REFLECTED BROWNIAN BRIDGE LOCAL TIME CONDITIONED ON ITS LOCAL TIME AT THE ORIGIN

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Abstract. The moments of the local time of a reflected Brownian bridge conditioned on its local time at the origin are computed by two different methods: First, by conditioning an identity for the unconditioned local time and second, by using a limit theorem for random forests.

1. Introduction

1.1. **Motivating remarks.** The analysis of construction costs of hash tables under the linear probing strategy is closely related to random forests (see e.g. [15]) as well as to functionals (in particular, the occupation time) of a reflected Brownian bridge conditioned on its local time at the origin (see [9, 14, 2, 3]). We are interested in the *local version* of this functional, i.e., the local time, which appears naturally as limiting profile of random forests. This process has been described on the one hand implicitly in terms of a stochastic differential equation (see [17]), on the other hand by a weak limit theorem for the finite-dimensional distributions (see [11]). In this note, we complement these results by presenting a moment convergence theorem.

1.2. **Notations.** Throughout this note, the standard Brownian motion (BM) will be denoted by $x(t)$. Other classical Brownian processes are the reflected BM: $x^+(t) := |x(t)|$, the Brownian Bridge (BB) on $[0,1]$: $B(t)$, the reflected BB on $[0,1]$: $B^+(t)$. The local time of $B^+(t)$ at level a and time t will be denoted by $t^+(t,a)$ and we define $t^+(t,a \mid b)$ to be the (total) local time at level a conditioned on having a local time at the origin equal to b.

We are interested in the moments of $t^+(1,y \mid b)$, i.e.,

$$\mu_k(y,b) := \mathbb{E}\left(t^+(1,y|b)^k\right),$$

and will present two approaches to compute these moments. One based on results of [12] and one based on a limit theorem for random forests (see [11]).

1.3. **Some known facts about normalized local time.** We will use the following abbreviations throughout the paper

$$\begin{aligned} Ch_i &:= \cosh(\sqrt{2\alpha}\rho_i) & Ch_{i,j} &:= \cosh(\sqrt{2\alpha}(\rho_i-\rho_j)) \\ Sh_i &:= \sinh(\sqrt{2\alpha}\rho_i) & Sh_{i,j} &:= \sinh(\sqrt{2\alpha}(\rho_i-\rho_j)) \\ E^i &:= e^{\sqrt{2\alpha}\rho_i)} & E^{i,j} &:= e^{\sqrt{2\alpha}(\rho_i-\rho_j)} \end{aligned}$$

From [12] (we take the opportunity to correct a misprint in [12]: F_5 has a $\sqrt{2}$ in the denominator), we know that, with $\tau^+(a) := t^+(1,a)$,

$$\begin{aligned} &\int_0^\infty e^{-\alpha t}\mathbb{E}\left[e^{-\beta_1\sqrt{t}\tau^+(\rho_1/\sqrt{t})-\beta_2\sqrt{t}\tau^+(\rho_2/\sqrt{t})}\right]\frac{dt}{\sqrt{2\pi t}} \\ &\quad= \frac{\alpha}{4Ch_1^2Sh_{2,1}^2\left(\beta_1+\sqrt{\frac{\alpha}{2}}\frac{Ch_2}{Ch_1Sh_{2,1}}\right)^2\left(\beta_2+\sqrt{\frac{\alpha}{2}}\frac{E^{2,1}}{Sh_{2,1}}\right)-2\alpha Ch_1^2\left(\beta_1+\sqrt{\frac{\alpha}{2}}\frac{Ch_2}{Ch_1Sh_{2,1}}\right)} \end{aligned} \tag{1}$$

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where β_1 and β_2 are arbitrary positive constants.

Remark. Note that since $\rho_1 < \rho_2$, the transform cannot be symmetric with respect to β_1 and β_2.

In order to find the distribution of the BB local time at the origin we may use the results in Louchard [13] or the path transformation techniques in Bertoin and Pitman [1]. But for illustration we use use $t^+(t,x) \overset{\mathcal{D}}{\equiv} \sqrt{t}.t^+(1, x/\sqrt{t})$ in conjunction with Equ. (1) to get

$$\mathbb{E}_0 \int_0^\infty e^{-\alpha t} \left[e^{-\beta t^+(t,0)} \Big| \, x(t) = 0 \right] \frac{1}{\sqrt{2\pi t}} \, dt = \frac{1}{2\beta + \sqrt{2\alpha}}.$$

Inverting on β leads to

$$\int_0^\infty e^{-\alpha t} Pr_0[\,[t^+(t,0) \big| \, x(t) = 0] \in db] \frac{1}{\sqrt{2\pi t}} \, dt = e^{-\sqrt{2\alpha}\, b/2} db$$

or

$$Pr_0[\,[t^+(t,0) \big| \, x(t) = 0] \in db] = \frac{b e^{-b^2/(8t)} \, db}{(4t)}$$

which is nothing but the Rayleigh density.

1.4. **Some known facts about random forests.** We consider the set $F(n,N)$ of random forests consisting of n vertices and N rooted trees which can be viewed as realizations of Galton-Watson branching processes with N initial particles and conditioned to have total progeny n. This means that we start with N particles each of which produces children. Then each of the children produces further descendants, and so on. The number of children of a particle is governed by a given distribution and all particles (of the whole tree) behave independently (i.e., their offsprings are *iid* random variables). Let $b_{n,N}$ denote the number of forests in $F(n,N)$, weighted according to the probability on $F(n,N)$, which is the above described probability conditioned on the total progeny. It is well known (see e.g. [16]) that the generating function for those forests is $b(z) = a(z)^N$ with

$$a(z) = z\varphi(a(z)). \tag{2}$$

Here $a(z)$ is the generating function for a single tree. Let $L_{n,N}(k)$ be the number of vertices at height k in a random forest in $F(n,N)$.

By [11, Theorem 1.3] we have for $n, N \to \infty$ such that $2N/\sigma\sqrt{n} \to \alpha > 0$

$$\mathbb{E} \left(\frac{2L_{n,N}(2\kappa\sqrt{n})}{\sigma\sqrt{n}} \right)^d \to \mathbb{E} \left(t^+(1, \kappa|\alpha)^d \right) \tag{3}$$

This can be used to compute the moments $\mu_k(y,b)$.

2. The moments

Theorem 1. *The moments of the local time of the reflected Brownian bridge conditioned to have local time b at the origin satisfy*

$$\mu_k(y,b) = 2\sqrt{2\pi} k! e^{b^2/8} \sum_{v=0}^{k-1} \frac{(-1)^v}{(k-v)!} \binom{k-1}{v} b^{k-v-1} \sum_{j=0}^{v} \binom{v}{j} (-1)^j \phi^{(v-1)} \left(-\frac{b + 4(k-j)y}{2} \right) \tag{4}$$

where $\phi^{(1)}(x) = \phi(x)$ denotes the Gaussian distribution function and $\phi^{(j+1)}(x) := \int_{-\infty}^x \phi^{(j)}(u)\,du$. $\phi^{(0)}(x)$ denotes the density of the standard normal distribution and $\phi^{(-1)}(x)$ its derivative.

2.1. **First approach.** In the same manner as in section 1.3 we analyze the reflected BB local time at x, conditioned on the local time at 0. Equ. (1) leads to

$$\begin{aligned} &\mathbb{E}_0 \int_0^\infty dt\, e^{-\alpha t} \int_0^\infty db\, e^{-\sqrt{t}\tau^+(\rho_1/\sqrt{t}\,|b)} b e^{-b^2/8} e^{-\delta\sqrt{t} b} \frac{1}{4\sqrt{2\pi t}} \\ &\quad = \frac{\alpha}{4Sh_1^2 \left(\delta + \sqrt{\frac{\alpha}{2}} \frac{Ch_1}{Sh_1} \right)^2 \left(1 + \sqrt{\frac{\alpha}{2}} \frac{E^1}{Sh_1} \right) - 2\alpha \left(\delta + \sqrt{\frac{\alpha}{2}} \frac{Ch_1}{Sh_1} \right)}. \end{aligned} \tag{5}$$

On setting $v := \sqrt{t}b$ the left-hand side becomes

$$\int_0^\infty db \int_0^\infty dv\, e^{-\alpha v^2/b^2} \mathbb{E}\left[e^{-v/b\tau^+(\rho_1 b/v\,|b)}\right] e^{-b^2/8} e^{-\delta v} \frac{1}{2\sqrt{2\pi}}$$

The right-hand side is of the form

$$\frac{A}{B(\delta+D)^2 + C(\delta+D)} = \left[\frac{A}{B}\cdot\frac{1}{s}\cdot\frac{1}{s+C/B}\right]_{s=\delta+D}$$

and thus inversion on δ using standard rules as listed, for example, in [6] yields

$$\frac{A}{C}\left(e^{-Dv} - e^{-v(C+BD)/B}\right).$$

Hence we obtain from (5)

$$\int_0^\infty db\, e^{-b^2/8} e^{-\alpha v^2/b^2} \mathbb{E}\left[e^{-v/b\tau^+(\rho_1 b/v\,|b)}\right] \frac{1}{2\sqrt{2\pi}} = F(v,\alpha,\rho_1,b) \tag{6}$$

with

$$F(v,\alpha,\rho_1,b) = \frac{1}{2}\left[\exp\left(-v\sqrt{\frac{\alpha}{2}}\frac{Ch_1 + \sqrt{\frac{\alpha}{2}}\frac{E^1 Ch_1 - 1}{Sh_1}}{Sh_1\left(1+\sqrt{\frac{\alpha}{2}}\frac{E^1}{Sh_1}\right)}\right) - \exp\left(-v\sqrt{\frac{\alpha}{2}}\frac{Ch_1}{Sh_1}\right)\right].$$

Substitute $w = \alpha v^2, \eta = \alpha^{-1/2}, y = \rho_1 b/v$, and $s := 1/b^2$. Then equ. (6) becomes

$$\int_0^\infty e^{-1/(8s)} e^{-ws} \mathbb{E}\left[e^{-\eta\sqrt{ws}\tau^+(y\,|b)}\right] \frac{ds}{4s^{3/2}\sqrt{2\pi}} = G(w,\eta) \tag{7}$$

with

$$G(w,\eta) = G(w,\eta,y,s) := F\left(\sqrt{\frac{w}{\alpha}}, \frac{1}{\eta^2}, \frac{yv}{b}, \frac{1}{\sqrt{s}}\right)$$

Expanding both sides of (7) w.r.t η , this gives

$$\int_0^\infty e^{-1/(8s)} e^{-ws} (-1)^k s^{k/2} \mu_k(y,b) \frac{ds}{4s^{3/2}\sqrt{2\pi}k!} = [\eta^k]\frac{G(w,\eta)}{w^{k/2}}. \tag{8}$$

The LHS of (8), L_1 say, is a Laplace transform, the RHS, $R_1(k)$, say, can be computed as follows. To simplify the notations set

$$X = \exp\left(-\sqrt{\frac{w}{2}}\right), \qquad Y = \exp\left(-\frac{2\sqrt{2wy}}{b}\right).$$

$R_1(k)$ becomes

$$[\eta^k]\frac{G_1(w,\eta)}{w^{k/2}}$$

with

$$\begin{aligned} G_1(w,\eta) &= \frac{1}{2}\exp\left(-\frac{\sqrt{w}}{\sqrt{2}}\frac{1+\eta(1+Y)/\sqrt{2}}{1+\eta(1-Y)/\sqrt{2}}\right) \\ &= \frac{X}{2}\exp\left(-\frac{\sqrt{w}}{\sqrt{2}}\frac{2\eta Y/\sqrt{2}}{1+\eta(1-Y)/\sqrt{2}}\right) \\ &= \frac{X}{2}\exp\left(\frac{\sqrt{w}}{\sqrt{2}}\left[\sum_1^\infty (-\eta/\sqrt{2})^i (1-Y)^{i-1} 2Y\right]\right) \end{aligned}$$

Setting $z := \sqrt{w}$ and expanding this (omitting the detailed derivation)

$$R_1(k) = \frac{X}{2}(-1)^k \sum_{v=0}^{k-1} \frac{(-1)^v}{(k-v)!}\frac{1}{(\sqrt{2}z)^v}\binom{k-1}{v}\sum_{j=0}^{v}\binom{v}{j}(-1)^j Y^{k-j} \tag{9}$$

Clearly, we need to invert Laplace transforms of the form $e^{-\sqrt{\gamma w}}/w^{(j+1)/2}$. But in [4] the following lemma is proved:

Lemma 1. *We have*

$$\int_0^\infty \phi^{(j)}(-b)e^{-ws}\frac{(2s)^{(j+1)/2}}{s}ds = \frac{e^{-\sqrt{2w}}}{w^{(j+1)/2}} \quad ,j \geq 1$$

where $b = 1/\sqrt{s}$.

It is also well known that

$$\int_0^\infty e^{-ws}e^{-1/(2s)}\frac{ds}{\sqrt{2\pi}s^{3/2}} = e^{-\sqrt{2w}},$$

$$\int_0^\infty e^{-ws}e^{-1/(2s)}\frac{ds}{\sqrt{\pi s}} = e^{-\sqrt{2w}}/\sqrt{w}.$$

Let us remark that $\phi^{(j)}(x)$ can be expressed in term of $\phi(x)$ and $e^{-x^2/2}/\sqrt{2\pi}$ in the form

$$\phi^{(j)}(x) = p_{1,j}(x)\phi(x) + p_{2,j}(x)e^{-x^2/2}/\sqrt{2\pi},$$

where $p_{1,j}(x)$ and $p_{2,j}(x)$ are polynomials of degree $j-1$ and $j-2$, respectively. In [4], we provide an efficient procedure to compute $\phi^{(j)}(x)$. Standard transformations lead to formulae inverting expressions of the form $e^{-\sqrt{\gamma w}}/w^{(j+1)/2}$. Now we note that each term of type XY^l in $R_1(.)$ leads to

$$\sqrt{\gamma} = \sqrt{2}/2 + l2\sqrt{2}y/b$$

Inverting (8) is now trivial (with Maple!). This gives, for instance,

$$\mu_1(y,b) = (b+4y)e^{-y(b+2y)} \tag{10}$$

$$\mu_2(y,b) = \left(4e^{-(4y+b)^2/8} + (b^2+8by-4)e^{-(8y+b)^2/8}\right)e^{b^2/8} \tag{11}$$

$$\begin{aligned}\mu_3(y,b) = \Bigg(&12\sqrt{2\pi}\left(\phi\left(-\frac{b+4y}{2}\right) - 2\phi\left(-\frac{b+8y}{2}\right) + \phi\left(-\frac{b+12y}{2}\right)\right) \\ &+ 12be^{-(8y+b)^2/8} + (b^3+12b^2y-12b)e^{-(12y+b)^2/8}\Bigg)e^{b^2/8}\end{aligned} \tag{12}$$

and, finally, application on (9) leads to (4).

Remark 1. As a check, we may compute $\mathbb{E}\int_0^1 B^+(t \mid b)dt = \int_0^\infty \mathbb{E}[\tau^+(x \mid b)]xdx$ and get exactly the expression we obtained in [4], $\phi(-b/2)\sqrt{2\pi}e^{b^2/8}/2$ (the local time at the origin used here is twice the local time at the origin considered in [4]).

Remark 2. Integrating the moments w.r.t. the b density gives the unconditioned moments as computed by Takacs in [18]. E.g., Maple gives

$$\begin{aligned}\mu_1(y) &= 2\sqrt{2\pi}(1-\phi(2y)) \\ \mu_2(y) &= -24y\sqrt{2\pi} + 4e^{-2y^2} + 4e^{-8y^2} + 16y\sqrt{2\pi}\phi(4y) + 8y\sqrt{2\pi}\phi(2y)\end{aligned}$$

higher moments can be integrated with Maple only numerically.

2.2. **Second approach.** Let $b_{k,m,n,N}$ denote the (weighted) number of forests in $F(n,N)$ with m nodes in stratum k. Then standard methods on generating functions (see [10] for a general introduction and [8] for the treatment of the particular case of random trees) give

$$\mathbf{P}\{L_{n,N}(k) = m\} = \frac{b_{k,m,n,N}}{b_{n,N}} = \frac{1}{b_{n,N}}[z^n u^m]y_k(z,ua(z))^N$$

where

$$y_0(z,u) = u, \qquad y_{i+1}(z,u) = z\varphi(y_i(z,u)), \quad i \geq 0.$$

Consequently, we have

$$\mathbb{E}L_{n,N}(k) = \frac{1}{b_{n,N}}[z^n]\frac{\partial}{\partial u}y_k(z,ua(z))^N|_{u=1}$$

and moreover

$$\left(\frac{2}{\sigma\sqrt{n}}\right)^m \mathbb{E}L_{n,N}(k)(L_{n,N}(k)-1)\cdots(L_{n,N}(k)-m+1) \sim \left(\frac{2}{\sigma\sqrt{n}}\right)^m \mathbb{E}L_{n,N}(k)^m.$$

Thus by (3) the moments can be calculated by

$$\mu_k(y,b) = \lim_{n\to\infty} \left(\frac{2}{\sigma\sqrt{n}}\right)^k \frac{1}{b_{n,N}} [z^n] \frac{\partial^k}{\partial u^k} y_\ell(z, ua(z))^N|_{u=1} \tag{13}$$

with $2N/\sigma\sqrt{n} \sim b$ and $\ell = \lfloor 2y\sqrt{n}/\sigma \rfloor$. The calculation of these coefficients is done by singularity analysis and contour integration. The relevant asymptotic expansions are provided in the following lemma:

Lemma 2. *Let $a = a(z)$ be the tree function (2) and $\alpha = \alpha(z) := z\varphi'(a(z))$. Furthermore, z_0 denotes the dominant singularity of $a(z)$ and $S := a(z_0)$. Then for $b = 2N/\sigma\sqrt{n}$ and $\ell = 2y\sqrt{n}/\sigma$ the following asymptotic expansion holds as $z \to z_0$ such that $z - z_0 \notin \mathbb{R}^+$:*

$$[z^n] \frac{1}{(1-\alpha)^m} \alpha^\ell a^N \sim \frac{S^N n^{(m-2)/2}}{\sigma^m z_0^n} \phi^{(m-1)}\left(-\frac{b+4y}{2}\right)$$

where m is a fixed nonnegative integer.

Proof. The lemma can be proved by an appropriate modification of the proof of [7, Theorem 4] □

When computing the derivatives in (13) only the main term of $\frac{\partial^k}{\partial u^k} y_\ell(z, ua(z))$ is relevant for the calculation of $\mu_k(y,b)$. Set $\beta = \beta(z) := z\varphi''(a(z))$. Then, using $\alpha(z) \sim 1$ for $z \to z_0$ (see [16] for a detailed expansion), we get by elementary calculations

$$\begin{aligned}
\frac{\partial}{\partial u} y_\ell(z,u)|_{u=a(z)} &= \alpha^\ell \\
\frac{\partial^2}{\partial u^2} y_\ell(z, ua(z))|_{u=a(z)} &= \beta\alpha^{\ell-1}\frac{1-\alpha^\ell}{1-\alpha} \\
\frac{\partial^3}{\partial u^3} y_\ell(z,u)|_{u=a(z)} &\sim 3\beta^2\alpha^{\ell-1}\frac{(1-\alpha^\ell)(1-\alpha^{\ell-1})}{(1-\alpha)(1-\alpha^2)}
\end{aligned}$$

Plugging these formulas into $\frac{\partial^j}{\partial u^j} y_\ell(z, ua(z))^N$, using $b_{n,N} \sim NS^N e^{-N^2/2n\sigma^2}/\sigma z_0^n \sqrt{2\pi n^3}$ and applying Lemma 2 gives (10)–(12).

This leads to the following lemma

Lemma 3. *If $z \to z_0$ such that $z - z_0 \notin \mathbb{R}^+$, then the following expansion holds:*

$$\frac{\partial^m}{\partial u^m} y_\ell(z,u)|_{u=a(z)} \sim \frac{m!\beta^{m-1}\alpha^\ell}{2^{m-1}} \left(\frac{1-\alpha^\ell}{1-\alpha}\right)^{m-1}$$

Proof. Note that Faà di Bruno's formula (see e.g. [5]) gives

$$\frac{\partial^{m+1} y_\ell}{\partial u^{m+1}}(z,1) = \sum_{\sum_{i=1}^m ik_i = m+1} \frac{(m+1)!}{k_1!\cdots k_m!} z\varphi^{(k_1+\cdots+k_m)}(a(z)) \prod_{j=1}^m \left(\frac{1}{j!}\frac{\partial^j y_{\ell-1}}{\partial u^j}\right)^{k_j} + \alpha(z)\frac{\partial^{m+1} y_{\ell-1}}{\partial u^{m+1}}(z,1).$$

By induction and using $\sum (j-1)k_j = m+1-\sum k_j$ we get

$$\prod_{j=1}^m \left(\frac{1}{j!}\frac{\partial^j y_{\ell-1}}{\partial u^j}\right)^{k_j} \sim \left(\frac{\beta}{2}\right)^{m+1-\sum k_j} (\alpha^{\ell-1})^{\sum k_j} \left(\frac{1-\alpha^{\ell-1}}{1-\alpha}\right)^{m+1-\sum k_j}.$$

The dominant ones of these terms are clearly those where $m+1-\sum k_j$ is maximal, which occurs if and only if $\sum k_j = 2$. This is equivalent to the existence of an i such that $k_\lambda = 1$ if $\lambda \in \{i, m+1-i\}$ and 0 else. In these cases we have $z\varphi^{(k_1+\cdots+k_m)}(a(z)) = \beta$ and thus

$$\begin{aligned}
\frac{\partial^{m+1} y_\ell}{\partial u^{m+1}}(z,1) &\sim \frac{(m+1)!\beta^m m}{2^m}\alpha^{2\ell-2}\left(\frac{1-\alpha^{\ell-1}}{1-\alpha}\right)^{m-1} + \alpha(z)\frac{\partial^{m+1} y_{\ell-1}}{\partial u^{m+1}}(z,1) \\
&\sim \frac{(m+1)!\beta^m\alpha^\ell}{2^m(1-\alpha)^{m-1}} m \sum_{i=1}^{\ell-1} \alpha^{\ell-i-1}(1-\alpha^{\ell-i})^{m-1}.
\end{aligned}$$

Using the binomial theorem and summing up w.r.t ℓ gives

$$\frac{\partial^{m+1} y_\ell}{\partial u^{m+1}}(z,1) \sim \frac{(m+1)!\beta^m\alpha^\ell}{2^m(1-\alpha)^m} m \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j \alpha^j \frac{1-\left(\alpha^{\ell-1}\right)^{j+1}}{j+1}$$
$$= \frac{(m+1)!\beta^m\alpha^\ell}{2^m(1-\alpha)^m} \left((1-\alpha^{\ell-1})^m - \frac{(1-\alpha)^m}{\alpha} \right) \sim \frac{(m+1)!\beta^m\alpha^\ell}{2^m} \left(\frac{1-\alpha^\ell}{1-\alpha} \right)^m$$

where we used $\alpha \sim 1$ in the last step. □

With the help of this lemma we can prove (4) now. By Faà di Bruno's formula we get

$$\left(\frac{\partial}{\partial u}\right)^m y_\ell^N = \sum_{\sum i k_i = m} \frac{m!}{k_1! \cdots k_m!} N(N-1)\cdots\left(N - \sum k_i + 1\right) y_\ell^{N-\sum k_i} \prod_{j=1}^m \left(\frac{1}{j!} \frac{\partial^j y_\ell}{\partial u^j} \right)^{k_j}.$$

Inserting $u = a(z)$ and using Lemma 3 we obtain

$$\left(\frac{\partial}{\partial u}\right)^m y_\ell^N \sim \sum_{\sum i k_i = m} \frac{m!}{k_1! \cdots k_m!} N(N-1)\cdots\left(N - \sum k_i + 1\right) a^{N+\sum(i-1)k_i} \prod_{j=1}^m \left(\frac{\beta^{j-1}\alpha^\ell}{2^{j-1}} \left(\frac{1-\alpha^\ell}{1-\alpha} \right)^{j-1} \right)^{k_j}$$
$$\sim \sum_{\sum i k_i = m} \frac{m!}{k_1! \cdots k_m!} N^{\sum k_i} \left(\frac{\beta}{2}\right)^{m-\sum k_i} (\alpha^\ell)^{\sum k_i} \left(\frac{1-\alpha^\ell}{1-\alpha} \right)^{m-\sum k_i} a^{N+m-\sum k_i}$$

Now apply Lemma 2 and use $\beta \sim \sigma^2/S$ and set as before $2N/\sigma\sqrt{n} = b$ and $\ell = \lfloor 2y\sqrt{n}/\sigma \rfloor$. This gives

$$[z^n] \left(\frac{\partial}{\partial u}\right)^m y_\ell^N \sim \sum_{\sum i k_i = m} \frac{m!}{k_1! \cdots k_m!} f\left(\sum k_i\right) \tag{14}$$

where

$$f(k) = \frac{n^{(m-2)/2} S^N}{z_0^n 2^m} b^k \sum_{i=0}^{m-k} \binom{m-k}{i} (-1)^i \phi^{(m-k-1)} \left(-\frac{b+4(k+i)y}{2} \right).$$

Rewriting (14) in the form

$$[z^n] \left(\frac{\partial}{\partial u}\right)^m y_\ell^N \sim \sum_{k=1}^{m-1} f(k) \sum_{\sum i k_i = m, \sum k_i = k} \frac{m!}{k_1! \cdots k_m!},$$

applying (see [5, p. 135])

$$\sum_{\sum i k_i = m, \sum k_i = k} \frac{m!}{k_1! \cdots k_m!} = \binom{m-1}{m-k} \frac{m!}{k!},$$

and substituting $k = m - v$ yields finally (4) and completes the proof.

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