_i \right),$$

where $\Delta$ and $\Delta_i$ are defined as stated in the theorem. Putting everything together, we obtain the claim of the theorem. □

*Remark:* Alternatively, one could also encode the simultaneous avoidance of patterns by a finite automaton. However, obtaining generating functions via this approach is quite costly in time and memory, since it requires the inversion of an $\ell \times \ell$ matrix, where $\ell = \sum_{i=1}^{m} |p_i|$ is the sum of the lengths of the forbidden patterns. The computation via formula (78) is algorithmically much more efficient. This formula can also be established via Goulden and Jackson's *cluster method*, which was established in [40] and generalized in [62].

**Definition 4.5.4.** *Define the denominator of $W(t,u)$ to be the* kernel *of the model*

$$K(t,u) := \mathit{denom}(W(t,u)) = (1 - tP(u))\Delta(t,u) + \sum_{i=1}^{m} t^{|p_i|} u^{\mathrm{alt}(p_i)} \Delta_i(t,u). \tag{81}$$

In addition to the mutual correlation matrix this kernel plays an important role in enumeration lattice paths avoiding several patterns. The kernel has $e$ distinct small roots $u_1, \ldots, u_e$ which, similarly as in the classical model of directed lattice paths, play an important role in expressing the generating function of lattice paths avoiding several patterns.

Here is what it gives for bridges:

**Theorem 4.5.5.** *The generating function for bridges avoiding the patterns $p_1, \ldots, p_m$ is*

$$B(t) = -\sum_{i=1}^{e} \frac{u_i'}{u_i} \frac{\Delta(t,u_i)}{K_t(t,u_i)} \tag{82}$$

*where $u_i$ are the small roots of the kernel.*

*Proof.* The proof uses a similar residue computation as 4.3.4. We have

$$B(t) = [u^0]W(t,u) = \frac{1}{2\pi i}\int_{|u|=\varepsilon} \frac{W(t,u)}{u}\,du = \sum_i^e \mathrm{Res}_{u=u_i(t)} \frac{W(t,u)}{u}.$$

The residues inside the small circle $|u| = \varepsilon$ are exactly the $e$ small roots $u_i(t)$ of $K(t,u)$. This leads to the theorem. ☐

**Theorem 4.5.6.** *The generating function for meanders avoiding the patterns $p_1, \ldots, p_m$ is*

$$M(t,u) = \frac{G(t,u)}{u^e K(t,u)} \prod_{i=1}^{e} (u - u_i(t)), \tag{83}$$

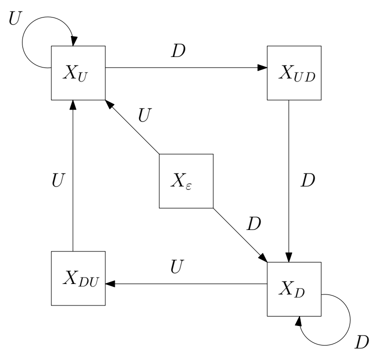
*where $u_1(t), \ldots, u_e(t)$ are all the small roots of the kernel $K(t,u)$ (defined in Formula (81)), and $G(t,u)$ is some formal power series in $t$ and polynomial in $u$ (as defined in (65)).*

*Proof.* The proof works exactly like the proof of Theorem 4.3.7, using the automaton that describes the avoidance of all patterns and its corresponding states. ☐

If $\Phi(t,u)$ is a monic polynomial of degree $e$, then we have the complete factorization $\Phi(t,u) = \prod_{i=1}^{e}(u-u_i)$ and thus $G(t,u) = 1$: this yields an explicit formula for $M(t,u)$ in terms of $K(t,u)$ and its small roots. It is shown in 4.3.10 and 4.3.11 that this happens for some natural cases of a single forbidden pattern.

When considering the avoidance of several patterns, it is generically not the case that $G(t,u) = 1$. In the next sections, we show how, in many cases, this factor $G(t,u)$ can be obtained. We also show how to obtain the formula for $E(t)$ without computing first $M(t,u)$.

**Example:** We illustrate the procedure outlined above by the example of Dyck paths avoiding the patterns UDU and DUD. These walks are encoded by the following automaton:




The corresponding adjacency matrix is (the states are ordered $X_\varepsilon, X_{\mathsf{U}}, X_{\mathsf{UD}}, X_{\mathsf{D}}, X_{\mathsf{DU}}$):

$$A = \begin{pmatrix} 0 & u & 0 & u^{-1} & 0 \\ 0 & u & u^{-1} & 0 & 0 \\ 0 & 0 & 0 & u^{-1} & 0 \\ 0 & 0 & 0 & u^{-1} & u \\ 0 & u & 0 & 0 & 0 \end{pmatrix}.$$

The kernel $K(t,u) = -u^{-1}(tu^2 - (1+t^2-t^4)u + t)$ can be calculated directly as $\det(I - tA)$, but also by Theorem 4.5.3 with the mutual correlation matrix

$$C = \begin{pmatrix} 1+t^2 & tu \\ t/u & 1+t^2 \end{pmatrix}.$$

The kernel has a unique small root, namely

$$u_1(t) = \frac{1+t^2-t^4-\sqrt{(1+t+t^2)(1+t-t^2)(1-t+t^2)(1-t-t^2)}}{2t}.$$

The functional equation for the generating function has the form

$$(M_1, M_2, M_3, M_4, M_5)(I-tA) = (1,0,0,0,0) - (\{u^{<0}\}t(M_1, M_2, M_3, M_4, M_5)A).$$

It is easy to see that

$$(\{u^{<0}\}t(M_1, M_2, M_3, M_4, M_5)A) = \frac{t}{u}(0,0,0,E(t),0)$$

because a path $w.a$ (where $w$ is a meander and $a$ is a step) goes below the $x$-axis if and only if $w$ is an excursion and the appended step $a$ is a $D$-step, and upon making a down-step the path enters the 4th state $X_{\mathsf{D}}$. Hence, we only need to compute the components $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_4$ of the autocorrelation vector $\vec{\mathbf{v}} = \operatorname{adj}(I-tA)\vec{\mathbf{1}}$ which are given by $\vec{\mathbf{v}}_1 = 1+t^2+t^4$ and $\vec{\mathbf{v}}_4 = 1+t^3u$. Thus

$$\Phi(t,u) = (1+t^2+t^4)u - tE(t)(1+t^3u). \tag{84}$$

Solving $\Phi(t,u)=0$ for $E(t)$ and keeping in mind that $u_1(t)$ is a root of $\Phi(t,u)=0$ we obtain

$$E(t) = \frac{u_1(t)(1+t^2+t^4)}{t(1+t^3u_1(t))} = \frac{1+t^2+t^4-\sqrt{(1+t+t^2)(1+t-t^2)(1-t+t^2)(1-t-t^2)}}{2t^2}.$$

Since $\Phi = G(t,u)(u-u_1)$ is a polynomial of degree 1 in $u$ we see by equating coefficients that $G(t)$ is the leading coefficient of $\Phi$ (as a polynomial in $u$). From (84) we obtain $G(t) = 1+t^2+t^4-t^4E(t)$. Using Theorem 4.5.6 we obtain the bivariate generating function for meanders, which, after inserting $u=1$ gives the univariate function

$$M(t) = -\left(\frac{1-t^3}{2t} - \frac{(1+t)\sqrt{(1+t+t^2)(1+t-t^2)(1-t+t^2)(1-t-t^2)}}{2t(1-t-t^2)}\right).$$

The enumerating sequence for meanders is the sequence A329703 from the On-Line Encyclopedia of Integer Sequences, and the one for excursions (counted by semilength) is A004148. The latter also counts some other constrained paths (like peakless Motzkin paths – see below), as well as some classes of RNA structures, ordered trees or permutations, see [26, 15, 45, 75].

*A multi-multivariate generating function for Motzkin paths with any set of forbidden patterns of length two*

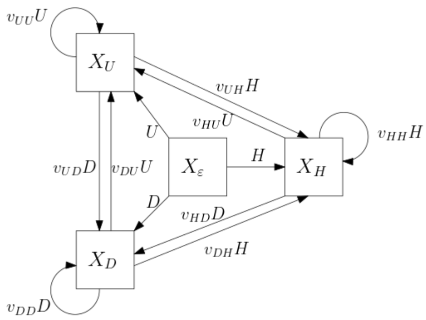
There is a vast amount of literature on Dyck or Motzkin lattice paths in which some combinations of forbidden patterns (like valleys or peaks) are considered. These works often rely on some ad-hoc context-free grammar decompositions; see e.g. [60, 31, 24]. Here, we show how our approach can extend and unify such results by directly finding a generating function with markers which indicate whether a certain pattern appears or not. For example, for Motzkin paths avoiding any combination of forbidden patterns of length 2, one introduces 9 *markers* – auxiliary variables $v_p$ that encode occurrences of all possible patterns $p$ of length 2 (marker $v_{\mathsf{UD}}$ for the pattern UD, etc.). This leads to the following theorem.

**Theorem 4.5.7.** *The generating function $E(t)$ of Motzkin excursions, where $v_p$ counts the number of occurrences of the pattern $p$, is*

$$\frac{(v_{\mathsf{DD}}-1) - t((v_{\mathsf{DD}}-1)v_{\mathsf{HH}} - (v_{\mathsf{DH}}-1)v_{\mathsf{HD}} - v_{\mathsf{DD}} + v_{\mathsf{DH}}) + \big(1+t(v_{\mathsf{DH}}-v_{\mathsf{HH}})\big)\frac{u\vec{\mathbf{v}}_1}{t\vec{\mathbf{v}}_4}\Big|_{u=u_1(t)}}{v_{\mathsf{DD}} + t(v_{\mathsf{DH}}v_{\mathsf{HD}} - v_{\mathsf{DD}}v_{\mathsf{HH}})}, \tag{85}$$

*where $u_1(t)$ is the unique small solution of $\det(I-tA)=0$ for the matrix $A$ defined below, and $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_4$ are the first and the fourth components of the autocorrelation vector $\vec{\mathbf{v}} := \operatorname{adj}(I-tA)\vec{\mathbf{1}}$.*

*Proof.* These paths are encoded by the following automaton:




The corresponding adjacency matrix is

$$A = \begin{pmatrix} 0 & u & 1 & u^{-1} \\ 0 & v_{\mathsf{UU}}u & v_{\mathsf{UH}} & v_{\mathsf{UD}}u^{-1} \\ 0 & v_{\mathsf{HU}}u & v_{\mathsf{HH}} & v_{\mathsf{HD}}u^{-1} \\ 0 & v_{\mathsf{DU}}u & v_{\mathsf{DH}} & v_{\mathsf{DD}}u^{-1} \end{pmatrix}.$$

From this matrix we can compute the kernel $K(t,u) = \det(I - tA)$ which is given by

$$\begin{aligned} K(t,u) = -\frac{1}{u}((&v_{\mathsf{HU}}v_{\mathsf{UH}} - v_{\mathsf{HH}}v_{\mathsf{UU}} + v_{\mathsf{UU}})t^2u^2 + (v_{\mathsf{DU}}v_{\mathsf{UD}} - v_{\mathsf{DD}}v_{\mathsf{UU}})t^2u + v_{\mathsf{HH}}tu - u \\ &+ (v_{\mathsf{DD}}v_{\mathsf{HH}}v_{\mathsf{UU}} - v_{\mathsf{DD}}v_{\mathsf{HU}}v_{\mathsf{UH}} - v_{\mathsf{DH}}v_{\mathsf{HD}}v_{\mathsf{UU}})t^3u \\ &+ (v_{\mathsf{DH}}v_{\mathsf{HU}}v_{\mathsf{UD}} + v_{\mathsf{DU}}v_{\mathsf{HD}}v_{\mathsf{UH}} - v_{\mathsf{DU}}v_{\mathsf{HH}}v_{\mathsf{UD}})t^3u \\ &+ (v_{\mathsf{DH}}v_{\mathsf{HD}} - v_{\mathsf{DD}}v_{\mathsf{HH}})t^2 + v_{\mathsf{DD}}t). \end{aligned}$$

Since only the second column of the adjacency matrix contains $u^1$, only the last column contains $u^{-1}$ and all other columns contain only powers $u^0$ we have that $uK(t,u)$ is a polynomial of degree (at most) 2 in $u$ and has one small root $u_1(t)$.

A path can cross the $x$-axis only by being on the $x$-axis and then taking a down step, hence entering the fourth state. Thus only the fourth component of $t\{u^{<0}\}\vec{M}A$ has terms with negative powers of $u$. Therefore, only the first and the fourth component of $\vec{F} = (1,0,0,0) - t\{u^{<0}\}\vec{M}A$ are nonzero and one has

$$\Phi(t,u) := \vec{\mathbf{v}}_1(t,u) - \vec{\mathbf{v}}_4(t,u)N(t,u) = 0, \tag{86}$$

where $\vec{\mathbf{v}}_1(t,u)$ and $\vec{\mathbf{v}}_4(t,u)$ are the first and the fourth components of $\operatorname{adj}(I - tA)\vec{\mathbf{1}}$, and $N(t,u)$ is the generating function for the terms with negative powers of $u$ in the fourth component of $t\{u^{<0}\}\vec{M}A$.

In order to use (86) for computing $E(t)$ we need to relate $N(t,u)$ to $E(t)$. This can be achieved in the following way: Let $E_H(t)$ and $E_D(t)$ be the generating functions for excursions whose last step is $H$ or $D$ respectively. Clearly we have

$$E(t) = 1 + E_H(t) + E_D(t).$$

Furthermore we have

$$N(t,u) = \frac{t}{u}(1 + v_{HD}E_H(t) + v_{DD}E_D(t))$$

and

$$E_H(t) = t(1 + v_{HH}E_H(t) + v_{DH}E_D(t)).$$

These three equations allow us to express $N$ in terms of $E$, namely

$$N(t,u) = \frac{t}{u}\left(1 + (v_{\mathsf{DH}} - v_{\mathsf{DD}}) \cdot \frac{t + tv_{\mathsf{DH}}(E(t) - 1)}{1 + tv_{\mathsf{DH}} - tv_{\mathsf{HH}}} + v_{\mathsf{DD}}E(t) - v_{\mathsf{DD}}\right).$$

Plugging this expression in (86) and using that $u_1(t)$ is a root of $\Phi$ we obtain the formula for $E(t)$ stated in the theorem. □

Setting $v_p = 1$ in (85) allows the pattern $p$, while setting $v_p = 0$ forbids it. An exhaustive analysis of all the $2^9 = 512$ cases leads to the following 75 distinct sequences for excursions, summarized in Table 3.

| Allowed patterns | OEIS[2] entry | GF | Growth rate | Allowed patterns | OEIS entry | GF | Growth rate | Allowed patterns | OEIS entry | GF | Growth rate |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 000000000 | A019590 | pol | 0 | 010011100 | A020711† | rat | ≈ 1.466 | 101010101 | A329696 | alg | 2 |
| 010001000 | A329670 | pol | 0 | 010011110 | A000930 | rat | ≈ 1.466 | 011110101 | A329695† | alg | 2 |
| 010001010 | A329677 | pol | 0 | 011110001 | A020711† | rat | ≈ 1.466 | 101010111 | A110199 | alg | 2 |
| 001000000 | A130716 | pol | 0 | 001110010 | A068921 | rat | ≈ 1.466 | 011110011 | A216604† | alg | 2 |
| 001000010 | A329678 | pol | 0 | 010101101 | A329687 | alg | ≈ 1.587 | 101110011 | A329698 | alg | 2 |
| 011001010 | A329679 | pol | 0 | 011100011 | A329688 | alg | ≈ 1.587 | 010111011 | A023432† | alg | ≈ 2.148 |
| 010001100 | A329680 | rat | 1 | 010101011 | A329689 | alg | ≈ 1.618 | 011111010 | A023432† | alg | ≈ 2.148 |
| 110001001 | A135528 | rat | 1 | 110011011 | A324969 | rat | ≈ 1.618 | 101001111 | A329699 | alg | ≈ 2.206 |
| 010001110 | A011655† | rat | 1 | 011101010 | A320690 | alg | ≈ 1.618 | 101100111 | A329700 | alg | ≈ 2.206 |
| 001000100 | A329681 | rat | 1 | 011011100 | A001611† | rat | ≈ 1.618 | 110011111 | A217282† | alg | ≈ 2.241 |
| 011100001 | A329682 | rat | 1 | 011011110 | A000045† | rat | ≈ 1.618 | 101111011 | A217282 | alg | ≈ 2.241 |
| 000010000 | A000012 | rat | 1 | 110001101 | A329691 | alg | ≈ 1.755 | 110101111 | A329676 | alg | ≈ 2.247 |
| 001100010 | A100063 | rat | 1 | 011100101 | A329692 | alg | ≈ 1.755 | 011101111 | A329666 | alg | ≈ 2.247 |
| 010011000 | A329683 | rat | 1 | 101100011 | A329693 | alg | ≈ 1.755 | 010111111 | A023431 | alg | ≈ 2.315 |
| 110011001 | A065033 | rat | 1 | 010101111 | A248100 | alg | ≈ 1.835 | 011111011 | A023431† | alg | ≈ 2.315 |
| 010011010 | A000027† | rat | 1 | 011101011 | A329694 | alg | ≈ 1.835 | 101011111 | A329701 | alg | ≈ 2.325 |
| 001010000 | A329684 | rat | 1 | 110001111 | A025250† | alg | ≈ 1.947 | 101110111 | A329702 | alg | ≈ 2.325 |
| 001010100 | A040001 | rat | 1 | 011100111 | A166289 | alg | ≈ 1.947 | 101101111 | A007477 | alg | ≈ 2.383 |
| 001010011 | A046698† | rat | 1 | 110101011 | A329664 | alg | 2 | 110111011 | A004149† | alg | ≈ 2.414 |
| 001010110 | A008619 | rat | 1 | 101000101 | A126120† | alg | 2 | 011111110 | A004149† | alg | ≈ 2.414 |
| 011011010 | A028310 | rat | 1 | 101000111 | A208355† | alg | 2 | 101111111 | A090344 | alg | ≈ 2.562 |
| 110001011 | A000931† | rat | ≈ 1.325 | 110011101 | A329695 | alg | 2 | 110111111 | A004148 | alg | ≈ 2.618 |
| 011001100 | A000931† | rat | ≈ 1.325 | 010111101 | A216604 | alg | 2 | 011111111 | A004148† | alg | ≈ 2.618 |
| 001100110 | A000931† | rat | ≈ 1.325 | 010111010 | A023426† | alg | 2 | 111101111 | A104545 | alg | ≈ 2.732 |
| 010101010 | A329686 | alg | ≈ 1.414 | 011101110 | A329671 | alg | 2 | 111111111 | A001006 | alg | 3 |

Table 3: Motzkin excursions avoiding a set of patterns of length 2. The allowed patterns are indicated via a binary word of length 9, whose bits correspond to the allowance (or not) of UU, UH, UD, HU, HH, HD, DU, DH, DD (in this order). The column GF indicates whether the generating function is polynomial (pol), rational but not polynomial (rat), or algebraic but not rational (alg).

This exhaustive analysis also shows that the 512 cases lead to 158 distinct sequences for meanders.

Via our approach, it is not difficult to get the explicit formulas for $E(t)$ and $M(t)$. These generating functions with all the markers $v_p$ are however quite lengthy (written out in the same font as used in this thesis, they take up almost 3 meters).

Thus we only give these explicit formulas when the set of patterns is a subset of $\{\mathsf{UU}, \mathsf{HH}, \mathsf{DD}\}$ or a subset of $\{\mathsf{UD}, \mathsf{HH}, \mathsf{DU}\}$, see Table 4.

[2] All the sequences labeled A329xxx are entries that we added to the On-Line Encyclopedia of Integer Sequences [63].The sequences marked by † are in the OEIS, but with a few terms of offset.

| Forbidden patterns | Generating functions of meanders and excursions | OEIS[3] | Growth rate |
|---|---|---|---|
| $\mathsf{UU}, \mathsf{HH}, \mathsf{DD}$ | $M = -(1+t)\left((1+t)(1-2t) - \sqrt{1-2t+t^2-4t^3+4t^4}\right)/\left(2t^2(1-2t)\right)$<br>$E = (1+t)\left(1-t^2-2t^3-(1+t)\sqrt{1-2t+t^2-4t^3+4t^4}\right)/(2t^4)$ | A329665<br>A329671 | 2 |
| $\mathsf{UU}, \mathsf{HH}$ | $M = -(1+t)\left(1-3t^2-t^3-\sqrt{1-2t^2-6t^3-3t^4+2t^5+t^6}\right)/\left(2t^2(1-2t-t^2)\right)$<br>$E = \left(1-t^2-t^3-\sqrt{1-2t^2-6t^3-3t^4+2t^5+t^6}\right)/(2t^3).$ | A329667<br>A329666 | $A_{\mathsf{UU},\mathsf{HH}} :=$<br>$\rho(1-t-2t^2+t^3)$ |
| $\mathsf{UU}, \mathsf{DD}$ | $M = -\left((1+t)(1-2t-t^2) - \sqrt{1-2t-t^2-t^4+2t^5+t^6}\right)/\left(2t^2(1-2t-t^2)\right)$<br>$E = \left(1-t-t^2-t^3-\sqrt{1-2t-t^2-t^4+2t^5+t^6}\right)/(2t^4)$ | A308435<br>A004149† | $1+\sqrt{2}$ |
| $\mathsf{HH}, \mathsf{DD}$ | $M = -\left(1-2t-3t^2-t^3-\sqrt{1-2t^2-6t^3-3t^4+2t^5+t^6}\right)/\left(2t(1+t)(1-2t-t^2)\right)$<br>$E = \left(1-t^2-t^3-\sqrt{1-2t^2-6t^3-3t^4+2t^5+t^6}\right)/(2t^3)$ | A329669<br>A329666 | $1+\sqrt{2}$<br>$A_{\mathsf{UU},\mathsf{HH}}$ |
| $\mathsf{UU}$ | $M = -(1+t)\left(1-t-3t^2-\sqrt{1-2t-t^2-2t^3+t^4}\right)/\left(2t^2(1-2t-2t^2)\right)$<br>$E = \left(1-t-t^2-\sqrt{1-2t-t^2-2t^3+t^4}\right)/(2t^3)$ | A329672<br>A004148† | $(3+\sqrt{5})/2$ |
| $\mathsf{HH}$ | $M = -\left(1-2t-2t^2-\sqrt{1-4t^2-8t^3-4t^4}\right)/\left(2t(1-2t-2t^2)\right)$<br>$E = \left(1-\sqrt{1-4t^2-8t^3-4t^4}\right)/\left(2t^2(1+t)\right)$ | A329673<br>A104545 | $1+\sqrt{3}$ |
| $\mathsf{DD}$ | $M = -\left(1-3t-t^2-\sqrt{1-2t-t^2-2t^3+t^4}\right)/\left(2t(1-2t-2t^2)\right)$<br>$E = \left(1-t-t^2-\sqrt{1-2t-t^2-2t^3+t^4}\right)/(2t^3)$ | A329674<br>A004148† | $1+\sqrt{3}$<br>$(3+\sqrt{5})/2$ |
| $\mathsf{UD}, \mathsf{HH}, \mathsf{DU}$ | $M = -(1+t)\left((1+t)(1-2t) - \sqrt{1-2t+t^2-4t^3+4t^4}\right)/\left(2t^2(1-2t)\right)$<br>$E = (1+t)\left(1-t-\sqrt{1-2t+t^2-4t^3+4t^4}\right)/(2t^3)$ | A329665<br>A329664 | 2 |
| $\mathsf{UD}, \mathsf{HH}$ | $M = -\left(1-2t-t^2+t^3-\sqrt{1-2t^2-6t^3-3t^4+2t^5+t^6}\right)/\left(2t(1-2t-t^2+t^3)\right)$<br>$E = \left(1+t^2+t^3-\sqrt{1-2t^2-6t^3-3t^4+2t^5+t^6}\right)/\left(2t^2(1+t)\right)$ | A329675<br>A329676 | $A_{\mathsf{UU},\mathsf{HH}}$ |
| $\mathsf{UD}, \mathsf{DU}$ | $M = -\left((1+t)(1-2t-t^2) - \sqrt{1-2t-t^2-t^4+2t^5+t^6}\right)/\left(2t^2(1-2t-t^2)\right)$<br>$E = \left(1-t-t^2-t^3-\sqrt{1-2t-t^2-t^4+2t^5+t^6}\right)/(2t^4)$ | A308435<br>A004149† | $1+\sqrt{2}$ |
| $\mathsf{HH}, \mathsf{DU}$ | $M = -(1+t)\left(1-t-3t^2+t^4-(1-t)\sqrt{1-2t^2-6t^3-3t^4+2t^5+t^6}\right)/\left(2t^2(1-2t-t^2+t^3)\right)$<br>$E = \left(1-t^2-t^3-\sqrt{1-2t^2-6t^3-3t^4+2t^5+t^6}\right)/(2t^3)$ | A329668<br>A329666 | $A_{\mathsf{UU},\mathsf{HH}}$ |
| $\mathsf{UD}$ | $M = -\left(1-3t+t^2-\sqrt{1-2t-t^2-2t^3+t^4}\right)/\left(2t(1-3t+t^2)\right)$<br>$E = \left(1-t+t^2-\sqrt{1-2t-t^2-2t^3+t^4}\right)/(2t^2)$ | A088518*<br>A004148† | $(3+\sqrt{5})/2$ |
| $\mathsf{DU}$ | $M = -\left((1+t)(1-3t+t^2)-(1-t)\sqrt{1-2t-t^2-2t^3+t^4}\right)/\left(2t^2(1-3t+t^2)\right)$<br>$E = \left(1-t-t^2-\sqrt{1-2t-t^2-2t^3+t^4}\right)/(2t^3)$ | A088518*<br>A004148† | $(3+\sqrt{5})/2$ |
| none | $M = -\left(1-3t-\sqrt{1-2t-3t^2}\right)/\left(2t(1-3t)\right)$<br>$E = \left(1-t-\sqrt{1-2t-3t^2}\right)/(2t^2).$ | A005773†<br>A001006 | 3 |

Table 4: Generating functions and growth rates of Motzkin excursions and meanders where the set of forbidden patterns is a subset of $\{\mathsf{UU}, \mathsf{HH}, \mathsf{DD}\}$ or a subset of $\{\mathsf{UD}, \mathsf{HH}, \mathsf{DU}\}$.

The asymptotic behavior of these sequences is $\frac{C}{\sqrt{\pi}} A^n n^\alpha$, where the constant $C$ and the growth rate $A$ are algebraic numbers depending on the model. The notation $\rho(P)$ used in the definition of $A_{\mathsf{UU},\mathsf{HH}}$ in the table stands for the largest positive root of the polynomial $P$.
The *drift* is the quantity

$$\delta := \lim_{n\to\infty} \frac{\text{average final altitude of walks on } \mathbb{Z} \text{ of length } n}{n}.$$

Unlike in the article [9], it is no longer the case that $\delta = P'(1)$ because there is an interplay between the forbidden patterns and the allowed steps $\mathcal{S}$.

The exponent $\alpha$ depends only on the sign of the drift. It is:

- $\alpha = -3/2$ for meanders with negative drift (above, when $\{\mathsf{UU}, \mathsf{HH}\}$ or $\{\mathsf{UU}\}$ are forbidden) and for excursions,
- $\alpha = 0$ for meanders with positive drift (above, when $\{\mathsf{HH}, \mathsf{DD}\}$ or $\{\mathsf{DD}\}$ are forbidden),
- $\alpha = -1/2$ for meanders with zero drift.

[3] See footnote 2 for the † symbol. Also, the sequences marked by * are bisections of A088518 that enumerates "symmetric secondary structures of RNA molecules with $n$ nucleotides", see [45].

Another interesting feature of the above tables is that they suggest there could be natural bijections between different classes of pattern-avoiding Motzkin paths, namely between:

- $\mathsf{UU}, \mathsf{HH}, \mathsf{DD}$ avoiding meanders and $\mathsf{UD}, \mathsf{HH}, \mathsf{DU}$ avoiding meanders
- $\mathsf{UU}, \mathsf{HH}$ avoiding excursions and $\mathsf{HH}, \mathsf{DD}$ avoiding excursions (time reversal)
- $\mathsf{UU}, \mathsf{DD}$ avoiding meanders and $\mathsf{UD}, \mathsf{DU}$ avoiding meanders
- $\mathsf{UU}, \mathsf{DD}$ avoiding excursions of length $n$ and $\mathsf{UD}, \mathsf{DU}$ avoiding excursions of length $n+1$
- $\mathsf{UU}$ avoiding excursions of length $n$, $\mathsf{DD}$ avoiding excursions of length $n$ (time reversal), $\mathsf{UD}$ avoiding excursions of length $n+1$ and $\mathsf{DU}$ avoiding excursions of length $n$

Some of these bijections are easy to construct, for example, UU-avoiding walks become DD avoiding walks under time reversal. Others however are not so straightforward, as some of the following bijections show.

**$\mathsf{DU}$ avoiding excursions of length $n$ and $\mathsf{UD}$-avoiding excursions of length $n+1$**

In a DU avoiding excursions, replace each occurrence of UD by $UHD$ and each occurrence of $\mathsf{UH}^k\mathsf{D}$ by $\mathsf{UH}^{k+1}\mathsf{D}$. This yields a longer walk (more precisely, a walk of length $n+m$, where $m$ is the number of *plateaus* $\mathsf{UH}^k\mathsf{D}$) which is still an excursion. But we want a walk of length $n+1$, thus some corrections have to be made. Note that for each $\mathsf{UH}^k\mathsf{D}$ (except the last one) there is a *trough* $\mathsf{DH}^\ell\mathsf{U}$ (with nonzero $\ell$, since the walk we started with avoids DU). Delete one of these horizontal steps, i.e. map $\mathsf{DH}^k\mathsf{U}$ to $\mathsf{DH}^{k-1}\mathsf{U}$. This way we end up with a walk of length $n+1$, because we expanded $m$ occurrences of $\mathsf{UH}^k\mathsf{D}$'s and shortened $m-1$ occurrences of $\mathsf{DH}^k\mathsf{U}$. Furthermore, we map the walk $\mathsf{H}^n$ to $\mathsf{H}^{n+1}$ (this is the only walk without any $\mathsf{UH}^k\mathsf{D}$'s, thus left unchanged by the above moves, hence we have to deal with it separately). We end up with a walk that avoids UD and has length $n+1$.

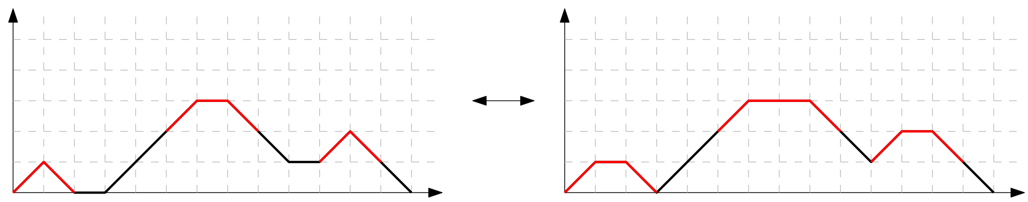


Figure 21: The bijection between DU avoiding excursions of length $n$ and UD-avoiding excursions of length $n+1$, illustrated with an example of length $n=13$. Plateaus and their expansions are marked in red.

The inverse mapping works similar: Shrink $\mathsf{UH}^{k+1}\mathsf{D}$ to $\mathsf{UH}^k\mathsf{D}$ and expand $\mathsf{DH}^k\mathsf{U}$ to $\mathsf{D}^{k+1}\mathsf{U}$. The new walk now avoids DU. Furthermore, we observe that it is one unit shorter than the original UD-avoiding excursion, since we shrank it $m$ times and expanded $m-1$ times.

Thus, we found a bijection between DU avoiding excursions of length $n$ and UD-avoiding excursions of length $n+1$.

**$\mathsf{DD}$ avoiding excursions of length $n$ and $\mathsf{DU}$ avoiding excursions of length $n$**

The main idea of the bijection between DD avoiding excursions and DU avoiding excursions is to leave excursions that avoid both DU and DD fixed and map excursions that contain DU to excursions containing DD.

Let $w$ be an excursion containing DU. Decompose it at the first occurrence in the following way:

$$(\text{prefix})\mathsf{DU}(\text{infix})\mathsf{D}(\text{suffix})$$

where the prefix is DU-free and the infix is an excursion. Furthermore, the prefix has to contain at least one U or starts at altitude greater than zero, because otherwise the D would cause the walk to drop to negative altitude and it would not be an excursion anymore. Prefix, infix, and suffix may be empty. This decomposition is unique, since the first occurrence of DU in a walk as well as the first return to the same altitude is well defined.

Now, consider the following map:

$$\phi : (\text{prefix})\mathsf{DU}(\text{infix})\mathsf{D}(\text{suffix}) \to (\text{prefix})\mathsf{U}(\text{infix})\mathsf{DD}(\text{suffix}). \tag{87}$$

The result now contains DD but might still contain DU. In order to get rid of DUs, recursively apply $\phi$ to $(\text{prefix})\mathsf{U}(\text{infix})\mathsf{D}$ and $\mathsf{D}(\text{suffix})$ (in case no DU is found in these subsequences, just leave it unchanged). This process terminates, since after each step the two subwalks to which $\phi$ is applied to is strictly shorter. Call the result $\Phi$. Then $\Phi(w)$ is DU-avoiding, because any occurrence of DU will be eliminated by successively applying $\phi$.

The inverse mapping works similar. Decompose at the first occurrence of DD and map:

$$\phi^{-1} : (\text{prefix})\mathsf{U}(\text{infix})\mathsf{DD}(\text{suffix}) \to (\text{prefix})\mathsf{DU}(\text{infix})\mathsf{D}(\text{suffix})$$

If the result still contains DD, apply $\phi^{-1}$ recursively to the subsequences $(\text{prefix})\mathsf{D}$ as well as $\mathsf{U}(\text{infix})\mathsf{D}(\text{sufffix})$.

This is indeed a bijection which can be shown via an inductive argument (over the length of the walk). However, the two segments from the decomposition are not excursions. But the bijection also works on the class of DU containing (and DD avoiding) walks that return to the same altitude as the valley DU to the class of DD containing (and DU avoiding) walks that have at leas one up-step earlier than the first occurrence of DD. Since this bijection also preserves start- and endpoints, excursions are mapped to excursions. This bijection however fails for general meanders, since not every DU containing meander has a return to the altitude as the valley (as already expected, since the generating functions for meanders are different).

**DD, HH avoiding excursions of length $n$ and DU, HH avoiding excursions of length $n$**

Since the previous bijection preserves H steps, it also preserves HH avoidance. Thus, we can use the same bijection as between DD avoiding excursions of length $n$ and DU avoiding excursions of length $n$ also for DD, HH avoiding excursions of length $n$ and DU, HH avoiding excursions of length $n$.

**UU, DD avoiding excursions of length $n$ and UD, DU avoiding excursions of length $n+1$**

The main idea behind this bijection is to combine the bijections "DU-avoiding, length $n \to$ UD-avoiding, length $n+1$", "DU-avoiding $\to$ DD-avoiding" and time reversal (which maps UU to DD) in a clever way to successively get rid of any pattern we do not want.

To do so, we also need a modified version of $\Phi$, which will be called $\tilde{\Phi}$. It maps a DU-containing walk to a DD-containing walk as described by the mapping $\phi$ only if the prefix in the decomposition for $\phi$ does not end in U, i.e.

$$\tilde{\phi} : p\mathsf{DU}i\mathsf{D}s \to p\mathsf{U}i\mathsf{DD}s \quad \text{if } p \neq \tilde{p}\mathsf{U}.$$

Recursively apply $\tilde{\phi}$ to $p\mathsf{U}i\mathsf{D}$ and $\mathsf{D}s$, until the only DU remaining are preceded by an U, and call the result upon termination $\tilde{\Phi}$ (this terminates for the same reasons $\Phi$ terminates). The mapping $\tilde{\Phi}^{-1}$ is defined similarly. The only remaining $DD$ after successively applying $\tilde{\Phi}^{-1}$ are DD which are preceded by U$\mathsf{U}e$, where $e$ is a (possibly empty) excursion.

If a walk contains no DU (or DU only as part of UDU) it remains invariant under $\tilde{\Phi}$.

The mapping $\tilde{\Phi}$ is a bijection from "HDU or DDU-containing, but DU$e$DD and HU$e$DD avoiding walks (where $e$ is any excursion)" to "DU$e$DD or HU$e$DD containing, but HDU and DDU avoiding walks".

**Lemma 4.5.8.** *The mappings $\tilde{\Phi}$ and $\tilde{\Phi}^{-1}$ both preserve* UU*-freeness and* UD*-freeness, i.e. if $w$ is* UU*-free, then $\tilde{\Phi}(w)$ is also* UU*-free (and analogously for $\tilde{\Phi}^{-1}$ as well as* UD*).*

*Proof.* This proof consists of four steps

1. First, we show that $\tilde{\phi}$ preserves UU-freeness. If a path $w = p\mathsf{DU}i\mathsf{D}s$ is UU-free then all its subpaths $p, i, s$ from the decomposition (87) are UU-free. Furthermore the infix $i$ does not start with U (otherwise U$i$ would create an UU) and the prefix $p$ does not end in $U$ (otherwise the path would remain unchanged under $\tilde{\phi}$ and thus remain UU-free). After applying $\tilde{\phi}$ the walk $w$ is mapped to $pUiDDs$, thus the only possible occurrences of an UU are from $p$U or U$i$. But since we already established that $p$ does not end in U and $i$ does not start with U, this is not possible.

2. Next, we show that $\tilde{\phi}$ also preserves UD-freeness. Similarly as above, we know that all the subpaths $p, i$ and $s$ are UD-free. Furthermore, since $i$ is an excursion, it does not start with D nor does it end with U. After applying $\tilde{\phi}$, the only possible occurrences of UD in $\tilde{\phi}$ are at U$i$ or $i$D, but we already know that $i$ does not begin with D, nor end with U, thus this is not possible.

3. Now we consider the inverse mapping $\tilde{\phi}^{-1}$. First we show that it preserves UU-freeness. We know that the walks $p, i$ and $s$ from the decomposition $\tilde{\phi}^{-1} : p\mathsf{U}i\mathsf{DD}s \to p\mathsf{DU}i\mathsf{D}s$ are UU-free. Furthermore, we know that $p$ does not end in $U$ (otherwise the mapping would not be applicable) and that $i$ does not start with U (otherwise U$i$ would create an occurrence of UU). After applying $\tilde{phi}^{-1}$ an occurrence of UU is only possible at U$i$, but we already established that $i$ does not start with U. Thus, the result is UU-free

4. Finally, we show that $\tilde{\phi}^{-1}$ also preserves UD-freeness. We know that $p, i$ and $s$ are UD-free. Furthermore we know that $p$ does not end in U (see above) and that $i$ does not start with D nor end with U (since $i$ is an excursion). After applying $\tilde{\phi}^{-1}$ an UD can only occur at $p$D, U$i$ or $i$D. But we already reasoned why all of these are not possible.

Since $\tilde{\Phi}$ is a concatenation of several applications of $\tilde{\phi}$, each preserving UU- and UD-freeness, the mapping $\tilde{\Phi}$ also preserves UU- and UD-freeness. Analogously, we obtain the same statement for $\tilde{\Phi}^{-1}$, which finishes the proof. □

Now that we have established all the needed properties, we can have a look at the actual bijection:

UU, DD avoiding excursions of length $n$ $\to$ UD, DU avoiding excursions of length $n+1$

This bijection consists of six steps (technically, the first one could be omitted, but applying time reversal twice results in better readability in which walk is mapped to which when representing them graphically).

1. Apply time reversal.
   The result is still an UU, DD avoiding walk of length $n$ (which might contain UD and DU)

2. Apply $\tilde{\Phi}$.
   The result is UU-free (it might still contain UD, DU and DD. We might have mapped some DUs to DDs in this step, we will deal with the remaining DUs later on).

3. Apply time reversal again.
   The result is DD-free (it might contain UU, UD or DU).

4. Apply $\Phi$ to get rid of DU.
   The result is DU-free (but might contain UU, UD or DD).

5. Expand plateaus and shrink troughs (as in the bijection ”DU-avoiding, length $n \leftrightarrow$ UD-avoiding, length $n+1$”). This is possible since the walk contains no DU, thus all troughs are of length at least one.
   The result is UD-free and of length $n+1$ (it might contain UU, DD or DU since shrinking troughs might have created some new DUs).

6. Finally, apply $\tilde{\Phi}$ to get rid of the remaining DUs. Each prefix $p$ from the decomposition (87) does not end in U, otherwise the path would have an instance of UD. Thus $\tilde{\Phi}$ coincides with $\Phi$ here and gets rid of any DU in the walk. Since $\tilde{\Phi}$ also preserves UD-freeness, we end up with a UD, DU-free path of length $n+1$, as desired (it may or may not contain UU or DD).

The inverse direction is basically this mapping read from bottom to top, as we will see in the next step.

UD, DU avoiding excursions of length $n+1 \to$ UU, DD avoiding excursions of length $n$

1. Start with an UD, DU-avoiding walk of length $n+1$. Apply $\tilde{\Phi}^{-1}$.
   The result is UD-free, since $\tilde{\Phi}^{-1}$ preserves UD-freeness (it might contain UU, DD or DU).

2. Expand troughs and shrink plateaus. We are allowed to do this, since the walk is UD-free. This way we got rid of DUs and shortened the length by 1.
   The result is DU-free and has length $n$ (it might contain UU, UD or DD).

3. Apply $\Phi^{-1}$ to get rid of all DDs.
   The result is a DD-avoiding walk of length $n$ (it might contain UU, UD or DU).

4. Apply time reversal. This way, DD is mapped to UU (and vice versa), DUs and UDs are left invariant however.
   The result is a UU-avoiding walk of length $n$ (it might contain DD, UD or DU).

5. Apply $\tilde{\Phi}^{-1}$ to get rid of remaining DDs. The mapping $\tilde{\Phi}^{-1}$ coincides here with $\tilde{\Phi}^{-1}$ because any prefix ending in U would create an UU. Thus the mapping gets indeed rid of any occurrence of DD. Since $\tilde{\Phi}^{-1}$ preserves UU-freeness, the result is UU-free, too.
   We end up with a UU and DD-free excursion of length $n$ (it might contain UD or DU).

6. Apply time reversal.
   We end up with a UU and DD-free excursion of length $n$ (it might contain UD or DU), as desired.
   (again, technically we could skip this step if we skipped the initial time reversal earlier.)

Finally we map $H^n$ to $H^{n+1}$ (and vice versa), since this is the only walk which has no plateaus (or troughs respectively).

To see that the above mapping is indeed a bijection, consider the properties of the intermediate results. They always coincide. Furthermore, each walk is either left invariant by the mappings involved, or, if it is changed, the change can be undone by the corresponding inverse mapping.

# 5 THE VECTORIAL KERNEL METHOD AND WATERMELONS

In this chapter we will combine the ideas and methods from Chapter 3 and 4 to analyze pattern avoidance as well as some other parameters in watermelons.

## 5.1 PATTERN AVOIDANCE IN WATERMELONS

After looking at pattern avoidance in one path, it might be interesting to have a look at pattern avoidance in pairs or tuples of paths. When looking at two or more paths, there are several ways to define pattern avoidance:

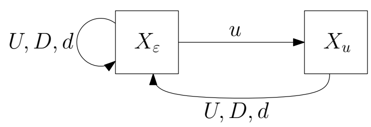
- Any simultaneous occurrence of the pattern in all paths is forbidden. E.g., it is forbidden that the upper and the lower path in a 2-watermelon both take three consecutive up-steps at the same time.
- One of the paths has to avoid a pattern, the other path, however, is unconstrained.
- All paths avoid the same pattern (or set of patterns) independently.

In the case of 2-watermelons without wall and arbitrary deviation it is possible to translate the problem of pattern avoidance in watermelons to pattern avoidance in one path. This can be done via the bijection from Lemma 3.2.3 with weighted Motzkin paths. Since this bijection is constructed step-by-step, patterns in the watermelon are more or less directly translated into patterns in the Motzkin path. However, there are some caveats and subtleties, which we are going to discuss in the following example.

**Example 5.1.1.** Pattern avoidance in watermelons. We will discuss all three possible definitions of pattern avoidance (simultaneous avoidance of a pattern, one of the paths avoids a pattern, both paths avoid the same pattern independently) in the case of the avoidance for the pattern UU in a 2-watermelon with arbitrary deviation and no wall.

Case 1: Simultaneous avoidance of the pattern UU.

If both paths of the watermelon simultaneously avoid the same pattern, this directly translates to the avoidance of a single pattern in the weighted Motzkin path (in the pattern in the weighted Motzkin path only level steps will appear, since both paths in the watermelon take the same step, thus their distance is unaffected, which results in a level step in the Motzkin path). Vice versa, the avoidance of one pattern in the weighted Motzkin paths corresponds uniquely to a sequence of pairs of steps in the watermelon. Since "both paths making an up step" corresponds to a level step weighted with $u$ in the Motzkin path, here the pattern $uu$ has to be avoided. This avoidance can be described by the following simple automaton:




Its adjacency matrix is

$$A = \begin{pmatrix} 1+z+z^{-1} & 1 \\ 1+z+z^{-1} & 0 \end{pmatrix}.$$

In order to avoid confusion with the level step with label $u$ the variable encoding the altitude of the Motzkin path is called $z$, whereas in the previous chapters it always was called $u$. Its kernel is given by

$$K(t,z) = -\frac{t^2z^2 + t^2z + tz^2 + t^2 + tz + t - z}{z}.$$

It has two roots, a large and a small one, the small one being

$$z_1(t) = \frac{1 - t - t^2 - \sqrt{1 - 2t - 5t^2 - 6t^3 - 3t^4}}{2}.$$

In order to compute the forbidden vector we have to consider

$$(M_1, M_2)A = ((1 + z + z^{-1})(M_1 + M_2), M_1)$$

and thus

$$z^{<0}\vec{M} \cdot A = (z^{-1}m, 0) \quad \text{and} \quad F = (1 - tz^{-1}m, 0)$$

where $m := [z^0](M_1 + M_2)$. Thus, we obtain that

$$\Phi(t,z) = z \cdot F(t,z)\,\mathrm{adj}(A)\vec{\mathbf{1}} = (t+1)z - (t+1)m.$$

Using $\Phi(t,z) = G(t,z)(z - z_1)$ and comparing coefficients we obtain $G(t,z) = t + 1$ and subsequently

$$M(t,z) = \frac{G(t,z)}{zK(t,z)}(z - z_1) = \frac{1 - (1+2z)(t^2+t) - \sqrt{1 - 2t - 5t^2 - 6t^3 - 3t^4}}{2t(t^2(1+z+z^2) + t(1+z+z^2) - z)}.$$

Since watermelons correspond to excursions under the bijection, we have to set $z = 0$ and obtain the following theorem:

**Theorem 5.1.2.** *The generating function of 2-watermelons without wall and arbitrary deviation where both paths simultaneously avoid the pattern* UU *is*

$$E(t) = M(t,0) = \frac{1 - t - t^2 - \sqrt{1 - 2t - 5t^2 - 6t^3 - 3t^4}}{2t^2(1+t)}.$$

The counting sequence of these objects is thus given by

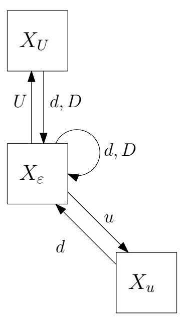
$$1, 2, 4, 11, 31, 92, 283, 893, 2875, 9407, 31189, 104555, 353794, 1206821, 4145350, 14326184, \ldots$$

as can be obtained by reading off coefficients. It has no OEIS [63] entry so far.

Case 2a: The upper path avoids UU.

Using the bijection with weighted Motzkin paths, we see that if the upper path takes two consecutive up steps, the Motzkin path has an occurrence of one of the following patterns: UU, Uu, uU, and uu. Thus, these patterns all have to be avoided. Conversely, if the Motzkin path has an occurrence of UU, Uu, uU or uu, the upper path always takes two consecutive up-steps. Thus, these four patterns cover exactly everything we want to avoid, but nothing extraneous. The avoidance of one pattern in the upper path thus translates into the avoidance of several patterns in the Motzkin-path. We already dealt with such problems in Section 4.5.

The avoidance of all these four patterns can be described by the following automaton:



Its adjacency matrix is (states ordered $\varepsilon, \mathsf{U}, u$)

$$A = \begin{pmatrix} 1+z^{-1} & z & 1 \\ 1+z^{-1} & 0 & 0 \\ 1+z^{-1} & 0 & 0 \end{pmatrix}.$$

Its kernel is

$$K(t,z) = -\frac{t^2z^2 + 2t^2z + tz - z + t^2 + t}{z}.$$

It has a large and a small root, the small root being

$$z_1 = \frac{1 - t - 2t^2 - \sqrt{1-2t-3t^2}}{2t^2}.$$

We have that

$$(M_1, M_2, M_3)A = ((1+z^{-1})(M_1 + M_2 + M_3), zM_1, M_1).$$

Its negative components are

$$\{z^{<0}\}\vec{M}A = (z^{-1}m, 0, 0)$$

where $m := [z^0]M_1 + M_2 + M_3$. Thus the forbidden vector is

$$F = (1 - tz^{-1}m, 0, 0)$$

and $\Phi$ can be computed to be

$$\Phi(t,z) = tz^2 + (1 + t - mt^2) - mt^2 - mt.$$

Via $\Phi = G(t,z)(z - z_1)$ and comparing coefficients we obtain that

$$G(t,z) = tz + t + 1.$$

Using this, we can compute

$$M(t,z) = \frac{G(t,z)}{zK(t,z)}(z - z_1) = \frac{(1+t+tz)(1-t-2t^2-2t^2z-\sqrt{1-2t-3t^2}}{2t^2((z+1)^2t^2 + (z+1)t - z)}$$

and subsequently we arrive at the following theorem:

**Theorem 5.1.3.** *The generating function of 2-watermelons with no wall and arbitrary deviation where the upper path avoids* UU *is*

$$E(t) = M(t,0) = \frac{1 - t - 2t^2 - \sqrt{1-2t-3t^2}}{2t^3}, \tag{88}$$

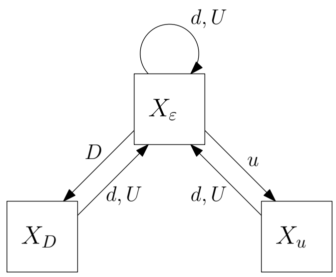
This generating function looks a lot like the generating function of Motzkin-numbers (A001006) which is given by

$$\frac{1 - t - \sqrt{1-2t-3t^2}}{2t^2}.$$

This suggests that there is a bijection between 2-watermelons with no wall and arbitrary deviation where the upper path avoids UU of length $n$ and (ordinary, unweighted) Motzkin paths of length $n+1$.

Case 2b: The lower path avoids UU.

This case is rather similar to the previous one. Here, the Motzkin path has to avoid the patterns DD, Du, uD, and uu. This is encoded by the automaton




Its adjacency matrix is given by

$$A = \begin{pmatrix} 1+z & z^{-1} & 1 \\ 1+z & 0 & 0 \\ 1+z & 0 & 0 \end{pmatrix}$$

and its kernel is

$$K(t,z) = -\frac{(t^2+t)z^2 + (2t^2+t-1)z + t^2}{z}.$$

It has one small root namely

$$z_1 = \frac{1-t-2t^2-\sqrt{1-2t-3t^2}}{2t(t+1)}.$$

We have that

$$(M_1, M_2, M_3)A = ((1+z)(M_1+M_2+M_3), z^{-1}M_1, M_1).$$

Its negative component is

$$\{z^{<0}\}\vec{M}A = (0, z^{-1}m_1, 0),$$

where $m_1 := [z^0]M_1$. Thus, the forbidden vector is given by

$$\vec{F} = (1, -tz^{-1}m_1, 0)$$

and

$$\Phi(t,z) = (t+1)z - m_1 t + t.$$

Equating coefficients gives us

$$G(t,z) = t+1.$$

From this we can then obtain the generating function for meanders:

$$M(t,z) = \frac{G}{zK}(z-z_1) = \frac{1-(1+2z)t-(2+2z)t^2-\sqrt{1-2t-3t^2}}{2t((1+z)^2t^2+(z^2+z)t-z)}$$

Considering the generating function for excursions gives us:

**Theorem 5.1.4.** *The generating function for 2-watermelons where the lower path avoids* UU *is*

$$E(t) = M(t,0) = \frac{1-t-2t^2-\sqrt{1-2t-3t^3}}{2t^3}.$$

Surprisingly, this coincides with the generating function (88) of 2-watermelons where the upper path avoids UU. On second glance, however, this is not surprising at all, since there is an easy bijection explaining this: First apply time reversal and obtain a watermelon where the upper path avoids DD (and the deviation has changed sign, but since we are considering watermelons with arbitrary deviation this does not matter), then apply a horizontal flip which reverses the role of upper and lower path as well as the role of up-steps and down-steps. Thus, we end up with a watermelon where the lower path avoids UU. This is illustrated in Figure 22.

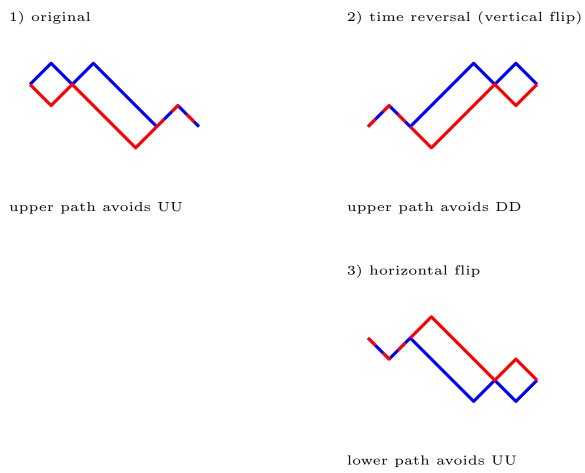
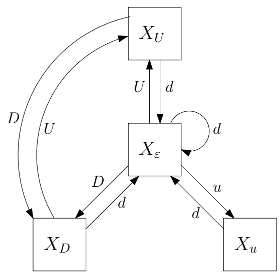



Figure 22: The bijection between 2-watermelons where the upper path avoids UU and 2-watermelons where the lower path avoids UU

Case 3: Both paths avoid UU independently.

Using the bijection with weighted Motzkin paths, we see that if the upper path takes two consecutive up steps, the Motzkin path has an occurrence of one of the following patterns: UU, Uu, uU, and uu. Thus, these patterns all have to be avoided. Similarly, if the lower path takes two consecutive up-steps the Motzkin path has an occurrence of one of the following patterns: DD, Du, uD, and uu. These have to be avoided as well. By a similar reasoning as in the previous cases we see that these patterns cover everything we want to avoid, but nothing extraneous.

Putting everything together, the Motzkin path has to avoid a total of seven patterns, namely: UU, Uu, uU, uu, DD, Du, and uD. The avoidance of all these patterns can be encoded by the following automaton:




Its adjacency matrix is given by

$$A = \begin{pmatrix} 1 & z & 1 & z^{-1} \\ 1 & 0 & 0 & z^{-1} \\ 1 & 0 & 0 & 0 \\ 1 & z & 0 & 0 \end{pmatrix}$$

(states ordered $\epsilon$, U, u, D). Thus, the kernel is given by

$$K(t,z) = \det(I - tA) = \frac{-t^2z^2 + (t^4 - t^3 - 2t^2 - t + 1)u - t^2}{u}.$$

It has two roots, a large one and a small one. The small root is given by

$$z_1 = \frac{1 - t - 2t^2 - t^3 + t^4 - \sqrt{1 - 2t - 3t^2 + 4t^4 + 2t^5 - 3t^6 - 2t^7 + t^8}}{2t^2}.$$

Furthermore we have that

$$(M_1, M_2, M_3, M_4) \cdot A = (M_1 + M_2 + M_3 + M_4, z(M_1 + M_4), M_1, z^{-1}(M_1 + M_2)).$$

Since $M_2$ counts all meanders ending in an up-step (of the Motzkin-path) we have that $[z^0]M_2 = 0$ and thus the only remaining negative powers are

$$\{z^{<0}\}(M_1, M_2, M_3, M_4) \cdot A = (0, 0, 0, -tz^{-1}m_1),$$

where $m_1 = [z^0]M_1$, i.e., the constant term of the generating function of all meanders ending in state $X_\epsilon$. Thus

$$\vec{F} = (1, 0, 0 - tz^{-1}m_1).$$

Hence $\Phi$ can be computed to be

$$\Phi(t, z) = tz^2 + (-t^2 m_1 - t^3 + t^2 + t + 1)z - tm_1 + t.$$

Using $\Phi(t, z) = G(t, z)(z - z_1)$ we can compute $G$ and obtain

$$G(t, z) = tz + \frac{1 + t + tz_1 - t^3}{1 + tz_1}.$$

From this we obtain:

**Theorem 5.1.5.** *The generating function of 2-watermelons without wall and with arbitrary deviation where both paths avoid the pattern* UU *independently is given by*

$$\begin{aligned} E(t) = M(t, 0) &= \frac{G}{zK}(z - z_1)\Big|_{z=0} \\ &= \frac{-1 + t + 2t^2 + t^3 - t^4 - 2t^5 + \sqrt{1 - 2t - 3t^2 + 2t^3 + 4t^4 + 2t^5 - 3t^6 - 2t^7 + t^8}}{2t^4(t^2 - 1)}. \end{aligned}$$

Reading off coefficients, we see that the counting sequence of 2-watermelons without wall and arbitrary deviation where both paths avoid UU is given by

$$1, 2, 4, 8, 17, 37, 82, 185, 423, 978, 2283, 5373, 12735, 30372, 72832, 175502, 424748, 1032004, \ldots$$

which is not listed in the OEIS [63] as of now.

## 5.2 THE VECTORIAL KERNEL METHOD AND HEIGHT-RELATED PARAMETERS

The vectorial method can not only be used to enumerate walks avoiding or containing a certain pattern, but other constraints that can be encoded by a finite automaton. The height of a walk is a classic example of this. In this section we will use the vectorial kernel method to re-derive the number of Dyck not exceeding height $h$ (which has already been studied in [59] and later in fuller generality for excursions with finite step set $\mathcal{S} \subset \mathbb{Z}$ not exceeding height $h$ in [22]) as an illustrating example for the ideas and methods and then use a similar approach to obtain results about the lower height in 2-watermelons. The latter one has only been done for some fixed values of $h$, since for arbitrary $h$ some difficulties not present in the Dyck path case arise.

### *Dyck paths of height $\leq h$*

The number of Dyck paths not exceeding a certain given height $h$ is an already well-known result. For example, it can be obtained as a special case of Theorem 2, Chapter 1.3 in [59], which counts paths with Dyck step set between two boundaries. There, Mohanty used a combination of a reflection-argument and inclusion-exclusion to obtain this result. Here we will re-derive this result with the vectorial kernel method before generalizing it to the lower height in 2-watermelons in the next subsection. We will start with a small value of $h$ as illustrating example before giving the general result.

*Dyck paths of length* $2n$ *height not exceeding 3*

Consider an automation with $h+1$ states $X_0, X_1, \ldots, X_h$ (here $h = 3$). The path is in state $X_i$ if it is at altitude $i$ currently. Taking an up step from there makes the path reach state $X_{i+1}$, taking a down step makes it reach state $X_{i-1}$. In state $X_h$ only down steps are allowed, otherwise the path would exceed height $h$. In state $X_0$ only up steps are allowed, otherwise the path would go below the $x$-axis.

For $h = 3$ the automaton looks like this:

$$\begin{tikzcd} X_0 \arrow[r, bend left, "U"] & X_1 \arrow[l, bend left, "D"] \arrow[r, bend left, "U"] & X_2 \arrow[l, bend left, "D"] \arrow[r, bend left, "U"] & X_3 \arrow[l, bend left, "D"] \end{tikzcd}$$

The adjacency matrix of this automaton is given by

$$A = \begin{pmatrix} 0 & u & 0 & 0 \\ u^{-1} & 0 & u & 0 \\ 0 & u^{-1} & 0 & u \\ 0 & 0 & u^{-1} & 0 \end{pmatrix}.$$

The kernel of the automation is

$$K(t,u) = \det(I - tA) = 1 - 3t^2 + t^4.$$

It has no zeroes (in $u$), thus the number $e$ of small roots of the kernel is $e = 0$.

Now apply the vectorial kernel method to find the generating function for those Dyck paths (excursions). Let $M_\alpha = M_\alpha(t,u)$ denote the bivariate generating function of the meanders that terminate in state $\alpha$ (for $\alpha = 0,1,2,3$). Obviously we have that $M(t,u) = \sum_\alpha M_\alpha(t,u)$, or, phrased differently $M(t,u) = (M_0, M_1, M_2, M_3) \cdot \vec{1}$, where $\vec{1}$ denotes the column vector $(1,1,1,1)^\top$. By a step-by-step construction we obtain the following functional equation

$$(M_0, M_1, M_2, M_3) = (1,0,0,0) + t(M_0, M_1, M_2, M_3)A - t\{u^{<0}\}((M_0, M_1, M_2, M_3)A),$$

where $\{u^{<0}\}$ denotes all terms in which the power of $u$ is negative. Rewriting this we obtain

$$(M_0, M_1, M_2, M_3)(I - tA) = (1,0,0,0) - t\{u^{<0}\}((M_0, M_1, M_2, M_3)A).$$

Define $F_\alpha$ to be the $\alpha$-th component of the vector on the right-hand side. With this notation the equation becomes

$$(M_0, M_1, M_2, M_3)(I - tA) = (F_0, F_1, F_2, F_3). \tag{89}$$

Let us have a closer look at $\{u^{<0}\}((M_0, M_1, M_2, M_3)A)$ in order to compute the vector $F$. We have that

$$M \cdot A = (u^{-1}M_1, u^{-1}M_2 + uM_0, u^{-1}M_3 + uM_1, uM_2).$$

Note that $M_i(t,u)$, the generating function of meanders ending in state $X_i$, only contains powers $u^i$, but no higher or lower powers of $u$. This is because the variable $u$ encodes the altitude of the walk and here the state $X_i$ denotes that the walk is at altitude $i$. Thus $MA$ contains no negative powers of $u$ and we have

$$\{u^{<0}\}((M_0, M_1, M_2, M_3)A) = (0,0,0,0)$$

and thus

$$(F_0, F_1, F_2, F_3) = (1,0,0,0).$$

Next, multiply equation (89) from the right with

$$(I - tA)^{-1}\vec{1} = \frac{(\operatorname{adj}(I - tA))\vec{1}}{\det(I - tA)},$$

where $\operatorname{adj}(I - tA)$ denotes the adjoint of $I - tA$. We already know that $\det(I - tA) = K(t,u) = 1 - 3t^2 + t^4$. Denote

$$\vec{v} = \vec{v}(t,u) := (\operatorname{adj}(I - tA))\vec{1}.$$

We obtain

$$M(t,u) = \frac{(F_0, F_1, F_2, F_3)\vec{v}}{K(t,u)}. \tag{90}$$

In our case

$$\operatorname{adj}(I - tA) = \begin{pmatrix} 1-2t & tu(1-t^2) & t^2u^2 & t^3u^3 \\ tu^{-1}(1-t^2) & 1-t^2 & tu & t^2u^2 \\ t^2u^{-2} & tu^{-1} & 1-t^2 & tu(1-t^2) \\ t^3u^{-3} & t^2u^{-2} & tu^{-1}(1-t^2) & 1-2t^2 \end{pmatrix},$$

and thus

$$\vec{v} = \begin{pmatrix} 1-2t^2+(t-t^3)u+t^2u^2+t^3u^3 \\ (t-t^3)u^{-1}+1-t^2+tu+t^2u^2 \\ t^2u^{-2}+tu^{-1}+1-t^2+(t-t^3)u \\ t^3u^{-3}+t^2u^{-2}+(t-t^3)u^{-1}+1-2t^2 \end{pmatrix}$$

Denote

$$\Phi(t,u) := u^e(F_0(t,u), F_1(t,u), F_2(t,u), F_3)\vec{v}(t,u).$$

From the proof of Theorem 4.4.3 we know that $\Phi$ is a polynomial in $u$ and that each small root $u_i(t)$ of the kernel $K(t,u)$ is also a root of $\Phi$. It follows that

$$\Phi(t,u) = G(t,u)\prod_{i=1}^{e}(u - u_i(t)) \tag{91}$$

for some $G(t,u)$ which is a polynomial in $u$ and a formal power series in $t$. Substituting this into (90) we obtain

$$M(t,u) = \frac{G(t,u)}{u^e K(t,u)}\prod_{i=1}^{e}(u - u_i(t)).$$

In our case $u^e = 1$ and the product is the empty product, because the kernel has no (small) roots in $u$. It remains to determine $G(t,u)$. Because the product $\prod_{i=1}^{e}(u - u_i(t))$ is empty, it follows that

$$G(t,u) = \Phi(t,u) = (F_0, F_1, F_1, F_3)\vec{v},$$

which after plugging in the known expressions for $F$ and $\vec{v}$ becomes

$$G(t,u) = 1 - 2t^2 + (t - t^3)u + t^2u^2 + t^3u^3.$$

Thus

$$M(t,u) = \frac{1 - 2t^2 + (t - t^3)u + t^2u^2 + t^3u^3}{1 - 3t^2 + t^4}.$$

For excursions we obtain

$$E(t) = M(t,0) = \frac{1 - 2t^2}{1 - 3t^2 + t^4}.$$

Transitioning to semilength, we obtain that the generating function for Dyck paths of height at most 3 and semilength $n$ is

$$E(x) = \frac{1 - 2x}{1 - 3x + x^2}.$$

*Dyck paths of length $2n$ height $\leq h$*

The same approach as in the previous subsection also works for arbitrary height $h$, but the computation of the determinant (the kernel) and the adjoint of the matrix $I - tA$ now becomes a bit trickier.

The automaton describing Dyck paths not exceeding height $h$ has $h+1$ states, in state $X_h$ only the down step is allowed, whereas in state $X_0$ only the up step is allowed:

$$\begin{tikzcd} X_0 \arrow[r, bend left, "U"] & X_1 \arrow[l, bend left, "D"] \arrow[r, bend left, "U"] & \cdots \arrow[l, bend left, "D"] \arrow[r, bend left, "U"] & X_{h-1} \arrow[l, bend left, "D"] \arrow[r, bend left, "U"] & X_h \arrow[l, bend left, "D"] \end{tikzcd}$$

The corresponding adjacency matrix $A_{h+1}$ is a $(h+1) \times (h+1)$ tridiagonal matrix with main diagonal entries zero, entries $u$ in the first diagonal above the main diagonal, and entries $u^{-1}$ in the first diagonal below the main diagonal, i.e.

$$A_{h+1} = \begin{pmatrix} 0 & u & 0 & \dots & 0 \\ u^{-1} & 0 & u & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & u^{-1} & 0 & u \\ 0 & \dots & 0 & u^{-1} & 0 \end{pmatrix}_{h+1 \times h+1}.$$

Thus

$$I_{h+1} - tA_{h+1} = \begin{pmatrix} 1 & -tu & 0 & \dots & 0 \\ -tu^{-1} & 1 & -tu & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & -tu^{-1} & 1 & -tu \\ 0 & \dots & 0 & -tu^{-1} & 1 \end{pmatrix}_{h+1 \times h+1}.$$

To compute the kernel $K_{h+1}(t,u) = \det(I_{h+1} - tA_{h+1})$, we compute its Laplace expansion along the first column and obtain

$$\begin{aligned} K_{h+1} &= 1 \cdot \det(I_h - tA_h) + tu^{-1} \cdot \det \begin{pmatrix} -tu & 0 & 0 & \dots & 0 \\ -tu^{-1} & 1 & -tu & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & -tu^{-1} & 1 & -tu \\ 0 & \dots & 0 & -tu^{-1} & 1 \end{pmatrix}_{h \times h} \\ &= K_d + tu^{-1} \cdot (-tu) \det \begin{pmatrix} 1 & -tu & 0 & \dots & 0 \\ -tu^{-1} & 1 & -tu & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & -tu^{-1} & 1 & -tu \\ 0 & \dots & 0 & -tu^{-1} & 1 \end{pmatrix}_{h-1 \times h-1} \\ &= K_h - t^2 K_{h-1}, \end{aligned}$$

where the second line comes from expanding the determinant of the matrix along the first row. Thus we obtain the recursion

$$K_{h+1} = K_h - t^2 K_{h-1}.$$

Solving it with the help of generating functions and reading off coefficients we obtain for the kernel $K_{h+1}$ of walks not exceeding height $h$ that

$$K_{h+1} = \sum_{j=0}^{h+1} \binom{h+1-j}{j} (-1)^j t^{2j}. \tag{92}$$

It does only depend on $t$, but not on $u$, thus it has no small roots in $u$, i.e. the number $e$ of small roots is zero. Writing $M_i := M_i(t,u)$ for the meanders that end in state $i$ (i.e. at height $i$) we obtain by a step by step construction of the walk the following vectorial functional equation

$$(M_0, M_1, \dots, M_h)(I_{h+1} - tA_{h+1}) = (1, 0, \dots, 0) - t\{u^{<0}\}((M_0, M_1, \dots, M_h)A_{h+1}). \tag{93}$$

Write $F := (F_0, F_1, \dots, F_h)$ for the right-hand side of (93). We have that

$$(M_0, M_1, \dots, M_h)A_{h+1} = (u^{-1}M_1, uM_0 + u^{-1}M_2, \dots, uM_{i-1} + u^{-1}M_{i+1}, \dots, uM_{h-1}).$$

Since $M_i$ is the generating function of meanders ending in height $i$ only powers $u^i$ occur in $M_i$. Thus

$$t\{u^{<0}\}((M_0, M_1, \dots, M_h)A_{h+1}) = (0, \dots, 0)$$

and

$$(F_0, F_1, \dots, F_h) = (1, 0, \dots, 0).$$

In order to compute the autocorrelation vector

$$\vec{v} = \operatorname{adj}(I_{h+1} - tA_{h+1}) \cdot (1, \dots, 1)^\top$$

we first need to compute $\operatorname{adj}(I_{h+1} - tA_{h+1})$. More specifically, we only need to compute the first row of $\operatorname{adj}(I_{h+1} - tA_{h+1})$ or the first entry of $\vec{v}$ since multiplication of $\vec{v}$ with $F$ from the left annihilates all but the first entry. The adjugate matrix $\operatorname{adj}(I_{h+1} - tA_{h+1}) = C_{h+1}^\top$ is defined as the transpose of the cofactor matrix, i.e. the matrix with entries $C_{ij} = (-1)^{i+j}M_{ij}$ where $M_{ij}$ is the minor obtained by deleting the $i$-th row and $j$-th column from $I_{h+1} - tA_{h+1}$. To compute the first row, i.e. the entries $\operatorname{adj}(I_{h+1} - tA_{h+1})_{1j}$ we need to compute the first column with entries $C_{j1}$ of the cofactor matrix. By deleting the first row and column we obtain that $M_{11} = \det A_h = K_h$.

Next, we want to show that all the entries $C_{j1}$ have order at least 1 in $u$, thus contribute nothing after plugging in $u = 0$ in the expression for $M(t, 0)$ which will be done later on.

**Lemma 5.2.1.** *For $j > 1$ the entries $C_{j1}$ have order at least 1 in $u$.*

*Proof.* Consider the minor obtained after deleting the $j$-th row and first column of $I_{h+1} - tA_{h+1}$, which is

$$M_{j1} = \det \begin{pmatrix}
-tu & 0 & \dots & & & & & \dots & 0 \\
1 & -tu & 0 & & & & & & \\
-tu^{-1} & 1 & -tu & 0 & & & & & \\
0 & \ddots & \ddots & \ddots & \ddots & & & & \\
\vdots & & -tu^{-1} & 1 & -tu & 0 & & & \\
 & & & 0 & -tu^{-1} & 1 & -tu & & \vdots \\
 & & & & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & & & & 0 & -tu^{-1} & 1 & -tu \\
0 & \dots & & & & \dots & 0 & -tu^{-1} & 1
\end{pmatrix}.$$

The first $j - 1$ rows have entries $-tu$ in the main diagonal, entries $1$ in the diagonal one unit below the main diagonal, and entries $-tu^{-1}$ in the diagonal two units below the main diagonal. We have at least one such row since $j > 1$. Now, we will use induction over the dimension of the matrix to complete the proof. Clearly, the claim of the lemma holds true for a one-dimensional matrix, which consists only of an entry $-tu$, thus has determinant $-tu$. Suppose the lemma holds for all matrices of the above type and dimension $h$. To compute the determinant of such a $h+1$-dimensional matrix, let us expand $M_{j1}$ by the first row. If $j > 2$ we obtain that

$$M_{ji}^{(h+1)} = (-tu) \cdot (\text{ the determinant of some matrix of the same type and dimension } h),$$

which by induction has order at least one in $u$. Multiplying with $(-tu)$ just increases the order further.

If $j = 2$ we only have one row with $-tu$ in the main diagonal, namely the first row. Thus, we obtain

$$M_{21}^{(h+1)} = (-tu) \cdot \det(I_h - tA_h).$$

We already computed $\det I_h - tA_h = K_h$ before and saw that it does not depend on $u$, thus has order and degree zero in $u$. Multiplying with $(-tu)$ gives us order 1 in $u$, which finishes the proof. □

The previous lemma gives us that $\Phi(t,u) = F \cdot \vec{v} = \sum_j C_{j1} = K_h + O(u)$. Since the number of small roots is zero we have $\prod(u - u_i) = 1$ and thus $G(t,u) = \Phi(t,u)$. We obtain for the generating function $M(t,u)$ of meanders

$$M(t,u) = \frac{G(t,u)}{K(t,u)} = \frac{K_h + O(u)}{K_{h+1}}.$$

Computing the generating function for excursions we need to plug in $u = 0$ and obtain

$$E(t) = M(t,0) = \frac{K_h}{K_{h+1}}$$

where

$$K_h = \sum_{j=0}^{h} \binom{h-j}{j} (-1)^j t^{2j}.$$

*Towards the lower height in 2-watermelons with wall*

There are two ways to define height in 2-watermelons. In Definition 3.1.18 we already encountered the definition of the (upper) height of a (2-)watermelon studied by Fulmek [38] and Feierl [33, 34]. Here, the condition that the path may not exceed height $h$ is imposed on the topmost path (and by the non-crossing condition implicitly also on all other paths). However, it is also possible to impose the height restriction on the lower path and let the upper path unconstrained. This leads us to the following definition:

**Definition 5.2.2.** *The lower height of a watermelon is the y-coordinate of the highest lattice point along the bottommost paths of the watermelon.*

Studying the lower height of a 2-watermelon with the same methods as in [38, 33, 34] did not turn out to be very fruitful. The main reason behind this is that a Lindström–Gessel–Viennot-approach does not work here. A crucial idea in the proof of the Lindström–Gessel–Viennot Lemma 3.1.7 was swapping the initial segments of intersecting paths. However, while swapping initial segments preserves (upper) height (since the upper height restriction also forces the same height restriction on the lower path), it does not preserve lower height, as can be seen in Figure 24.

For studying the lower height in 2-watermelons, the bijection from Lemma 3.2.3 with weighted Motzkin paths again turns out to be helpful. The bijection itself does not encode the wall condition. But the following automation will take care of the lower height as well as the wall condition. Its states correspond to the altitude of the lower path.

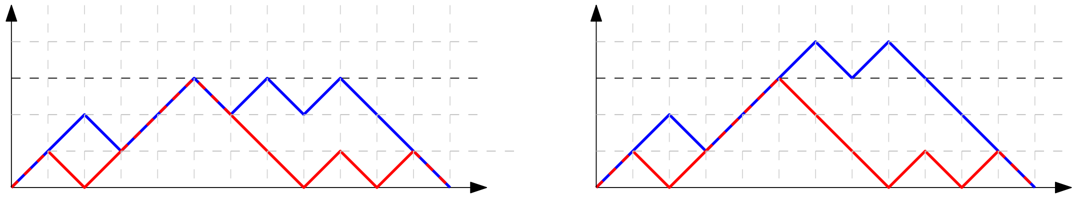


Figure 23: A 2-watermelon of (upper) height not exceeding 3 (left) and a 2-watermelon of lower height not exceeding 3 (height 5, if we use the definition with non-touching paths starting at (0,0) and (0,2) respectively). In the latter case, the upper path, however, may exceed height 3.

Figure 24: A schematic illustration why the Lindström–Gessel–Viennot Lemma does not work for counting watermelons not exceeding a certain lower height: the swapping of initial segments used in the proof of the lemma may violate the lower height condition.

$$\begin{tikzcd} X_0 \arrow[r, bend left, "{u,D}"] & X_1 \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & X_2 \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & \cdots \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & X_{h-1} \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & X_h \arrow[l, bend left, "{U,d}"] \end{tikzcd}$$

Note that the lower height does not correspond to the height of the weighted Motzkin path obtained via the bijection. Thus, applying the bijection and then the results from [22] does not work out.

Again, we will first have a look at small cases before discussing how it could be done for arbitrary $h$. Note that the latter is not completely proven, however.

*2-watermelons with wall with lower height not exceeding height 2*

For lower height not exceeding 2, the automaton encoding both the wall condition as well as the restriction on the lower height is

$$\begin{tikzcd} X_0 \arrow[r, bend left, "{u,D}"] & X_1 \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & X_2 \arrow[l, bend left, "{U,d}"] \end{tikzcd}$$

The adjacency matrix of this automaton is given by

$$A = \begin{pmatrix} 0 & 1+z^{-1} & 0 \\ 1+z & 0 & 1+z^{-1} \\ 0 & 1+z & 0 \end{pmatrix}.$$

and thus

$$I - tA = \begin{pmatrix} 1 & -t-tz^{-1} & 0 \\ -t-tz & 1 & -t-tz^{-1} \\ 0 & -t-tz & 1 \end{pmatrix}.$$

The kernel is given by

$$K(t,z) = \det(I - tA) = -2t^2 z + 1 - 4t^2 - 2t^2 z^{-1}$$

and its roots are

$$z_{1/2}(t) = \frac{1 - 4t^2 \pm \sqrt{1 - 8t^2}}{4t^2}.$$

The root with minus is the small root, thus we have that the number of small roots is $e = 1$.

Denote $M_i(t, z)$ the generating function of meanders that end in state (i.e. at lower height) $i$. We want to compute the generating function of watermelons where both paths end at the $x$-axis. The condition that the Motzkin path gives us that both watermelon paths end in the same point and the condition that the lower path ends in state $X_0$ and thus at lower height zero gives us additionally that both of the watermelon paths end at height zero. Thus, the object we are interested in is $M_0(t, 0)$.

Via a step-by-step construction we obtain the following functional equation

$$(M_0, M_1, M_2)(I - tA) = (1, 0, 0) - t\{z^{<0}\}((M_0, M_1, M_2)A).$$

Let

$$F := (F_0, F_1, F_2) := (1, 0, 0) - t\{z^{<0}\}((M_0, M_1, M_2)A).$$

We have

$$\{z^{<0}\}((M_0, M_1, M_2)A) = (0, z^{-1}m_0, z^{-1}m_1)$$

where $m_0 = [z^0]M_0(t, z) = M_0(t, 0)$ and $m_1 = [z^0]M_1(t, z) = M_1(t, 0)$. Thus

$$(F_0, F_1, F_2) = (1, -tz^{-1}m_0, -tz^{-1}m_1).$$

The autocorrelation vector of this automaton is

$$\vec{v} = \operatorname{adj}(I - tA) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} t^2z^{-2} - t^2z + t^2z^{-1} - t^2 + tz^{-1} + t + 1 \\ tz + 2t + 1 + tz^{-1} \\ t^2z^2 + t^2z - t^2z^{-1} - t^2 + tz + t + 1 \end{pmatrix}.$$

When plugging in these expressions for $F$ and $\vec{v}$ into

$$\Phi(t, z) = z^e F\vec{v} = z(1, -tz^{-1}m_0, -tz^{-1}m_1)\vec{v}$$

we would a priori obtain negative powers in $z$, namely

$$\{z^{<0}\}\Phi(t, z) = t^2z^{-1} - t^2z^{-1}m_0 + t^3z^{-1}m_1 = t^2z^{-1}(1 - m_0 + tm_1). \tag{94}$$

But because of the reasoning in the proof of Theorem 4.3.7 $\Phi(t, z)$ has to be a polynomial in $z$, thus these negative powers should not occur. In fact, they actually cancel out because of some relations between $m_0$ and $m_1$: A meander ending in state $X_0$ is either an empty walk or a meander ending in state $X_1$ and a step that takes it to state $X_0$, or, phrased differently,

$$M_0(t, z) = 1 + tM_1(t, z)(1 + z).$$

Reading off the coefficient $[z^0]$ we obtain

$$[z^0]M_0(t, z) = 1 + t[z^0]M_1(t, z),$$

which is

$$m_0 = 1 + tm_1. \tag{95}$$

This causes the negative powers in (94) to vanish, thus $\Phi$ is actually a polynomial in $z$. More precisely, after plugging $m_0 = 1 + tm_1$ in the expression for $\Phi$ we obtain

$$\Phi(t, z) = -t^3m_1z^2 - 2\,t^3m_1z - m_1t^3 - m_1t^2z - t^2z^2 - 2\,m_1t^2 - 2\,t^2z - tm_1 - t^2 + tz + z.$$

Because

$$\Phi(t, z) = G(t, z)(z - z_1)$$

and $\Phi$ has degree 2 in $z$, we know that $\deg_z(G) = 1$, i.e. $G = az + b$ with $a$ and $b$ unknown. Reading off coefficients from $\Phi$ we obtain the following linear system of equations

$$\begin{aligned} -z_1 b &= m_1 t^3 - 2m_1 t^2 - mt - t^2 \\ b - z_1 a &= -2m_1 t^3 - m_1 t^2 - 2t^2 + t + 1 \\ a &= m_1 t^3 - t^2 \end{aligned}$$

in the three unknowns $a, b$, and $m_1$. Solving this system we obtain

$$\begin{aligned} a &= -\frac{t^2 \left(2\,tz_1 + 2\,t + z_1 + 1\right)}{t^2 {z_1}^2 + 2\,t^2 z_1 + t^2 + tz_1 + 2\,t + 1} \\ b &= -\frac{2\,t^3 z_1 + 2\,t^3 + t^2 z_1 - t^2 - 3\,t - 1}{t^2 {z_1}^2 + 2\,t^2 z_1 + t^2 + tz_1 + 2\,t + 1} \\ m_1 &= -\frac{t^2 {z_1}^2 + 2\,t^2 z_1 + t^2 - tz_1 - z_1}{t\left(t^2 {z_1}^2 + 2\,t^2 z_1 + t^2 + tz_1 + 2\,t + 1\right)} \end{aligned}$$

The generating function of 2-watermelons with wall and deviation zero where the lower path does not exceed height 2 is given by $m_0$ which can be computed via

$$m_0 = 1 + tm_1 = \frac{\left(2\,z_1 + 2\right)t + z_1 + 1}{1 + \left(z_1 + 1\right)^2 t^2 + \left(z_1 + 2\right)t}.$$

After plugging in the small root $z_1$ and some simplifications we obtain that

$$m_0 = \frac{3 - \sqrt{1 - 8t^2}}{2(1 + t^2)}.$$

Transitioning to semilength gives us the following theorem:

**Theorem 5.2.3.** *The number of watermelons of lenght* $2n$ *not exceeding lower height 2 is given by*

$$E(x) = \frac{3 - \sqrt{1 - 8x}}{2(1 + x)}.$$

Expanding $E(x)$ into a series we obtain that the first few terms are

$$1, 1, 3, 13, 67, 381, 2307, 14589, 95235, \ldots$$

This series coincides with the generalized Catalan numbers $C(2; n)$ (A064062). Since the number of watermelons of lower height not exceeding 1 is given by the (ordinary) Catalan numbers $C(n) = C(1; n)$ (as can easily be seen from the fact that the lower path has to be the zigzag path and the upper path can be any Dyck path) this might suggest the conjecture that watermelons of lower height not exceeding $h$ is given by the generalized Catalan numbers $C(h; n)$.

The generalized Catalan numbers are defined as

$$C(h; n) = [x^n]\frac{2h - 1 - \sqrt{1 - 4hx}}{2(h - 1 + x)}, \tag{96}$$

for explicit forms and more information about them see [1].

This conjecture, however, is not true as can be seen in the next subsection.

*2-watermelons with wall with lower height not exceeding height 3*

For lower height not exceeding 3, the automaton encoding both the wall condition as well as the restriction on the lower height is

$$\begin{tikzcd} X_0 \arrow[r, bend left, "{u,D}"] & X_1 \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & X_2 \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & X_3 \arrow[l, bend left, "{U,d}"] \end{tikzcd}$$

Its adjacency matrix is

$$A = \begin{pmatrix} 0 & 1+z^{-1} & 0 & 0 \\ 1+z & 0 & 1+z^{-1} & 0 \\ 0 & 1+z & 0 & 1+z^{-1} \\ 0 & 0 & 1+z & 0 \end{pmatrix}$$

and its kernel is

$$K(t,z) = \det(I - tA) = \frac{t^4z^4 + 4t^4z^3 + 6t^4z^2 + 4t^4z - 3t^2z^3 + t^4 - 6t^2z^2 - 3t^2z + z^2}{z^2}.$$

It has four roots, namely

$$\begin{aligned} z_1(t) &= \frac{-4\,t^2 + \sqrt{5} + 3}{4t^2} + \frac{\sqrt{-8\,\sqrt{5}t^2 + 14 - 24\,t^2 + 6\,\sqrt{5}}}{4t^2} \\ z_2(t) &= \frac{-4\,t^2 + \sqrt{5} + 3}{4t^2} - \frac{\sqrt{-8\,\sqrt{5}t^2 + 14 - 24\,t^2 + 6\,\sqrt{5}}}{4t^2} \\ z_3(t) &= -\frac{4\,t^2 + \sqrt{5} - 3}{4t^2} + \frac{\sqrt{4\,\sqrt{5}t^2 - 12\,t^2 - 3\,\sqrt{5} + 7}\sqrt{2}}{4t^2} \\ z_4(t) &= -\frac{4\,t^2 + \sqrt{5} - 3}{4t^2} - \frac{\sqrt{4\,\sqrt{5}t^2 - 12\,t^2 - 3\,\sqrt{5} + 7}\sqrt{2}}{4t^2} \end{aligned}$$

Upon computing their limits for $t \to 0$, we see that $z_2$ and $z_4$ are the small roots.

Denote $M_i(t,z)$ the generating function of meanders that end in state $i$ (for $i = 0, \dots 3$). By a similar reasoning as in the previous example, we are interested only in $M_0(t,0)$.

A step-by-step construction gives us the functional equation

$$(M_0, M_1, M_2, M_3)(I - tA) = (1,0,0,0) - t\{z^{<0}\}((M_0, M_1, M_2, M_3)A).$$

We have

$$\{z^{<0}\}\vec{M} \cdot A = (0, z^{-1}m_0, z^{-1}m_1, z^{-1}m_2),$$

where $m_i := [z^0]M_i(t,z) = M_i(t,0)$. Thus

$$F = (1,0,0,0) - t\{z^{<0}\}\vec{M} \cdot A = (1, -tz^{-1}m_0, -tz^{-1}m_1, -tz^{-1}m_2).$$

We then can compute $\Phi = z^2F\vec{v}$ (where $\vec{v} = \operatorname{adj}(I - tA) \cdot \vec{\mathbf{1}}$) and obtain

$$\begin{aligned} \Phi(t,z) = &- z^4t^4m_2 + z^3(-2\,t^2 - t^3m_1 - t^3 - 2\,t^4m_2 - t^3m_2 + t^4m_0) \\ &+ z^2(-3\,t^2 + 1 + t - 2\,t^3 - t^2m_2 + 3\,t^4m_0 - t^2m_1 - t^2m_0 + t^4m_1 + t^3m_0 - t^3m_1) \\ &+ z\alpha_1(t) \\ &+ 2\,t^3 + t^4m_0 - t^2m_1 - t^2m_0 + 3\,t^4m_1 - t^3m_0 + t^3m_1 + 2\,t^3m_2 + t^4m_2 + t^2 \\ &+ \frac{t^4m_1}{z} - \frac{t^3m_0}{z} + \frac{t^3}{z}, \end{aligned}$$

where $\alpha_1(t) = ((3m_0 + 2m_2 + 3m_1)t^4 + (3m_2 + m_1 + m_0)t^3 - (m_2 + 2m_1 + 2m_0)t^2 + (1 - m_2 - m_1 - m_0)t)$. Again, it looks as if $\Phi$ had negative $z$-powers (namely $\{z^{<0}\}\Phi = \frac{t^3}{z}(tm_1 + 1 - m_0)$), but relation (95) holds here, too (for the same reason as in the previous example) and cancels out these negative powers. Thus, $\Phi$ only contains non-negative $z$-powers, as it has to be.

From $\Phi = G(z)(z-z_2)(z-z_4)$ where $G = az^2 + bz + c$ (for some unknowns $a, b$, and $c$) we obtain by equating coefficients the following system of equations

$$\begin{aligned}
m_0t^4 + 3m_1t^4 + m_2t^4 - m_0t^3 + m_1t^3 + 2m_2t^3 - m_0t^2 - m_1t^2 + 2t^3 + t^2 &= cz_2z_4 \\
\alpha_1(t) &= -cz_2 + bz_2z_4 - cz_4, \\
3m_0t^4 + m_1t^4 + m_0t^3 - m_1t^3 - m_0t^2 - m_1t^2 - m_2t^2 - 2t^3 - 3t^2 + t + 1 &= -bz_2 - bz_4 + c + az_2z_4, \\
m_0t^4 - 2m_2t^4 - m_1t^3 - m_2t^3 - t^3 - 2t^2 &= -az_2 - az_4 + b, \\
-m_2t^4 &= a, \\
m_0 &= 1 + tm_1
\end{aligned}$$

The last line in the above system is not obtained from equating coefficients, but from relation (95). It is a linear system of six equations in the six unknowns $a, b, c, m_0, m_1, m_2$. Solving it for $m_0$ (as established earlier, we are not really interested in the other unknowns, only in $m_0 = M_0(t, 0)$) we obtain

$$\begin{aligned}
M(t,0) =& \frac{1}{80t^6 + 64t^4 + 16t^2 + 16} \times \Bigg( ((-2t^2+1)\sqrt{2(4\sqrt{5}t^2 - 12t^2 - 3\sqrt{5} + 7} \\
&+ (8t^4 + 4t^2 + 1)\sqrt{5} - 20t^4 - 4t^2 - 3)\sqrt{-8\sqrt{5}t^2 + 14 - 24t^2 + 6\sqrt{5}} \\
&- \Big((8t^4 + 4t^2 + 1)\sqrt{5} + 20t^4 + 4t^2 + 3\Big)\sqrt{2(4\sqrt{5}t^2 - 12t^2 - 3\sqrt{5} + 7} \\
&+ 72t^4 + 24t^2 + 20\Bigg).
\end{aligned}$$

By transitioning to semilength we obtain the following theorem:

**Theorem 5.2.4.** *The generating function of watermelons of semilength $n$ and lower height not exceeding 3 is*

$$\begin{aligned}
E(x) =& \frac{1}{80x^3 + 64x^2 + 16x + 16} \times \Bigg( ((-2x+1)\sqrt{2(4\sqrt{5}x - 12x - 3\sqrt{5} + 7} \\
&+ (8x^2 + 4x + 1)\sqrt{5} - 20x^2 - 4x - 3)\sqrt{-8\sqrt{5}x + 14 - 24x + 6\sqrt{5}} \\
&- \Big((8x^2 + 4x + 1)\sqrt{5} + 20x^2 + 4x + 3\Big)\sqrt{2(4\sqrt{5}x - 12x - 3\sqrt{5} + 7} \\
&+ 72x^2 + 24x + 20\Bigg).
\end{aligned}$$

By reading off coefficients, we see that their counting sequence is

$$1, 1, 3, 14, 83, 567, 4236, 33605, 278169, 2376153, 20793323, 185463380, 1679954956, \ldots$$

This sequence is not yet listed in the OEIS [63]. For comparison's sake, the generalized Catalan numbers $C(3; n)$ are listed as A064063 and start with

$$1, 1, 4, 25, 190, 1606, 14506, 137089, 1338790, 13403950, 136846144, 1419257434, \ldots$$

Thus the earlier mentioned conjecture that 2-watermelons with wall and deviation zero which do not exceed lower height $h$ are counted by the generalized Catalan numbers $C(h; n)$ does not hold.

*Arbitrary lower height*

Now we well try to extend this to arbitrary height and describe what difficulties are encountered here. The automaton describing 2-watermelons with wall not exceeding lower height $h$ is given by

$$
\begin{tikzcd}
X_0 \arrow[r, bend left, "{u,D}"] & X_1 \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & X_2 \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & \dots \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & X_{h-1} \arrow[l, bend left, "{U,d}"] \arrow[r, bend left, "{u,D}"] & X_h \arrow[l, bend left, "{U,d}"]
\end{tikzcd}
$$

Let $z$ be the variable encoding the height of the Motzkin path. Then the adjacency matrix of the above automaton is given by

$$
A_{h+1} = \begin{pmatrix} 0 & 1+z^{-1} & 0 & \dots & 0 \\ 1+z & 0 & 1+z^{-1} & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & 1+z & 0 & 1+z^{-1} \\ 0 & \dots & 0 & 1+z & 0 \end{pmatrix}_{(h+1)\times(h+1)},
$$

i.e. a tridiagonal matrix with main diagonal entries zero, entries in the first diagonal above the main being $1+z^{-1}$ and in the first diagonal below the main $1+z$. All other entries are zero. Thus

$$
I_{h+1} - tA_{h+1} = \begin{pmatrix} 1 & -t-tz^{-1} & 0 & \dots & 0 \\ -t-tz & 1 & -t-tz^{-1} & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & -t-tz & 1 & -t-tz^{-1} \\ 0 & \dots & 0 & -t-tz & 1 \end{pmatrix}_{(h+1)\times(h+1)}.
$$

In order to compute the kernel $K_{h+1} = K_{h+1}(t,z) = \det(I_{h+1} - tA_{h+1})$ we compute the Laplace expansion of the above matrix along the first column and obtain a sum of two matrices, the first being $I_h - tA_h$ the second one having only one nonzero entry in the first row. Computing its determinant by expanding along the first row, we obtain the following recursion for the kernel

$$
K_{h+1} = K_h - t^2 z^{-1}(1+z)^2 K_{h-1} \tag{97}
$$

for $h \geq 1$ and $K_1 = K_0 = 1$. Denote $K(x) := \sum_{h\geq 0} K_h x^h$ the generating function of this series. Then, after some straightforward manipulations, the recurrence relation translates into

$$
K(x) = \frac{1}{1-(x-t^2z^{-1}(1+z)^2x^2)}.
$$

Expanding this into a binomial series and reading off coefficients we obtain

$$
K_h = [x^h]K(x) = \sum_{k=0}^{h} (-1)^k \frac{t^{2k}(1+z)^{2k}}{z^k} \binom{h-k}{k}. \tag{98}
$$

Next, we would have to compute the roots of the kernel and have a look at which ones are small roots. This turns out to be a much more complicated task than in the case of one Dyck path where the kernel (92) did not depend on the variable of interest at all.

Seeing that (97) is a linear recursion of second order of (Laurent) polynomials in $z$, this suggests that a similar approach as Hoggatt and Bicknell [44] used for computing the zeroes of Fibonacci polynomials given by the linear recursion

$$
F_{n+1}(z) = zF_n(z) + F_{n-1} \quad \text{for } n \geq 2, \quad F_1(z) = 1, F_2(z) = z
$$

might turn out to be fruitful.

We will illustrate the method used to find the zeroes of Fibonacci polynomials (a clever substitution that enabled the use of some identities between hyperbolic functions) and try to mimic it for the zeroes of the kernel side by side:

Recurrence and initial conditions:

Fibonacci-polynomials
$F_{n+1}(z) = zF_n(z) + F_{n-1}(z)$
$F_1(z) = 1, F_2(z) = z$

Kernel-polynomials
$K_{n+1}(z) = K_n(z) - t^2z^{-1}(1+z)^2K_{n-1}$
$K_0(z) = 1, K_1(z) = 1,$
$K_2(z) = 1 - tz^{-1} - 2t^2 - t^2z$

Note: technically, $K_n(z) = K_n(t,z)$, but since we are interested in its zeroes in $z$, we omit the dependency on $t$.

Auxiliary equation:

$Y^2 - zY - 1 = 0$

$Y^2 - Y + t^2z^{-1}(1+z)^2 = 0$

Zeroes of the auxiliary equation:

$\alpha = \frac{z+\sqrt{z^2+4}}{2}$
$\beta = \frac{z-\sqrt{z^2+4}}{2}$

$\alpha = \frac{1+\sqrt{1-4t^2z^{-1}(1+z)^2}}{2}$
$\beta = \frac{1-\sqrt{1-4t^2z^{-1}(1+z)^2}}{2}$

Expressing $F_n$ (or $K_n$) as $c_1\alpha^n + c_2\beta^n$ ($c_1$ and $c_2$ can be determined from the initial conditions):

$F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$

$K_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$

Using a substitution (specified below) and the identities $\sinh(c) = \frac{e^c-e^{-c}}{2}, \cosh(c) = \frac{e^c+e^{-c}}{2}$ as well as $\cosh^2(c) - \sinh^2(c) = 1$ to simplify $\alpha$ and $\beta$:

substitution: $z =: 2\sinh(c)$
this implies $\sqrt{z^2+4} = 2\cosh(c)$
$\alpha = \frac{z+\sqrt{z^2+4}}{2} = \sinh(c) + \cosh(c) = e^c$
$\beta = \frac{z-\sqrt{z^2+4}}{2} = \sinh(c) - \cosh(c) = -e^{-c}$

substitution: $z =:???$
(some options are discussed on the following pages)

Using this $F_n$ (respectively $K_n$) becomes:

$F_n = \frac{e^{cn}-(-1)^ne^{-nc}}{e^c-e^{-c}}$
This implies:
$F_{2n} = \frac{\sinh(2nc)}{\cosh(c)}$
$F_{2n+1} = \frac{\cosh((2n+1)c)}{\cosh(c)}$

$K_n = \dots$

From this representation we then can find the zeroes of the Fibonacci polynomials. Clearly the polynomial equals zero when the corresponding hyperbolic functions vanish. For $c = a + ib$ we have

$$|\sinh(c)|^2 = \sinh^2 a + \sin b,$$

$$|\cosh(c)|^2 = \sinh^2 a + \cos^2 b.$$

For real $a$ the function $\sinh a$ is zero if and only if $a = 0$, which implies that the zeroes of $\sinh c$ are those of $\sinh ib = i \sin b$ and the zeroes of $\cosh c$ are those of $\cosh ib = \cos b$. From this we can easily find the $c$'s necessary and sufficient for $F_n(z) = 0$. We have to distinguish two cases:

1. $F_{2n} = 0$ implies that $\sinh(2nc) = 0$ and $\cosh c \neq 0$. From $c = a + ib$ and the above reasoning this implies that $\sin(2nb) = 0$ and $\cos b \neq 0$. Hence $b = \frac{k\pi}{2n}$ and $c = ib$. Thus the zeroes of the Fibonacci polynomials $F_{2n}$ are $z = \pm 2i \sin \frac{k\pi}{2n}$ for $k = 0, 1, \ldots, n-1$.

2. $F_{2n+1} = 0$ implies that $\cosh(2n+1)c = 0$ and $\cosh c \neq 0$. From $c = a + ib$ and the above reasoning this implies that $\cosh((2n+1)ib) = \cos((2n+1)b) = 0$ and $\cos b \neq 0$. Hence $b = \frac{(2k+1)\pi}{2(2n+1)}$ and $c = ib$. Thus the zeroes of the Fibonacci polynomials $F_{2n+1}$ are $z = \pm 2i \sin \frac{(2k+1)\pi}{2(2n+1)}$ for $k = 0, 1, \ldots, n-1$.

For the kernel polynomials, however, it is unclear what to use as substitution. The straightforward approach of setting $\alpha = e^c$ and $\beta = -e^{-c}$ and trying to derive a relationship between $z$ and $c$ works out fine for the Fibonacci polynomials, however, it does not seem to work for the kernel polynomials. Setting

$$\alpha = \frac{1 + \sqrt{1 - 4t^2 z^{-1}(1+z)^2}}{2} = e^c$$
$$\beta = \frac{1 - \sqrt{1 - 4t^2 z^{-1}(1+z)^2}}{2} = -e^{-c}$$

gives us, upon adding these two equations

$$1 = e^c - e^{-c} = 2\sinh(c),$$

which does not relate $z$ and $c$.

Another option would be the substitution $\frac{1+z}{\sqrt{z}} =: \frac{1}{2t} \sin c$ (suggested by Bernhard Gittenberger, private communication) which gives us

$$\alpha = \frac{1 + \sqrt{1 - \sin^2 c}}{2} = \frac{1 + \cos c}{2} = \cos\left(\frac{c}{2}\right)^2$$

and, by a similar computation,

$$\beta = \sin\left(\frac{c}{2}\right)^2.$$

Thus

$$K_n = \frac{\cos\left(\frac{c}{2}\right)^{2n+2} - \sin\left(\frac{c}{2}\right)^{2n+2}}{\cos c}.$$

In order to have $K_n = 0$ we need $\cos c \neq 0$ and $\cos\left(\frac{c}{2}\right)^{2n+2} - \sin\left(\frac{c}{2}\right)^{2n+2} = 0$. The first equation gives us

$$c \neq \frac{\pi}{2} + k\pi \quad \text{for } k \in \mathbb{Z}.$$

The second equation is equivalent to

$$\cos\left(\frac{c}{2}\right)^{2n+2} = \sin\left(\frac{c}{2}\right)^{2n+2}. \tag{99}$$

As long as $\cos\left(\frac{c}{2}\right) \neq 0$ this is equivalent to

$$1 = \left(\frac{\sin\left(\frac{c}{2}\right)}{\cos\left(\frac{c}{2}\right)}\right)^{2n+2} = \tan\left(\frac{c}{2}\right)^{2n+2}$$

or

$$c = 2\arctan\left(e^{\frac{2\pi k i}{2n+2}}\right)$$

for $k = 0, 1, \ldots, 2n+1$.

For $k = 0$ we have $\frac{c}{2} = \arctan(1) = \frac{\pi}{4} + 2k\pi$ which would lead to $\cos c = 0$. For $k = n+1$ we have $\frac{c}{2} = \arctan(-1) = \frac{-3\pi}{4} + 2k\pi$ which also would lead to $\cos c = 0$. For these values denominator and numerator of $K_n$ vanish simultaneously. A computation of the limit $\lim_{c \to \frac{\pi}{2}} K_n(c)$ via L'Hôpital's rule gives us

$$\lim_{c \to \frac{\pi}{2}} K_n(c) = \frac{n+1}{2^n}.$$

Similarly

$$\lim_{c \to -\frac{3\pi}{2}} K_n(c) = \frac{n+1}{2^n}.$$

Thus we have to rule out these values as zeroes of $K_n$.

For all other values of $k$ we have that $\cos c \neq 0$ by the following reasoning: For $\cos c = 0$ we have $c = \frac{\pi}{2} + \ell\pi$ thus $\cos\left(\frac{c}{2}\right) = \pm\frac{\sqrt{2}}{2}$. On the other hand $\frac{c}{2} = \arctan\left(e^{\frac{2\pi k i}{2n+2}}\right)$ and by $\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$, we see that $\cos(\arctan(x)) = \pm\frac{\sqrt{2}}{2}$ only for $x = \pm 1$, i.e. $k = 0$ and $n+1$.

For $\frac{c}{2} = \arctan\left(e^{\frac{2\pi k i}{2n+2}}\right)$ we have that $\cos\left(\frac{c}{2}\right) = \frac{1}{\sqrt{1+\left(e^{\frac{2\pi k i}{2n+2}}\right)^2}} \neq 0$, thus we do not lose any further zeroes.

For $z$ this gives us

$$z_{1/2} = \frac{\arctan\left(e^{\frac{2\pi k i}{2n+2}}\right)^2 - 2t^2 \pm \arctan\left(e^{\frac{2\pi k i}{2n+2}}\right)\sqrt{\arctan\left(e^{\frac{2\pi k i}{2n+2}}\right)^2 - 4t^2}}{2t^2} \tag{100}$$

for $k = 1, \ldots, n$ and $k = n+2, \ldots, 2n+1$. Some of these solutions, however, coincide since the solution with $+$ for $k$ and the solution with $-$ for $k+n+1$ are the same. Thus, we can consider only the solutions with $+$. For different values of $k$ we obtain different roots, thus we have in total $2n$ roots for $K_n$.

This is a lot more than expected. Looking at (98), we see that $(-1)^k \frac{t^{2k}(1+z)^{2k}}{z^k}\binom{n-k}{k}$ only contributes to $K_n$ if the binomial coefficient is nonzero, i.e. if $n-k \geq k$, which is equivalent to $\lfloor \frac{n}{2} \rfloor \geq k$. Thus, the sum describing $K_n$ can be rewritten as

$$K_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{t^{2k}(1+z)^{2k}}{z^k}\binom{h-k}{k}.$$

From this, we see that only $2 \cdot \lfloor \frac{n}{2} \rfloor$ roots of $K_n$ are to be expected.

But even after finding those zeroes the work is not fully done, because we still need to compute the entries adjoint of $I - tA$ and the autocorrelation vector in a similar vein as Lemma 5.2.1.

# 6 CONCLUSION

In this thesis several parameters, namely area, contacts, and returns, related to non-crossing pairs of lattice paths (2-watermelons) were studied. One of the key ingredients for these studies was a bijection between 2-watermelons and weighted Motzkin-paths. We saw that the area as well as the number of contacts and returns between two non-crossing paths behaves similarly as the area or the contacts and returns between one path and the $x$-axis. This is because the bijection preserves many parameters.

Furthermore, pattern avoidance in lattice paths was examined. We generalized the vectorial kernel method developed in [3], a very powerful tool for dealing with pattern avoidance in lattice paths, in two directions, namely for the avoidance of several patterns at once as well as for walks with longer steps. Using these methods we gave a full classification of Motzkin-paths avoiding any set of patterns of length two. It turned out that some of these objects are counted by the same sequence and we were able to give several explicit bijections between them.

Finally these two topics were combined to study pattern avoidance in watermelons. The bijection with weighted Motzkin paths allows us to see these objects as just one path, which fits in the framework of the vectorial kernel method. This chapter also suggests that the vectorial kernel method turns out to be helpful to study other parameters of 2-watermelons, e.g. lower height.

There are also several possibilities for follow-up research questions, e.g. finding some of the bijections between Motzkin paths with pattern constraints which are still missing or studying parameters like area or contacts with the $x$-axis in lattice paths avoiding patterns.

# 7 APPENDIX

## NOTATIONS

| | |
|---|---|
| $\mathbb{N}$ | natural numbers. In this thesis we follow the convention $0 \in \mathbb{N}$. |
| $\mathbb{Z}$ | integers |
| $\mathbb{R}$ | real numbers |
| $\mathbb{C}$ | complex numbers |
| $n!$ | factorial of a natural number: $0! := 1, n! := n \cdot (n-1)! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$. |
| $\Gamma(z)$ | Gamma function: $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ (defined for $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$). For non-negative integers $n$ we have that $\Gamma(n) = (n-1)!$. |
| $a^{\underline{k}}$ | falling factorial: $a^{\underline{0}} := 1, a^{\underline{k}} := a \cdot (a-1)^{\underline{k-1}} = a \cdot (a-1) \cdot \ldots \cdot (a-k+1)$ for $a \in \mathbb{R}$ and $k \in \mathbb{N}$ |
| $a^{\overline{k}}$ | rising factorial: $a^{\overline{0}} := 1, a^{\overline{k}} := a \cdot (a+1)^{\overline{k-1}} = a \cdot (a+1) \cdot \ldots \cdot (a+k-1)$ for $a \in \mathbb{R}$ and $k \in \mathbb{N}$ |
| $\binom{a}{k}$ | binomial coefficient: $\binom{a}{k} = \frac{a^{\underline{k}}}{k!}$ for $a \in \mathbb{R}, k \in \mathbb{N}$. For $n \in \mathbb{N}$ we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. |
| $\mathbb{P}(X = k)$ | probability that a random variable $X$ has value $k$ |
| $\mathbb{E}(X)$ | expected value of a random variable $X$ |
| $\mathbb{V}(X)$ | variance of a random variable $X$ |
| $m_r(X)$ | $r$-th moment of a random variable $X$ |
| $\operatorname{sgn}(\sigma)$ | sign of a permutation $\sigma$ defined by $\operatorname{sgn}(\sigma) := (-1)^{\lvert\operatorname{inv}(\sigma)\rvert}$ where $\operatorname{inv}(\sigma) := \{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$ |
| $\det(A)$ | determinant of a matrix $A = (a_{ij})_{i,j=1}^n$ defined by $\det(A) := \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$. |
| $\operatorname{adj}(A)$ | adjoint of a matrix $A$. It can be obtained from $\operatorname{adj}(A) = \det(A) \cdot A^{-1}$, where $A^{-1}$ is the inverse matrix of $A$. |
| $f(z)\rvert_{z=a}$ | function $f$ evaluated at $z = a$. |
| $\operatorname{Res}_a(f(z))$ | residue of the function $f$ at the point $z = a$. |
| $[z^n]$ | coefficient of $z^n$ in a formal power series $A(z)$ |
| $R[z]$ | polynomials in $z$ with coefficients in the ring $R$ |
| $R[[z]]$ | formal power series in $z$ with coefficients in the ring $R$ |
| $R[z, z^{-1}]$ | Laurent polynomials in $z$ with coefficients in the ring $R$ |
| $R((z))$ | formal Laurent series in $z$ with coefficients in the ring $R$ |
| $\mathsf{U}, \mathsf{D}, \mathsf{H}, \mathsf{F}$ | shorthand for up-step $(1,1)$, down-step $(1,-1)$, horizontal step $(1,0)$ and forward step $(2,0)$ |

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