A Note on a Model in Ruin Theory using Derivative Securities *

Bernhard Gittenberger

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Abstract

This paper is intended as a note on [13], where risk processes related to storm damage insurance are investigated. We consider a risk process in a Markovian environment with n states where changes from one state to another are caused either by a claim or by external events. We offer an asymptotic approach by adapting the results in [2] which works for a general claim size distributions.

1 Introduction

The aim of this note is to examine the effect of catastrophe bonds or bonds with embedded options on risk processes in a Markovian environment. The motivation for studying such risk processes comes from recent developments in the insurance business, namely the increasing use of alternative reinsurance concepts. Examples include the Winterthur CAT-Coupon "Hagel" [16] which has been analyzed by Schmock [12], the USAA hurricane bond [17], and the Swiss Re. earthquake bond [15].

We will assume that the model which fits to the situation of the insurance company is the following: there are time intervals in which an insurance is not faced with an adverse claim situation, and then there will be comparatively short periods of time where the number and size of the claim will make a reinsurance necessary. We assume that if such a state of nature j is entered from a state i the insurance will receive a sum of $K_{i,j}$ out of exercising the embedded option. Also the insurance will receive a sum of $L_{i,j}$ if this state transition is accompanied by a claim. We assume that all payments are non-negative, and thus contain only opportunity and no risk for the insurance company. This potential profit has to be paid for by option premiums. These payments are assumed to occur with sufficiently high frequency and can therefore be modeled by a continuous payment.

Siegl and Tichy [13] studied this model for exponential claim distributions. They provided a general model with n states and for the case of two states they could prove existence and uniqueness of the solution of the system of equations describing the probability of survival. But it seems that their method is not easily modifyable in order to work for other types of distributions. Thus we will use a method of Asmussen [2] and give an asymptotic solution to this model for arbitrary claim size distributions.

2 The Model

Basically, what we are modeling is a time-nonhomogeneous process with a possibility for a deterministic jump which is governed by a Markov process: Denote by C_i the size of the i-th claim. Furthermore let $(Z_t, t \geq 0)$ denote the underlying Markov jump process whose value is the current

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state. Call the state space $E = \{1, \ldots, n\}$. The jumps arrive according to a Poisson process which we denote by N_t. Thus the risk process $(R_t, t \geq 0)$ can be written in the form

$$R_t = \bar{c}_t t - \sum_{n=1}^{N_t} C_t \tag{1}$$

where $\bar{c}_t$ denotes the average income up to time t, i.e.,

$$\bar{c}_t = \sum_{i=1}^{n} T_i(t) c_i$$

when $T_i(t)$ is the total time of the interval $[0, t]$ the process has spent in state i and c_i is the intensity of the income (which is equal to premium minus payment for the embedded option) in state i. Note that for technical reasons the process is written as if there is no initial reserve. However, this can be coped with by redefining the ruin event to occur if $R_t < -x$.

For the case of two different states and state independent claim size distribution and no jumps the model has already been discussed by Reinhard [11]. Some further generalizations of this idea are due to Björk and Grandell [5], Asmussen [2], Asmussen and Højgaard [3], and Fuh [6]. For further literature see [7, 8, 13]. Related models were treated by Klüppelberg [9], Promislov [10], and Berg and Haberman [4].

Storm periods: We assume that the duration of each period is not known a priori. Let us moreover assume that the duration of each period is exponentially distributed. The premium in state i is c_i and the claim size is assumed to be gamma distributed with parameters β_i and α. The claim intensity is given by the matrix $\Lambda = (\lambda_{i,j})$ and the state change intensity is given by $\Gamma = (\gamma_{i,j})$. These two matrices represent the two possible ways that a state change from a state i to a state j may occur. Either the change occurs after an exponentially distributed claim with parameter β_i (intensity $\lambda_{i,j}$) or by a change without claim (intensity $\gamma_{i,j}$, where we define $\gamma_{ii} := -\sum_{j \neq i} \gamma_{ij}$ for technical reasons). Conditional on a state change from state i to j due to a claim the insurance receives a payment of $L_{i,j}$ and conditional on a state change without claim a payment of size $K_{i,j}$.

In Asmussen [2] a Markov modulated risk process was studied and a corrected diffusion approximation in the sense of Siegmund [14] for the ruin probabilities was derived. That paper deals with a model without payments upon state change and without the possibility that a state change and a claim may coincide. However, Asmussen's methods can be adapted to our situation and we can get results for the general model. In the following we will present the relevant theorems for our model, but we confine ourselves with sketching the proofs (for details consult [2]).

Set $\hat{F}_t(i, j; \alpha) := \mathbf{E}_i[e^{-\alpha R_t}, Z_t = j]$ (R_t defined in (1)) where probabilities, expected values, and so on, with subscript i denote expressions conditioned on $Z_0 = i$. We will need a representation for the matrix $\hat{\mathbf{F}}_t(\alpha) := (\hat{F}_t(i, j; \alpha))_{i,j \in E}$. Assuming that $Z_t = i$ and restricting the process to the time interval $[0, h]$ we get according to the different possibilities of what can happen (each, a claim, a state change, and a claim accompanied by a state change, may occur or not) the following relation:

$$\begin{aligned} \hat{F}_{t+h}(i, j; \alpha) &= (1 - \lambda_{ii} h + \gamma_{ii} h) \hat{F}_t(i, j; \alpha) e^{-\alpha h c_i} + \lambda_{ii} h \hat{B}_i(\alpha) \hat{F}_t(i, j; \alpha) \\ &\quad + h \left(\sum_{k \neq i} \gamma_{ik} \hat{F}_t(k, j; \alpha) e^{-\alpha K_{ik}} + \sum_{k \neq i} \lambda_{ik} \hat{F}_t(k, j; \alpha) e^{-\alpha L_{ik}} \right) + O\left(h^2\right) \end{aligned}$$

where $\hat{B}_i(\alpha)$ is the m.g.f. of the claim size, i.e., $\hat{B}_i(\alpha) = \int_0^\infty e^{\alpha x}\, dB_i(x)$. The above equation transforms to

$$\begin{aligned} \frac{d}{dt} \hat{F}_t(i, j; \alpha) &= (-\lambda_{ii} + \gamma_{ii} - \alpha c_i) \hat{F}_t(i, j; \alpha) + \lambda_{ii} \hat{B}_i(\alpha) \hat{F}_t(i, j; \alpha) \\ &\quad + \sum_{k \neq i} \gamma_{ik} \hat{F}_t(k, j; \alpha) e^{-\alpha K_{ik}} + \sum_{k \neq i} \lambda_{ik} \hat{F}_t(k, j; \alpha) e^{-\alpha L_{ik}} \end{aligned}$$

and this is equivalent to

$$\frac{d}{dt}\hat{\mathbf{F}}_t(\alpha) = (\mathbf{S}(\alpha) + \Gamma + \mathbf{G}(\alpha) + \Lambda + \mathbf{L}(\alpha) - \alpha\mathbf{C})\hat{\mathbf{F}}_t(\alpha)$$

where

$$\mathbf{G}(\alpha) = (\gamma_{ij}(e^{-\alpha K_{ij}} - 1))_{i,j\in E}, \qquad \mathbf{L}(\alpha) = (\lambda_{ij}(e^{-\alpha L_{ij}} - 1))_{i,j\in E}$$

and $\mathbf{S}$ and $\mathbf{C}$ are diagonal matrices defined by

$$\mathbf{S}(\alpha) = \mathrm{diag}(\lambda_{ii}(\hat{B}_i(\alpha) - 2))_{i\in E}, \qquad \mathbf{C} = \mathrm{diag}(c_i)_{i\in E}.$$

Therefore by $\hat{\mathbf{F}}_0(\alpha) = \mathbf{I}$ we immediately have

Theorem 1 *The following relation holds:*

$$\hat{\mathbf{F}}_t(\alpha) = e^{t(\mathbf{S}(\alpha)+\Gamma+\mathbf{G}(\alpha)+\Lambda+\mathbf{L}(\alpha)-\alpha\mathbf{C})}$$

In the following we will need a version of Wald's fundamental identity of sequential analysis and therefore we have to embed our process into an exponential family. Let $e^{\kappa(\alpha)}$ denote the spectral radius of $\hat{\mathbf{F}}_1(\alpha)$ and $h^{(\alpha)}$ a corresponding right eigenvector, that means we have $\hat{\mathbf{F}}_1(\alpha)h^{(\alpha)} = e^{\kappa(\alpha)}h^{(\alpha)}$. Furthermore, set $F_t(i,j;A) = \mathbf{P}\{R_t \in A, Z_t = j\}$. Then we define the exponential family via the measure-valued matrix $\mathbf{F}_t^{(\theta)}$ with elements

$$F_t^{(\theta)}(i,j,;dx) = \frac{h^{(\theta-\theta_0)}(j)}{h^{(\theta-\theta_0)}(i)} e^{(\theta-\theta_0)x - t\kappa(\theta-\theta_0)} F_t(i,j;dx) \tag{2}$$

where we choose (following [1, Ch. XII]) the location parameter $\theta_0 := -\gamma_0$ with $\kappa'(\gamma_0) = 0$ (a solution to this equation is assumed to exist). These probability measures can be interpreted as follows

Theorem 2 *The probabilities $\mathcal{P}_{\theta;i}$ defined by $\mathbf{P}_{\theta;i}\{R_t \in A, Z_t = j\} = F_t^{(\theta)}(i,j,;A)$ correspond to a risk process R_t in a Markovian environment with initial condition $Z_0 = i$ and with parameters*

$$\lambda_{\theta;ii} = \lambda_{ii}\hat{B}_i(\theta-\theta_0), \quad \hat{B}_{\theta;i}(\alpha) = \frac{\hat{B}_i(\alpha+\theta-\theta_0)}{\hat{B}_i(\theta-\theta_0)}$$

and

$$\hat{\mathbf{F}}_t^{(\theta)}(\alpha) = e^{t(\mathbf{S}_\theta(\alpha)+\Psi_\theta(\alpha)-\alpha C)}$$

where $\mathbf{S}_\theta(\alpha) = \mathbf{S}(\alpha+\theta-\theta_0) - \mathbf{S}(\theta-\theta_0)$ and

$$\Psi_\theta(\alpha) = \mathbf{S}(\theta-\theta_0) + \Delta_\theta^{-1}\Psi(\alpha)\Delta_\theta - (\kappa(\theta-\theta_0) + \theta - \theta_0)C$$

with $\Delta_\theta = diag(h^{(\theta-\theta_0)}(i))_{i\in E}$ and $\Psi(\alpha) = \Gamma + G(\alpha) + \Lambda + L(\alpha)$.

Proof. (2) translates to

$$\begin{aligned} \hat{\mathbf{F}}_t^{(\theta)}(\alpha) &= e^{-t\kappa(\theta-\theta_0)}\Delta_\theta^{-1}\hat{\mathbf{F}}_t(\alpha+\theta-\theta_0)\Delta_\theta \\ &= e^{-t\kappa(\theta-\theta_0)}\exp\left(t\left(\mathbf{S}(\alpha+\theta-\theta_0) + \Delta_\theta^{-1}\Psi(\alpha)\Delta_\theta - (\alpha+\theta-\theta_0)C\right)\right). \end{aligned}$$

Moreover, because of $\hat{\mathbf{F}}_t^{(\theta)}(0)e = e$ (where e is the row vector with each entry equal to 1) and $\hat{\mathbf{F}}_{s+t}^{(\theta)} = \hat{\mathbf{F}}_s^{(\theta)}\hat{\mathbf{F}}_t^{(\theta)}$ we have indeed a new Markov-modulated random walk. □

The following lemma of Asmussen [2] (a generalized Wald's identity) remains unchanged for our model:

Lemma 1 *Let τ be a stopping time w.r.t. the filtration $\mathcal{F}_t = \sigma(Z_s, R_s : s \le t)$ and $F \in \mathcal{F}_t$ an event satisfying $F \subseteq \{\tau < \infty\}$. Then for any i, θ we have*

$$\begin{aligned} \mathbf{P}_i F &= \mathbf{P}_{\theta_0;i} F \\ &= h^{(\theta-\theta_0)}(i)\mathbf{E}_{\theta;i}\left[h^{(\theta-\theta_0)}(Z_\tau)^{-1}\exp\left((\theta_0-\theta)R_\tau + \tau\kappa(\theta-\theta_0)\right); F\right]. \end{aligned}$$

For a proof see [1, p. 266].

Following Asmussen [2] we use uniformisation, i.e. we choose $\eta > \sum_j \lambda_{ij} - \gamma_{ii}$ for all i and a Poisson process $(N_t^*; t \geq 0)$ with rate η and construct (Z_t) and (N_t) in the following way: each arrival of (N_t^*) causes an arrival of (N_t) w.p. λ_{ii}/η, a jump of (Z_t) to state j w.p. γ_{ij}/η, a jump to state j accompanied by an arrival of (N_t) w.p. λ_{ij}/η, and a dummy event w.p. $(\eta+\gamma_{ii}-\sum_j \lambda_{ij})/\eta$. Set $\sigma(0) := 0$ and $\sigma(n)$ equal to the n-th arrival epoch of N_t^*, i.e., $\sigma(n) := \min\{t : N_t^* = n\}$ and let $S_n = R_{\sigma(n)}$, $X_n = S_n - S_{n-1}$, and $J_n = Z_{\sigma(n)}$. Then J_n and Z_t have the same stationary distribution π and we have a Markov modulated random walk in the sense that (J_n, X_n) is a Markov chain with transition function depending only on the first coordinate. Let $F(i,j;A) := \mathbf{P}_i\{J_1 = j, X_1 \in A\}$. This in conjunction with the initial conditions ($S_0 = X_0 = 0$) completely specifies the random walk. Let us further define the time-reversed process (J_n^*, X_n^*) given by the transition function

$$F^*(i,j;A) = \mathbf{P}_\pi\{J_0 = i, X_1 \in A | J_1 = i\} = \frac{\pi(j)}{\pi(i)} F(j,i;A).$$

Moreover, let

$$\begin{aligned}
\tau_+ &= \inf\{n \geq 1 : S_n > 0\}, \qquad G_+(i,j;A) = \mathbf{P}_i\{S_{\tau_+} \in A, J_{\tau_+} = j\}, \\
\tau_- &= \inf\{n \geq 1 : S_n^* \leq 0\}, \qquad G_-(i,j;A) = \frac{\pi(j)}{\pi(i)} \mathbf{P}_j\{S_{\tau_-}^* \in A, J_{\tau_-}^* = i\}
\end{aligned}$$

and denote the corresponding matrices by $\mathbf{F}(A)$, $\mathbf{G}_+(A)$, and $\mathbf{G}_-(A)$.

If the premium is sufficient, then by [2, Lemma 5.1] $\mathbf{G}_+(\mathbf{R})$ has positive left eigenvector $\pi_+ = \pi(\mathbf{I}-\mathbf{G}_-(\mathbf{R}))$, where $\pi = \pi^{(\theta)}$ is the Perron-Frobenius left eigenvector of $\mathbf{S}_0(\theta)+\Psi_0(\theta)-\theta\mathbf{C}$. Now we are in position to formulate the corrected diffusion approximation result (cf. [2, Theorem 7.1]):

Theorem 3 *As $\theta_0 \uparrow 0$, $x \to \infty$ in such a way that $\xi = x\theta_0 < 0$ remains fixed, we have*

$$U_i(x) \approx 1 - e^{-\gamma x - \gamma c_1(i)}$$

where $\gamma > 0$ is the solution of the Lundberg equation $\kappa(\gamma) = 0$ and $c_1(i) = \pi_+^{(0)}\mathbf{M}_+^{(2)}e/2\pi_+^{(0)}\mathbf{M}_+^{(1)}e - h'(i) + \pi_+^{(0)}h'$ with $h' = (e\pi^{(0)} - \Gamma - \Lambda)^{-1}(\mathbf{S}_0'(0) + \Psi_0'(0) - \mathbf{C})e$ and

$$M_+^{(k)}(i,j) = \int_0^\infty x^k G_+(i,j;dx) = \hat{G}_+^{(k)}(i,j;0).$$

Proof. The proof is essentially the same as that of [2, Theorem 7.1], so we will give a very brief sketch: It relies on the fact that

$$\left(\frac{1}{\sqrt{x^2}\kappa''(0)} R_{tx^2} - \kappa'(0)tx^2\right)_{t\geq 0}$$

has a standard Brownian limit and that (by similar arguments as in [14])

$$\mathbf{E}_{\theta_0;i}\left[e^{-\lambda\tau(x)\kappa_0''(0)/x}; \tau(x) < \infty\right] \to e^{-h(\lambda,\xi)} \tag{3}$$

where $\tau(x)$ is the time of ruin under initial reserve x, i.e.

$$\tau(x) = \inf\{t \geq 0 : R_t < -x\},$$

$\kappa_0''(0) = \kappa''(\gamma_0)$ (which follows from the exponential family setup by $\kappa_\theta(\alpha) = \kappa_{\theta_0}(\alpha+\theta-\theta_0)-\kappa_{\theta_0}(\theta-\theta_0)$), and $h(\lambda,\xi) = \sqrt{2\lambda+\xi^2} - \xi$ is the Laplace transform of the inverse Gaussian distribution $G(\cdot;\xi,1)$ with $G(T;\xi,c) = 1 - \Phi\left(c/\sqrt{T} - \xi\sqrt{T}\right) + e^{2\xi c}\Phi\left(-c/\sqrt{T} - \xi\sqrt{T}\right)$ (which is the first passage time distribution of Brownian motion with drift ξ to level c). The idea is to use more accurate asymptotics in (3) and do a formal inversion. The crucial point in the proof is to apply Lemma 1. Here we take 0 instead of θ_0 and get thus the Perron-Frobenius eigenvector $h^{(\theta)}$ of $M(\theta) := \mathbf{S}_0(\theta) + \Psi_0(\theta) - \theta C$. Let $\pi^{(\theta)}$ denote the left eigenvector normalized by $\pi_\theta e = \pi_\theta h(\theta) = 1$. Now by differentiating $\pi_\theta h(\theta) = 1$ we get

$$(\pi^{(0)})' e = -\pi^{(0)} (h^{(0)})' = 0$$

since $h^{(0)} = e$ and $(\pi^{(0)})' e = (\pi^{(0)} e)'$. Next differentiating $M(\theta) h^{(\theta)} = \kappa(\theta) h^{(\theta)}$ yields

$$-M'(0) e = M(0)(h^{(0)})' = (M(0) - e\pi^{(0)})(h^{(0)})'$$

where we applied $\kappa(0) = \kappa'(0) = 0$ and $\pi^{(0)}(h^{(0)})' = 0$. Thus we get

$$(h^{(0)})' = -(M(0) - e\pi^{(0)})^{-1} M'(0) e$$

which equals the expression above.

The remainder of the proof is the same as the proof of [2, Theorem 7.1] and consists of tedious calculations in order to derive an approximation for the finite horizon ruin probability (up to time T). The statement in the Theorem is then obtained by performing the limit for $T \to \infty$. $\square$

Finally, the matrices $M_+^{(i)}$ can be calculated as in [2, Lemma 5.2 and Theorem 5.2], where $H(\theta)$ has to be substituted by $H(\theta) := \theta \mathbf{C} - \mathbf{S}(\theta) - \mathbf{L}'(0) - \mathbf{G}'(0)$ and Λ by $\mathbf{L}'(0)$.

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Address:
Bernhard Gittenberger
Institut für Geometrie
TU Wien
Wiedner Hauptstr. 8-10/113
A-1040 Wien
Austria