ON THE PROFILE OF RANDOM TREES

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Abstract. Let T be a plane rooted tree with n nodes which is regarded as family tree of a Galton-Watson branching process conditioned on the total progeny. The profile of the tree may be described by the number of nodes or the number of leaves in layer $t\sqrt{n}$, respectively. It is shown that these two processes converge weakly to Brownian excursion local time. This is done via characteristic functions which are obtained by means of generating functions arising from the combinatorial setup and complex contour integration. Besides, an integral representation for the two dimensional density of Brownian excursion local time is derived.

1. Introduction

Consider a class $\mathcal{A}$ of plane rooted trees. Define for each $T \in \mathcal{A}$ the size $|T|$ by the number of nodes T consists of and a weight

$$\omega(T) = \prod_{k\geq 0} \varphi_k^{n_k(T)}, \tag{1.1}$$

where $(\varphi_k; k \geq 0)$ are non-negative numbers and $n_k(T)$ is the number of nodes $v \in T$ with out-degree k. Furthermore set

$$a_n = \sum_{T:|T|=n} \omega(T).$$

Then the corresponding generating function (GF) $a(z) = \sum_{n\geq 0} a_n z^n$ satisfies the functional equation

$$a(z) = z\varphi(a(z)), \tag{1.2}$$

where $\varphi(t) = \sum_{k\geq 0} \varphi_k t^k$. According to Meir and Moon [20] we will call such a family of trees *simply generated*. Now equip the sets $\mathcal{A}_n = \{T \in \mathcal{A} : |T| = n\}$ with the probability distribution induced by the weight function $\omega(T)$. Then we call each tree $T \in \mathcal{A}$ a random tree.

As Aldous [1] pointed out there is a natural correspondence between simply generated random trees and Galton-Watson branching processes: Let X be a branching process with offspring distribution ξ determined by

$$\mathbf{P}\{\xi = k\} = \frac{\tau^k \varphi_k}{\varphi(\tau)},$$

where τ is an arbitrary nonnegative number within the circle of convergence of $\varphi(t)$. Then the distribution of X conditioned on the total progeny $|X|$ is determined by $\mathbf{P}\{X = T | |X| = n\}$ and it is easily seen that this distribution coincides with that induced by (1.1). Furthermore it is obvious to see that there occurs no loss of generality if only critical branching processes are considered. The condition for a branching process to be critical, $\mathbf{E}\xi = 1$, translated into the "language of trees" is $\tau\varphi'(\tau) = \varphi(\tau)$ and the variance of ξ is given by

$$\sigma^2 = \frac{\tau^2\varphi''(\tau)}{\varphi(\tau)}. \tag{1.3}$$

Date: October 14, 1996.
1991 *Mathematics Subject Classification.* Primary: 60J80, Secondary: 05C05.
Key words and phrases. random trees, branching processes, Brownian excursion, local time.
This research was supported by the Austrian Science Foundation FWF, grant P10187-MAT.

Consider a simply generated tree $T \in \mathcal{A}_n$. We denote by $L_T(k)$ the number of nodes at distance k from the root where the distance of two nodes v and w is defined as usual by the number of edges of the path connecting v and w. If T is a random tree then $L_T(k)$ becomes a random variable denoted by $L_n(k)$. For non-integer k we define $L_n(k)$ by linear interpolation:

$$L_n(t) = (\lfloor t \rfloor + 1 - t) L_n(\lfloor t \rfloor) + (t - \lfloor t \rfloor) L_n(\lfloor t \rfloor + 1), \quad t \geq 0.$$

We will show that the scaled process

$$l_n(t) = \frac{1}{\sqrt{n}} L_n(t\sqrt{n}), \quad t \geq 0,$$

weakly converges to Brownian excursion local time as n tends to infinity. This proves a conjecture stated by Aldous [1, Conjecture 4].

Theorem 1.1. *Let $\varphi(t)$ be the GF of a family of random trees. Besides, let $W(s)$ denote Brownian excursion of duration 1 and $l(t)$ its (total) local time at level t, i.e.*

$$l(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 I_{[t, t+\varepsilon]}(W(s))\, ds$$

Assume that $\varphi(t)$ has a positive or infinite radius of convergence R and $d = \gcd\{i | \varphi_i > 0\} = 1$. Furthermore suppose that the equation

$$t\varphi'(t) = \varphi(t)$$

has a minimal positive solution $\tau < R$ and that σ^2 defined by (1.3) is finite. Then the process $l_n(t)$ converges weakly to Brownian excursion local time, exactly that means

$$l_n(t) \xrightarrow{w} \frac{\sigma}{2} l\left(\frac{\sigma}{2} t\right)$$

in $C[0, \infty)$, as $n \to \infty$.

Remark 1. The case $d > 1$ can be treated analogously. All weak limit theorems throughout this paper remain unchanged except that we have to require $n \equiv 1 \mod d$. In case of local limit theorems the only difference is a factor d in the density of the limiting distribution. Thus we may restrict ourselves to $d = 1$.

Remark 2. Originally Aldous [1] formulated his conjecture in terms of the step function process $\frac{1}{\sqrt{n}} L_n(\lfloor t\sqrt{n} \rfloor)$. The reason that we decided to work with a linear interpolated process instead of a step function process is that the proof of tightness (section 6) is essentially shorter for the first one since all "trajectories" of the process are continuous functions in $C[0, \infty)$. More precisely, there is a similar tightness condition for the space $D[0, \infty)$ in which step functions are allowed (see [4]) and in fact, by using direct (but messy) extensions of the method presented in section 6 we are also able to prove the "original" conjecture.

Since the distribution of $\sup_{t \geq 0} l(t)$ is the same as that of $2 \sup_{0 \leq t \leq 1} W(t)$ (see [3] or [1]), which has been shown to be

$$\mathbf{P}\left\{ \sup_{0 \leq t \leq 1} W(t) \leq x \right\} = 1 - 2 \sum_{k \geq 1} (4x^2k^2 - 1) e^{-2x^2k^2} \tag{1.4}$$

by Kennedy [16], Theorem 1.1 immediately implies the following property for the width of trees:

Corollary. *Under the assumption of Theorem 1.1 we have*

$$\sup_{t \geq 0} l_n(t) \xrightarrow{w} \sigma \sup_{0 \leq t \leq 1} W(t)$$

as $n \to \infty$.

The process $l_n(t)$ may be viewed as the "local" process corresponding to the search depth process (for details see Aldous [1]) which is obtained by traversal of the contour of the tree. Aldous [2] proved that this process converges weakly to Brownian excursion. A process similar to the search depth process is the contour process introduced by Gutjahr and Pflug [12] which is constructed of the leaf heights of the tree. This process also converges to Brownian excursion (see [10]) and so it is natural to expect that the corresponding local process converges to local time, too: Let $\hat{L}_T(k)$ denote the number of leaves at distance k from the root and $\hat{L}_n(k)$ the random variable we get if T is a random tree. Then the following theorem holds:

Theorem 1.2. *Under the assumptions of Theorem 1.1 the scaled process*

$$\hat{l}_n(t) = \frac{1}{\sqrt{n}}\Big((\lfloor t\sqrt{n}\rfloor + 1 - t\sqrt{n})\hat{L}_n(\lfloor t\sqrt{n}\rfloor) + (t\sqrt{n} - \lfloor t\sqrt{n}\rfloor)\hat{L}_n(\lfloor t\sqrt{n}\rfloor + 1)\Big), \quad t \geq 0,$$

converges weakly to Brownian excursion local time: Precisely we have

$$\hat{l}_n(t) \xrightarrow{w} \frac{\varphi_0}{\varphi(\tau)}\frac{\sigma}{2} l\left(\frac{\sigma}{2}t\right)$$

in $C[0,\infty)$, as $n \to \infty$.

The density of local time. The one dimensional density of $(l(t); t \geq 0)$ at $t = \rho$ is well studied. There are several representations available in the literature: Using the theory of branching processes Kennedy [15] and Kolchin [18, Theorem 2.5.6] obtained

$$f_\rho(x) = \frac{x}{4}\int_0^1 (1-s)^{-\frac{3}{2}} e^{-\frac{x^2\rho^2}{8(1-s)}} g_{2\rho}\left(\frac{x}{2}, s\right) ds, \tag{1.5}$$

where $g_r(z,s)$ is the density of a distribution given by its characteristic function:

$$\psi_r(\theta_1,\theta_2) = \left[\frac{\sinh(r\sqrt{-2i\theta_2})}{r\sqrt{-2i\theta_2}} - i\theta_1\left(\frac{\sinh(r\sqrt{-i\theta_2/2})}{r\sqrt{-i\theta_2/2}}\right)^2\right]^{-1}$$

Takács [23] calculated this density by means of a generating function approach

$$f_\rho(x) = 2\sum_{j\geq 1}\sum_{k=1}^{j}\binom{j}{k} e^{-(x+2\rho j)^2/2}\frac{(-x)^k}{(k-1)!}H_{k+2}(x+2\rho j). \tag{1.6}$$

$H_k(z)$ are the Hermite polynomials defined by

$$H_k(z) = (-1)^k e^{z^2/2}\frac{d^k}{dz^k}e^{-z^2/2}.$$

Knight [17] worked directly with Brownian excursion and obtained

$$f_\rho(x) = 2^{-1/2}\pi^{5/2}\rho^{-3}\int_0^1 f^*\left(\frac{\pi^2(1-s)}{2\rho^2}\right) h(s,x)\, ds \tag{1.7}$$

with

$$f^*(z) = 4\sqrt{2}\pi^3\sum_{k\geq 1} k^2\frac{d}{dz}\left(\frac{e^{-k^2\pi^2/z}}{\sqrt{2\pi z^3}}\right) \quad \text{and}$$

$$h(s,x) = -\frac{1}{2\rho\sqrt{2\pi s}}\sum_{i\geq 0}\frac{1}{i!}\frac{d^{i-1}}{dx^{i-1}}\left(x^i\frac{d^2}{dx^2}e^{-2\rho^2(x+i)^2/s}\right),$$

where $\frac{d^{-1}}{dx^{-1}} = \left(\frac{d}{dx}\right)^{-1}$.

Getoor and Sharpe [9] also used a direct approach and derived a double Laplace transform of the density:

$$\int_0^\infty e^{-\alpha t} \int_0^t \frac{1}{\sqrt{2\pi s}} \int_{t-s}^\infty \frac{1}{\sqrt{2\pi r^3}} \mathbf{E}\left[e^{-\beta l(\rho/\sqrt{r})\sqrt{r}}\right] dr\, ds\, dt = \phi_\rho(\alpha,\beta) \tag{1.8}$$

where

$$\phi_\rho(\alpha,\beta) = \frac{1}{\alpha}\frac{\sqrt{2\alpha}+\beta(1+e^{-2\rho\sqrt{2\alpha}})}{\sqrt{2\alpha}+\beta(1-e^{-2\rho\sqrt{2\alpha}})} - \frac{\beta\sqrt{2}}{(1+2\rho\beta)\alpha^{3/2}}.$$

This formula was also shown by Louchard [19] who found a considerably shorter proof via Kac's formula for Brownian functionals.

When studying $M/M/1$-queues Cohen and Hooghiemstra [5] got an integral representation for the above density:

$$f_\rho(x) = \frac{1}{i\sqrt{2\pi}} \int_\gamma \frac{-se^{-s}}{\sinh^2(\rho\sqrt{-2s})} \exp\left(-\frac{x}{\sqrt{2}}\frac{\sqrt{-s}e^{\rho\sqrt{-2s}}}{\sinh(\rho\sqrt{-2s})}\right) ds, \tag{1.9}$$

where γ is the straight line $\{z : \Re z = -1\}$. They also derived the Laplace transform of the two dimensional densities of local and occupation time. We will get an integral representation for the density of the two dimensional distribution as a side result.

Finally it should be mentioned that Hooghiemstra [13] found a direct proof for the equivalence of (1.7) and (1.9).

Remark. Note that the expressions (1.5)–(1.9) are only representations of the continuous part of the local density. Obviously the local time density has a jump of magnitude $\mathbf{P}\left\{\sup_{0\le t\le 1} W(t) < \rho\right\}$ at 0. This quantity is given by (1.4).

2. Plan of the proof

In order to prove Theorems 1.1 and 1.2 we will use [4, Theorem 12.3]. Thus we have to show that the finite dimensional distributions (fdd's) of $l_n(t)$ and $\hat{l}_n(t)$, respectively, converge weakly to those of Brownian excursion local time and that these sequences are tight. The proof of tightness is presented in section 6 and so we turn now to the weak limit theorems.

Consider a random tree $T \in \mathcal{A}_n$ and set

$$a_{dmn} = \sum_{T\in\mathcal{A}_n, L_T(d)=m} \omega(T).$$

Thus the distribution of $L_n(d)$ is given by

$$\mathbf{P}\left\{L_n(d) = m | T \in \mathcal{A}_n\right\} = \frac{a_{dmn}}{a_n}.$$

In order to obtain this distribution we use the immediate translation technique of combinatorial constructions into GFs which is widely used in combinatorial enumeration (for a description see e.g. [24]): If we denote vertices by $\circ$, then $\mathcal{A}$ may be described by the symbolic recursion

$$\mathcal{A} = \Psi(\mathcal{A}),$$

where the operator Ψ is defined by

$$\Psi(\mathcal{X}) = \varphi_0 \cdot \{\circ\} \,\cup\, \varphi_1 \cdot \{\circ\} \times \mathcal{X} \,\cup\, \varphi_2 \cdot \{\circ\} \times \mathcal{X} \times \mathcal{X} \,\cup\, \cdots$$

As $\cup$ and $\times$ correspond to sum and product we immediately get the functional equation (1.2).

Now let us mark all nodes of layer d and denote marked nodes by $\bullet$. Call the tree family obtained from $\mathcal{A}$ in that way $\mathcal{B}$ and its GF $a_d(z,u)$. Then it is easy to see that

$$\mathcal{B} = \Psi^d(\{\bullet\} \times \mathcal{A})$$

holds. Due to the correspondence

$$\circ \leftrightarrow z$$
$$\bullet \leftrightarrow uz$$

we immediately get

$$a_d(z,u) = y_d(z, ua(z))$$

where

$$\begin{aligned} y_0(z,u) &= u \\ y_{i+1}(z,u) &= z\varphi(y_i(z,u)), \quad i \geq 0. \end{aligned} \tag{2.1}$$

Further applications of these marking techniques can be found in [7].

Obviously the GF corresponding to $\mathcal{B}$ satisfies

$$a_d(z,u) = \sum_{n,m\geq 0} a_{dmn} u^m z^n.$$

Thus the distribution of $L_n(d)$ can be obtained by extracting the coefficient of $z^n u^m$ of $a_d(z,u)$ which we denote by $[z^n u^m]a_d(z,u)$. In order to prove Theorems 1.1 and 1.2 we need only weak limit theorems and thus it suffices to work with characteristic functions. The characteristic function of $\frac{1}{\sqrt{n}} L_n(k)$ is

$$\phi_{kn}(t) = \frac{1}{a_n}[z^n] y_k\left(z, e^{it/\sqrt{n}} a(z)\right).$$

The characteristic function of $\frac{1}{\sqrt{n}} \hat{L}_n(k)$ can be derived analogously using combinatorial constructions as described above:

$$\hat{\phi}_{kn}(t) = \frac{1}{a_n}[z^n] y_k\left(z, \varphi_0 z(e^{it/\sqrt{n}} - 1) + a(z)\right).$$

In a similar way we get the characteristic function for the higher dimensional distributions: The characteristic function of $\left(\frac{1}{\sqrt{n}} L_n(k_1), \dots, \frac{1}{\sqrt{n}} L_n(k_p)\right)$ is given by

$$\phi_{k_1\cdots k_p n}(t_1,\dots,t_p) = \frac{1}{a_n}[z^n] y_{k_1}\left(z, e^{it_1/\sqrt{n}} y_{k_2-k_1}\left(z, \dots y_{k_p-k_{p-1}}\left(z, e^{it_p/\sqrt{n}} a(z)\right) \dots\right)\right.$$

and that of $\left(\frac{1}{\sqrt{n}} \hat{L}_n(k_1), \dots, \frac{1}{\sqrt{n}} \hat{L}_n(k_p)\right)$ by

$$\begin{aligned} \hat{\phi}_{k_1\cdots k_p n}(t_1,\dots,t_p) =& \frac{1}{a_n}[z^n] y_{k_1}\left(z, \varphi_0 z\left(e^{it_1/\sqrt{n}} - 1\right)\right. \\ &\left. + y_{k_2-k_1}\left(z, \dots + y_{k_p-k_{p-1}}\left(z, \varphi_0 z\left(e^{it_p/\sqrt{n}} - 1\right) + a(z)\right) \dots\right)\right. \end{aligned}$$

In order to extract the desired coefficient asymptotically we will use Cauchy's integral formula with a suitably chosen integration contour and approximate the integrand there. Therefore we need a detailed knowledge of the behaviour of the recursion (2.1):

Lemma 2.1. *Let z_0 be the point on the circle of convergence of $a(z)$ which lies on the positive real axis. Set $z = z_0\left(1 + \frac{x}{n}\right)$ and $\alpha = z\varphi'(a(z))$. Furthermore assume that $|u - a(z)| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ and $\frac{x}{n} \to 0$ in such a way that $|\arg(-x)| < \pi$ and*

$$\left|1 - \sqrt{\frac{-x}{n}}\right| \leq 1 + \frac{C}{\sqrt{n}}$$

are satisfied. Then $y_k(z,u)$ admits the local representation

$$y_k(z,u) = a(z) + \frac{(u-a(z))\alpha^k}{\frac{\sqrt{-x/n} + \sigma(\tau-u)/\tau\sqrt{2}}{2\sqrt{-x/n}} + \frac{\sqrt{-x/n} - \sigma(\tau-u)/\tau\sqrt{2}}{2\sqrt{-x/n}}\alpha^k + \mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)},$$

uniformly for $k = \mathcal{O}(\sqrt{n})$.

Remark. As $a(z)$ has only non-negative coefficients and $d = \gcd\{i|\varphi_i > 0\} = 1$ the function $a(z)$ has only one singularity on the circle of convergence which is located at z_0 (for more details see section 3). $y_k(z,u)$ is closely related to $a(z)$ and hence it has also exactly one singularity near z_0 the nature of which is described by the above result.

By means of this lemma we will be able to derive the characteristic function of the limiting distribution:

Theorem 2.1. *Let* $k_i = \kappa_i\sqrt{n}, i = 1,\dots,p$ *where* $0 < \kappa_1 < \cdots < \kappa_p$. *Then the characteristic function* $\phi_{\kappa_1\dots\kappa_p}(t_1,\dots,t_p) = \lim_{n\to\infty} \phi_{k_1\cdots k_p n}(t_1,\dots,t_p)$ *of the limiting distribution of* $\left(\frac{1}{\sqrt{n}}L_n(k_1),\dots,\frac{1}{\sqrt{n}}L_n(k_p)\right)$ *satisfies*

$$\phi_{\kappa_1\dots\kappa_p}(t_1,\dots,t_p) = 1 + \frac{\sigma}{i\sqrt{2\pi}}\int_\gamma f_{\kappa_1,\dots,\kappa_p,\sigma}(x,t_1,\dots,t_p)e^{-x}\,dx, \tag{2.2}$$

where

$$\begin{aligned} &f_{\kappa_1,\dots,\kappa_p,\sigma}(x,t_1,\dots,t_p) = \\ &\quad \Psi_{\kappa_1\sigma}(x, it_1 + \Psi_{\kappa_2-\kappa_1,\sigma}(\dots\Psi_{\kappa_{p-1}-\kappa_{p-2},\sigma}(x, it_{p-1} + \Psi_{\kappa_p-\kappa_{p-1},\sigma}(x,it_p))\cdots) \end{aligned} \tag{2.3}$$

with

$$\Psi_{\kappa\sigma}(x,t) = \frac{t\sqrt{-x}e^{-\kappa\sigma\sqrt{-x/2}}}{\sqrt{-x}e^{\kappa\sigma\sqrt{-x/2}} - t\frac{\sigma}{\sqrt{2}}\sinh\left(\kappa\sigma\sqrt{-\frac{x}{2}}\right)} \tag{2.4}$$

and γ *is the Hankel-like contour*[1] $\gamma_1\cup\gamma_2\cup\gamma_3$ *defined by*

$$\begin{aligned} \gamma_1 &= \left\{s\big||s| = 1 \text{ and } \Re s \le 0\right\}, \\ \gamma_2 &= \left\{s\big|\Im s = 1 \text{ and } \Re s \ge 0\right\}, \\ \gamma_3 &= \overline{\gamma}_2. \end{aligned} \tag{2.5}$$

Remark 1. Note that for $p = 1,2$ Cohen and Hooghiemstra [5] established the above representation with a straight line parallel to the imaginary axis as integration path. However, it is easy to see that bending the path does not change the value of the integral. Choosing the line means that we need a more detailed asymptotic expansion of the integrand when estimating the error term in section 5. Thus we use the Hankel contour instead.

Remark 2. Note that by using the method presented in this section the above theorem can only be established for the step function process $\frac{1}{\sqrt{n}}L_n\left(\lfloor t\sqrt{n}\rfloor\right)$. However, in section 6 we will show the inequality

$$\mathbf{P}\left\{\left|L_n\left(\rho\sqrt{n}\right) - L_n\left((\rho+\theta)\sqrt{n}\right)\right| \ge \varepsilon\sqrt{n}\right\} \le C\frac{\theta^2}{\varepsilon^4}$$

for some $C > 0$. This implies

$$\mathbf{P}\left\{\left|L_n(\rho\sqrt{n}) - L_n\left(\lfloor\rho\sqrt{n}\rfloor\right)\right| \ge \varepsilon\sqrt{n}\right\} \le \frac{C}{\varepsilon^\beta n} \to 0, \quad \text{as } n\to\infty.$$

Therefore it suffices to prove the theorem for the step function process.

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[1] The names "Hankel contour", "Hankel integral", etc. originate from Hankel's representation of the Gamma function,

$$\frac{1}{2\pi i}\int_\gamma (-s)^{-\alpha}e^{-s}\,ds = \frac{1}{\Gamma(\alpha)},$$

and have become usual due to the quite frequent occurrence of integration contours similar to γ in asymptotical problems in combinatorics.

In the cases $p = 1$ and $p = 2$ the representation (2.2) coincides with the characteristic functions of Brownian excursion local time (cf. [5]). But we still have to show that this is also true for higher dimensions. Thus we have to determine the higher dimensional distributions of local time. We will present two different methods for obtaining these distributions: First we indicate how the generating function approach can be used in order to derive the fdd's of local time. As this approach allows no direct computation we have to do a detour via occupation times: We will derive the characteristic function $\Phi_{\kappa_1\cdots\kappa_p\eta}(t_1,\dots,t_p)$ of the joint distribution of Brownian excursion occupation time for the sets $[\kappa_1,\kappa_1+\eta]\cup\cdots\cup[\kappa_p,\kappa_p+\eta]$. Then we get the characteristic function of the fdd's of local time by

$$\bar{\phi}_{\kappa_1\cdots\kappa_p}(t_1,\dots,t_p) = \lim_{\eta\to 0}\Phi_{\kappa_1\cdots\kappa_p\eta}\left(\frac{t_1}{\eta},\dots,\frac{t_p}{\eta}\right). \tag{2.6}$$

Second, we will present a direct computation of these fdd's using excursion theory. Here we compute the expected value of a suitably chosen random variable with respect to Itô's measure and then take the inverse Laplace transform. The evaluation of Itô's measure is based on a decomposition of the Brownian excursion sample path and Ray-Knight theorems.

It turns out that

$$\bar{\phi}_{\sigma\kappa_1/2,\dots,\sigma\kappa_p/2}\left(\frac{\sigma t_1}{2},\dots,\frac{\sigma t_p}{2}\right) = \phi_{\kappa_1\cdots\kappa_p}(t_1,\dots,t_p)$$

which completes the first step in the proof of Theorem 1.1, i.e. the weak limit law for the fdd's. The proof of Theorem 1.2 runs along analogous lines due to the similarity of the involved GF's and thus we will omit the details.

The next section deals with the asymptotic solution of the recursion (2.1). Section 4 contains the proof of Theorem 2.1 and section 5 provides the fdd's of Brownian excursion occupation time and its local time. Afterwards we will present the proof of tightness. The final section is devoted to an integral representation of the two dimensional local time density which extends the one of Cohen and Hooghiemstra [5].

3. Asymptotic solution of the main recursion

We will now establish an asymptotic solution of $y_i(z,u)$ of the recursion (2.1). Of course the limit of interest is $a(z)$, the analytic solution of $a(z) = z\varphi(a(z))$. It is well known that $a(z)$ has a local expansion of the form

$$a(z) = \tau - \frac{\tau\sqrt{2}}{\sigma}\sqrt{1-\frac{z}{z_0}} + \mathcal{O}\left(\left|1-\frac{z}{z_0}\right|\right) \tag{3.1}$$

around its singulariy $z_0 = 1/\varphi'(\tau)$. The assumption $d = 1$ ensures that $|z\varphi'(a(z))| < 1$ for $|z| = z_0$, $z \neq z_0$. Hence, by the implicit function theorem $a(z)$ has an analytic continuation to the region $|z| < z_0+\delta$, $\arg(z-z_0) \neq 0$ for some $\delta > 0$. Furthermore it follows that $\alpha = z\varphi'(a(z))$ has similar analytic properties, especially it has the local expansion

$$\alpha = 1 - \sigma\sqrt{2}\sqrt{1-\frac{z}{z_0}} + \mathcal{O}\left(\left|1-\frac{z}{z_0}\right|\right). \tag{3.2}$$

The first step in establishing asymptotic solutions for $y_i(z,u)$ is to derive proper a-priori estimates for

$$w_i = w_i(z,u) = y_i(z,u) - a(z).$$

Lemma 3.1. *Set $\alpha = z\varphi'(a(z))$ and suppose that $w_0 = u - a(z) = \mathcal{O}(1)$ and $1/2 \le |\alpha| \le 1 + \mathcal{O}(|w_0|)$. If $i = \mathcal{O}\left(|w_0|^{-1}\right)$ then*

$$w_i = \mathcal{O}\left(w_0\alpha^i\right).$$

Proof. Set $N = |w_0|^{-1}$. Then we have $i \le C_1 N$ and $|\alpha| \le 1 + C_2/N$ for some constants $C_1, C_2 > 0$. Furthermore there exists a constant C_3 such that

$$\max_{|y|\le 1}|z\varphi''(a(z)+y)| \le C_3.$$

If $N < e^{C_1(C_2+C_3)}$ then $i \le C_1 N$ is absolutely bounded and there is nothing to prove. Therefore we may assume that $N \ge e^{C_1(C_2+C_3)}$. We will proceed by induction and we will show in a first step that $|w_i| \le (1+(C_2+C_3)/N)^i |w_0|$. By using the local expansion

$$\begin{aligned} y_{i+1}(z,u) &= z\varphi(y_i(z,u)) = z\varphi(a(z)+w_i) \\ &= a(z) + z\varphi'(a(z))w_i + z\varphi''(a(z)+\theta_i)w_i^2/2 \\ &= a(z) + \alpha w_i + z\varphi''(a(z)+\theta_i)w_i^2/2 \end{aligned} \tag{3.3}$$

we immediately obtain

$$\begin{aligned} |w_{i+1}| &\le (1 + C_2|w_0| + C_3|w_i|)\,|w_i| \\ &\le \left(1 + C_2/N + C_3(1+C_3/N)^i|w_0|\right)(1+(C_2+C_3)/N)^i|w_0| \\ &\le \left(1 + C_2/N + C_3 e^{C_3C_1}/N\right)(1+(C_2+C_3)/N)^i|w_0| \\ &= (1+(C_2+C_3)/N)^{i+1}|w_0|. \end{aligned}$$

Hence $|w_i| \le e^{C_1(C_2+C_3)}/N \le 1$ for $i \le C_1 N$.

Finally, $|\alpha| \ge 1/2$ gives

$$\begin{aligned} |w_{i+1}| &\le (|\alpha| + C_3|w_i|)|w_i| \\ &\le |\alpha|\left(1 + 2(C_2+C_3)e^{C_1(C_2+C_3)}/N\right)|w_i| \end{aligned}$$

which implies

$$|w_i| \le e^{2C_1(C_2+C_3)e^{C_1(C_2+C_3)}}|\alpha|^i|w_0|.$$

□

With help of this a-priori estimate we are now able to obtain a significantely better estimate for w_i via bootstrapping. This will prove Lemma 2.1. (A similar procedure has been used in [6]). Note that under the assumptions of Lemma 2.1 $w_0 = \mathcal{O}(\sqrt{n})$ and $\alpha = 1 + \mathcal{O}\left(n^{-1/2}\right)$. Hence we can apply Lemma 3.1 for $i = \mathcal{O}(\sqrt{n})$. The asymptotic relation

$$w_{i+1} = \alpha w_i + \beta w_i^2 + \mathcal{O}\left(|w_i|^3\right),$$

in which $\beta = z\varphi''(a(z))/2$, can be rewritten to

$$\begin{aligned} \frac{1}{w_{i+1}} &= \frac{1}{\alpha w_i}\frac{1}{1+\beta w_i/\alpha + \mathcal{O}(|w_i|^2)} \\ &= \frac{1}{\alpha w_i} - \frac{\beta}{\alpha} + \mathcal{O}\left(\frac{|w_i|}{|\alpha|}\right). \end{aligned}$$

If we set $q_i = \frac{\alpha^i}{w_i}$ then

$$q_{i+1} = q_i - \beta\alpha^{i-1} + \mathcal{O}\left(|w_i||\alpha|^i\right)$$

which provides

$$q_i = \frac{1}{w_0} - \frac{\beta}{\alpha}\frac{1-\alpha^i}{1-\alpha} + \mathcal{O}\left(|w_0|\left|\frac{1-\alpha^{2i}}{1-\alpha^2}\right|\right).$$

Recall that we use the representation $z = z_0\left(1+\frac{x}{n}\right)$. Thus

$$\begin{aligned} w_0 &= u - a(z) = u - \tau + \frac{\tau\sqrt{2}}{\sigma}\sqrt{-\frac{x}{n}} + \mathcal{O}\left(\frac{|x|}{n}\right) \\ \beta &= \frac{z_0\varphi''(\tau)}{2}\left(1 + \mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)\right) = \frac{\sigma^2}{2\tau}\left(1 + \mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)\right). \end{aligned}$$

which gives

$$-\frac{w_0\beta}{\alpha(1-\alpha)} = \frac{\sigma^2}{2\tau}\frac{\tau - u - \frac{\tau\sqrt{2}}{\sigma}\sqrt{-x/n}}{\sigma\sqrt{-2x/n}} + \mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)$$

and proves Lemma 2.1.

4. The finite dimensional limiting distributions

We will use the results of the previous section to prove Theorem 2.1. Let us first consider the two dimensional case:

Proposition 4.1. *Let k and h be nonnegative integers and denote by $\phi_{k,k+h,n}(s,t)$ the characteristic function of the joint distribution of $\frac{1}{\sqrt{n}}L_n(k)$ and $\frac{1}{\sqrt{n}}L_n(k+h)$. Furthermore let $\phi_{\kappa,\kappa+\eta}(s,t)=\lim_{n\to\infty}\phi_{k,k+h,n}(s,t)$ denote the characteristic function of the limiting distribution of $\left(\frac{1}{\sqrt{n}}L_n(k),\frac{1}{\sqrt{n}}L_n(k+h)\right)$. Then $\phi_{\kappa,\kappa+\eta}(s,t)$ admits the following representation:*

$$\phi_{\kappa,\kappa+\eta}(s,t)=1+\frac{\sigma}{i\sqrt{2\pi}}$$
$$\times\int_\gamma \frac{\sqrt{-x}e^{-x-\kappa\sigma\sqrt{-x/2}}\left(is+\dfrac{it\sqrt{-x}e^{-\eta\sigma\sqrt{-x/2}}}{\sqrt{-x}e^{\eta\sigma\sqrt{-x/2}}-it\frac{\sigma}{\sqrt{2}}\sinh\left(\eta\sigma\sqrt{-\frac{x}{2}}\right)}\right)dx}{\sqrt{-x}e^{\kappa\sigma\sqrt{-x/2}}-\left(is+\dfrac{it\sqrt{-x}e^{-\eta\sigma\sqrt{-x/2}}}{\sqrt{-x}e^{\eta\sigma\sqrt{-x/2}}-it\frac{\sigma}{\sqrt{2}}\sinh\left(\eta\sigma\sqrt{-\frac{x}{2}}\right)}\right)\sinh\left(\kappa\sigma\sqrt{-\frac{x}{2}}\right)}, \tag{4.1}$$

where γ is the Hankel contour (2.5).

Proof. Obviously we have

$$\begin{aligned}\phi_{k,k+h,n}(s,t)&=\frac{1}{a_n}[z^n]y_k\left(z,e^{is/\sqrt{n}}y_h\left(z,e^{it/\sqrt{n}}a(z)\right)\right)\\&=\frac{1}{2\pi i a_n}\int_\Gamma y_k\left(z,e^{is/\sqrt{n}}y_h\left(z,e^{it/\sqrt{n}}a(z)\right)\right)\frac{dz}{z^{n+1}}\end{aligned}$$

by Cauchy's integral formula. As integration path we lay a truncated Hankel contour $\gamma'=\Gamma_1\cup\Gamma_2\cup\Gamma_3$ around the singularity (cf. remark after lemma 2.1) closed by a circular arc Γ_4:

$$\begin{aligned}\Gamma_1&=\left\{z=z_0\left(1+\frac{x}{n}\right)\middle|\,\Re x\le 0\text{ and }|x|=1\right\}\\ \Gamma_2&=\left\{z=z_0\left(1+\frac{x}{n}\right)\middle|\,\Im x=1\text{ and }0\le\Re x\le\log^2 n\right\}\\ \Gamma_3&=\overline{\Gamma}_2\\ \Gamma_4&=\left\{z\,\middle|\,|z|=z_0\left|1+\frac{\log^2 n+i}{n}\right|\text{ and }\arg\left(1+\frac{\log^2 n+i}{n}\right)\le|\arg(z)|\le\pi\right\}.\end{aligned}\tag{4.2}$$

If $z\in\gamma'$, then substitute $z=z_0\left(1+\frac{x}{n}\right)$ and set $\bar{x}=\frac{x}{n}$ and $\alpha=z\varphi'(a(z))$. Besides, let $u=e^{is/\sqrt{n}}$ and $v=e^{it/\sqrt{n}}$. Now, applying lemma 2.1 yields

$$y_h(z,va(z))=a(z)+R_h(v,z)$$

where

$$R_h(v,z)=\frac{(v-1)a(z)\alpha^h}{\dfrac{\sqrt{-\bar{x}}+\frac{\sigma}{\tau\sqrt{2}}(\tau-va(z))}{2\sqrt{-\bar{x}}}-\dfrac{\sqrt{-\bar{x}}+\frac{\sigma}{\tau\sqrt{2}}(\tau-va(z))}{2\sqrt{-\bar{x}}}\alpha^h+\mathcal{O}\left(\sqrt{|\bar{x}|}\right)}$$

and

$$y_k(z,uy_h(z,va(z)))=a(z)+R_k(u,v,z)$$

where

$$R_k(u,v,z)=\frac{(a(z)(u-1)+uR_h(v,z))\alpha^k}{\frac{1}{2}\left(1+\alpha^k+\dfrac{\sigma}{\tau\sqrt{-2\bar{x}}}(\tau-ua(z)-uR_h(v,z))(1-\alpha^k)\right)+\mathcal{O}\left(\sqrt{|\bar{x}|}\right)}$$

Now fix s and t and set $k = \lfloor \kappa\sqrt{n} \rfloor$. Then it follows from (3.1) and (3.2) that, as $n \to \infty$, the following asymptotic expansions apply:

$$
\begin{aligned}
a(z)(u-1) &= \frac{is\tau}{\sqrt{n}} + \mathcal{O}\left(\frac{\sqrt{|x|}}{n}\right) \\
\alpha^k &= \exp\left(-2\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)\left(1+\mathcal{O}\left(\frac{|x|}{\sqrt{n}}\right)\right) \\
\tau - ua(z) &= \tau(1-u) + \frac{u\tau\sqrt{2}}{\sigma}\sqrt{-\frac{x}{n}} + \mathcal{O}\left(\frac{|x|}{n}\right) \\
&= -\frac{is\tau}{\sqrt{n}} + \frac{\tau\sqrt{2}}{\sigma}\sqrt{-\frac{x}{n}} + \mathcal{O}\left(\frac{|x|}{n}\right)
\end{aligned}
$$

Hence we have

$$
R_k(u,v,z) \sim \frac{\tau}{\sqrt{n}} \frac{\sqrt{-x}\exp\left(-\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)\left(is\tau + \frac{\sqrt{n}}{\tau}R_h\right)}{\sqrt{-x}\exp\left(\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right) - \left(is\tau + \frac{\sqrt{n}}{\tau}R_h\right)\frac{\sigma}{\sqrt{2}}\sinh\left(\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)}. \tag{4.3}
$$

and in an analogous way it can be shown that for $h = \lfloor \eta\sqrt{n} \rfloor$

$$
R_h \sim \frac{\tau}{\sqrt{n}} \frac{it\sqrt{-x}\exp\left(-\eta\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)}{\sqrt{-x}\exp\left(\eta\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right) - it\frac{\sigma}{\sqrt{2}}\sinh\left(\eta\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)}, \tag{4.4}
$$

Note that for $x = y \pm i$ we have $R_k(u,v,z) \sim 1$ as $y \to \infty$ and therefore the substitution of the integration path γ' by γ is justified by the dominated convergence theorem.

What remains to be done is to estimate the contribution of Γ_4. It is easy to see that

$$
\left[\frac{\partial}{\partial x_2} y_h(x_1,x_2)\right]_{x_1=z,x_2=a(z)} = \alpha^h,
$$

where $\alpha = z\varphi'(a(z))$. Thus by Taylor's theorem we have

$$
\begin{aligned}
y_k(z, uy_h(z,va(z))) &= a(z) + \alpha^k(uy_h(z,va(z)) - a(z)) + \mathcal{O}\left((uy_h(z,va(z)) - a(z))^2\right) \\
&= a(z) + a(z)\alpha^k\left(u - 1 + \alpha^h(v-1)u + \mathcal{O}\left((v-1)^2\right)\right) \\
&\quad + \mathcal{O}\left(\left(u-1+\alpha^h(v-1)u\right)^2\right)
\end{aligned}
$$

The first term satisfies

$$
\frac{1}{2\pi i a_n}\int_{\gamma'\cup\Gamma_4} a(z)\frac{dz}{z^{n+1}} = 1
$$

and hence there is nothing more to do. In order to estimate the remainder observe that due to $d = 1$ the maximum of α on Γ_4 is attained for $z \in \gamma' \cap \Gamma_4$. Then

$$
\alpha^k \sim \exp\left(-\kappa\sigma\sqrt{-2x}\right)
$$

and as $x = \log^2 n \pm i$ we have

$$
\Re\sqrt{-x} = \Re\sqrt{-\log^2 n + i} = \cos\left(\frac{\pi}{2} - \frac{1}{2}\arctan\frac{1}{\log^2 n}\right) \sim \frac{1}{2\log n}.
$$

This implies $\alpha^k = \mathcal{O}(1)$ for $z \in \Gamma_4$. On the other hand for $z \in \gamma'$ α reaches its minimum if $z = 1 - 1/n$. Therefore

$$
\min_{z\in\gamma'}|\alpha^k| \sim \exp\left(-\kappa\sigma\sqrt{2}\right)
$$

and hence

$$
\max_{z\in\Gamma_4}|\alpha^k| = \mathcal{O}\left(\min_{z\in\gamma'}|\alpha^k|\right).
$$

Finally, the facts that $|z^{-n-1}| \sim e^{-\log^2 n}$ for $z \in \Gamma_4$ and that the length of γ' is $\mathcal{O}\left(\log^2 n/n\right)$ give

$$\left|\int_{\Gamma_4} R_k(u,v,z)\frac{dz}{z^{n+1}}\right| = \mathcal{O}\left(\left|\int_{\gamma'} R_k(u,v,z)\frac{dz}{z^{n+1}}\right| \frac{n}{\log^2 n}\, e^{-\log^2 n}\right).$$

Thus the contribution of Γ_4 is negligibly small and using

$$a_n = \frac{\tau}{\sigma z_0^n\sqrt{2\pi n^3}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)$$

we get (4.1) and the proof is complete. □

Now, iterating the steps of the above proof yields Theorem 2.1 as an immediate corollary.

5. Finite dimensional distributions of occupation and local time

In this section we will determine the fdd's of Brownian excursion local time. This theorem in conjunction with tightness of $L_n(t)$ implies Theorem 1.1.

Theorem 5.1. *Let $\bar\phi_{\kappa_1\ldots\kappa_p}(t_1,\ldots,t_p)$ denote the characteristic function of the joint distribution of $(l(\kappa_1),\ldots,l(\kappa_p))$. Then we have*

$$\bar\phi_{\kappa_1\ldots\kappa_p}(t_1,\ldots,t_p) = 1 + \frac{\sqrt{2}}{i\sqrt{\pi}}\int_\gamma f_{\kappa_1,\ldots,\kappa_p,2}(x,t_1,\ldots,t_p)e^{-x}\,dx,$$

where $f_{\kappa_1,\ldots,\kappa_p,2}(x,t_1,\ldots,t_p)$ is given by (2.3).

We offer two different proofs of Theorem 5.1. The first one is the generating function approach we already used in the previous section and the second one is a direct computation by means of probabilistic arguments from excursion theory.

Proposition 5.1. *The characteristic function of the joint distribution of $L([\kappa_1,\kappa_1+\eta]),\ldots,$ $L([\kappa_p,\kappa_p+\eta])$ satisfies*

$$\Phi_{\kappa_1\ldots\kappa_p\eta}(t_1,\ldots,t_p) = 1 + \frac{1}{i\sqrt{\pi}}\int_\gamma F_{\kappa_1,\ldots,\kappa_p,\eta}(x,t_1,\ldots,t_p)e^{-x}\,dx, \tag{5.1}$$

where

$$\begin{aligned}&F_{\kappa_1,\ldots,\kappa_p,\eta}(x,t_1,\ldots,t_p) = \\ &\qquad \Xi_{\kappa_1,\eta}(x,t_1,\Xi_{\kappa_2-\kappa_1,\eta}(\ldots\Xi_{\kappa_{p-1}-\kappa_{p-2},\eta}(x,t_{p-1},\Xi_{\kappa_p-\kappa_{p-1},\eta}(x,t_p,0))\cdots)\end{aligned}$$

with

$$\begin{aligned}\Xi_\kappa(x,t,y) = &\Big(\sqrt{-x}e^{-\kappa\sqrt{-2x}}\Big(\big(it+y\sqrt{-x}\big)\sinh\Big(\eta\sqrt{-2(x+it)}\Big)\\ &\qquad -y\sqrt{-x-it}\cosh\Big(\eta\sqrt{-2(x+it)}\Big)\Big)\Big)\\ &\times\Big[\sqrt{-x(-x-it)}e^{\kappa\sqrt{-2x}}\cosh\Big(\eta\sqrt{-2(x+it)}\Big)\\ &\quad-\Big(\big(it+y\sqrt{-x}\big)\sinh\Big(\eta\sigma\sqrt{(-x-it)/2}\Big)\\ &\qquad -y\sqrt{-x-it}\cosh\Big(\eta\sqrt{-2(x+it)}\Big)\Big)\sinh\big(\kappa\sqrt{-2x}\,\big)\\ &\quad+\sqrt{-x}\,\big(\sqrt{-x}+y\big)\,e^{\kappa\sqrt{-2x}}\sinh\Big(\eta\sqrt{-2(x+it)}\Big)\Big]^{-1}\end{aligned}$$

This proposition immediately implies Theorem 5.1 by means of (2.6).

Proof. Denote Brownian excursion occupation time of $[a,b]$ by

$$L([a,b]) = \int_0^1 I_{[a,b]}(W(s))\,ds.$$

In order to derive the fdd's of occupation time we observe that the process

$$H_n(t) = \frac{1}{n} \sum_{k \le t\sqrt{n}} L_n(k), \quad t \ge 0$$

weakly converges to occupation time (see Aldous [1, Corollary 3]): Exactly, we have

$$H_n(t) \xrightarrow{w} L\left(\left[0, \frac{\sigma t}{2}\right]\right) \tag{5.2}$$

in $C[0,\infty)$, as $n \to \infty$. Hence the characteristic function of the fdd's of occupation time satisfies

$$\Phi_{\sigma\kappa_1/2,\dots,\sigma\kappa_p/2,\sigma\eta/2}(t_1,\dots,t_p) = \\ \lim_{n\to\infty} \frac{1}{a_n} [z^n]\, y_{k_1}(z, y_h(u_1 z, u_1 y_{k_2}(\dots, u_{p-1} y_{k_p}(z, y_h(u_p z, u_p a(z))) \cdots) \tag{5.3}$$

where $u_j = e^{it_j/n}, j = 1,\dots,p$, $k_j = \lfloor \kappa_j \sqrt{n} \rfloor$ and $h = \lfloor \eta\sqrt{n} \rfloor$. For calculating the coefficient on the right-hand side we use again Cauchy's integral formula with the integration contour Γ given by (4.2).

Note that (5.2) holds for any choice of $\varphi(t)$. In order to abbreviate our calculations we set $\varphi(t) = 1/(1-t)$. In this case the recursion (2.1) can be solved exactly: By elementary considerations we get

$$y_h(z,u) = z\frac{d_h(z) - u d_{h-1}(z)}{d_{h+1}(z) - u d_h(z)}, \tag{5.4}$$

where

$$d_h(z) = \left(\frac{1+\sqrt{1-4z}}{2}\right)^h - \left(\frac{1-\sqrt{1-4z}}{2}\right)^h. \tag{5.5}$$

Let us first consider the case $p = 1$. For notational convenience we introduce the following abbreviations:

$$\begin{aligned} w &= \sqrt{1-4z}, & \tilde{w} &= \sqrt{1-4uz}, \\ a &= \frac{1+w}{2}, & \tilde{a} &= \frac{1+\tilde{w}}{2}, \\ b &= \frac{1-w}{2}, & \tilde{b} &= \frac{1-\tilde{w}}{2}, \\ c &= \frac{b}{a}, & \tilde{c} &= \frac{\tilde{b}}{\tilde{a}}. \end{aligned} \tag{5.6}$$

The GF we are dealing with now is $y_k(z, y_h(uz, ua(z)))$. Besides, note that according to the above convention $a(z) = b$. So we have

$$\begin{aligned} y_k(z, y_h(uz, ua(z))) &= z\frac{d_k(z) - \frac{uz}{\tilde{a}} B d_{k-1}(z)}{d_{k+1}(z) - \frac{uz}{\tilde{a}} B d_k(z)} \\ &= b\left(1 - \frac{c^{k-1}(1-c)\,(\tilde{a}c - ubB)}{\tilde{a} - ubB - c^k\,(\tilde{a}c - ubB)}\right) = b(1-R) \end{aligned} \tag{5.7}$$

where

$$B = 1 - \frac{\tilde{w}\tilde{b}^h\left(1 - ub/\tilde{b}\right)}{\tilde{a}^{h+1}\,(1 - ub/\tilde{a}) - \tilde{b}^{h+1}\left(1 - ub/\tilde{b}\right)} = 1 - \bar{R} \tag{5.8}$$

Now set $u = e^{it/n}$ and consider the case $z \in \gamma'$. Then $z = z_0\left(1 + \frac{x}{n}\right)$ where $x = o(\sqrt{n}\,)$. This implies

$$1 - u\frac{b}{\tilde{b}} \sim 1 - \frac{1-w}{1-\tilde{w}} \sim \sqrt{-\frac{x}{n}} - \sqrt{\frac{-x-it}{n}}$$
$$1 - u\frac{b}{\tilde{a}} \sim 1 - \frac{1-w}{1+\tilde{w}} \sim \sqrt{-\frac{x}{n}} + \sqrt{\frac{-x-it}{n}}$$

and

$$a^k \sim e^{\kappa\sqrt{-x}}, \qquad b^k \sim e^{\kappa\sqrt{-x}},$$
$$\tilde{a}^h \sim e^{\eta\sqrt{-x-it}}, \qquad \tilde{b}^h \sim e^{\eta\sqrt{-x-it}}.$$

Hence

$$\bar{R} \sim \frac{1}{\sqrt{n}}\frac{2\sqrt{-x-it}\left(\sqrt{-x}-\sqrt{-x-it}\right)e^{-\eta\sqrt{-x-it}}}{\left(\sqrt{-x}+\sqrt{-x-it}\right)e^{\eta\sqrt{-x-it}} - \left(\sqrt{-x}-\sqrt{-x-it}\right)e^{-\eta\sqrt{-x-it}}}.$$

Combining (5.7) and (5.8) gives

$$R = \frac{wb^k\left(\tilde{a}/a - u + u\bar{R}\right)}{a^{k+1}\left(\tilde{a}/a - ub/a + ub\bar{R}/a\right) - b^{k+1}\left(\tilde{a}/a - u + u\bar{R}\right)}.$$

Using

$$\frac{\tilde{a}}{a} - u \sim \frac{1+\tilde{w}}{1+w} - 1 \sim \sqrt{\frac{-x-it}{n}} - \sqrt{-\frac{x}{n}}$$
$$\frac{\tilde{a}}{a} - u\frac{b}{a} \sim \frac{1+\tilde{w}}{1+w} - \frac{1-w}{1+w} \sim \sqrt{\frac{-x-it}{n}} + \sqrt{-\frac{x}{n}}$$

we get

$$R \sim \frac{2}{\sqrt{n}}\frac{\sqrt{-x}(\sqrt{-x-it}-\sqrt{-x}+\bar{R}\sqrt{n}\,)e^{-\kappa\sqrt{-x}}}{(\sqrt{-x}+\sqrt{-x-it}+\bar{R}\sqrt{n}\,)e^{\kappa\sqrt{-x}} - (\sqrt{-x-it}-\sqrt{-x}+\bar{R}\sqrt{n}\,)e^{-\kappa\sqrt{-x}}}. \tag{5.9}$$

In a similar way as in the previous section it can be shown that

$$\left|\int_{\Gamma_4} a(z)R(z)\frac{dz}{z^{n+1}}\right| = \mathcal{O}\left(\left|\int_{\gamma'} a(z)R(z)\frac{dz}{z^{n+1}}\right|\frac{n^{3/2}}{\log n}\,e^{-\log^2 n}\right)$$

and so we are able to compute the right-hand side of (5.3) now: In case of $\varphi(t) = 1/(1-t)$ we have

$$a_n = \frac{1}{n}\binom{2n-2}{n-1} \sim \frac{4^{n-1}}{\sqrt{\pi n^3}} \tag{5.10}$$

and $\sigma^2 = 2$. Due to (5.2) the considered process converges to $L([\kappa/\sqrt{2}, (\kappa+\eta)/\sqrt{2}])$. Thus we have to perform the substitutions $\kappa \to \kappa\sqrt{2}$ and $\eta \to \eta\sqrt{2}$ in order to get the desired formula. This yields

$$\Phi_{\kappa\eta}(t) = 1 + \frac{1}{i\sqrt{\pi}}\int\limits_{\gamma}\frac{N(t,x)}{D(t,x)}e^{-x}\,dx, \tag{5.11}$$

where

$$N(t,x) = \sqrt{-x}e^{-\kappa\sqrt{-2x}}it\sinh\left(\eta\sqrt{-2(x+it)}\right)$$
$$D(t,x) = \sqrt{-x(-x-it)}e^{\kappa\sqrt{-2x}}\cosh\left(\eta\sqrt{-2(x+it)}\right) - xe^{\kappa\sqrt{-2x}}\sinh\left(\eta\sqrt{-2(x+it)}\right)$$
$$- it\sinh\left(\kappa\sqrt{-2x}\,\right)\sinh\left(\eta\sqrt{-2(x+it)}\right).$$

Note that the numerator is bounded by $\exp(C\sqrt{|x|})$ and for $x = y + i$ the denominator satisfies

$$D(t,x) \sim -ye^{\kappa\sqrt{-2(y+i)}+\eta\sqrt{-2(y+(1+t)i)}}, \quad y \to \infty.$$

Due to the fact that the real part of the exponent converges to zero as $y \to \infty$ the denominator is bounded from below by a positive constant. Therefore the substitution of the integration path γ' by γ is justified by the dominated convergence theorem.

In the case $p = 2$ we have to consider the GF $y_k(z, y_h(u_1 z, u_1 y_l(z, y_h(u_2 z, u_2 a(z)))))$. Proceeding analogously to the case $p = 1$ gives

$$\Phi_{\kappa\lambda\eta}(s,t) = 1 + \frac{1}{i\sqrt{\pi}} \int\limits_{\gamma} \frac{N(s,t,x)}{D(s,t,x)} e^{-x}\, dx \tag{5.12}$$

where

$$\begin{aligned}
N(x,s,t) &= \sqrt{-x}e^{-\kappa\sqrt{-2x}} \Big(\big(is + \sqrt{-x}R\sqrt{n}\big) \sinh\Big(\eta\sqrt{-2(x+is)}\Big) \\
&\qquad - \sqrt{-x-is}R\sqrt{n}\cosh\Big(\eta\sqrt{-2(x+is)}\Big)\Big) \\
D(x,s,t) &= \sqrt{-x(-x-is)}e^{\kappa\sqrt{-2x}}\cosh\Big(\eta\sqrt{-2(x+is)}\Big) - \big(\big(is + \sqrt{-x}R\sqrt{n}\big) \\
&\qquad \times \sinh\Big(\lambda\sqrt{-2(x+is)}\Big) - \sqrt{-x-is}R\sqrt{n}\cosh\Big(\lambda\sqrt{-2(x+is)}\Big)\Big) \\
&\qquad \times \sinh\big(\kappa\sqrt{-2x}\big) + \sqrt{-x}\big(\sqrt{-x} + R\sqrt{n}\big)e^{\kappa\sqrt{-2x}}\sinh\Big(\lambda\sqrt{-2(x+is)}\Big)
\end{aligned}$$

Finally, from (5.11) and (5.12) we immediately obtain (5.1) by induction. □

We now turn to a probabilistic proof of Theorem 5.1. In order to simplify notation set for a probability measure P and a random variable X

$$P[X] = \int_{\Omega} X\, dP.$$

Proposition 5.2. *Let π_r denote the law of Brownian excursion of length r and $l_r(t)$ the corresponding (total) local time. Then we have for $\Re x < 0$*

$$\int\limits_0^\infty \frac{1}{2\sqrt{2\pi r^3}} \pi_r \left[\left(1 - e^{i(t_1 l_r(\kappa_1)+\cdots+t_p l_r(\kappa_p))}\right)\right] e^{xr}\, dr = -f_{\kappa_1,\dots,\kappa_p,2}(x, t_1, \dots, t_p).$$

Since

$$\bar{\phi}_{\kappa_1\cdots\kappa_p}(t_1,\dots,t_p) = \pi_1\left[e^{(i(t_1 l(\kappa_1)+\cdots+t_p l(\kappa_p))}\right] \tag{5.13}$$

Theorem 5.1 follows directly by taking the inverse Laplace transform and setting $r = 1$. (Note that the integration path of the inverse Laplace transform is a straight line parallel to the imaginary axis. However, bending the line to a Hankel contour does not change the value of the integral, cf. Remark 2 after Theorem 2.1.)

Proof. Recall that the Itô measure of positive excursions (for details in excursion theory see e.g. [22, Ch. XII]) is given by

$$n^+ = \int\limits_0^\infty \frac{1}{2\sqrt{2\pi r^3}} \pi_r\, dr.$$

This means that we have to compute

$$\begin{aligned}
&n^+ \left[(1 - \exp\left(i(t_1 l_\zeta(\kappa_1) + \cdots + t_p l_\zeta(\kappa_p))\right) e^{x\zeta}\right] \\
&\quad = \int\limits_0^\infty \frac{1}{2\sqrt{2\pi r^3}} \pi_r \left[1 - \exp\left(i(t_1 l_r(\kappa_1) + \cdots + t_p l_r(\kappa_p))\right)\right] e^{xr}\, dr,
\end{aligned}$$

where ζ is the life time of the excursion.

Denote by $T_a = \inf\{t > 0 : W(t) = a\}$, $a \geq 0$, the first hitting time of a. Then the sample path of a Brownian excursion of length ζ may be decomposed into the two processes $(W(t); 0 \leq t \leq T_{\kappa_1})$ and $(W(t); T_{\kappa_1} \leq t \leq \zeta)$, provided that $T_{\kappa_1} < \zeta$. By Williams' path decomposition theorem (see [25, 26] or [22, Ch. VII, Theorem (4.9)]) the first one is a three dimensional Bessel process started at 0 and stopped at T_{κ_1} and the second one is a Brownian motion started at κ_1 and killed at 0. Furthermore these two processes are independent under n^+ conditionally on $T_{\kappa_1} < \zeta$ (for details see [22, Ch. XII, sect. 4]). Let R_0 denote the law of a Bessel 3 process, P_{κ_1} the law of Brownian motion started at κ_1, T_0 the first hitting time of 0 of Brownian motion started at κ_1 (i.e. $\zeta = T_{\kappa_1} + T_0$), and $l_{T_0}(t)$ the local time of Brownian motion started at κ_1 and killed at 0. Then we have

$$\begin{aligned} n^+ &\left[(1 - \exp\left(i(t_1 l_\zeta(\kappa_1) + \cdots + t_p l_\zeta(\kappa_p))\right) e^{x\zeta}\right] \\ &= n^+(T_{\kappa_1} < \zeta) R_0\left[e^{xT_{\kappa_1}}\right] P_{\kappa_1}\left[(1 - \exp\left(i\left(t_1 l_{T_0}(\kappa_1) + \cdots + t_p l_{T_0}(\kappa_p)\right)\right) e^{xT_0}\right] \\ &= \frac{\sqrt{-2x}}{2\sinh\left(\kappa_1\sqrt{-2x}\right)} P_{\kappa_1}\left[(1 - \exp\left(i\left(t_1 l_{T_0}(\kappa_1) + \cdots + t_p l_{T_0}(\kappa_p)\right)\right) e^{xT_0}\right], \end{aligned} \tag{5.14}$$

where we used

$$n^+(T_{\kappa_1} < \zeta) = \frac{1}{2\kappa_1} \quad \text{and} \quad R_0\left[e^{xT_{\kappa_1}}\right] = \frac{\kappa_1\sqrt{-2x}}{\sinh\left(\kappa_1\sqrt{-2x}\right)}.$$

In order to evaluate (5.14) we will use the Ray-Knight theorems: Denote by $\mathrm{BESQ}^d(x)$ the square of a Bessel process of dimension d started at x. Then the process $(l_{T_0}(x); 0 \leq x \leq \kappa_1)$ and $\mathrm{BESQ}^2(0)$ (restricted on the interval $[0, \kappa_1]$) have the same law. Moreover, if we set $\tau_y = \inf\{t : l_t(0) = y\}$ then $(l_{\tau_y}(x); x \geq 0)$ and $\mathrm{BESQ}^0(y)$ have the same law, i.e. $(l_{T_0}(x); x \geq \kappa_1)$ is a $\mathrm{BESQ}^0(l_{T_0}(x))$. Due to the Markov property the two processes $(l_{T_0}(x); 0 \leq x \leq \kappa_1)$ and $(l_{T_0}(x); x \geq \kappa_1)$ are independent, conditionally on $l_{T_0}(\kappa_1)$.

Now, observe that

$$T_0 = \int_0^\infty l_{T_0}(a)\,da = \int_0^{\kappa_1} l_{T_0}(a)\,da + \int_{\kappa_1}^{\kappa_2} l_{T_0}(a)\,da + \cdots + \int_{\kappa_{p-1}}^{\kappa_p} l_{T_0}(a)\,da + \int_{\kappa_p}^\infty l_{T_0}(a)\,da$$

Thus we have

$$\begin{aligned} P_{\kappa_1} &\left[\left(1 - e^{i(t_1 l_{T_0}(\kappa_1) + \cdots + t_p l_{T_0}(\kappa_p))}\right) e^{xT_0}\right] \\ &= \mathbf{E}\left[\left(1 - e^{i(t_1 X_{\kappa_1} + \cdots + t_p X_{\kappa_p}}\right) \exp\left(x\int_0^{\kappa_1} X_u\,du + x\int_{\kappa_1}^\infty X_u\,du\right)\right] \end{aligned}$$

where $(X_u; 0 \leq u \leq \kappa_1)$ is a $\mathrm{BESQ}^2(0)$ and $(X_u; u \geq \kappa_1 | X_{\kappa_1} = x)$ is a $\mathrm{BESQ}^0(x)$.

If X_t is a BESQ^0 then [21, formula (2.k)] yields for $\kappa' < \kappa''$

$$\begin{aligned} \mathbf{E} &\left[\exp\left(itX_{\kappa''} + x\int_{\kappa'}^{\kappa''} X_u\,du\right)\middle| X_{\kappa'}\right] \\ &= \exp\left(-X_{\kappa'}\sqrt{-\frac{x}{2}}\,\frac{1 - it\sqrt{-2/x}\coth\left((\kappa'' - \kappa')\sqrt{-2x}\right)}{\coth\left((\kappa'' - \kappa')\sqrt{-2x}\right) - it\sqrt{-2/x}}\right) \end{aligned} \tag{5.15}$$

and

$$\mathbf{E}\left[\exp\left(x\int_{\kappa''}^\infty X_u\,du\right)\middle| X_{\kappa''}\right] = \exp\left(-X_{\kappa''}\sqrt{-\frac{x}{2}}\right). \tag{5.16}$$

Hence, equations (5.15), (5.16), and the Markov property imply

$$
\begin{aligned}
&\mathbf{E}\left[\exp\left(it_pX_{\kappa_p}+x\int_{\kappa_{p-1}}^{\infty}X_u\,du\right)\Big|\,X_{\kappa_{p-1}}\right]\\
&\quad=\mathbf{E}\left[\exp\left(it_pX_{\kappa_p}+x\int_{\kappa_{p-1}}^{\kappa_p}X_u\,du\right)\mathbf{E}\left[\exp\left(x\int_{\kappa_p}^{\infty}X_u\,du\right)\Big|\,X_{\kappa_p}\right]\Big|\,X_{\kappa_{p-1}}\right]\\
&\quad=\mathbf{E}\left[\exp\left(\left(it_p-\sqrt{-\frac{x}{2}}\right)X_{\kappa_p}+x\int_{\kappa_{p-1}}^{\kappa_p}X_u\,du\right)\Big|\,X_{\kappa_{p-1}}\right]\\
&\quad=\exp\left(-X_{\kappa_{p-1}}\left(\sqrt{-\frac{x}{2}}-\Psi_{\kappa_p-\kappa_{p-1},2}(x,it_p)\right)\right)
\end{aligned}
$$

where $\Psi(x,t)$ is defined by (2.4). Consequently we have

$$
\begin{aligned}
&\mathbf{E}\left[\exp\left(it_{p-1}X_{\kappa_{p-1}}+it_pX_{\kappa_p}+x\int_{\kappa_{p-2}}^{\infty}X_u\,du\right)\Big|\,X_{\kappa_{p-2}}\right]\\
&\quad=\mathbf{E}\left[\exp\left(it_{p-1}X_{\kappa_{p-1}}+x\int_{\kappa_{p-2}}^{\kappa_{p-1}}X_u\,du\right)\right.\\
&\qquad\qquad\left.\times\,\mathbf{E}\left[\exp\left(\left(it_p-\sqrt{-\frac{x}{2}}\right)X_{\kappa_p}+x\int_{\kappa_{p-1}}^{\kappa_p}X_u\,du\right)\Big|\,X_{\kappa_{p-1}}\right]\Big|\,X_{\kappa_{p-2}}\right]\\
&\quad=\mathbf{E}\left[\exp\left(\left(it_{p-1}+\Psi_{\kappa_p-\kappa_{p-1}}(x,it_p)-\sqrt{-\frac{x}{2}}\right)X_{\kappa_{p-1}}+x\int_{\kappa_{p-2}}^{\kappa_{p-1}}X_u\,du\right)\Big|\,X_{\kappa_{p-2}}\right]\\
&\quad=\exp\left(-X_{\kappa_{p-2}}\left(\sqrt{-\frac{x}{2}}-\Psi_{\kappa_{p-1}-\kappa_{p-2},2}\left(x,it_{p-1}+\Psi_{\kappa_p-\kappa_{p-1},2}(x,it_p)\right)\right)\right)
\end{aligned}
$$

and proceeding analogously we obtain after all

$$
\begin{aligned}
&\mathbf{E}\left[\exp\left(it_2X_{\kappa_2}+\cdots+it_pX_{\kappa_p}+x\int_{\kappa_1}^{\infty}X_u\,du\right)\Big|\,X_{\kappa_1}\right]\\
&\qquad=\exp\left(-X_{\kappa_1}\left(\sqrt{-\frac{x}{2}}-\tilde f_{\kappa_2\cdots\kappa_p}(x,t_2,\dots,t_p)\right)\right)
\end{aligned}
$$

where

$$
\tilde f_{\kappa_2,\dots,\kappa_p}(x,t_1,\dots,t_p)=\Psi_{\kappa_2-\kappa_1,2}(\dots\Psi_{\kappa_{p-1}-\kappa_{p-2},2}(x,it_{p-1}+\Psi_{\kappa_p-\kappa_{p-1},2}(x,it_p))\cdots).
$$

In order to complete the computation of (5.14) we use that $(X_u;0\le u\le\kappa_1)$ is a $\mathrm{BESQ}^2(0)$ and apply again [21, formula (2.k)] with $d=2$ and $x=0$ and get

$$
\begin{aligned}
&\mathbf{E}\left[\exp\left(it_1X_{\kappa_1}+\cdots+it_pX_{\kappa_p}+x\int_0^{\infty}X_u\,du\right)\right]\\
&=\mathbf{E}\left[\exp\left(itX_{\kappa_1}+x\int_0^{\kappa_1}X_u\,du\right)\mathbf{E}\left[\exp\left(it_2X_{\kappa_2}+\cdots+it_pX_{\kappa_p}+x\int_{\kappa_1}^{\infty}X_u\,du\right)\Big|\,X_{\kappa_1}\right]\right]\\
&=\mathbf{E}\left[\exp\left(\left(it-\sqrt{-\frac{x}{2}}+\tilde f_{\kappa_2\cdots\kappa_p}(x,t_2,\dots,t_p)\right)X_{\kappa_1}+x\int_0^{\kappa_1}X_u\,du\right)\right]\\
&=\frac{1}{e^{\kappa_1\sqrt{-2x}}-2\left(it_1+\tilde f\right)\sinh\left(\kappa_1\sqrt{-2x}\,\right)/\sqrt{-2x}}.
\end{aligned}
$$

Hence

$$\begin{aligned}
&\mathbf{E}\left[\exp(xT_0) - \exp\left(it_1X_{\kappa_1} + \cdots + it_pX_{\kappa_p} + xT_0\right)\right] \\
&= \frac{1}{e^{\kappa_1\sqrt{-2x}}} - \frac{1}{e^{\kappa_1\sqrt{-2x}} - 2\left(it_1 + \tilde{f}\right)\sinh\left(\kappa_1\sqrt{-2x}\,\right)/\sqrt{-2x}} \\
&= \frac{-\sqrt{2}\left(it_1 + \tilde{f}\right)\sinh\left(\kappa_1\sqrt{-2x}\,\right)}{\sqrt{-x}e^{\kappa_1\sqrt{-2x}} - \left(it_1 + \tilde{f}\right)\sqrt{2}\sinh\left(\kappa_1\sqrt{-2x}\,\right)} \\
&= -2\frac{\sinh\left(\kappa_1\sqrt{-2x}\,\right)}{\sqrt{-2x}} f_{\kappa_1\cdots\kappa_p 2}(x, t_1, \ldots, t_p)
\end{aligned}$$

which completes the proof of Proposition 5.2. □

6. Tightness

In this section we will show that the sequence of random variables $l_n(t) = n^{-1/2}L_n(t\sqrt{n})$, $t \geq 0$, is tight in $C[0,\infty)$. Since a sequence of stochastic processes $X_n(t)$, $t \geq 0$ is tight in $C[0,\infty)$ if and only if $X_n(t)$, $0 \leq t \leq T$ is tight in $C[0,T]$ for all $T > 0$ (see [14, p. 63]) we may restrict ourselves to finite intervals, i.e. it suffices to consider $L_n(t)$, $0 \leq t \leq A\sqrt{n}$, where $A > 0$ is an arbitrary real constant.

By [4, Theroem 12.3] tightness of $l_n(t) = n^{-1/2}L_n(t\sqrt{n})$, $0 \leq t \leq A$, follows from tightness of $L_n(0)$ (which is obvioulsy satisfied) and from an estimate of the form

$$\mathbf{P}\left\{|L_n(\rho\sqrt{n}) - L_n((\rho+\theta)\sqrt{n})| \geq \varepsilon\sqrt{n}\right\} \leq C\frac{\theta^\alpha}{\varepsilon^\beta} \tag{6.1}$$

for some $\alpha > 1$, $\beta \geq 0$, and $C > 0$ uniformly for $0 \leq \rho \leq \rho + \theta \leq A$. We will derive (6.1) from the following property:

Theorem 6.1. *There exists a constant $C > 0$ such that*

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^4 \leq C\,h^2 n \tag{6.2}$$

holds for all non-negative integers n, r, h.

Obviously Theorem 6.1 proves (6.1) for $\alpha = 2$ and $\beta = 4$ if $\rho\sqrt{n}$ and $\theta\sqrt{n}$ are non-negative integers. However, in the case of linear interpolation it is an easy exercise (see [12] or [10]) to extend (6.1) to arbitrary $\rho, \theta \geq 0$ (probably with a different constant C).

Remark. It should be mentioned that it is not sufficient to consider the second moment $\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^2$. The optimal upper bound is given by

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^2 \leq C\,h\sqrt{n}$$

which provides (6.1) just for $\alpha = 1$.

It remains to prove Theorem 6.1. Since the coefficient

$$a_{nkl,rh} = [z^n u^k v^l] y_r(z, u y_h(z, v a(z)))$$

is the (weighted) number of trees of size n with k nodes in layer r and l nodes in layer $r+h$, i.e.

$$\mathbf{P}\left\{L_n(r) = k, L_n(r+h) = l\right\} = \frac{a_{nkl,rh}}{a_n},$$

we obtain

$$\mathbf{P}\left\{L_n(r) - L_n(r+h) = m\right\} = \frac{1}{a_n}[z^n u^m] y_r(z, u y_h(z, u^{-1}a(z)))$$

and consequently

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^4 = \frac{1}{a_n}[z^n] H_{rh}(z), \tag{6.3}$$

in which

$$H_{rh}(z) = \left(\frac{\partial}{\partial u} + 7\frac{\partial^2}{\partial u^2} + 6\frac{\partial^3}{\partial u^3} + \frac{\partial^4}{\partial u^4}\right) y_r(z, uy_h(z, u^{-1}a(z)))\Big|_{u=1}. \tag{6.4}$$

In order to prove Theorem 6.1 we have to use a proper representation of $H_{rh}(z)$.

Proposition 6.1. *Set* $\alpha = z\varphi'(a(z))$ *and*

$$\Delta = \{z : |z| < z_0 + \eta,\ |\arg(z - z_0)| > \vartheta\}, \tag{6.5}$$

in which $0 < \vartheta < \pi/2$ *is arbitrary but fixed. Then* $H_{rh}(z)$ *can be represented as*

$$H_{rh}(z) = G_{1,rh}(z)\frac{(1-\alpha^h)^2}{(1-\alpha)^3} + G_{2,rh}(z)\frac{1-\alpha^h}{(1-\alpha)^2} + G_{3,rh}(z)\frac{1}{1-\alpha} + G_{4,rh}(z), \tag{6.6}$$

in which $G_{j,rh}(z)$, $1 \le j \le 4$, *are uniformly bounded for* $z \in \Delta$ *and* $r, h \ge 0$.

The proof of Proposition 6.1 requires the following formulas.

Lemma 6.1. *Let* $\alpha = z\varphi'(a(z))$, $\beta = z\varphi''(a(z))$, $\gamma = z\varphi'''(a(z))$, *and* $\delta = z\varphi''''(a(z))$. *Then we have*

$$\begin{aligned}
\frac{\partial y_r}{\partial u}(z,1) &= \alpha^r,\\
\frac{\partial^2 y_r}{\partial u^2}(z,1) &= \frac{\beta}{\alpha}\alpha^r\frac{1-\alpha^r}{1-\alpha},\\
\frac{\partial^3 y_r}{\partial u^3}(z,1) &= \frac{\gamma}{\alpha}\alpha^r\frac{1-\alpha^{2r}}{1-\alpha^2} + 3\frac{\beta^2}{\alpha}\alpha^r\frac{(1-\alpha^r)(1-\alpha^{r-1})}{(1-\alpha)(1-\alpha^2)},\\
\frac{\partial^4 y_r}{\partial u^4}(z,1) &= \frac{\delta}{\alpha}\alpha^r\frac{1-\alpha^{3r}}{1-\alpha^3}\\
&\quad + \left(2\beta\gamma(2+5\alpha+5\alpha^r+3\alpha^{r+1}) + 3\beta^3/\alpha\right)\alpha^r\frac{(1-\alpha^r)(1-\alpha^{r-1})}{(1-\alpha^2)(1-\alpha^3)}\\
&\quad + 3\beta^3(1+5\alpha)\alpha^r\frac{(1-\alpha^r)(1-\alpha^{r-1})(1-\alpha^{r-2})}{(1-\alpha)(1-\alpha^2)(1-\alpha^3)}.
\end{aligned}$$

Proof. From $y_{r+1}(z,u) = z\varphi(y_r(z,u))$ we directly obtain the recurring relations

$$\begin{aligned}
\frac{\partial y_{r+1}}{\partial u} &= z\varphi'(y_r)\frac{\partial y_r}{\partial u},\\
\frac{\partial^2 y_{r+1}}{\partial u^2} &= z\varphi''(y_r)\left(\frac{\partial y_r}{\partial u}\right)^2 + z\varphi'(y_r)\frac{\partial^2 y_r}{\partial u^2},\\
\frac{\partial^3 y_{r+1}}{\partial u^3} &= z\varphi'''(y_r)\left(\frac{\partial y_r}{\partial u}\right)^3 + 3z\varphi''(y_r)\frac{\partial y_r}{\partial u}\frac{\partial^2 y_r}{\partial u^2} + z\varphi'(y_r)\frac{\partial^3 y_r}{\partial u^3},\\
\frac{\partial^4 y_{r+1}}{\partial u^4} &= z\varphi''''(y_r)\left(\frac{\partial y_r}{\partial u}\right)^4 + 6z\varphi'''(y_r)\left(\frac{\partial y_r}{\partial u}\right)^2\frac{\partial^2 y_r}{\partial u^2}\\
&\quad + 3z\varphi''(y_r)\left(\frac{\partial^2 y_r}{\partial u^2}\right)^2 + 4z\varphi''(y_r)\frac{\partial y_r}{\partial u}\frac{\partial^3 y_r}{\partial u^3} + z\varphi'(y_r)\frac{\partial^4 y_r}{\partial u^4}.
\end{aligned}$$

Since $y_r(z,1) = a(z)$ for all $r \ge 0$ this system of recurring relations has the explicit solutions stated in Lemma 6.1 for $u = 1$. □

Proof. (*Proposition 6.1*) First we can use Lemma 6.1 to make (6.4) more explicit. Since

$$\begin{aligned}
\frac{\partial}{\partial u}y_r(z, uy_h(z, u^{-1}a(z))) &= \frac{\partial y_r}{\partial u}(z, uy_h(z, u^{-1}a(z)))\\
&\quad \times \left(y_h(z, u^{-1}a(z)) - u^{-1}a(z)\frac{\partial y_h}{\partial u}(z, u^{-1}a(z))\right).
\end{aligned}$$

we obtain

$$\frac{\partial}{\partial u} y_r(z, u y_h(z, u^{-1} a(z)))\Big|_{u=1} = a(z)\alpha^r(1-\alpha^h).$$

Similarly

$$\begin{aligned}
\frac{\partial^2}{\partial u^2} y_r(z, u y_h(z, u^{-1} a(z)))\Big|_{u=1} &= a(z)^2 \frac{\partial^2 y_r}{\partial u^2}(z,1)(1-\alpha^h)^2 + a(z)^2\alpha^r \frac{\partial^2 y_h}{\partial u^2}(z,1),\\
\frac{\partial^3}{\partial u^3} y_r(z, u y_h(z, u^{-1} a(z)))\Big|_{u=1} &= a(z)^3 \frac{\partial^3 y_r}{\partial u^3}(z,1)(1-\alpha^h)^3 + 3a(z)^3 \frac{\partial^2 y_r}{\partial u^2}(z,1)\frac{\partial^2 y_h}{\partial u^2}(z,1)(1-\alpha^h),\\
&\quad - 3a(z)^2\alpha^r \frac{\partial^2 y_h}{\partial u^2}(z,1) - a(z)^3\alpha^r \frac{\partial^3 y_h}{\partial u^3}(z,1),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^4}{\partial u^4} y_r(z, u y_h(z, u^{-1} a(z)))\Big|_{u=1} &= a(z)^4 \frac{\partial^4 y_r}{\partial u^4}(z,1)(1-\alpha^h)^4 + 7a(z)^4 \frac{\partial^3 y_r}{\partial u^3}(z,1)\frac{\partial^2 y_h}{\partial u^2}(z,1)(1-\alpha^h)^2\\
&\quad - 12a(z)^4 \frac{\partial^2 y_r}{\partial u^2}(z,1)\frac{\partial^2 y_h}{\partial u^2}(z,1)(1-\alpha^h)\\
&\quad + 3a(z)^4 \frac{\partial^2 y_r}{\partial u^2}(z,1)\left(\frac{\partial^2 y_h}{\partial u^2}(z,1)\right)^2\\
&\quad - 4a(z)^4 \frac{\partial^2 y_r}{\partial u^2}(z,1)\frac{\partial^3 y_h}{\partial u^3}(z,1)(1-\alpha^h) + 12a(z)^2\alpha^r \frac{\partial^2 y_h}{\partial u^2}(z,1)\\
&\quad + 8a(z)^3\alpha^r \frac{\partial^3 y_h}{\partial u^3}(z,1) + a(z)^4\alpha^r \frac{\partial^4 y_h}{\partial u^4}(z,1),
\end{aligned}$$

yielding an explicit expression of $H_{rh}(z)$ in terms of $a(z)$.

Now notice that

$$\sup_{z\in\Delta} |\alpha| = 1 \tag{6.7}$$

since $\alpha = z\varphi'(a(z))$ has the local expansion (3.2). Hence, a representation of the form (6.6) follows immediately with functions $G_{j,rh}(z)$, $1 \le j \le 4$, which are uniformly bounded for $z \in \Delta$. □

The final step of the proof of Theorem 6.1 is to use (6.6) and the following lemma from singularity analysis [8]:

Lemma 6.2. *Let $F(z)$ be analytic in Δ (defined in (6.5)) in which z_0 and η are positive real numbers and $0 < \vartheta < \pi/2$. Furthermore suppose that there exists a real number β such that*

$$F(z) = \mathcal{O}\left((1 - z/z_0)^{-\beta}\right) \qquad (z \in \Delta).$$

Then

$$[z^n]F(z) = \mathcal{O}\left(z_0^{-n} n^{\beta-1}\right).$$

Proof. (*Theorem 6.1*) Since $a_n \sim (\tau/\sqrt{2\pi\sigma^2}) z_0^{-n} n^{-3/2}$ Theorem 6.1 is equivalent to

$$[z^n]H_{rh}(z) = \mathcal{O}\left(z_0^{-n}\frac{h^2}{\sqrt{n}}\right) \tag{6.8}$$

uniformly for all $r, h \ge 0$. Note that $H_{r0}(z) \equiv 0$. So we may assume that $h \ge 1$.

First, let us consider the first term of $H_{rh}(z)$ (in the representation (6.6)):

$$\begin{aligned}
G_{1,rh}(z)\frac{(1-\alpha^h)^2}{(1-\alpha)^3} &= G_{1,rh}(z)\frac{1}{1-\alpha}\sum_{i=0}^{h-1}\alpha^i \sum_{j=0}^{h-1}\alpha^j\\
&= \sum_{i,j=0}^{h-1} G_{1,rh}(z)\frac{\alpha^{i+j}}{1-\alpha} = \mathcal{O}\left(h^2 \frac{1}{|1-\alpha|}\right).
\end{aligned}$$

Since

$$\frac{1}{1-\alpha} = \mathcal{O}\left((1-z/z_0)^{-1/2}\right).$$

we can apply Lemma 6.2 with $\beta = 1/2$ and obtain

$$G_{1,rh}(z)\frac{(1-\alpha^h)^2}{(1-\alpha)^3} = \mathcal{O}\left(z_0^{-n}h^2n^{-1/2}\right).$$

The coefficient of the second term is even smaller:

$$\begin{aligned}[z^n]G_{2,rh}(z)\frac{(1-\alpha^h)}{(1-\alpha)^2} &= [z^n]G_{2,rh}(z)\frac{1}{1-\alpha}\sum_{i=0}^{h-1}\alpha^i \\ &= \mathcal{O}\left(z_0^{-n}hn^{-1/2}\right) = \mathcal{O}\left(z_0^{-n}h^2n^{-1/2}\right).\end{aligned}$$

Similarly we can treat the remaining terms:

$$[z^n]G_{3,rh}(z)\frac{1}{1-\alpha} = \mathcal{O}\left(z_0^{-n}n^{-1/2}\right) = \mathcal{O}\left(z_0^{-n}h^2n^{-1/2}\right)$$

and

$$[z^n]G_{4,rh}(z) = \mathcal{O}\left(z_0^{-n}n^{-1}\right) = \mathcal{O}\left(z_0^{-n}h^2n^{-1/2}\right).$$

Thus we have proved (6.8) which is equivalent to (6.2). □

Remark. The proof of tightness of $\hat{l}_n(t)$ runs along the same lines as that for $l_n(t)$.

7. The two dimensional density of Brownian excursion local time

In this section we indicate how our method can be used to obtain an integral represention of the two dimensional density of local time (extending the result of Hooghiemstra [13]) by showing a local limit theorem for planted plane trees.

We want to note that it is also possible to derive this integral representation for this densitiy directly by probabilistic means in a similar fashion as in section 5. However, the multidimensional local time density will be treated in a forthcoming paper [11], where also a direct probabilistic proof is offered.

Proposition 7.1. *Let $\kappa, \lambda, \rho, \theta > 0$. Besides, set $k = \lfloor \kappa\sqrt{n} \rfloor$, $l = \lfloor \lambda\sqrt{n} \rfloor$, $r = \lfloor \rho\sqrt{n} \rfloor$ and $h = \lfloor \theta\sqrt{n} \rfloor$. Denote by $a_{nkl,rh}$ number of all planted plane trees of size n having k nodes in layer r and l nodes in layer $r+h$. Then the following limit theorem holds:*

$$\begin{aligned}\lim_{n\to\infty} n\frac{a_{nkl,rh}}{a_n} &= \lim_{n\to\infty}\frac{n}{a_n}[u^kv^lz^n]y_r(z,uy_h(z,bv)) \\ &= \frac{1}{i\sqrt{\pi}}\int_\gamma \frac{\kappa x^2}{\sinh^2\left(\theta\sqrt{-x}\right)\sinh^2\left(\rho\sqrt{-x}\right)}\exp\left(-\sqrt{-x}\left(\frac{\lambda e^{\theta\sqrt{-x}}}{\sinh\left(\theta\sqrt{-x}\right)}\right.\right. \\ &\qquad \left.\left.+\frac{\kappa\sinh\left((\rho+\theta)\sqrt{-x}\right)}{\sinh\left(\rho\sqrt{-x}\right)\sinh\left(\theta\sqrt{-x}\right)}\right)\right)\sum_{j\ge0}\frac{(-\kappa\lambda x)^j}{j!(j+1)!\sinh^{2j}\left(\theta\sqrt{-x}\right)}e^{-x}\,dx,\end{aligned} \tag{7.1}$$

where γ is the straight line $\{z : \Re z = -1\}$. Furthermore,

$$\lim_{n\to\infty}\sqrt{n}\frac{a_{nk0,rh}}{a_n} = \frac{1}{i\sqrt{\pi}}\int_\gamma \frac{-xe^{-x}}{\sinh^2\left(\rho\sqrt{-x}\right)}\exp\left(-\sqrt{-x}\frac{\kappa\sinh\left((\rho+\theta)\sqrt{-x}\right)}{\sinh\left(\rho\sqrt{-x}\right)\sinh\left(\theta\sqrt{-x}\right)}\right)dx, \tag{7.2}$$

and

$$\lim_{n\to\infty}\frac{a_{n00,rh}}{a_n} = -\frac{1}{i\sqrt{\pi}}\int_\gamma \frac{\sqrt{-x}\cosh\left(\rho\sqrt{-x}\right)}{\sinh\left(\rho\sqrt{-x}\right)}e^{-x}\,dx \tag{7.3}$$

Denote the joint density of $l(\rho)$ and $l(\rho+\theta)$ by $f_{\rho,\theta}(\kappa,\lambda)$. Set $r=\rho\sqrt{n}$, $h=\theta\sqrt{n}$, $k=\kappa\sqrt{n}$ and $l=\lambda\sqrt{n}$. Then $f_{\rho,\theta}(\kappa,\lambda)$ satisfies

$$\frac{4}{\sigma^2} f_{\frac{\sigma}{2}\rho,\frac{\sigma}{2}\theta}\left(\frac{2}{\sigma}\kappa,\frac{2}{\sigma}\lambda\right)=\lim_{n\to\infty}\frac{n}{a_n}[z^n u^k v^l] y_r(z, u y_h(z, v a(z))). \tag{7.4}$$

Hence (7.1) yields the continuous part of local time density. (Note that $\sigma^2=2$ for $\varphi(t)=1/(1-t)$.) It remains to determine the discrete parts. Obviously there are Brownian excursion sample paths which do not reach level $\rho+\theta$ or not even level ρ and each of these situations occurs with positive probability. Equivalently, this means $\mathbf{P}\{l(\rho)=0, l(\rho+\theta)=0\}>0$ and $\mathbf{P}\{l(\rho)\in[a,b], l(\rho+\theta)=0\}>0$ for an arbitrary interval $[a,b]$. Thus there is a point mass at the origin and each interval on the κ-axis contains mass, too. In order to determine these masses we can use (7.2) and (7.3).

Theorem 7.1. *The joint distribution of $l(\rho)$ and $l(\rho+\theta)$ is absolutely continuous w.r.t. Lebesgue measure on the set $\{(\kappa,\lambda):\kappa\ge 0$ and $\lambda>0\}$. Its density is given by*

$$\begin{aligned} f_{\rho,\theta}(\kappa,\lambda) &= \\ &\frac{1}{i2\sqrt{2\pi}}\int_\gamma \frac{\kappa x^2}{\sinh^2\left(\theta\sqrt{-2x}\right)\sinh^2\left(\rho\sqrt{-2x}\right)}\exp\left(-\sqrt{\frac{-x}{2}}\left(\frac{\lambda e^{\theta\sqrt{-2x}}}{\sinh\left(\theta\sqrt{-2x}\right)}\right.\right.\\ &\left.\left.+\frac{\kappa\sinh\left((\rho+\theta)\sqrt{-2x}\right)}{\sinh\left(\rho\sqrt{-2x}\right)\sinh\left(\theta\sqrt{-2x}\right)}\right)\right)\sum_{j\ge 0}\frac{(-\kappa\lambda x)^j}{j!(j+1)!2^j\sinh^{2j}\left(\theta\sqrt{-2x}\right)}e^{-x}\,dx, \end{aligned}$$

Along the line $\lambda=0$ and the distribution function has a jump of height

$$\frac{1}{i\sqrt{2\pi}}\int_\gamma \frac{-xe^{-x}}{\sinh^2\left(\rho\sqrt{-2x}\right)}\exp\left(-\sqrt{\frac{-x}{2}}\frac{\kappa\sinh\left((\rho+\theta)\sqrt{-2x}\right)}{\sinh\left(\rho\sqrt{-2x}\right)\sinh\left(\theta\sqrt{-2x}\right)}\right)dx.$$

and at the origin one of height

$$-\frac{1}{i\sqrt{\pi}}\int_\gamma \frac{\sqrt{-x}\cosh\left(\rho\sqrt{-2x}\right)}{\sinh\left(\rho\sqrt{-2x}\right)}e^{-x}\,dx.$$

Remark. Note that the infinite sum appearing in the representation of $f_{\rho,\theta}(\kappa,\lambda)$ in Theorem 7.1 is related to the first Bessel function $J_1(z)$.

Proof. (*Sketch*) Obviously, we have

$$\begin{aligned} y_r(z, u y_h(z, bv)) &= z\frac{d_r - u y_h(z,bv) d_{r-1}}{d_{r+1}-u y_h(z,bv) d_r} \\ &= z\frac{d_r - u y_h(z,bv) d_{r-1}}{d_{r+1}}\sum_{i\ge 0}\left(\frac{d_r}{d_{r+1}} y_h(z,bv)\right)^i u^i, \end{aligned}$$

where $d_r=d_r(z)$ is defined by (5.5) and $b=a(z)$ (again we use notation (5.6)). A similar representation holds for $y_h(z,bv)$ and thus we have

$$\begin{aligned} [u^k v^l] y_r(z, u y_h(z,bv)) &= z\left(1-\frac{d_{r-1}d_{r+1}}{d_r^2}\right)\left(\frac{d_r}{d_{r+1}}\right)^{k+1}\left(\frac{z d_h}{d_{h+1}}\right)^k\left(\frac{b d_h}{d_{h+1}}\right)^l \\ &\times\sum_{i=0}^{\min(k,l)-1}\binom{k}{i+1}\binom{l-1}{i}\left(1-\frac{d_{h-1}d_{h+1}}{d_h^2}\right)^{i+1} \end{aligned} \tag{7.5}$$

Now, we have to determine the coefficient of z^n in the above expression. In order to do this we use Cauchy's integral formula choosing a truncated line normal to the real axis and complemented by a circular arc as integration path. To be precise, we integrate along $\Gamma = \gamma' \cup \Gamma'$ given by

$$\gamma' = \left\{ z : z = \frac{1}{4}\left(1 - \frac{1+it}{n}\right) \text{ and } |t| \le \sqrt{2n+1} \right\}$$

$$\Gamma' = \left\{ z : |z| = \frac{1}{4} \text{ and } \arctan \frac{\sqrt{2n+1}}{n-1} \le |\arg z| \le \pi \right\}$$

Evaluating each factor in (7.5) asymptotically gives on γ' by means of the substitution $z = \frac{1}{4}\left(1 + \frac{x}{n}\right)$ we get

$$1 - \frac{d_{h-1}d_{h+1}}{d_h^2} \sim \frac{1}{n} \frac{-x}{\sinh^2\left(\theta\sqrt{-x}\right)} \tag{7.6}$$

$$\left(\frac{zd_h}{d_{h+1}}\right)^k \sim 2^{-k} \exp\left(-\frac{\kappa\sqrt{-x}\cosh\left(\theta\sqrt{-x}\right)}{\sinh\left(\theta\sqrt{-x}\right)}\right) \tag{7.7}$$

$$\left(\frac{bd_h}{d_{h+1}}\right)^l \sim \exp\left(-\frac{\lambda\sqrt{-x}e^{\theta\sqrt{-x}}}{\sinh\left(\theta\sqrt{-x}\right)}\right) \tag{7.8}$$

$$\left(\frac{d_r}{d_{r+1}}\right)^{k+1} \sim 2^{k+1} \exp\left(-\frac{\kappa\sqrt{-x}\cosh\left(\rho\sqrt{-x}\right)}{\sinh\left(\rho\sqrt{-x}\right)}\right) \tag{7.9}$$

With the help of Stirling's formula it can be shown that for any $\varepsilon > 0$ we have

$$\sum_{i \le n^{1/4-\varepsilon}} \binom{k}{i+1}\binom{l-1}{i}\left(1 - \frac{d_{h-1}d_{h+1}}{d_h^2}\right)^i \sim \sqrt{n} \sum_{i \le n^{1/4-\varepsilon}} \frac{\kappa^{i+1}}{(i+1)!}\frac{\lambda^i}{i!}\left(\frac{-x}{\sinh^2\left(\theta\sqrt{-x}\right)}\right)^i \tag{7.10}$$

and that the remainder of the sum is $\mathcal{O}\left(e^{-Cm\log n}\right)$ for a suitable constant C. Collecting the constants in (7.6)–(7.10) and applying (5.10) yields (7.1).

Finally, it can be shown in a similar way as in section 5 that the contributions of Γ' and $\gamma \setminus \gamma'$ are negligibly small and the proof is complete.

Similary we obtain

$$\begin{aligned}
[z^n u^k v^0] y_r(z, uy_h(z, vb)) &= [z^n] z \left(1 - \frac{d_{r-1}d_{r+1}}{d_r^2}\right)\left(\frac{d_r}{d_{r+1}}\right)^{k+1}\left(\frac{zd_h}{d_{h+1}}\right)^k \\
&\sim \frac{4^{n-1}}{i\pi n^2}\int_\gamma \frac{-xe^{-x}}{\sinh^2\left(\rho\sqrt{-x}\right)} \exp\left(-\sqrt{-x}\frac{\kappa \sinh\left((\rho+\theta)\sqrt{-x}\right)}{\sinh\left(\rho\sqrt{-x}\right)\sinh\left(\theta\sqrt{-x}\right)}\right) dx,
\end{aligned}$$

for $k > 0$ and

$$\begin{aligned}
[z^n u^0 v^0] y_r(z, uy_h(z, bv)) = [z^n] z \frac{d_r}{d_{r+1}} &\sim \frac{4^{n-1}}{i\pi n}\int_\gamma \left(1 - \frac{\sqrt{-x}\cosh\left(\rho\sqrt{-x}\right)}{\sqrt{n}\sinh\left(\rho\sqrt{-x}\right)}\right) e^{-x}\, dx \\
&= -\frac{4^{n-1}}{i\pi n^{3/2}}\int_\gamma \frac{\sqrt{-x}\cosh\left(\rho\sqrt{-x}\right)}{\sinh\left(\rho\sqrt{-x}\right)} e^{-x}\, dx
\end{aligned}$$

which completes the proof of Proposition 7.1. □

Acknowledgment. The authors are indebted to Guy Louchard for pointing out some references dealing with the one dimensional local time density as well as to an anonymous referee for indicating a direct proof of Theorem 5.1.

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