CONVERGENCE OF BRANCHING PROCESSES TO THE LOCAL TIME OF A BESSEL PROCESS

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Abstract. We study Galton-Watson branching processes conditioned on the total progeny to be n which are scaled by a sequence c_n tending to infinity as $o(\sqrt{n})$. It is shown that this process weakly converges to the totallocal time of a two-sided three-dimensional Bessel process. This is done by means of characteristic functions and a generating function approach.

1. Introduction

Let $(\varphi_k; k \geq 0)$ be a sequence of non-negative numbers and set $\varphi(t) = \sum_{k \geq 0} \varphi_k t^k$. Consider a Galton-Watson branching process with offspring distribution ξ given by

$$\mathbf{P}\{\xi = k\} = \frac{\tau^k \varphi_k}{\varphi(\tau)}, \tag{1.1}$$

where τ is an arbitrary nonnegative number within the circle of convergence of $\varphi(t)$. Without loss of generality we may restrict ourselves to critical branching processes, i.e. we may assume $\mathbf{E}\xi = 1$ which equivalently means that τ satisfies $\tau\varphi'(\tau) = \varphi(\tau)$. The variance of ξ can also be expressed in terms of $\varphi(t)$ and is given by

$$\sigma^2 = \frac{\tau^2 \varphi''(\tau)}{\varphi(\tau)}. \tag{1.2}$$

Now consider the family tree T of such a process conditioned on the total progeny to be n and let n_k be the number of nodes of this tree which have out-degree k. Then the offspring distribution (1.1) corresponds to assigning the weight

$$\omega(T) = \prod_{k \geq 0} \varphi_k^{n_k(T)}$$

to T. Denote by $|T|$ the number of nodes of such a tree and let a_n be the (weighted) number of all trees with n nodes, i.e.

$$a_n = \sum_{T:|T|=n} \omega(T).$$

Then the corresponding generating function (GF) $a(z) = \sum_{n \geq 0} a_n z^n$ which will play a key role throughout this paper satisfies the functional equation

$$a(z) = z\varphi(a(z)).$$

Denote by $(L_n(t), t \geq 0)$ a Galton-Watson process the total progeny of which is n, i.e. for integer t $L_n(t)$ is the size of the t-th generation. For non-integer t we define $L_n(t)$ by linear interpolation:

$$L_n(t) = (\lfloor t \rfloor + 1 - t) L_n(\lfloor t \rfloor) + (t - \lfloor t \rfloor) L_n(\lfloor t \rfloor + 1), \quad t \geq 0.$$

Furthermore, let $(c_n, n \geq 0)$ be a sequence of positive numbers satisfying the conditions

$$c_n \to \infty \text{ and } c_n = o(\sqrt{n}).$$

Date: February 26, 1998.
1991 *Mathematics Subject Classification.* 60J80, 60F05, 05A16.
Key words and phrases. branching processes, random trees, Bessel processes, local time.
This research was supported by the Austrian Science Foundation FWF, grant P10187-MAT.

We will show that the scaled process

$$l_n(t) = \frac{1}{c_n} L_n(c_n t), \quad t \geq 0, \tag{1.3}$$

weakly converges to total local time of a three-dimensional Bessel process which proves a conjecture stated by Aldous [1, Conjecture 7].

Theorem 1.1. *Let $\varphi(t)$ be the GF of a sequence of non-negative numbers. Besides, let $(B(s), -\infty < s < \infty)$ denote a two sided Bessel 3 process, that means $(B(s), s \geq 0)$ and $(B(-s), s \geq 0)$ are both three-dimensional Bessel processes. Denote by $l(t)$ the total local time at level t of $B(s)$, i.e.*

$$l(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} I_{[t,t+\varepsilon]}(B(s))\, ds$$

where I_A is the indicator function of the set A. Furthermore, assume that $\varphi(t)$ has a positive or infinite radius of convergence R and $\zeta = \gcd\{i | \varphi_i > 0\} = 1$. Suppose that the equation

$$t\varphi'(t) = \varphi(t)$$

has a minimal positive solution $\tau < R$ and that σ^2 defined by (1.2) is finite. Then the process $l_n(t)$ defined by (1.3) converges weakly to local time of $B(s)$, exactly that means

$$l_n(t) \xrightarrow{w} \frac{\sigma}{2} l\left(\frac{\sigma}{2} t\right) \stackrel{d}{=} \frac{\sigma^2}{4} l(t)$$

in $C[0, \infty)$, as $n \to \infty$.

Remark 1. The case $\zeta > 1$ can be treated analogously, but is technically more complicated. However, the weak limit theorem remains unchanged except that we have to require $n \equiv 1 \mod \zeta$. Thus the restriction to $\zeta = 1$ is justified.

Remark 2. Note that Aldous [1] formulated his conjecture for the step function process $\frac{1}{c_n} L_n(\lfloor c_n t \rfloor)$. But in this case the proof of tightness would be much more involved and thus we decided to work with the interpolated process. However, there is a similar tightness condition for the space $\mathrm{D}[0, \infty)$ (see [2]) and our method can be directly extended to prove the "original" conjecture which would require much more technical effort without gaining any further insight into the structure of the problem.

Remark 3. The average extinction time of a branching process conditioned on the total progeny to be n is proportional to $\sqrt{n}$. Thus the behavior changes if we choose $c_n = \sqrt{n}$ as scaling factor. In this case Brownian excursion local time is obtained as limit process as was shown in [4].

The proof of Theorem 1.1 is divided into two parts: First, we have to show that the finite dimensional distributions (fdd's) of $l_n(t)$ converge weakly to those of $l(t)$, which is done in the next section. The one-dimensional limit theorem has been established by Kennedy [13, Theorem 1] and Kolchin [14, Theorem 2.5.4]. The second quality we have to prove is that the sequence $l_n(t)$ is tight. This proof is based on [2, Theorem 12.3] and is deferred to the last section.

Remark. We would like to mention that there are other approaches to related problems in the literature: Lamperti and Ney [15] proved finite-dimensional convergence results of a similar type for branching processes conditioned to have infinite total progeny. Perhaps it is possible to use their ideas to obtain a different way of attacking this problem. Finally, note that recently Pitman [17] reproved the results in [4, 5] by means of an approach via stochastic differential equations.

2. Finite Dimensional Convergence

2.1. **The limiting distributions of l_n.** We will prove the convergence of the fdd's to those of local time by computing the (weighted) number $a_{k_1m_1k_2m_2\cdots k_dm_dn}$ of trees of size n with m_i nodes in layer k_i, $i=1,\dots,d$. Then the desired distribution is given by

$$\mathbf{P}\left\{L_n(k_1)=m_1,\dots,L_n(k_d)=m_d\right\}=\frac{a_{k_1m_1k_2m_2\cdots k_dm_dn}}{a_n}.$$

In order to determine this distribution we use a generating function approach:We have (for details see [4], for general background we refer to [6])

$$\sum_{m_1,\dots,m_d,n\geq 0} a_{k_1m_1k_2m_2\cdots k_dm_dn}u_1^{m_1}\cdots u_d^{m_d}z^n = y_{k_1}\left(z,u_1y_{k_2-k_1}\left(z,\dots y_{k_d-k_{d-1}}\left(z,u_da(z)\right)\dots\right)\right.$$

where

$$\begin{aligned} y_0(z,u)&=u\\ y_{i+1}(z,u)&=z\varphi(y_i(z,u)),\quad i\geq 0. \end{aligned}$$

Consequently the characteristic function of the joint distribution of $\frac{1}{c_n}L_n(k_1),\dots,\frac{1}{c_n}L_n(k_d)$ is given by

$$\phi_{k_1\cdots k_dn}(t_1,\dots,t_d)=\frac{1}{a_n}[z^n]y_{k_1}\left(z,e^{it_1/c_n}y_{k_2-k_1}\left(z,\dots y_{k_d-k_{d-1}}\left(z,e^{it_d/c_n}a(z)\right)\dots\right)\right. \tag{2.1}$$

where $[z^n]f(z)$ denotes the coefficient of z^n in the power series of $f(z)$.

In order to extract the desired coefficient we will use the following

Lemma 2.1. *Let z_0 be the point on the circle of convergence of $a(z)$ which lies on the positive real axis. Set $w=u-a(z)$, $\alpha=z\varphi'(a(z))$ and $\beta=z\varphi''(a(z))/2$. If $|w|=\mathcal{O}\left(\frac{1}{c_n}\right)$ and $z-z_0\to 0$ in such a way that $\arg(z-z_0)\neq 0$ and $|1-\sqrt{z-z_0}|\leq 1+\frac{C}{\sqrt{n}}$, then $y_k(z,u)$ admits the local representation*

$$y_k(z,u)=a(z)+\frac{\alpha^k w}{1-\frac{\beta}{\alpha}\frac{1-\alpha^k}{1-\alpha}w+\mathcal{O}\left(\left|\frac{1-\alpha^{2k}}{1-\alpha^2}\right||w|^2\right)} \tag{2.2}$$

uniformly for $k=\mathcal{O}(c_n)$.

Proof. The lemma looks very similar to [4, Lemma 2.1] except that there the assumption $|w|=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ is required. But investigating the proof of this lemma shows that the crucial part is [4, Lemma 3.1] which states that under the assumptions $w=\mathcal{O}(1)$ and $1/2\leq|\alpha|\leq 1+\mathcal{O}(|w|)$ we have for $k=\mathcal{O}\left(|w|^{-1}\right)$

$$y_k(z,u)-a(z)=\mathcal{O}\left(|w\alpha^k|\right). \tag{2.3}$$

Note that it is well known (see e.g. [16]) that $a(z)$ has a local expansion of the form

$$a(z)=\tau-\frac{\tau\sqrt{2}}{\sigma}\sqrt{1-\frac{z}{z_0}}+\mathcal{O}\left(\left|1-\frac{z}{z_0}\right|\right) \tag{2.4}$$

around its singulariy $z_0=1/\varphi'(\tau)$ (This can e.g. be easily derived by direct application of [7, Theorem 7.1]).

The assumption $\zeta=1$ ensures that $|z\varphi'(a(z))|<1$ for $|z|=z_0$, $z\neq z_0$. Hence, by the implicit function theorem $a(z)$ has an analytic continuation to the region $|z|<z_0+\delta$, $\arg(z-z_0)\neq 0$ for some $\delta>0$. Furthermore, it follows that $\alpha=z\varphi'(a(z))$ has similar analytic properties, especially it has the local expansion

$$\alpha=1-\sigma\sqrt{2}\sqrt{1-\frac{z}{z_0}}+\mathcal{O}\left(\left|1-\frac{z}{z_0}\right|\right). \tag{2.5}$$

Therefore the assumptions of Lemma 2.1 imply $1/2 \le \alpha \le 1 + \mathcal{O}(1/\sqrt{n})$ which is even more than necessary. Due to (2.3) equation (2.2) holds if $kw = \mathcal{O}(1)$ which is indeed the case. $\square$

Theorem 2.1. *Let $k_i = \kappa_i c_n, i = 1, \dots, d$ where $0 < \kappa_1 < \cdots < \kappa_d$. Then the characteristic function $\phi_{\kappa_1 \dots \kappa_d}(t_1, \dots, t_d) = \lim_{n\to\infty} \phi_{k_1 \cdots k_d n}(t_1, \dots, t_d)$ of the limiting distribution of $\left(\frac{1}{c_n} L_n(k_1), \dots, \frac{1}{c_n} L_n(k_d)\right)$ satisfies*

$$\phi_{\kappa_1 \dots \kappa_d}(t_1, \dots, t_d) = \frac{1}{\left(1 - i\frac{\sigma^2}{2}\left(t_1 + \frac{t_2}{A_2}\right)\kappa_1\right)^2 A_2^2 \cdots A_d^2} \tag{2.6}$$

where for $d = 1$ we set $A_2^2 \cdots A_d^2 := 1$ and

$$A_j = \begin{cases} 1 - i\frac{\sigma^2}{2}\left(t_j + \frac{t_{j+1}}{A_{j+1}}\right)(\kappa_j - \kappa_{j-1}), & j = 2, \dots, d-1, \\ 1 - i\frac{\sigma^2}{2} t_d(\kappa_d - \kappa_{d-1}), & j = d. \end{cases}$$

Remark 1. Note that by means of the generating function approach we get only a proof of this theorem for integer k_i and thus a limit theorem for the step function process $L_n(\lfloor tc_n \rfloor)/c_n$. However, a direct application of the tightness inequality (Theorem 3.1) shows that the difference $L_n(\lfloor tc_n \rfloor)/c_n - l_n(t)$ converges to zero in probability and thus the theorem is correct as stated.

Remark 2. It should be mentioned that, following [3], this approach in conjunction with multivariate saddle point asymptotics would allow to establish a corresponding local limit theorem, too.

Proof. Let us apply Cauchy's integral formula on (2.1) with the integration contour $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ where

$$\begin{aligned} \Gamma_1 &= \left\{ z = z_0\left(1 + \frac{x}{n}\right) \middle| \Re x \le 0 \text{ and } |x| = 1 \right\} \\ \Gamma_2 &= \left\{ z = z_0\left(1 + \frac{x}{n}\right) \middle| \Im x = 1 \text{ and } 0 \le \Re x \le \log^2 n \right\} \\ \Gamma_3 &= \overline{\Gamma}_2 \\ \Gamma_4 &= \left\{ z \,\middle|\, |z| = z_0 \left|1 + \frac{\log^2 n + i}{n}\right| \text{ and } \arg\left(1 + \frac{\log^2 n + i}{n}\right) \le |\arg(z)| \le \pi \right\}. \end{aligned} \tag{2.7}$$

The case $d = 1$. Set $\gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. It will turn out that the main contribution of the integral comes from γ. Expanding (2.2) into a series yields

$$\phi_{kn}(s) = 1 + \frac{1}{2\pi i a_n} \int_\Gamma \sum_{m\ge1} (w\beta)^m \frac{\alpha^k}{\beta}\left(\frac{1-\alpha^k}{\alpha(1-\alpha)}\right)^{m-1} \frac{dz}{z^{n+1}} \left(1 + \mathcal{O}\left(\left|\frac{1-\alpha^{2k}}{1-\alpha^2}\right| |w|^2\right)\right)$$

with $w = (e^{is/c_n} - 1)a(z)$. We will investigate the contribution of each term in the above sum when integrating over γ. If we fix m, then substituting $z = z_0\left(1 + \frac{x}{n}\right)$ on γ and applying (2.4), (2.5) and

$$\beta = \frac{\sigma^2}{2\tau} + \mathcal{O}\left(\sqrt{\left|1 - \frac{z}{z_0}\right|}\right).$$

leads to

$$\begin{aligned} (w\beta)^m \frac{\alpha^k}{\beta\alpha^{m-1}}\left(\frac{1-\alpha^k}{1-\alpha}\right)^{m-1} &= \left(\frac{is\sigma^2}{2c_n}\right)^m \frac{(\sqrt{n})^{m-1}\tau}{\sigma^{m+1}(\sqrt{2})^{m-3}} \cdot \frac{e^{-\lambda\sqrt{-x}}\left(1 - e^{-\lambda\sqrt{-x}}\right)^{m-1}}{(\sqrt{-x})^{m-1}} \\ &\quad \times \left(1 + \mathcal{O}\left(\sqrt{\left|\frac{x}{n}\right|}\right) + \mathcal{O}\left(\frac{1}{c_n}\right)\right) \end{aligned} \tag{2.8}$$

where $k = \kappa c_n$ and $\lambda = \sigma\sqrt{2}\kappa\frac{c_n}{\sqrt{n}}$. Moreover, one easily derives

$$e^{-\lambda\sqrt{-x}}(1-e^{-\lambda\sqrt{-x}})^{m-1} = \left[\frac{d^{m-1}}{dy^{m-1}}e^{-y\sqrt{-x}}\right]_{y=0}(-1)^{m-1}\lambda^{m-1} + \left[\frac{d^{m}}{dy^{m}}e^{-y\sqrt{-x}}\right]_{y=0}\frac{(-1)^{m+1}(m+1)\lambda^m}{2} + \mathcal{O}\left(\lambda^{m+1}\right)$$

for $\lambda \to 0$. Hence (2.8) becomes

$$(w\beta)^m\frac{\alpha^k}{\beta\alpha^{m-1}}\left(\frac{1-\alpha^k}{1-\alpha}\right)^{m-1} = \left(\left(\frac{is\kappa\sigma^2}{2}\right)^m\frac{2\tau}{\kappa\sigma^2 c_n} - \left(\frac{is\kappa\sigma^2}{2}\right)^m\frac{\tau\sqrt{2}(m+1)}{\sigma\sqrt{n}}\sqrt{-x}\right) \times\left(1+\mathcal{O}\left(\frac{c_n}{\sqrt{n}}\right)+\mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right)\right). \tag{2.9}$$

Let us extend Γ_2 and Γ_3 to infinity and denote the integration contour obtained in this way γ'. Note that the right-hand side in (2.9) is $\mathcal{O}\left(\sqrt{-x}\right)$ and besides, we have on γ'

$$\frac{dz}{z^{n+1}} = \frac{dx}{z_0^n n}\left(1+\frac{x}{n}\right)^{-n} = \frac{dx}{z_0^n n}e^{-x}\left(1+\mathcal{O}\left(\frac{\log^4 n}{n}\right)\right) \tag{2.10}$$

Thus the integrand remains bounded and we may substitute γ by γ' due to the dominated convergence theorem. Observing that

$$\frac{1}{2\pi i}\int_{\gamma'}(-s)^{-\alpha}e^{-s}\,ds = \frac{1}{\Gamma(\alpha)}$$

implies that the first term in (2.9) vanishes and we get

$$(w\beta)^m\frac{\alpha^k}{\beta\alpha^{m-1}}\left(\frac{1-\alpha^k}{1-\alpha}\right)^{m-1} = \left(\frac{is\kappa\sigma^2}{2}\right)^m\frac{(m+1)\tau}{\sigma\sqrt{2\pi n}}\left(1+\mathcal{O}\left(\frac{c_n}{\sqrt{n}}\right)+\mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right)\right). \tag{2.11}$$

Summing up over m, (2.11) in conjunction with (2.10) and

$$a_n = \frac{\tau}{\sigma z_0^n\sqrt{2\pi n^3}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)$$

would give (2.6) for $d = 1$ provided that we can keep the errors small.

Error estimates. Let us first have a look at the errors occurring when summing up: Note that (2.11) only holds for $m = o(\min(c_n, \sqrt{n}/c_n))$. Thus we have to split the sum at $M = o(\min(c_n, \sqrt{n}/c_n))$. Now observe that

$$\left|w\beta\left(\frac{1-\alpha^k}{(1-\alpha)\alpha}\right)\right| \sim \left|\frac{is\kappa\sigma^2}{2}\right| < 1$$

for sufficiently small s and therefore we obtain

$$\sum_{m>M}(w\beta)^m\left(\frac{1-\alpha^k}{(1-\alpha)\alpha}\right)^{m-1} = o\left(\max\left(\frac{1}{c_n},\frac{c_n}{\sqrt{n}}\right)\right)$$

and so summing up does not cause any problems.

What remains to be done is to estimate the contribution of Γ_4. In order to do this we turn back to (2.1). Obviously, we have

$$\left[\frac{\partial}{\partial x_2}y_k(x_1,x_2)\right]_{x_1=z,x_2=a(z)} = \alpha^k$$

and Taylor's theorem yields

$$y_k(z, e^{is/c_n}) = a(z) + \alpha^k a(z)w + \mathcal{O}\left(\left|w^2 a(z)^2\right|\right).$$

The first term satisfies

$$\frac{1}{2\pi i a_n}\int_{\gamma'\cup\Gamma_4} a(z)\frac{dz}{z^{n+1}} = 1$$

and hence does not produce any errors. In order to estimate the remainder observe that due to $\zeta = 1$ and (2.5) the maximum of $|\alpha|$ on Γ_4 is attained for $z \in \gamma \cap \Gamma_4$. Then

$$\alpha^k \sim \exp\left(-\kappa\sigma\sqrt{2}\frac{c_n}{\sqrt{n}}\sqrt{-x}\right)$$

and thus $\alpha^k = \mathcal{O}(1)$ for $z \in \Gamma_4$. Finally, the fact that $|z^{-n-1}| \sim e^{-\log^2 n}$ for $z \in \Gamma_4$ shows the contribution of Γ_4 to be negligibly small and completes the proof for $d = 1$.

The case $d > 1$. Let us first consider the case $d = 2$. So we have to deal with the generating function $y_k(z, uy_h(z, va(z)))$ with $k = \kappa c_n$, $h = \eta c_n$, and $u = e^{is/c_n}$, $v = e^{it/c_n}$. If $z \in \gamma$, then (2.2) and (2.9) imply

$$y_h(z, va(z)) = a(z) + R_h(v, z)$$

where

$$R_h = \left(\frac{it\tau/c_n}{1-\frac{it\eta\sigma^2}{2}} - \frac{\tau\sqrt{2}}{\sigma\sqrt{n}}\left(\frac{1}{\left(1-\frac{it\eta\sigma^2}{2}\right)^2} - 1\right)\sqrt{-x}\right)\left(1 + \mathcal{O}\left(\frac{c_n}{\sqrt{n}}\right) + \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right)\right)$$

and $y_k(z, uy_h(z, va(z))) = a(z) + R_k(u, v, z)$ where R_k is obtained from (2.9) by substituting w by $w + R_h$. Due to the fact that $R_h = A + B\lambda\sqrt{-x} + \mathcal{O}(\lambda^2)$ and that the left-hand side of (2.9) has a similar form, say $w^m\left(P + Q\lambda\sqrt{-x} + \mathcal{O}(\lambda^2)\right)$, where $\lambda = \sigma\sqrt{2}\kappa\frac{c_n}{\sqrt{n}}$, we have to compute

$$\begin{aligned}
\phi_{khn}(s,t) =& 1 + \frac{1}{2\pi i n a_n z_0^n}\int_\Gamma \sum_{m\ge 1}\left(w + A + B\lambda\sqrt{-x} + \mathcal{O}(\lambda^2)\right)^m\left(P + Q\lambda\sqrt{-x} + \mathcal{O}(\lambda^2)\right)e^{-x}\,dx \\
=& 1 + \frac{1}{2\pi i n a_n z_0^n}\int_\Gamma \sum_{m\ge 1}\left((w+A)^m P + ((w+A)^m Q + m(w+A)^{m-1}BP)\lambda\sqrt{-x}\right. \\
&\left. + \mathcal{O}(\lambda^2)\right)e^{-x}\,dx \\
=& 1 - \frac{\sigma^2\kappa c_n}{\tau}\sum_{m\ge 1}\left((w+A)^m P + ((w+A)^m Q + m(w+A)^{m-1}BP)\right)\left(1 + \mathcal{O}\left(\frac{c_n}{\sqrt{n}}\right)\right).
\end{aligned}$$

Then we insert

$$A = \frac{it\tau/c_n}{1-\frac{it\eta\sigma^2}{2}}, \qquad B = -\frac{\tau}{\kappa c_n\sigma^2}\left(\frac{1}{\left(1-\frac{it\eta\sigma^2}{2}\right)^2} - 1\right)$$

and

$$P = \frac{2\tau\beta^m c_n^{m-1}\kappa^{m-1}}{\sigma^2}, \qquad Q = -\frac{(m+1)\tau\beta^m c_n^{m-1}\kappa^{m-1}}{\sigma^2}$$

and use

$$w = \frac{is\tau}{c_n} + \mathcal{O}\left(\frac{1}{c_n^2}\right).$$

Taking the limit for $n \to \infty$ we directly obtain (2.6) as desired.

The error estimates can be done in the same way as in the case $d = 1$ and the cases $d > 2$ follow immediately from the above considerations by induction. □

2.2. **The fdd's of Bessel 3 local time.** In order to complete the weak limit theorem contained in Theorem 1.1 we have to identify the distributions in Theorem 2.1 as those of the local time of a two-sided three-dimensional Bessel process. This is done by the following

Proposition 2.1. *The characteristic function of the joint distribution of* $l(\kappa_1), \ldots, l(\kappa_d)$ *is given by*

$$\mathbf{E}\left[\exp(it_1 l(\kappa_1) + \cdots + it_d l(\kappa_d))\right] = \frac{1}{(1 - 2i(t_1 + t_2/\tilde{A}_2)\kappa_1)^2 \tilde{A}_2^2 \cdots \tilde{A}_d^2} \tag{2.12}$$

with

$$\tilde{A}_j = \begin{cases} 1 - 2i\left(t_j + \frac{t_{j+1}}{\tilde{A}_{j+1}}\right)(\kappa_j - \kappa_{j-1}), & j = 2, \ldots, d-1, \\ 1 - 2it_d(\kappa_d - \kappa_{d-1}), & j = d. \end{cases}$$

As in (2.1) the empty product occurring for $d = 1$ *on the right hand side has to be set equal to 1.*

Proof. By the Ray-Knight theorem the considered local time process is the square of a Bessel 4 process (see [19, p. 38]). These processes are well studied: Let $\mathrm{BESQ}_x^4(t)$ denote the square of a Bessel 4 process started at $x \geq 0$. Then [18, Chap. XI, Corollary 1.4] tells us that for $t > 0$ the Feller semi-group of $\mathrm{BESQ}_x^4(t)$ has a density in y equal to

$$q_t(x,y) = \begin{cases} \dfrac{1}{2t}\sqrt{\dfrac{y}{x}}\exp\left(-\dfrac{x+y}{2t}\right) I_1\left(\dfrac{\sqrt{xy}}{t}\right), & x > 0, \\ \dfrac{1}{(2t)^2}\, y \exp\left(-\dfrac{y}{2t}\right), & x = 0, \end{cases}$$

where I_1 denotes the first Bessel function. Due to the Markov property we have

$$\begin{aligned} &\mathbf{E}\left[e^{it_1 l(\kappa_1) + \cdots + it_d l(\kappa_d)}\right] \\ &\quad = \int \cdots \int e^{it_1 x_1 + \cdots + it_d x_d} q_{\kappa_1}(0, x_1) q_{\kappa_2 - \kappa_1}(x_1, x_2) \cdots q_{\kappa_d - \kappa_{d-1}}(x_{d-1}, x_d)\, dx_1 \cdots dx_d \end{aligned}$$

and as the Laplace transform of $q_t(x,y)$ satisfies $\mathcal{L}_s(q_t(x,y)) = (1+2st)^{-2}\exp(-sx/(1+2st))$, it is easy to derive (2.12) from the above formula. □

3. Tightness

In this section we will show that the sequence of random variables $l_n(t) = c_n^{-1} L_n(c_n t)$, $t \geq 0$, is tight in $\mathrm{C}[0,\infty)$. By [12, Theorem 4.10] it suffices to establish tightness in $\mathrm{C}[0,T]$ for all $T > 0$ and thus we may confine ourselves with considering $L_n(t)$, $0 \leq t \leq Ac_n$, where $A > 0$ is an arbitrary real constant.

[2, Theorem 12.3] tells us that we only have to show that $L_n(0)$ is tight (this is obvious) and that we have to find $\alpha > 1$, $\beta \geq 0$, and $C > 0$ such that

$$\mathbf{P}\left\{|L_n(\rho c_n) - L_n((\rho+\theta)c_n)| \geq \varepsilon c_n\right\} \leq C\frac{\theta^\alpha}{\varepsilon^\beta} \tag{3.1}$$

holds uniformly for $0 \leq \rho \leq \rho + \theta \leq A$. In order to derive (3.1) we need a slightly sharpened version of [4, Theorem 6.1]:

Theorem 3.1. *There exist constants* $C > 0$ *and* $D > 0$ *such that*

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^4 \leq C\, h^2 r^2 \tag{3.2}$$

holds for all positive integers n, r, h *with* $r, h \leq D\sqrt{n}$.

Obviously Theorem 3.1 proves (3.1) for $\alpha = 2$ and $\beta = 4$ if ρc_n and θc_n are non-negative integers and ρ and $\rho + \theta$ are bounded (which is satisfied). However, in the case of linear interpolation it is an easy exercise (see [11] or [10]) to extend (3.1) to non-integer ρc_n and θc_n (possibly with a different constant C).

Remark. As in the case $c_n = \sqrt{n}$ which was treated in [4] it is also not sufficient to consider the second moment $\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^2$.

The proof of Theorem 3.1 is essentially a slight modification of that of [4, Theorem 6.1]. The two-dimensional distribution of the number of nodes in layer r and $r+h$ is given by

$$\mathbf{P}\left\{L_n(r) = k, L_n(r+h) = l\right\} = \frac{1}{a_n}[z^n u^k v^l] y_r(z, u y_h(z, v a(z)))$$

and thus

$$\mathbf{P}\left\{L_n(r) - L_n(r+h) = m\right\} = \frac{1}{a_n}[z^n u^m] y_r(z, u y_h(z, u^{-1} a(z))).$$

Consequently

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^4 = \frac{1}{a_n}[z^n] H_{rh}(z),$$

where

$$H_{rh}(z) = \left[\left(\frac{\partial}{\partial u} + 7\frac{\partial^2}{\partial u^2} + 6\frac{\partial^3}{\partial u^3} + \frac{\partial^4}{\partial u^4}\right) y_r(z, u y_h(z, u^{-1} a(z)))\right]_{u=1}. \tag{3.3}$$

Since $a_n \sim (\tau/\sqrt{2\pi\sigma^2}) z_0^{-n} n^{-3/2}$, (3.2) is valid if

$$[z^n] H_{rh}(z) = \mathcal{O}\left(z_0^{-n} \frac{h^2 r^2}{n^{3/2}}\right) \tag{3.4}$$

holds uniformly for $r, h = o\left(\sqrt{n}\,\right)$.

The proof is based on two lemmata. The first one is the well-known transfer lemma of Flajolet and Odlyzko [8]:

Lemma 3.1. *Let $F(z)$ be analytic in Δ defined by*

$$\Delta = \{z : |z| < z_0 + \eta,\ |\arg(z - z_0)| > \vartheta\},$$

where z_0 and η are positive real numbers and $0 < \vartheta < \pi/2$. Furthermore suppose that there exists a real number β such that

$$F(z) = \mathcal{O}\left((1 - z/z_0)^{-\beta}\right) \qquad (z \in \Delta).$$

Then

$$[z^n] F(z) = \mathcal{O}\left(z_0^{-n} n^{\beta - 1}\right).$$

Lemma 3.2. *Set $\alpha = z\varphi'(a(z))$. Then for $n \to \infty$, $r = \rho c_n$ and any fixed positive integers k, l and $k_i, i = 1 \dots l$, we have*

$$[z^n]\alpha^r = \mathcal{O}\left(z_0^{-n} \frac{r}{n^{3/2}}\right) \tag{3.5}$$

$$[z^n]\frac{\alpha^r}{1 + \alpha + \alpha^2 + \cdots + \alpha^k} = \mathcal{O}\left(z_0^{-n} \frac{r}{n^{3/2}}\right) \tag{3.6}$$

$$[z^n]\frac{\alpha^r}{\prod_{i=1}^{l}(1 + \alpha + \alpha^2 + \cdots + \alpha^{k_i})} = \mathcal{O}\left(z_0^{-n} \frac{r}{n^{3/2}}\right) \tag{3.7}$$

uniformly for $\rho = \mathcal{O}(1)$.

Proof. As usual we evaluate the coefficient by means of Cauchy's integral formula using the integration contour (2.7). By (2.5) we have on γ

$$\alpha^r = \exp\left(-r\sigma\sqrt{2}\sqrt{\frac{-x}{n}}\right)\left(1 + \mathcal{O}\left(\frac{\log^2 n}{n}\right)\right)$$

and thus

$$
\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \alpha^r \frac{dz}{z^{n+1}} &= \frac{1}{2\pi i z_0^n n}\int_{\gamma'} \exp\left(-r\sigma\sqrt{2}\sqrt{\frac{-x}{n}}-x\right)dx\left(1+\mathcal{O}\left(\frac{\log^2 n}{n}\right)\right)+\mathcal{O}\left(e^{-\log^2 n}\right)\\
&= \frac{r\sigma}{\sqrt{2\pi n^3}z_0^n}\exp\left(-\frac{r^2\sigma^2}{2n}\right)\left(1+\mathcal{O}\left(\frac{\log^2 n}{n}\right)\right)\\
&= \mathcal{O}\left(z_0^{-n}\frac{r}{n^{3/2}}\right)
\end{aligned}
$$

which proves (3.5) by keeping in mind that the contribution of Γ_4 is $\mathcal{O}\left(e^{-\log^2 n}\right)$ due to the fact that the maximum on Γ_4 is attained for $z\in\Gamma_4\cap\gamma$.

In order to prove (3.6) we employ the same argument and immediately obtain that the contribution of Γ_4 is $\mathcal{O}\left(\log n\cdot e^{-\log^2 n}\right)$. Moreover, on γ we have

$$
\frac{1}{1+\alpha+\alpha^2+\cdots+\alpha^k} = \frac{1}{k+1}\left(1+\frac{\sigma\sqrt{2}\sum_{i=1}^k i}{k+1}\sqrt{\frac{-x}{n}}\right)\left(1+\mathcal{O}\left(\frac{\log^2 n}{n}\right)\right). \tag{3.8}
$$

Now using

$$
\frac{1}{2\pi i}\int_{\gamma'}\sqrt{-x}e^{-\lambda\sqrt{-x}-x}\,dx = \frac{1}{4\sqrt{\pi}}(2-\lambda^2)\exp\left(-\frac{\lambda^2}{4}\right)
$$

it is easily seen that the second term in (3.8) yields a coefficient of order $o(r/n^{3/2})$ and the contribution of the first term is corvered by (3.5). The proof of (3.7) is now immediate. □

Proof. (*Theorem 3.1*) With $\alpha = z\varphi'(a(z))$, $\beta = z\varphi''(a(z))$, $\gamma = z\varphi'''(a(z))$, and $\delta = z\varphi''''(a(z))$ we have (for details see [4, Lemma 6.1])

$$
\begin{aligned}
\frac{\partial y_r}{\partial u}(z,1) =& \alpha^r,\\
\frac{\partial^2 y_r}{\partial u^2}(z,1) =& \frac{\beta}{\alpha}\alpha^r\frac{1-\alpha^r}{1-\alpha},\\
\frac{\partial^3 y_r}{\partial u^3}(z,1) =& \frac{\gamma}{\alpha}\alpha^r\frac{1-\alpha^{2r}}{1-\alpha^2}+3\frac{\beta^2}{\alpha}\alpha^r\frac{(1-\alpha^r)(1-\alpha^{r-1})}{(1-\alpha)(1-\alpha^2)},\\
\frac{\partial^4 y_r}{\partial u^4}(z,1) =& \frac{\delta}{\alpha}\alpha^r\frac{1-\alpha^{3r}}{1-\alpha^3}+\left(2\beta\gamma(2+5\alpha+5\alpha^r+3\alpha^{r+1})+3\beta^3/\alpha\right)\alpha^r\frac{(1-\alpha^r)(1-\alpha^{r-1})}{(1-\alpha^2)(1-\alpha^3)}\\
&+3\beta^3(1+5\alpha)\alpha^r\frac{(1-\alpha^r)(1-\alpha^{r-1})(1-\alpha^{r-2})}{(1-\alpha)(1-\alpha^2)(1-\alpha^3)}.
\end{aligned}
$$

Setting $Y_{rh}(z,u) = y_r(z, uy_h(z,u^{-1}a(z)))$ and evaluating the terms in (3.3) gives

$$
\frac{\partial}{\partial u}Y_{rh}(z,1) = a(z)\alpha^r(1-\alpha^h), \tag{3.9}
$$

$$
\frac{\partial^2}{\partial u^2}Y_{rh}(z,1) = a(z)^2\frac{\beta}{\alpha}\alpha^r\frac{1-\alpha^r}{1-\alpha}(1-\alpha^h)^2+a(z)^2\alpha^{r+h}\frac{\beta}{\alpha}\frac{1-\alpha^r}{1-\alpha}, \tag{3.10}
$$

$$
\begin{aligned}
\frac{\partial^3}{\partial u^3}Y_{rh}(z,1) =& a(z)^3\alpha^r\left(\frac{\gamma}{\alpha}\frac{1-\alpha^{2r}}{1-\alpha^2}+3\frac{\beta^2}{\alpha}\frac{(1-\alpha^r)(1-\alpha^{r-1})}{(1-\alpha)(1-\alpha^2)}(1-\alpha^h)^3\right)\\
&+3a(z)^3\frac{\beta^2}{\alpha^2}\alpha^{r+h}\frac{(1-\alpha^r)(1-\alpha^h)^2}{(1-\alpha)^2}-3a(z)^2\alpha^{r+h}\frac{\beta}{\alpha}\frac{1-\alpha^h}{1-\alpha}\\
&-a(z)^3\alpha^{r+h}\left(\frac{\gamma}{\alpha}\frac{1-\alpha^{2h}}{1-\alpha^2}+3\frac{\beta^2}{\alpha}\frac{(1-\alpha^h)(1-\alpha^{h-1})}{(1-\alpha)(1-\alpha^2)}\right),
\end{aligned} \tag{3.11}
$$

and

$$\begin{aligned}\frac{\partial^4}{\partial u^4}Y_{rh}(z,1) &= a(z)^4\alpha^r\left(\frac{\delta}{\alpha}\frac{1-\alpha^{3r}}{1-\alpha^3} + \big(2\beta\gamma(2+5\alpha+5\alpha^r+3\alpha^{r+1})\right.\\
&\quad \left. + 3\beta^3/\alpha\big)\frac{(1-\alpha^r)(1-\alpha^{r-1})}{(1-\alpha^2)(1-\alpha^3)} + 3\beta^3(1+5\alpha)\frac{(1-\alpha^r)(1-\alpha^{r-1})(1-\alpha^{r-2})}{(1-\alpha)(1-\alpha^2)(1-\alpha^3)}\right)(1-\alpha^h)^4\\
&\quad + 7a(z)^4\alpha^{r+h}\left(\frac{\gamma}{\alpha}\frac{1-\alpha^{2r}}{1-\alpha^2} + 3\frac{\beta^2}{\alpha}\frac{(1-\alpha^r)(1-\alpha^{r-1})}{(1-\alpha)(1-\alpha^2)}\right)\frac{\beta}{\alpha}\frac{1-\alpha^h}{1-\alpha}(1-\alpha^h)^2\\
&\quad - 12a(z)^4\alpha^{r+h}\frac{\beta^2}{\alpha^2}\frac{(1-\alpha^r)(1-\alpha^h)}{(1-\alpha)^2} + 3a(z)^4\alpha^{r+2h}\frac{\beta^3}{\alpha^3}\frac{(1-\alpha^r)(1-\alpha^h)^2}{(1-\alpha)^3}\\
&\quad - 4a(z)^4\alpha^{r+h}\frac{\beta}{\alpha}\frac{1-\alpha^r}{1-\alpha}\left(\frac{\gamma}{\alpha}\frac{1-\alpha^{2h}}{1-\alpha^2} + 3\frac{\beta^2}{\alpha}\frac{(1-\alpha^h)(1-\alpha^{h-1})}{(1-\alpha)(1-\alpha^2)}\right)(1-\alpha^h)\\
&\quad + 12a(z)^2\alpha^{r+h}\frac{\beta}{\alpha}\frac{1-\alpha^h}{1-\alpha} + 8a(z)^3\alpha^{r+h}\left(\frac{\gamma}{\alpha}\frac{1-\alpha^{2h}}{1-\alpha^2} + 3\frac{\beta^2}{\alpha}\frac{(1-\alpha^h)(1-\alpha^{h-1})}{(1-\alpha)(1-\alpha^2)}\right)\\
&\quad + a(z)^4\alpha^{r+h}\left(\frac{\delta}{\alpha}\frac{1-\alpha^{3h}}{1-\alpha^3} + \big(2\beta\gamma(2+5\alpha+5\alpha^h+3\alpha^{h+1}) + 3\beta^3/\alpha\big)\frac{(1-\alpha^h)(1-\alpha^{h-1})}{(1-\alpha^2)(1-\alpha^3)}\right.\\
&\quad \left. + 3\beta^3(1+5\alpha)\frac{(1-\alpha^h)(1-\alpha^{h-1})(1-\alpha^{h-2})}{(1-\alpha)(1-\alpha^2)(1-\alpha^3)}\right). \qquad (3.12)\end{aligned}$$

Now observe that (by noting that in the dominating part of the integration contour the functions $a(z), \beta, \gamma, \delta$ are asymptotically equal to constants) all terms occurring in the above expressions have the form

$$f(z) = C\alpha^{m_0}(1-\alpha)^{l_1}\prod_{i=1}^{l_2}(1+\alpha+\cdots+\alpha^{m_i})\prod_{i=1}^{l_3}\frac{1}{1+\alpha+\cdots+\alpha^{k_i}}.$$

where $m_i = o(\sqrt{n})$ and the other quantities are fixed constants. If $l_1 > 0$ we will apply Lemma 3.1. We have by (2.5) $\alpha^r = \mathcal{O}(1)$ in Δ and also $(1+\alpha+\cdots+\alpha^{k_i})^{-1} = \mathcal{O}(1)$ and thus Lemma 3.1 can be directly applied and yields

$$[z^n]f(z) = \mathcal{O}\left(z_0^{-n}n^{-1-l_1/2}\prod_{i=1}^{l_2}m_i\right). \qquad (3.13)$$

In the case $l_1 = 0$ we use Lemma 3.2 and get

$$[z^n]f(z) = \mathcal{O}\left(z_0^{-n}n^{-3/2}\left.\frac{\partial}{\partial\alpha}\alpha^{m_0}\prod_{i=1}^{l_2}(1+\alpha+\cdots+\alpha^{m_i})\right|_{\alpha=1}\right). \qquad (3.14)$$

Applying (3.13) and (3.14) to (3.9)–(3.12) we obtain

$$\begin{aligned}[z^n]H_{rh}(z) &= \mathcal{O}\left(z_0^{-n}\left(\frac{r^3+r^2h^2+rh^3+h^4}{n^{3/2}} + \frac{r^2h^3}{n^2} + \frac{r^2h^3}{n^{5/2}} + \frac{r^3h^4}{n^3}\right)\right)\\
&= \mathcal{O}\left(z_0^{-n}\frac{r^3+r^2h^2+rh^3+h^4}{n^{3/2}}\right) + o\left(z_0^{-n}\frac{r^2h^2}{n^{3/2}}\right)\end{aligned}$$

which implies (3.4) and completes the proof. □

Acknowledgment. The author wishes to thank Jean-François Le Gall for pointing out some useful references.

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