Asymptotic analysis of generating functions arising from dynamic graphs

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Abstract

We study generating functions arising from sequentially growing labeled graphs where at each step either a new vertex is created or a new edge between two existing vertices is added. We provide explicit representations of the generating functions and derive asymptotic formulas for their coefficients using Laplace's method and Bessel function approximations in the undirected model, and Hayman admissibility combined with the saddle point method in the directed model. Finally, we study a natural parameter of each model and indicate further parameter studies.

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1 Introduction

Dynamic random graph processes are a natural model for evolving networks, but in many settings the network is not only random: it is genuinely time-varying, and the interaction structure co-evolves with the dynamics taking place on it [6, 17]. Such sequentially growing graphs are also studied as abstractions in their own right, for instance, in evolutionary dynamics on growing populations [20], and random graph models are widely used as benchmarks and null models for hypothesis testing and synthetic data generation [11]. This motivates the study of analytically tractable growth rules. For such models, one can ask both how many objects exist after n steps and what a typical object looks like. In this paper, we answer these questions by deriving sharp coefficient asymptotics and limit laws for natural parameters. While many prominent random graph models are specified via a random construction rule (e.g., edge addition as in Erdős–Rényi [12], or vertex-by-vertex growth as in preferential attachment [2]), analytic combinatorics suggests a complementary viewpoint: for each n, fix the *class* of objects obtainable after n steps and study the *uniform* random element of that class through its generating function [15].

We consider labeled graphs built in discrete time by iterating one of two operations: either creating a new vertex or adding an edge between existing vertices. As a starting point, we analyze two particular variants that we call Model A and Model B. In Model A edges are

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undirected and loops are forbidden; in Model B edges are directed and loops are allowed. Both models are variations of the diagram families introduced in [7] and briefly discussed in [15]. However, the focus in [7] was different and not on enumeration *per se* as in our paper. For a related variation of these diagram families, a random generation scheme could be deduced, see [8], but this is neither what we (currently) aim for in this paper.

Our graph processes are linked to well-known number sequences: In the directed case, the counting sequence is the sequence of the generalized Bell numbers (OEIS A135920), which are related to the Legendre–Stirling numbers introduced in [13]. These link our directed model to the closely related classical number sequences that occur in the framework of change of polynomial bases that goes back to Carlitz [9] and also to the Legendre-Stirling framework developed by Andrews and Littlejohn [3, 4]. In particular, all these families of number sequences satisfy the same recurrence relation and functional equation and differ only by the value of a single parameter. Nevertheless, each particular sequence exhibits its own mathematical challenges. Likewise, the two models of dynamic graph processes we are studying here, though defined in a very similar way, require a different mathematical treatment after all.

Beyond counting, the generating function framework allows the investigation of typical parameters under the uniform distribution on graphs generated in n steps, like, for instance, the number of vertices or the number of edges created after the creation of a specified vertex, or the number of loops. We will present distributional results on the number of vertices here. These can also be seen as examples for a multivariate generalization of the diagram families in [7].

This is complementary to works that analyze the time evolution and probabilistic dynamics of the growth rule itself (for instance, degree evolution and structural properties along the process), such as [14]. Their work focuses on graphs constructed by successively sampling from a finite, known pool of virtual vertices and edges. Similarly, our process can be viewed as a sequential growth mechanism where the probabilities of adding an edge or a vertex depend on the current configuration. However, our focus lies on the typical shape of a graph arising from the uniform distribution of the resulting structures.

Plan of the paper: We start with defining Model A, undirected dynamic graphs without loops, and deriving an explicit series representation for the generating function. Next we analyze this series, which is not a power series, with the discrete Laplace method, yielding the asymptotic number of graphs that can be generated in a fixed number of steps of the evolution process. Section 3 is devoted to parameter analysis and takes as most natural example the number of vertices of a graph generated in a fixed number of steps. A tilted version of the discrete Laplace method applies and eventually, we prove a central limit law for this parameter.

Section 4 introduces Model B, directed dynamic graphs with loops allowed. After setting up the generating function, we derive the asymptotic number of graphs that can be generated in a fixed number of steps. To do so, we show that the generating function is Hayman admissible and so an automatic saddle point method applies. In Section 5 we derive the number of vertices in a generated graph. It turns out that the generating function is bivariate Hayman admissible, yielding a normal limit law and we derive asymptotic mean and variance. The final section deals with a further parameter that exhibits a phase transition.

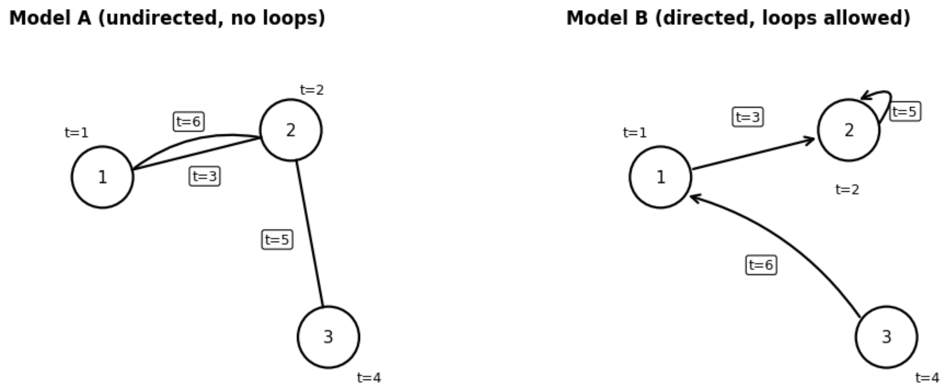

Figure 1 Example histories for both models. The label $t = \tau$ indicates the step at which the corresponding object was created.

2 Model A: Undirected dynamic graphs without loops

We start to describe the graph process we want to analyze. At each discrete time step we create a vertex or an edge and each created item gets a time stamp. Precisely, either (i) a new vertex is created, or (ii) a new undirected edge is added between two *distinct* existing vertices. These two vertices may already be adjacent, *i.e.*, we do only forbid loops, but allow multiple edges. We investigate the bivariate generating function that counts all histories generated by this process, keeping track of the number of steps and the number of vertices.

To this end, let the variable t mark time (number of steps) and u mark the number of vertices. A step creating a vertex corresponds to a factor ut in the generating function, as we add a vertex and a step. When creating an edge, we must add a step and then choose two distinct vertices. This amounts to two pointing-and-removing operations and then reverting the removals. Moreover, as the edge is undirected, the order of the vertices must not be taken into account. After all, this yields the following differential equation for the generating function $T(t,u)$:

$$T(t,u) = 1 + ut\,T(t,u) + \frac{u^2 t}{2}\,\frac{\partial^2}{\partial u^2} T(t,u).$$

Equivalently, as a formal power series in t,

$$T(t,u) = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{t\,u}{1-\binom{k}{2}t}. \tag{1}$$

This can be seen, as each vertex creation yields a factor tu and after having created k vertices so far, there are $\binom{k}{2}$ possibilities for each edge that is created before the creation of the $(k+1)$-th vertex. As we focus on the dynamics, the coefficient $[t^n u^m]T(t,u)$ is the number of graphs that can be generated in n steps and have m vertices (and hence $n-m$ edges). We are primarily interested in asymptotics of

$$[t^n]T(t,1), \qquad \text{as } n \to \infty.$$

The first few coefficients are

$$T(t,1) = 1 + t + t^2 + 2t^3 + 6t^4 + 25t^5 + 135t^6 + 909t^7 + \mathcal{O}\left(t^8\right).$$

▶ Remark 1. The product in (1) has poles at $t = 2/(k(k-1))$ for $k \geq 2$. These poles accumulate at $t = 0$, hence the radius of convergence of the series $T(t, 1)$ is 0. In particular, asymptotic analysis must be performed at the coefficient level rather than by classical singularity analysis at a dominant singularity.

2.1 Explicit coefficient representations and asymptotics

Our goal here is to get an intermediate and usable representation for the coefficients of $T(t, 1)$ that will enable us to get their asymptotics. The following proposition gives a representation as an infinite sum. It will turn out that only a partial sum yields already the asymptotics and that the terms of the sum follow a Gaussian law, making them amenable to the discrete Laplace method.

▶ **Proposition 2.** *We have $[t]T(t, 1) = 1$. Moreover, for $n \geq 2$, the coefficients of the generating function* (1) *for graphs from Model A with respect to the number of steps are given by*

$$[t^n]T(t, 1) = \frac{1}{\sqrt{2}} \sum_{k \geq 2} \binom{k}{2}^{n-1} (2k-1) J_{2k-1}(2\sqrt{2}), \tag{2}$$

where

$$J_\nu(z) = \sum_{\ell \geq 0} \frac{(-1)^\ell \left(\frac{z}{2}\right)^{2\ell+\nu}}{\Gamma(\nu + \ell + 1)\ell!} \tag{3}$$

is the Bessel function of the first kind.

Proof. We start with analyzing the products in (1). Fix $r \geq 1$ and consider the term

$$\prod_{k=1}^{r} \frac{tu}{1 - \binom{k}{2} t} = u^r \, t^r \prod_{k=1}^{r} \frac{1}{1 - \binom{k}{2} t}.$$

A partial fraction decomposition gives

$$\prod_{k=1}^{r} \frac{1}{1 - \binom{k}{2} t} = \sum_{k=1}^{r} \frac{\binom{k}{2}^{r-1} A_{k,r}}{1 - \binom{k}{2} t}$$

where

$$A_{k,r} = \prod_{\substack{s=1 \\ s \neq k}}^{r} \frac{1}{\binom{k}{2} - \binom{s}{2}}.$$

Therefore,

$$T(t, u) = 1 + \sum_{r \geq 1} u^r \, t^r \sum_{k=1}^{r} \frac{\binom{k}{2}^{r-1} A_{k,r}}{1 - \binom{k}{2} t}. \tag{4}$$

The factor $A_{k,r}$ can be simplified using standard Gamma function identities:

$$A_{k,r} = \frac{(-1)^{k-r} \, 2^{r-1} (2k-1)}{\Gamma(k+r) \, \Gamma(r+1-k)} = \frac{(-1)^{k-r} \, 2^{r-1} (2k-1)}{(k+r-1)! \, (r-k)!}.$$

Extracting coefficients in (4) gives, for $n \geq 1$,

$$[t^n]T(t,u) = \sum_{r=1}^{n} u^r \sum_{k=1}^{r} \binom{k}{2}^{n-1} A_{k,r} \tag{5}$$

and, in particular, an explicit double sum if we set $u = 1$. Now, exchanging the order of summation, summing the inner sum in (5) and using a standard Bessel function identity we obtain the claimed representation, after all. As $\binom{1}{2} = 0$ the inner sum in (5) actually starts with $k = 2$, and so does the sum in (2). ◀

2.2 The asymptotic number of graphs generated in n steps

We analyze (2) using the discrete Laplace method, see [15, Appendix B.6], for the general saddle point/Laplace framework in analytic combinatorics. Define

$$S_n := \sum_{k=2}^{2n} \exp\left(\Phi_n(k)\right) \quad \text{where} \quad \Phi_n(k) := \log\left(\binom{k}{2}^{n-1} \frac{2k-1}{\sqrt{2}} J_{2k-1}(2\sqrt{2})\right). \tag{6}$$

▶ **Lemma 3.** *We have $[t^n]T(t,1) = S_n + \mathcal{O}\left((C/n)^{2n}\right)$, as $n \to \infty$, where $C > 0$ is a computable constant.*

Proof. By Proposition 2, $[t^n]T(t,1)$ is obtained by extending the range of summation of S_n to infinity. From (3) we infer that for any fixed $z > 0$ the bound

$$|J_\nu(z)| \leq \frac{|z|^\nu}{2^\nu \Gamma(\nu+1)}$$

holds for sufficiently large ν. Therefore, if $k > n$ then (6) and Stirling's formula yield $e^{\Phi_n(k)} = \mathcal{O}\left(C^k / k^{2k-2n}\right)$ with $C > e^2/2$. Eventually, this implies that the tail of (2) is of the claimed order, hence super-exponentially small and thus negligible. ◀

Now we are ready to apply the discrete Laplace method and derive the first main result.

▶ **Theorem 4** (Undirected model: asymptotic number of graphs)**.** *The number of undirected dynamic graphs that can be generated in exactly n steps is asymptotically given by*

$$[t^n]T(t,1) \sim \sqrt{2}\, \frac{W\left(n\sqrt{2}\right)}{n 2^{2n} \sqrt{1 + W\left(n\sqrt{2}\right)}} \exp\left(2n\left(W\left(n\sqrt{2}\right) - 1 + \frac{1}{W\left(n\sqrt{2}\right)}\right)\right)$$

where W denotes the Lambert W function.

▶ Remark 5. The form of this asymptotics is to some extent reminiscent of the classical asymptotics of the Bell numbers, *cf.* [15, Example VIII.6].

Proof. We start from (6), expand the logarithm and analyze term by term. Let $f(x) := J_{2x-1}(2\sqrt{2})$ for real $x \geq 1$. Using the series definition (3) of J_ν, we obtain

$$f(x) = \sum_{\ell \geq 0} \frac{(-1)^\ell (\sqrt{2})^{2\ell+2x-1}}{\ell!\, \Gamma(\ell+2x)} = \frac{2^{x-\frac{1}{2}}}{\Gamma(2x)} \left(1 - \frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right), \qquad x \to \infty.$$

From the well-known asymptotic expansion for the digamma function, namely

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + \mathcal{O}\left(\frac{1}{z^2}\right), \qquad z \to \infty, \tag{7}$$

we get

$$\frac{f'(x)}{f(x)} = \log 2 - 2\log(2x) + \frac{1}{2x} + \mathcal{O}\left(\frac{\log x}{x^2}\right), \quad x \to \infty. \tag{8}$$

Next, we determine the location of the saddle point: Differentiating $\Phi_n(x)$ and using (8) yields

$$\Phi_n'(x) = \frac{2n}{x} - 2\log x - \log 2 - \frac{1}{2x} + \mathcal{O}\left(\frac{n}{x^2}\right).$$

From this, it is easily seen that there is a unique maximum $x^\star$ satisfying

$$\frac{n}{x^\star} = \log\left(x^\star\sqrt{2}\right) + \frac{1}{4x^\star} + \mathcal{O}\left(\frac{n}{(x^\star)^2}\right),$$

and therefore, with $W = W\left(n\sqrt{2}\right)$,

$$x^\star = \frac{n}{W} + \mathcal{O}(1). \tag{9}$$

To derive this, first ignore the lower-order corrections and solve

$$\frac{n}{x} = \log(x\sqrt{2}).$$

Now write $v = \log(x\sqrt{2})$. Then $ve^v = n\sqrt{2}$. By definition of the Lambert W-function, $v = W(n\sqrt{2})$, and therefore

$$x\sqrt{2} = \frac{n\sqrt{2}}{W(n\sqrt{2})}.$$

The omitted terms only produce a lower-order perturbation, which yields (9). As is well known, the width of the Gaussian curve is dictated by the second derivative

$$\Phi_n''(x) = -\frac{2(n-1)}{x^2} - \frac{2}{x} + \mathcal{O}\left(\frac{n}{x^3}\right) + \mathcal{O}\left(\frac{1}{x^2}\right), \quad \text{as } x \to \infty.$$

Standard arguments from the Laplace method (Gaussian approximation locally around $x^\star$ and exponential tail bounds by unimodality) yield the following result:

$$S_n \sim \exp\left(\Phi_n(x^\star)\right)\sqrt{\frac{2\pi}{-\Phi_n''(x^\star)}}.$$

Inserting (9), we have

$$\log x^\star = W - \tfrac{1}{2}\log 2 + \mathcal{O}\left(\frac{W}{n}\right),$$

using the defining relation $We^W = n\sqrt{2}$. The approximation for $f(x)$ together with Stirling's formula for $\log\Gamma(2x)$ gives a refined expansion for $\Phi_n(x)$ which, evaluated at $x = x^\star$, yields

$$\Phi_n(x^\star) = -2n\log 2 + 2n\left(W - 1 + \tfrac{1}{W}\right) - \tfrac{3}{2}W + \mathcal{O}(1).$$

Finally, evaluating the above expansion of $\Phi_n''(x)$ at $x = x^\star$, quantities involving $1/x^\star$ and $1/(x^\star)^2$ can be replaced by their leading W-terms up to a negligible error and we obtain the curvature estimate

$$\sqrt{\frac{2\pi}{-\Phi_n''(x^\star)}} = \sqrt{\frac{\pi n}{W(W+1)}}\,\left(1 + \mathcal{O}\left(\tfrac{W}{n}\right)\right).$$

Plugging these into the Laplace approximation for S_n and simplifying again with $We^W = n\sqrt{2}$ yields the claimed asymptotic. ◀

3 Parameter analysis for Model A: the number of vertices

We now extend the previous counting result by keeping track of the number of vertices. We encode the distribution of the random variable U_n, the number of vertices in a uniformly random graph generated in n steps, via its probability generating function

$$M_n(u) = \frac{[t^n]T(t,u)}{[t^n]T(t,1)}.$$

This representation allows us to extract moments from derivatives at $u = 1$ and to prove a central limit theorem via the characteristic function. Analogous to the derivation of (2) we obtain

$$[t^n]T(t,u) = \sum_{k\geq 2} \binom{k}{2}^{n-1} \frac{(2k-1)}{\sqrt{2}} \sqrt{u}\, J_{2k-1}(2\sqrt{2u}) \tag{10}$$

and, like before, the sum may be truncated at $k \leq 2n$ with a negligible tail, provided that u is near 1.

To analyze the sum, we extend the index to a real variable $x \geq 1$ and set

$$\Phi_{n,u}(x) := \log\left(\binom{x}{2}^{n-1} \frac{2x-1}{\sqrt{2}} \sqrt{u}\, J_{2x-1}\left(2\sqrt{2u}\right)\right).$$

Using the estimate

$$J_{2x-1}(2\sqrt{2u}) = \frac{(2u)^{x-\frac{1}{2}}}{\Gamma(2x)}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right), \quad \text{as } x \to \infty,$$

we obtain

$$\Phi_{n,u}(x) = \Phi_n(x) + x\log u + \mathcal{O}\left(\frac{1}{x}\right),$$

uniformly for u in a compact neighborhood of 1. Differentiating yields

$$\Phi'_{n,u}(x) = \frac{2n}{x} - 2\log x - \log 2 + \log u + \mathcal{O}\left(\frac{1}{x}\right) + \mathcal{O}\left(\frac{n}{x^2}\right),$$

which implies that the unique saddle point is asymptotically given by

$$x^\star(u) = \frac{n}{W\left(n\sqrt{2/u}\right)} + \mathcal{O}\left(1\right).$$

Using these ingredients, a Gaussian limit law can be shown after all.

▶ **Theorem 6** (Gaussian limit law for the number of vertices). *Let U_n denote the number of vertices in a graph of Model A that is randomly generated in n steps. Then U_n is asymptotically normally distributed where mean and variance satisfy, as $n \to \infty$,*

$$\mathbb{E}[U_n] \sim \frac{n}{W(n\sqrt{2})} \quad \textit{and} \quad \operatorname{Var}(U_n) \sim \frac{n}{2W(n\sqrt{2})^2}.$$

4 Model B: Directed dynamic graphs with loops allowed

4.1 The generating function

Consider the directed variant where either a new vertex is created or a new directed edge is added between existing vertices at each step, allowing loops as well. Therefore, when adding a vertex, we must increase the number of steps, too, which amounts to a factor tu in the generating function. Adding an edge means that we perform two successive pointing operations on the vertices. The generating function thus satisfies the differential equation

$$T(t,u) = 1 + utT(t,u) + u\,\frac{\mathrm{d}}{\mathrm{d}u}\left(ut\,\frac{\mathrm{d}}{\mathrm{d}u}T(t,u)\right).$$

Equivalently, we can express it as the formal power series

$$T(t,u) = \sum_{n,m\geq 0} c_{n,m}\, t^n u^m = \sum_{n=0}^{\infty}\prod_{k=1}^{n}\frac{tu}{1-k^2t}, \tag{11}$$

as we can see similarly to the explanation after (1). Here $c_{n,m}$ denotes the number of graphs that can be generated in n steps and have m vertices (and hence $n-m$ edges). Let us also set $c_n := [t^n]T(t,1)$ for the number of graphs that can be generated in n steps. At $u=1$ the first terms are

$$T(t,1) = 1 + t + 2t^2 + 7t^3 + 37t^4 + 264t^5 + 2433t^6 + 27913t^7 + \mathcal{O}\left(t^8\right).$$

Our next goal would be to get an explicit closed-form expression for $T(t,u)$. This seems out of reach, but we can get a closed form for its quasi-Borel transform.

▶ **Definition 7.** *Let $A(t,u) = \sum_{n,k\geq 0} a_{n,k}t^n u^k$ be a formal power series in two variables. We define the quasi-Borel transform of $A(t,u)$ by*

$$\mathcal{B}(A(t,u)) := \sum_{n,k\geq 0} a_{n,k}\,\frac{(2t)^n}{(2n)!}u^k.$$

With this modified concept of the classical Borel transform (see [18, pp.79] or [10]), where we replaced $n!$ by $(2n)!$ and introduced a factor 2^n in the numerator, we can show the following result.

▶ **Theorem 8.** *Let $B(t,u)$ denote the quasi-Borel transform of $T(t,u)$. Then we have*

$$B(t,u) = E\left(u\left(E(t)-1\right)\right) \quad \textit{where} \quad E(y) := \cosh\left(\sqrt{2y}\right). \tag{12}$$

▶ Remark 9. Note that the univariate counting sequence $c_n = [t^n]T(t,1)$ coincides with the generalized Bell numbers (OEIS A135920). This is reflected by the closed form of $B(t,u)$ which resembles the exponential generating function of the Stirling numbers of the second kind, $\exp\left(u\left(e^t-1\right)\right)$: If we replace $e^{(\cdot)}$ by $\cosh\left(\sqrt{2(\cdot)}\right)$ in both occurrences, we get $B(t,u)$.

Proof. Decomposing (11) into partial fractions gives

$$\begin{aligned} T(t,u) &= 1 + \sum_{n\geq 1} u^n \sum_{k=1}^{n}\frac{t}{1-k^2t}\prod_{s=1}^{k-1}\frac{1}{k^2-s^2}\prod_{s=k+1}^{n}\frac{1}{k^2-s^2} \\ &= 1 + \sum_{n\geq 1} u^n \sum_{k=1}^{n}\left(\frac{1}{1-k^2t}-1\right)\frac{2(-1)^{n-k}}{(n-k)!(n+k)!}. \end{aligned}$$

Now, we observe that

$$-\sum_{n\geq 0} u^n \sum_{k=1}^{n} \frac{2(-1)^{k-n}}{(n-k)!(n+k)!} = J_0\left(2\sqrt{u}\right)$$

by (3). Next, we apply the quasi-Borel transform and obtain

$$B(t,u) = 1 + \sum_{n\geq 0} u^n \sum_{k=1}^{n} \frac{2\left(-1+\cosh\left(k\sqrt{2t}\right)\right)(-1)^{n-k}}{(n-k)!(n+k)!}.$$

By the definition of the series (3), the inner sum equals

$$[x^{2n}]2\sum_{k\geq 0}\left(\cosh(kt)-1\right)J_{2k}(2x).$$

By the well-known identity $\sum_{n=-\infty}^{\infty} J_n(z)t^n = \exp\left(\frac{z}{2}\left(t-\frac{1}{t}\right)\right)$, see [1, Formula 9.1.41],

$$\sum_{k\in\mathbb{Z}} J_{2k}(2x)e^{kt} = \cosh\left(2x\sinh\left(t/2\right)\right),$$

and hence

$$2\sum_{k\geq 0}\left(\cosh(kt)-1\right)J_{2k}(2x) = \cosh\left(2x\sinh\left(t/2\right)\right)-1.$$

Extracting $[x^{2n}]$ and using $2\sinh\left(t/2\right)^2 = \cosh t - 1$ gives

$$\sum_{k=0}^{n} \frac{2\left(\cosh(kt)-1\right)(-1)^{n-k}}{(n-k)!(n+k)!} = \frac{2^n\left(\cosh(t)-1\right)^n}{(2n)!}.$$ ◀

4.2 The asymptotic number of graphs

We will show that our generating function $B(t,1) = E(E(t)-1)$ is Hayman admissible (also known as $\mathcal{H}$-admissible) in the sense of [19] (see also [16], where the concept is revisited and generalized to the multivariate setting). The $\mathcal{H}$-admissible functions are designed to be amenable to the saddle point method. Their definition relies on properties of their logarithmic derivatives of first and second order. The next auxiliary lemma collects all the relevant information (which we call Hayman data) for all admissible functions we need in this paper. As both the Hayman data for the outer and that for the inner function prove important in our proofs, we set $g(t) := E(t)-1$ in order to simplify notation.

▶ **Lemma 10** (Hayman data). *Define*

$$A_y(z) := \frac{z\,y'(z)}{y(z)}, \quad b_y(z) := zA_y'(z).$$

Then we have, as $t\to\infty$,

$$A_E(t) = \frac{\sqrt{2t}}{2}\tanh\left(\sqrt{2t}\right), \qquad b_E(t) = \frac{\sqrt{2t}}{4}\tanh\left(\sqrt{2t}\right) + \frac{t}{2\cosh\left(\sqrt{2t}\right)^2},$$

and

$$A_g(t) = \sqrt{\frac{t}{2}}\coth\left(\sqrt{\frac{t}{2}}\right), \qquad b_g(t) = \frac{1}{2}\sqrt{\frac{t}{2}}\coth\left(\sqrt{\frac{t}{2}}\right) - \frac{t}{4\sinh\left(\sqrt{t/2}\right)^2}.$$

Moreover, as $z \to \infty$ and $t \to \infty$,

$$A_E(z) \sim \sqrt{\frac{z}{2}}, \qquad E(z) \sim \tfrac{1}{2} e^{\sqrt{2z}}, \qquad A_g(t) \sim \sqrt{\frac{t}{2}}, \qquad g(t) \sim \tfrac{1}{2} e^{\sqrt{2t}}.$$

▶ **Theorem 11.** *The entire function $t \mapsto B(t,1)$ given by (12) is $\mathcal{H}$-admissible in the univariate sense. Let W denote the Lambert W-function and set $w_n := W(2n)$. Then, as $n \to \infty$,*

$$c_n \sim \frac{1}{\sqrt{w_n + 1}} \frac{\exp\left(2n\left(w_n + \frac{1}{w_n}\right)\right)}{(2e)^{2n}}.$$

Proof. Write $F(t) := B(t,1) = E(g(t))$. Since E and g are entire and $g(r) > 0$ for $r > 0$, the function F is entire and $F(r) > 0$ for $r > 0$. By Lemma 10 and the chain rule, the univariate Hayman data are given by

$$a(r) = A_g(r)\, A_E(g(r)), \qquad b(r) = b_g(r)\, A_E(g(r)) + A_g(r)^2\, b_E(g(r)). \tag{13}$$

Set

$$\delta(r) := b(r)^{-2/5}.$$

$\mathcal{H}$-admissible functions are defined by three conditions, see [19, (2.4)–(2.6)]: (I) Local expansion for $|\theta| \leq \delta(r)$, (II) tail bound for $\delta(r) \leq |\theta| \leq \pi$ and (III) divergence of the variance $b(r)$.

We start with the third condition, as this is used in the proof of the other two conditions.

Condition (III): Using (13) and Lemma 10, we have $A_g(r) \sim \sqrt{r/2}$, $g(r) \sim \frac{1}{2} e^{\sqrt{2r}}$, and hence $b_E(g(r)) = \Theta\big(e^{\sqrt{r/2}}\big)$. Since the second term dominates we have

$$b(r) \sim A_g(r)^2\, b_E(g(r)) = \Theta\left(r e^{\sqrt{r/2}}\right) \xrightarrow[r \to \infty]{} \infty,$$

so in particular $\delta(r) = b(r)^{-2/5} \to 0$.

Condition (I): Set

$$L_r(\theta) := \log F(re^{i\theta}).$$

Since $F(r) > 0$ for $r > 0$ and F is analytic, for r sufficiently large, $F\left(re^{i\theta}\right) \neq 0$ holds throughout the (shrinking) window $|\theta| \leq \delta(r)$. Hence, the function L_r is well-defined and analytic there. Taylor's theorem with remainder yields, uniformly for $|\theta| \leq \delta(r)$,

$$L_r(\theta) = L_r(0) + i\theta a(r) - \frac{\theta^2}{2} b(r) \;+\; \mathcal{O}\left(|\theta|^3 \sup_{|\varphi| \leq |\theta|} \left|re^{i\varphi} b'(re^{i\varphi})\right|\right).$$

For the remainder, since b' is analytic and $|\varphi| \leq \delta(r)$ is a shrinking window, we have

$$\sup_{|\varphi| \leq \delta(r)} \left|re^{i\varphi} b'\left(re^{i\varphi}\right)\right| = \mathcal{O}\left(r\, |b'(r)|\right).$$

By Lemma 10 and (13) we have $r\, b'(r) = \mathcal{O}\left(\sqrt{r}\, b(r)\right)$, hence

$$\sup_{|\varphi| \leq \delta(r)} \left|re^{i\varphi} b'\left(re^{i\varphi}\right)\right| = \mathcal{O}\left(\sqrt{r}\, b(r)\right).$$

It is now straightforward to see that, uniformly for $|\theta| \leq \delta(r)$, the remainder is bounded by $\mathcal{O}\left(\sqrt{r}\, b(r)^{-1/5}\right) \xrightarrow[r\to\infty]{} 0$. This justifies the local Gaussian approximation.

Condition (II): Let $z = re^{i\theta}$ with $\delta(r) \leq |\theta| \leq \pi$. Set

$$s(z) := \sqrt{2z}, \qquad u(z) := \sqrt{2g(z)}.$$

Then $g(z) = \cosh(s(z)) - 1$ and $F(z) = \cosh(u(z))$. *(a) For $\delta(r) \leq |\theta| \leq 1$.* Here, $\Re s = \sqrt{2r}\cos(\theta/2) \to \infty$, hence

$$\cosh s - 1 = \tfrac{1}{2}e^{s}\left(1 + \mathcal{O}\left(e^{-\Re s}\right)\right),$$

and therefore $|u(z)| \sim \exp\left(\sqrt{r/2}\cos(\theta/2)\right)$. Since $|\cosh w| \leq e^{|w|}$, $\log|F(z)| \leq |u(z)|$. Since $\log F(r) \sim \exp\left(\sqrt{r/2}\right)$ and $1 - \cos(\theta/2) \geq \theta^2/12$ for $|\theta| \leq 1$, it follows that for some $c > 0$,

$$\log\left|F\left(re^{i\theta}\right)\right| - \log F(r) \leq -c\, e^{\sqrt{r/2}} \min\{\theta^2\sqrt{r},\, 1\} \leq -c\, e^{\sqrt{r/2}}\, \delta(r)^2\sqrt{r}.$$

Using $b(r) \asymp re^{\sqrt{r/2}}$, we have

$$e^{\sqrt{r/2}}\delta(r)^2\sqrt{r} = e^{\sqrt{r/2}}b(r)^{-4/5}\sqrt{r} \to \infty,$$

hence

$$\left|F\left(re^{i\theta}\right)\right| = o\left(\frac{F(r)}{\sqrt{b(r)}}\right).$$

(b) Large angles $1 \leq |\theta| \leq \pi$. Then $\Re s = \sqrt{2r}\cos(\theta/2) \leq \sqrt{2r}\cos(1/2)$. Thus

$$|u(z)| = \mathcal{O}\left(e^{\sqrt{r/2}\cos(1/2)}\right) \qquad \text{and} \qquad \log|F(z)| = \mathcal{O}\left(e^{\sqrt{r/2}\cos(1/2)}\right).$$

It follows that $\log|F(z)| = o(\log F(r))$. Therefore,

$$\log\left(\frac{|F(z)|\sqrt{b(r)}}{F(r)}\right) \leq o(\log F(r)) + \tfrac{1}{2}\log b(r) - \log F(r) \xrightarrow[r\to\infty]{} -\infty.$$

Since $F(t)$ is $\mathcal{H}$-admissible, we apply Corollary 2 of [16]: letting ρ_n be the unique solution of $a(\rho_n) = n$, we have

$$c_n \sim \frac{(2n)!}{2^n}\,\frac{F(\rho_n)}{\rho_n^n\sqrt{2\pi b(\rho_n)}}.$$

Solving the saddle point equation yields $\rho_n = 2w_n^2\,(1 + o(1))$ and evaluating $F(\rho_n)$ and $b(\rho_n)$ with Lemma 10 followed by Stirling's formula for $(2n)!$ gives the stated asymptotic. ◀

5 Parameter analysis for Model B: the number of vertices

▶ **Proposition 12.** *The generating function $B(t,u) = E(ug(t))$ is $\mathcal{H}$-admissible in the sense of the multivariate Hayman scheme (see [16, Def. 2]) in any range $(R_0,\infty)^2$ with $R_0 > 0$.*

▶ Remark 13. This result is not surprising in view of Theorem 11. It is remarkable, though, that already $E(t)$ is Hayman admissible which is to our knowledge the first time that we encounter a Hayman admissible function of sub-exponential growth. And the fact that $g(t)$ is $\mathcal{H}$-admissible and also $E(g(t))$ and $E(ug(t))$ resembles some closure properties known for $\mathcal{H}$-admissible functions, *cf.* [19, Theorem VI] and [16, Theorem 12]. It is likely that our proofs can be adapted to prove closure properties with $E(t)$ instead of e^t as the outer function.

Note that bivariate Hayman admissibility as defined in [16, Def. 2] imposes certain asymptotic relations on the function, say $f(t,u)$, which hold for $|t| \to \infty$ and $|u| \to \infty$ simultaneously. The interesting range for the inferred limit theorems for $[t^n u^m]f(t,u)$ is when n and m tend to infinity in such a way that m is around the average number of the considered parameter. But in our case, the saddle point is then at a point $(\rho_{t,n}, \rho_{u,n})$ with $\rho_{u,n}$ staying bounded.

However, the definition of $\mathcal{H}$-admissibility can be weakened at the price of losing the closure properties satisfied by the class of Hayman admissible functions in the original sense. But we do not care for closure properties here, and moreover, we do not even lose them, as the function is $\mathcal{H}$-admissible anyway. The results actually needed from [16] for our limit theorem are those which guarantee asymptotic normality; and they hold for this weaker kind of admissibility as well. The next result makes this precise.

▶ **Proposition 14.** *Let a function $f(t,u)$ be called quasi-$\mathcal{H}$-admissible if it satisfies only [16, Def. 2, (I)–(III)] and not all five conditions, and for $|t| \to \infty$ and $|u| = \Theta(1)$ ($\mathcal{H}$-admissibility requires an asymptotics for both tending to infinity). Then, the generating function $B(t,u) = E(ug(t))$ is quasi-$\mathcal{H}$-admissible in any range $(R_0, \infty) \times [R_1, R_2]$ with R_0, R_1, R_2 being fixed positive constants.*

Proof. The proof is only a slight variation of the proof of Proposition 12. ◀

▶ **Corollary 15.** *Let $f(t,u)$ be quasi-$\mathcal{H}$-admissible in some range $(R_0, \infty) \times [R_1, R_2]$, and let $a(t,u) = (a_1(t,u), a_2(t,u))$ and $B_f(t,u)$ denote its multivariate Hayman data. Denote the eigenvalues of $B_f(t,u)$ by $\lambda_1(t,u)$ and $\lambda_2(t,u)$ and set $f_{n,m} = [t^n u^m]f(t,u)$.*

Then, the coefficients $f_{n,m}$ follow asymptotically a multivariate Gaussian density, i.e., they satisfy the uniform asymptotic formula stated in [16, Theorem 4], but for $n \to \infty$ and $m = \Theta(1)$. Moreover, the partial sums of its coefficients satisfy a normal limit law: Let $A(t,u)$ be the orthogonal matrix that diagonalizes $B_f(t,u)$, i.e., $A(t,u)B_f(t,u)A(t,u)^T$ is a diagonal matrix. Set $\tilde{a}(r_t, r_u) = a(r_t, r_u)A(r_t, r_u)^T$ and $(\tilde{n}, \tilde{m}) = (n,m)A(r_t, r_u)^T$. Then, for $r_t \to \infty$ and $r_u = \Theta(1)$, we have

$$\sum_{\substack{i,j \ s.t. \\ \tilde{i} \le \tilde{a}_1(r_t,r_u) + \omega_1 \sqrt{\lambda_1(r_t,r_u)}, \\ \tilde{j} \le \tilde{a}_2(r_t,r_u) + \omega_2 \sqrt{\lambda_2(r_t,r_u)}}} f_{i,j}\, r_t^i r_u^j \sim \int_{-\infty}^{\omega_1} \int_{-\infty}^{\omega_2} e^{-(s^2+y^2)/2}\, \mathrm{d}s\, \mathrm{d}y. \tag{14}$$

Proof. Weakening [16, Def. 2] to quasi-$\mathcal{H}$-admissible functions and inspecting the proofs of [16, Theorem 4] shows that the assertion of the theorem, adapted as in the formulation of Corollary 15, is still valid in the weaker setting. Likewise, (14) is the corresponding adaptation of [16, Theorem 5] to the weaker setting, which is also valid. ◀

▶ **Theorem 16 (Normal limit law for the number of vertices).** *Let M_n be the number of vertices in a uniform random directed graph generated in n steps, i.e.*

$$\mathbb{P}(M_n = m) = \frac{c_{n,m}}{c_n} \qquad (m \ge 0).$$

Then, as $n \to \infty$, M_n is asymptotically normally distributed, where mean and variance are given asymptotically by

$$\mathbb{E}[M_n] \sim \frac{n}{w_n} \quad \textit{and} \quad \mathrm{Var}(M_n) \sim \frac{n}{2w_n}.$$

Proof. Corollary 15 implies that M_n satisfies a central limit theorem. It remains to determine the asymptotic mean and the asymptotic variance.

Since the factor $\frac{(2n)!}{2^n}$ is independent of m, it cancels in coefficient ratios. We determine

$$\mathbb{E}[M_n] = \frac{[t^n]\partial_u B(t,u)|_{u=1}}{[t^n]B(t,1)}, \qquad \mathbb{E}[M_n(M_n-1)] = \frac{[t^n]\partial_u^2 B(t,u)|_{u=1}}{[t^n]B(t,1)}.$$

Using $B(t,u) = E(ug(t))$, we have

$$\partial_u B(t,u)\big|_{u=1} = g(t)\, E'(g(t)) = B(t,1)\, A_E(g(t)),$$

and

$$\partial_u^2 B(t,u)\big|_{u=1} = g(t)^2\, E''(g(t)) = B(t,1)\left(A_E(g(t))^2 - A_E(g(t)) + b_E(g(t))\right),$$

where we used the identity $z^2E''(z)/E(z) = A_E(z)^2 - A_E(z) + b_E(z)$. Let ρ_n be the saddle point for $t \mapsto B(t,1)$ from the proof of the previous theorem; then $\rho_n \sim 2w_n^2$ and $g(\rho_n) \to \infty$. We apply Corollary 2 again, but here note that multiplying an admissible function by an analytic factor that is asymptotically constant on the Hayman window only multiplies the coefficient asymptotics by that factor evaluated at the saddle, see [5, Thm. 3]. But note that the assumptions of that theorem are not entirely satisfied, as $A_E(g(t))/A_E(g(|t|))$ is not bounded if $g(t)$ is near the imaginary axis. However, in this range $B(t,1)\, A_E(g(t))/(B(|t|,1)\, A_E(g(|t|)))$ can be shown to be small such that the assertion of [5, Thm. 3] is still true. This leads us to

$$\mathbb{E}[M_n] \sim A_E(g(\rho_n)) \qquad \text{and} \qquad \mathbb{E}[M_n(M_n-1)] \sim A_E(g(\rho_n))^2 - A_E(g(\rho_n)) + b_E(g(\rho_n)).$$

Hence $\mathbb{E}[M_n^2] = \mathbb{E}[M_n(M_n-1)] + \mathbb{E}[M_n] \sim A_E(g(\rho_n))^2 + b_E(g(\rho_n))$, and therefore

$$\mathrm{Var}(M_n) \sim b_E(g(\rho_n)).$$

Let $h_n := \sqrt{\rho_n/2}$. From $a(\rho_n) = n$ and Lemma 10 we have

$$n = a(\rho_n) = A_g(\rho_n)A_E(g(\rho_n)) \sim \sqrt{\frac{\rho_n}{2}} \cdot \frac{1}{2} e^{\sqrt{\rho_n/2}} = \frac{1}{2}\, h_n e^{h_n},$$

hence $h_n e^{h_n} = 2n(1+o(1))$ and therefore

$$h_n = W(2n(1+o(1))) = w_n + o(1).$$

Again by Lemma 10, $g(\rho_n) \sim \frac{1}{2}e^{2h_n}$, and hence

$$A_E(g(\rho_n)) \sim \sqrt{\frac{g(\rho_n)}{2}} \sim \frac{1}{2}e^{h_n} \sim \frac{1}{2}e^{w_n} = \frac{n}{w_n}, \qquad b_E(g(\rho_n)) \sim \frac{1}{2}A_E(g(\rho_n)) \sim \frac{n}{2w_n},$$

using $w_n e^{w_n} = 2n$. ◀

6 A parameter exhibiting a phase transition

Fix an integer $N \geq 1$ and some $\gamma \in (0, 1]$ such that γN is an integer. Consider Model B conditioned on having exactly N vertex creation steps and mark the edges that are generated as long as the graph has exactly γN vertices. Then, the corresponding generating function is (see the explanation after (1) that has to be adapted appropriately)

$$F(t, u) = t^N \left(\prod_{k=1}^{N} \frac{1}{1 - k^2 t} \right) \cdot \frac{1 - (\gamma N)^2 t}{1 - u(\gamma N)^2 t}.$$

Let us define the random variable U_n (depending on N and γ) by

$$\mathbb{P}(U_n = m) = \frac{[t^n u^m] F(t, u)}{[t^n] F(t, 1)}, \qquad m \geq 0.$$

U_n equals the number of edges that are created after the (γN)-th vertex has been created, but before the $(\gamma N + 1)$-th vertex will be created on a total of n steps. This parameter exhibits two different regimes, where the limiting distribution is either geometric or exponential.

Subcritical regime: If $\gamma < 1$, then the dominant singularity of $F(t, u)$ in t is the pole at $t_0 = 1/N^2$ coming from the product, while the factor $(1 - u(\gamma N)^2 t)^{-1}$ is analytic at t_0. A routine coefficient ratio argument yields

$$\frac{[t^n] F(t, u)}{[t^n] F(t, 1)} \longrightarrow \frac{1 - \gamma^2}{1 - u\gamma^2}, \quad \text{as } n \to \infty,$$

and so, $U_n \xrightarrow{d} \mathrm{Geom}(1 - \gamma^2)$ with $\mathbb{P}(U_n = m) \to (1 - \gamma^2)\gamma^{2m}$.

Critical regime: If $\gamma = 1$, the factor $(1 - N^2 t)$ cancels one pole in the product, and the remaining marked pole at $(1 - uN^2 t)^{-1}$ becomes the dominant singularity of the system if u is sufficiently close to 1. Specifically, the radius of convergence $\rho(u) = (uN^2)^{-1}$ now depends directly on u, whereas it was previously fixed at $1/N^2$.

Computing the expectation in the usual way by $[z^n] F_u(t, 1)/[t^n] F(t, 1)$ gives

$$\mathbb{E}[U_n] = \frac{[t^n] \left(F(t, 1) N^2 t / (1 - N^2 t) \right)}{[t^n] F(t, 1)} \sim n$$

as the functions in the numerator and the denominator have the same dominant pole, but of second order in the numerator. So, in the critical case the mean grows linearly in n whereas it is bounded in the subcritical case.

To proceed, let us rescale the random variable: Set $X_n := U_n/n$. Then, the tail probability for $x \geq 0$ is given by

$$\mathbb{P}(X_n > x) = \mathbb{P}(U > xn) = \sum_{m > \lfloor xn \rfloor} \mathbb{P}(U = m) = \frac{[t^n] \sum_{m > \lfloor xn \rfloor} u^m F(t, u) \big|_{u=1}}{[t^n] F(t, 1)}.$$

A standard transfer for $F(t, u)$ near $t = 1/N^2$ yields $\lim_{n\to\infty} \mathbb{P}\left(\frac{U}{n} > x\right) = e^{-x}$. Eventually, we obtain that U_n is exponentially distributed. Precisely, we have $U_n/n \xrightarrow{d} \mathrm{Exp}(1)$.

This illustrates a singularity cancellation mechanism yielding a genuine phase transition in limit laws.

7 Conclusion

We studied two graph evolution models which complement the investigations in [7] and derived the asymptotic number of graphs at time n. In both cases, we eventually arrive at the results by utilizing a saddle point approach, nevertheless the mathematical challenges to be overcome are different. We also showed on examples how to enhance the generating functions to the bivariate setting and study parameters.

Interesting links to classical number sequences were discovered: For the directed model (Model B), the coefficients admit an additional purely combinatorial interpretation. Let $T(n,k)$ denote the number of graphs with n steps and k vertices, and let $P(n,k)$ be the number of set partitions of $\{1,1',\dots,n,n'\}$ into k blocks of even size. Then one has

$$T(n,k) = \frac{P(n,k)}{(2k-1)!!},$$

as recorded in OEIS A156289. Moreover, the numbers $P(n,k)$ satisfy the recurrence

$$P(n,k) = (2(k-1)+1)\,P(n-1,k-1) + (k(k-1)+k)\ P(n-1,k).$$

Combinatorially, the two terms correspond to inserting the new pair $\{n,n'\}$ either by creating a new block (and placing one or both of n,n' into it), or by distributing n and n' among the existing k blocks; in the split cases, even block sizes are restored by transferring the current maximum element from one affected block to the other. Under the associated mapping from even-block partitions to graphs (blocks $\leftrightarrow$ vertices, and each pair $\{m,m'\}$ yielding either a loop or a directed edge according to whether it lies in one or two blocks), each graph with k vertices arises from exactly $(2k-1)!!$ such partitions: for a fixed vertex set $\{1,\dots,k\}$ there are $(2k-1)!!$ ways to pair the $2k$ distinguished elements that realize the block minima (hence the vertex labels), while the remaining elements encode the edge/loop labels.

Further parameters fall into a similar scheme. For instance, consider the random variable X_n equal to the number of loops in a graph of Model B, which is associated to the bivariate generating function

$$F(t,u) := \sum_{n\geq 0} \frac{t^n}{\prod_{k=1}^{n}(1-(k(k-1)+uk)t)}.$$

Clearly, $\mathbb{E}\left[u^{X_n}\right] = [t^n]F(t,u)/[t^n]F(t,1)$. By partial fraction decomposition and identities for the Gamma function, one can show that

$$[t^n]F(t,u) = \sum_{k\geq 1}(k^2+(u-1)k)^{n-1}\beta_k(u),$$

where $\beta_k(u)$ is analytic near $u=1$ and satisfies a uniform asymptotics of the form $\beta_k(u) \sim \frac{2^{k-1}}{\Gamma(2k+u-1)}$ near $u=1$. As before, we write the coefficient in the form $[t^n]F(t,u) = \sum_{k\geq 1}\exp(\Phi_n(k,u))$. Again, the key ingredients are a discrete Laplace analysis around the saddle point $x^* \sim n/W(n\sqrt{2})$ together with uniform control for u in a neighborhood of 1. Finally, a quasi-powers argument yields a logarithmic scale asymptotic normality: $\mathbb{E}[X_n] \sim W(n\sqrt{2})$, $\mathrm{Var}(X_n) \sim W(n\sqrt{2})$.

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