THE NUMBER OF DESCENDANTS IN SIMPLY GENERATED RANDOM TREES

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Abstract. We derive asymptotic results on the distribution of the number of descendants in simply generated trees. Our method is based on a generating function approach and complex contour integration.

1. Introduction

The aim of this note is to generalize some recent results for binary trees by Panholzer and Prodinger [15] to a larger class of rooted trees. The number of descendants of a node j is the number of nodes in the subtree rooted at j, and the number of ascendants is the number of nodes between j and the root. Recently, Panholzer and Prodinger [15] studied the behavior of these parameters in binary trees during various traversal algorithms. The case of binary search trees was treated by Martínez, Panholzer and Prodinger [14]. In this paper we will study the number of descendants in simply generated trees (defined below). The number of ascendants is already treated in [1] and [10].

Let us start with a description of the traversal algorithms we will investigate. In the binary case there are basically three traversal algorithms. All of them are recursive algorithms treating the left subtree before the right subtree. They differ with respect to the visit of the root: first (*preorder*), middle (*inorder*), and last (*postorder*). We will study the number of descendants in simply generated trees during preorder and postorder traversal. Since the outdegree of any node in a simply generated tree need not be equal to zero or two, inorder traversal cannot be well defined for that class of trees.

Let us recall the definition of simply generated trees. Let $\mathcal{A}$ be a class of plane rooted trees and define for $T \in \mathcal{A}$ the size $|T|$ by the number of nodes of T. Furthermore there is assigned a weight $\omega(T)$ to each $T \in \mathcal{A}$. Let a_n denote the quantity

$$a_n = \sum_{|T|=n} \omega(T)$$

Besides, let us define the generating function (GF) corresponding to $\mathcal{A}$ by $a(z) = \sum_{n\ge 0} a_n z^n$. According to Meir and Moon [13] we call a family of trees *simply generated* if its GF satisfies a functional equation of the form $a(z) = z\varphi(a(z))$, where $\varphi(t) = \sum_{i\ge 0} \varphi_i t^i$ with $\varphi_i \ge 0, \varphi_0 > 0$.

Let $n_k(T)$ denote the number of nodes $v \in T$ with outdegree k (the outdegree of v is the number of edges incident with v that lead away from the root). Then we can equivalently define simply generated trees as trees with weight

$$w(T) = \prod_{k\ge 0} \varphi_k^{n_k(T)}. \tag{1.1}$$

Another correspondence which was pointed out by Aldous [1] is considering simply generated trees as representations of Galton-Watson branching processes conditioned on the total progeny. Under this point of view the offspring distribution induces the weights (1.1) (for more details see [1] or also [4]).

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In order to prove our results we will employ a generating function approach and singularity analysis in a similar fashion as used in [7]. For an introduction to the combinatorial techniques see e.g. [8, 11]. For an extensive presentation of marking techniques in combinatorial constructions with applications to random mappings see [5, 6]. Random mapping statistics similar to the tree statistics studied in this paper can be found in [2, 9].

2. Main results and Preliminaries

Choose a tree with n nodes at random (according to the distribution induced by (1.1)) and let $\alpha_j(n)$ and $\omega_j(T)$ denote the number of descendants of the jth node during preorder and postorder traversal, respectively, of the tree. We will study the distributions of these random variables and prove the following theorem:

Theorem 2.1. *Assume that $\varphi(t)$ has a positive radius of convergence R and that the equation $t\varphi'(t) = \varphi(t)$ has a minimal positive solution $\tau < R$. Then we have for $j \sim \rho n$:*

$$\mathbf{E}\alpha_j(n) \sim \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \sqrt{n} \quad \textit{and} \quad \mathbf{E}\omega_j(n) \sim \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \frac{\sqrt{\rho}}{\sqrt{1-\rho}} \sqrt{n}$$

where $\sigma^2 = \tau^2\varphi''(\tau)/\varphi(\tau)$. The variances satisfy the asymptotic relations

$$\mathbf{Var}\alpha_j(n) \sim \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}} - \arcsin\sqrt{1-\rho} \right) n^{3/2}$$

and

$$\mathbf{Var}\omega_j(n) \sim \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \left(\frac{\sqrt{\rho}}{\sqrt{1-\rho}} + \arcsin\sqrt{1-\rho} - \frac{\pi}{2} \right) n^{3/2}.$$

Furthermore a local limit theorem holds: Let the singularity of $a(z)$ on the circle of convergence be denoted by $z_0 = 1/\varphi'(\tau)$, then we have

(2.1)
$$\mathbf{P}\{\alpha_j(n) = m\} = \frac{a_m z_0^m}{\tau} \left(1 + O\left(\frac{m\log^2 n}{n}\right)\right) = \frac{1}{\sigma\sqrt{2\pi m^3}} \left(1 + O\left(\frac{1}{m}\right) + O\left(\frac{m\log^2 n}{n}\right)\right)$$

$$\mathbf{P}\{\omega_j(n) = m\} = \frac{a_m z_0^m}{\tau} \left(1 + O\left(\frac{m\log^2 n}{n}\right)\right), \quad m \le j,$$

$$= \frac{1}{\sigma\sqrt{2\pi m^3}} \left(1 + O\left(\frac{1}{m}\right) + O\left(\frac{m\log^2 n}{n}\right)\right), \quad m \le j, \tag{2.2}$$

uniformly for $m \ll n/\log^2 n$.

Remark 1. Note that if simply generated trees are viewed as conditioned branching processes, then σ^2 is just the variance of the offspring distribution.

Remark 2. Note that here an interesting phenomenon occurs: the distributions in the local limit theorem do not depend on j. This is no contradiction to the formulas for expectation and variance, since on the one hand the variances are very large ($\mathbf{Var}\alpha_j(n) \gg (\mathbf{E}\alpha_j(n))^2$) and thus the knowledge of the expectation tells us only little about the distribution. On the other hand due to the heavy tail in (2.1) and (2.2) the local limit theorem cannot be used to derive expressions for the moments.

Let us first set up the generating functions for the preorder case. Therefore denote the by a_{nkm} the (weighted) number of trees with n nodes such that the jth node x_j has m descendants. We are interested in the generating function

$$a_1(z,u,v) = \sum_{n,j,m\ge 0} a_{nkm} z^n u^j v^m.$$

It is easier to work with

$$a_1^{(m)}(z,u) = [v^m] a_1(z,u,v),$$

where the symbol $[x^n]f(x)$ denotes the coefficient of x^n in the formal power series $f(x)$. Thus we will build this function now: Note that there is a unique path connecting x_j with the root. To each of these nodes there are attached subtrees of the whole tree. The path itself and those subtrees which lie left from the path contains only nodes which are traversed before x_j, while the nodes in the subtrees on the right-hand side from the path are traversed after x_j. Thus a node with degree i on this path and j_1 subtrees on the left-hand side and j_2 subtrees on the right-hand side contributes $zu\varphi_i a(zu)^{j_1}a(z)^{j_2}$ to the generating function. Summing up over all possible configurations we get

$$a_1^{(m)}(z,u) = \frac{uz^m a_m}{1-\phi_1(z,u,1)},$$

where

$$\phi_1(z,u,v) = zu\sum_{i\geq 1}\varphi_i \sum_{j_1+j_2=i-1} a(zu)^{j_1}a(zv)^{j_2} = \frac{a(zu)-ua(zv)/v}{a(zu)-a(zv)}$$

The postorder case can be treated in an analogous way. In this case we get

$$\tilde{a}_1^{(m)}(z,u) = \frac{u^{m+1}z^m a_m}{1-\phi_1(z,u,1)/u}$$

3. Proof of Theorem 2.1

Since the generating functions for the preorder and the postorder case are so closely related it suffices to consider the preorder case.

3.1. The Expected value of $\alpha_j(n)$.

We have

$$\mathbf{E}\alpha_j(n) = \frac{1}{a_n}[z^n u^j]\sum_{m\geq 0} m a_1^{(m)}(z,u) = \frac{1}{a_n}[z^n u^j]\frac{zu(a(zu)-a(z))a'(z)}{a(z)(u-1)}$$

$$= \frac{1}{a_n}[z^n u^j]\frac{u(a(zu)-a(z))}{(u-1)(1-z\varphi'(a(z)))} \tag{3.1}$$

In order to compute this coefficient we will use Cauchy's integral formula with the following integration contour. Let z run through the contour $\Gamma_0 = \Gamma_{01}\cup\Gamma_{02}\cup\Gamma_{03}\cup\Gamma_{04}$ defined by

$$\Gamma_{01} = \left\{ z = z_0\left(1+\frac{t}{n}\right) \middle| \Re t \leq 0 \text{ und } |t| = 1\right\}$$

$$\Gamma_{02} = \left\{ z = z_0\left(1+\frac{t}{n}\right) \middle| \Im t = 1 \text{ und } 0 \leq \Re t \leq \log^2 n\right\}$$

$$\Gamma_{03} = \overline{\Gamma}_{02}$$

$$\Gamma_{04} = \left\{ z \middle| |z| = z_0\left|1+\frac{\log^2 n + i}{n}\right| \text{ und } \arg\left(1+\frac{\log^2 n+i}{n}\right) \leq |\arg(z)| \leq \pi\right\}.$$

and since the location of the singularity changes when z varies, the appropriate contour for u is $\Gamma_1 = \Gamma_{11}\cup\Gamma_{12}\cup\Gamma_{13}\cup\Gamma_{14}$ defined by

$$\Gamma_{11} = \left\{ u = \left(1+\frac{s}{j}\right) \middle| \Re s \leq -R(t) \text{ and } |s+R(t)+I(t)i| = 1\right\}$$

$$\Gamma_{12} = \left\{ u = \left(1+\frac{s}{j}\right) \middle| \Im s = -I(t)+1, -R(t)\leq \Re s \text{ and } |u| \leq \left|1+\frac{\log^2 j + i}{j}\right|\right\}$$

$$\Gamma_{13} = \left\{ u = \left(1+\frac{s}{j}\right) \middle| \Im s = -I(t)-1, -R(t)\leq \Re s \text{ and } |u| \leq \left|1+\frac{\log^2 j + i}{j}\right|\right\}$$

$$\Gamma_{14} = \left\{ u \middle| |u| = \left|1+\frac{\log^2 j + i}{j}\right| \text{ and } \arg u \in [-\pi, \arg z_{13}]\cup[\arg z_{12},\pi]\right\},$$

where

$$R(t)=\max\left(0,\frac{j}{n}\Re t\right)\quad\text{and}\quad I(s,\cdots,s_p,t)=\max\left(n^{2/3},\frac{j}{n}\Im t\right)$$

and z_{1k} denotes the point of Γ_{1k} with maximal absolute value. For convenience, set $\gamma_0=\Gamma_{01}\cup\Gamma_{02}\cup\Gamma_{03}$ and $\gamma_1=\Gamma_{11}\cup\Gamma_{12}\cup\Gamma_{13}$.

Now we use well known expansions (see e.g. [13]) for the tree function $a(z)$ and related functions in order to get the local behaviour of the integrand near its singularity: we have for $z\to z_0$ inside the domain $\{z:|z|\le z_0+\varepsilon,\arg(1-z/z_0)\ne\pi\}$ for some $\varepsilon>0$ the local expansions

$$a(z)=\tau-\frac{\tau\sqrt{2}}{\sigma}\sqrt{1-\frac{z}{z_0}}+O\left(\left|1-\frac{z}{z_0}\right|\right) \tag{3.2}$$

and

$$z\varphi'(a(z))=1-\sigma\sqrt{2}\sqrt{1-\frac{z}{z_0}}+O\left(\left|1-\frac{z}{z_0}\right|\right) \tag{3.3}$$

Inserting this into (3.1) yields for $z\in\gamma_0$ and $u\in\gamma_1$

$$\begin{aligned}&\frac{1}{a_n(2\pi i)^2}\int_{\gamma_0}\int_{\gamma_1}\frac{u(a(zu)-a(z))}{(u-1)(1-z\varphi'(a(z)))}\frac{du}{u^{j+1}}\frac{dz}{z^{n+1}}\\&=\frac{1}{a_nz_0^n(2\pi i)^2}\int_{\gamma_0}\int_{\gamma_1}\frac{\left(\sqrt{-\frac{t}{n}}-\sqrt{-\frac{t}{n}-\frac{s}{j}}\right)\frac{\tau\sqrt{2}}{\sigma}}{\frac{s}{j}\sigma\sqrt{2}\sqrt{-\frac{t}{n}}}e^{-t-s}\frac{dt\,ds}{nj}\left(1+O\left(\left|\frac{t}{n}\right|+\left|\frac{s}{j}\right|\right)\right)\\&=\frac{\tau}{a_nz_0^n\sigma^2\sqrt{n}(2\pi i)^2}\int_{\gamma_0}\int_{\gamma_0}\frac{\sqrt{-\frac{t}{n}}-\sqrt{-\frac{v}{j}}}{\left(v-\frac{tj}{n}\right)\sqrt{-t}}e^{-t(1-j/n)-v}\,dt\,dv\left(1+O\left(\frac{\log^2 n}{n}+\frac{\log^2 j}{j}\right)\right)\end{aligned}$$

Extending the integration contour to ∞ (call the new contour γ and expanding the denominator into a series and using the fact (Hankel's representation of the Gamma function, see e.g. [16]) that for any positive constant A and integers k,l, one of which is nonnegative, we have

$$\int_\gamma\int_\gamma t^kv^le^{-tA-v}\,dt\,dv=0,$$

yields after some elementary calculations

$$\begin{aligned}&\frac{\tau}{a_nz_0^n\sigma^2\sqrt{n}(2\pi i)^2}\int_{\gamma_0}\int_{\gamma_0}\frac{\sqrt{-\frac{t}{n}}-\sqrt{-\frac{v}{j}}}{\left(v-\frac{tj}{n}\right)\sqrt{-t}}e^{-t(1-j/n)-v}\,dt\,dv\\&=\frac{\tau}{a_nz_0^n\sigma^2\sqrt{jn}(2\pi i)^2}\sum_{k\ge0}\left(\frac{j}{n}\right)^k\left(1-\frac{j}{n}\right)^{-k-1/2}\int_\gamma\int_\gamma(-w)^{k-1/2}(-v)^{-k-1/2}e^{-w-v}\,dw\,dv\\&\quad\times\left(1+O\left(e^{-\frac12\log^2 n}\right)\right)\\&=\frac{\tau}{a_nz_0^n\sigma^2\pi\sqrt{jn}}\sum_{k\ge0}\left(\frac{j}{n}\right)^k\left(1-\frac{j}{n}\right)^{-k-1/2}(-1)^k\left(1+O\left(e^{-\frac12\log^2 n}\right)\right)\\&=\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\frac{\sqrt{1-j/n}}{\sqrt{j/n}}\sqrt{n}\left(1+O\left(e^{-\frac12\log^2 n}\right)\right),\end{aligned} \tag{3.4}$$

where we used again Hankel's representation

$$\frac{1}{2\pi i}\int_\gamma(-s)^{-\alpha}e^{-s}\,ds=\frac{1}{\Gamma(\alpha)}$$

as well as

$$\frac{1}{\Gamma(-k+1/2)\Gamma(k+1/2)}=\frac{\sin\pi(k+1/2)}{\pi}=\frac{(-1)^k}{\pi}$$

and $a_n = \tau/\sigma z_0^n \sqrt{2\pi n^3}(1 + O(1/\sqrt{n}))$ which can be easily obtained by applying [7, Theorem 3.1] to (3.2). (3.4) is already the desired expression, thus what remains to be shown is that the integrals where $z \in \Gamma_{04}$ or $u \in \Gamma_{14}$ are negligibly small. On Γ_{04} and Γ_{14} the estimates $|u|^{-j-1} \ll e^{-\log^2 j}$ and $|z|^{-n-1} \ll z_0^{-n} e^{-\log^2 n}$, respectively, hold. Moreover, observe that we have $1/|u-1| \le 1/j \ll 1/n$ along the integration contour. Furthermore, $a(z)$ (and hence $1/(1 - z\varphi'(a(z)))$ is analytic in the set surrounded by the integration contour. This in conjunction with (3.3) yields $1/(1 - z\varphi'(a(z)) \ll n$ and therefore

$$\begin{aligned} &\frac{1}{a_n} \iint\limits_{(z,u)\in\Gamma_0\times\Gamma_1\setminus\gamma_0\times\gamma_1} \frac{u(a(zu) - a(z))}{(u-1)(1 - z\varphi'(a(z)))} \frac{du}{u^{j+1}} \frac{dz}{z^{n+1}} \\ &\quad \ll n^{7/2} e^{-\log^2 n - \log^2 j} \\ &\quad = o\left(\frac{1}{a_n} \iint\limits_{(z,u)\in\gamma_0\times\gamma_1} \frac{u(a(zu) - a(z))}{(u-1)(1 - z\varphi'(a(z)))} \frac{du}{u^{j+1}} \frac{dz}{z^{n+1}} \right) \end{aligned}$$

which completes the proof.

3.2. **The Variance (sketch).** We need an expression for the second moment. We have

$$\mathbf{E}\alpha_j(n)^2 = \frac{1}{a_n}[z^n u^j] \frac{uz^2 a''(z)(a(zu) - a(z))}{(u-1)a(z)}$$

By elementary calculations we get

$$a''(z) = \frac{2\varphi'(a(z))\varphi(a(z))}{(1 - z\varphi'(a(z)))^2} + \frac{z\varphi''(a(z))\varphi^2(a(z))}{(1 - z\varphi'(a(z)))^3}$$

and thus

$$\mathbf{E}\alpha_j(n)^2 = \frac{1}{a_n}[z^n u^j] \left(\frac{2uz\varphi'(a(z))(a(zu) - a(z))}{(u-1)(1 - z\varphi'(a(z)))^2} + \frac{uza(z)\varphi''(a(z))(a(zu) - a(z))}{(u-1)(1 - z\varphi'(a(z)))^3} \right)$$

Obviously, the dominant singularity in this expression comes from the second term. Proceeding as in the previous section and using $z\varphi''(a(z)) \sim \sigma^2/\tau$ for $z \to z_0$ gives

$$\begin{aligned}
&\frac{1}{a_n}[z^n u^j]\frac{2uz\varphi'(a(z))(a(zu)-a(z))}{(u-1)(1-z\varphi'(a(z)))^2} \\
&\quad\sim \frac{\sigma^2}{\tau a_n z_0^n (2\pi i)^2}\int_{\gamma_0}\int_{\gamma_1}\frac{\tau\frac{\tau\sqrt{2}}{\sigma}\left(\sqrt{-\frac{t}{n}}-\sqrt{-\frac{t}{n}-\frac{s}{j}}\right)}{\frac{s}{j}\cdot 2\sqrt{2}\sigma^3\left(-\frac{t}{n}\right)^{3/2}}e^{-t-s}\frac{dt\,ds}{nj} \\
&\quad= \frac{\tau\sqrt{n}}{2a_n z_0^n\sigma^2(2\pi i)^2}\int_{\gamma_0}\int_{\gamma_0}\frac{\sqrt{-\frac{t}{n}}-\sqrt{-\frac{v}{j}}}{\left(v-\frac{tj}{n}\right)(-t)^{3/2}}e^{-t(1-j/n)-v}\,dt\,dv \\
&\quad\sim \frac{\tau}{2a_n z_0^n\sigma^2(2\pi i)^2}\sum_{k\ge 0}\left(\frac{j}{n}\right)^k\int_\gamma(-t)^{k-1}e^{-t(1-j/n)}\,dt\int_\gamma(-1)(-v)^{-k-1}e^{-v}\,dv \\
&\qquad+\frac{\tau\sqrt{n}}{2a_n z_0^n\sigma^2(2\pi i)^2\sqrt{j}}\sum_{k\ge 0}\left(\frac{j}{n}\right)^k\int_\gamma(-t)^{k-3/2}e^{-t(1-j/n)}\,dt\int_\gamma(-1)(-v)^{-k-1/2}e^{-v}\,dv \\
&\quad= -\frac{\tau}{2a_n z_0^n\sigma^2}+\frac{\tau\sqrt{n}}{2a_n z_0^n\sigma^2\pi\sqrt{j}}\sum_{k\ge 0}\left(\frac{j}{n}\right)^k\frac{(-1)^k}{(-k+1/2)}\left(1-\frac{j}{n}\right)^{-k-1/2} \\
&\quad= -\frac{\tau}{2a_n z_0^n\sigma^2}+\frac{\tau}{a_n z_0^n\sigma^2\pi\sqrt{j/n}}\sqrt{1-j/n}\left(1+\arctan\frac{\sqrt{j/n}}{\sqrt{1-j/n}}\right) \\
&\quad= \frac{\tau}{a_n z_0^n\sigma^2\pi}\left(\frac{\sqrt{1-j/n}}{\sqrt{j/n}}+\arctan\frac{\sqrt{j/n}}{\sqrt{1-j/n}}-\frac{\pi}{2}\right) \\
&\quad= \frac{n^{3/2}\sqrt{2}}{\sigma\sqrt{\pi}}\left(\frac{\sqrt{1-j/n}}{\sqrt{j/n}}-\arcsin\sqrt{1-j/n}\right)
\end{aligned}$$

and we are done.

3.3. **The distribution (sketch).** We need to evaluate

$$\mathbf{P}\left\{\alpha_j(n)=m\right\}=\frac{1}{a_n}[z^n u^j v^m]a_1(z,u,v)=\frac{1}{a_n}[z^n u^j]\frac{uz^m a_m(a(zu)-a(z))}{a(z)(u-1)}.$$

We use the same integration contour as in the previous sections and get for $m \ll n/\log^2 n$

$$
\begin{aligned}
\mathbf{P}\{\alpha_j(n)=m\} &= \frac{a_m\sqrt{2}}{\sigma z_0^{n-m}a_n(2\pi i)^2}\int_\gamma\int_\gamma \frac{\left(\sqrt{-\frac{t}{n}}-\sqrt{-\frac{v}{n}}\right)}{v-\frac{tj}{n}}e^{-v-t(1-j/n)}\,dt\,dv \\
&\quad\times\left(1+O\left(e^{-\log^2 n/2}\right)+O\left(\frac{m\log^2 n}{n}\right)\right) \\
&= -\frac{a_m\sqrt{2}\left(1+O\left(m\log^2 n/n\right)\right)}{\sigma z_0^{n-m}a_n n^{3/2}(2\pi i)^2} \\
&\quad\times\sum_{k\ge 0}\left(\frac{j}{n}\right)^k\int_\gamma\int_\gamma(-t)^{k+1/2}(-v)^{-k-1}e^{-t(1-j/n)-v}\,dt\,dv \\
&= -\frac{a_m\sqrt{2}\left(1+O\left(m\log^2 n/n\right)\right)}{\sigma z_0^{n-m}a_n n^{3/2}(1-j/n)^{3/2}}\sum_{k\ge 0}\left(\frac{j/n}{1-j/n}\right)^k\frac{1}{\Gamma(-k-1/2)\Gamma(k+1)} \\
&= \frac{a_m\left(1+O\left(m\log^2 n/n\right)\right)}{\sqrt{2\pi}\sigma z_0^{n-m}a_n n^{3/2}(1-j/n)^{3/2}}\sum_{k\ge 0}\left(\frac{j/n}{1-j/n}\right)^k\binom{-3/2}{k} \\
&= \frac{a_m}{\sqrt{2\pi}\sigma z_0^{n-m}a_n n^{3/2}}\left(1+O\left(m\log^2 n/n\right)\right) \\
&= \frac{1}{\sigma\sqrt{2\pi m^3}}\left(1+O\left(\frac{1}{m}\right)+O\left(\frac{m\log^2 n}{n}\right)\right)
\end{aligned}
$$

as desired.

4. Concluding remarks

It would be interesting to get also expressions for the joint distributions of $(\alpha_{j_1}(n),\dots,\alpha_{j_d}(n))$ and joint moments, as were derived in [1, 10] for the number of ascendants. But since an invariance property similar to [10, Lemma 3.3]) is not true in this case, we are not able to derive a general and simple shape for the generating functions which occur when we compute these joint distributions. The method presented here only in principle allows us to compute these joint distributions and joint moments, but the expressions we would encounter are terribly involved.

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