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MINIMAL SURFACES AND CRYSTALLOGRAPHY

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ABSTRACT. This survey reports on some directions in the theory of triply periodic minimal surfaces (TPMS) where the computer gives new possibilities in the graphical presentation and in the exact numerical calculations as well. Some numerical data and statements, in some remarks, are mentioned first time here as partly joint results with Jenö SZIRMAI and Attila BÖLCSKEI, which will be published later. A classification strategy is formulated at the end, where we prove, respectively disprove the existence of some TPMS's.

Dedicated to Professor Johannes BÖHM on the occasion of his 75th birthday

1. INTRODUCTION

We first start with the famous SCHWARZ' *D (diamond) minimal surface* described in Fig. 1 *by the pair of space groups* $\mathbf{Pn\bar{3}m/Fd\bar{3}m}$.

Here in picture (a) we have indicated the surface patch $R_0R_1R_2$ and its image BR_1R_2 under the halfturn about R_1R_2. The patch $R_0R_1R_2$ will be extended under the crystallographic space group $\mathbf{Pn\bar{3}m}$ (No. 224 in [5]), which is generated by

$$\begin{aligned}\mathbf{Pn\bar{3}m}: \quad & \mathbf{r_1}: (x,y,z) \mapsto (-x+1, y, -z)(R_1R_0),\\ & \mathbf{r_2}: (x,y,z) \mapsto (-z+\frac{1}{2}, -y+\frac{1}{2}, -x+\frac{1}{2})(R_1R_2),\\ & \mathbf{m}: (x,y,z) \mapsto (y,x,z) \quad \text{(in the plane } OR_0AR_2), \quad (1)\end{aligned}$$

i.e., by two halfturns ($\mathbf{2}$) and a plane reflection ($\mathbf{m}$). The surface is incident to the halfturn axes and perpendicular to the reflection plane. The last transform preserves the sides of the surface, the halfturns change them.

Considering only the side preserving transforms of the surface in the space group $\mathbf{Pn\bar{3}m}$, we obtain its subgroup $\mathbf{Fd\bar{3}m}$ (No. 227) of index

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two [5], generated by

$$\mathbf{Fd\bar{3}m:}\quad \begin{aligned} &(x,y,z)\mapsto(y,x,z)\quad (0R_0AR_2),\\ &(x,y,z)\to(x,z,y)\quad (OAB),\\ &(x,y,z)\to(z+\frac{1}{2},-y+\frac{1}{2},x-\frac{1}{2}). \end{aligned} \tag{2}$$

The last halfturn axis is orthogonal to the plane $R_0R_1R_2$ in R_1. The reflection planes OR_0A, OBA and their images under the latter halfturn form a so-called *sphenoid*, i.e., a tetrahedron of congruent faces with opposite rectangles at AR_0 and OB, respectively, and $\frac{\pi}{3}$ angles at the remaining 4 edges.

The space group pair $\mathbf{Pn\bar{3}m/Fd\bar{3}m}$ exactly describes the whole minimal D-surface if the patch $R_0R_1R_2$ will correctly be adjusted by the complex parameter domain $\omega_0\omega_1\omega_2$ in Fig. 1b (to be explained later in Sect. 4).

In Fig. 1c we have symbolically drawn the surface fundamental domains of the hyperbolic plane group pair, induced by our space group pair. $\mathbf{Pn\bar{3}m}$ induces a hyperbolic plane reflection group $\star$**2,4,6** in John Conway's notation [13]. That means, the reflection lines have the angles $\frac{\pi}{2}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$ at R_1 (**222**), R_0 ($\mathbf{\bar{4}m2}$) and R_2 ($\mathbf{\bar{3}m}$), respectively. In parentheses we have inserted the stabilizers of the points in crystallographic notations, respectively. The side preserving space group $\mathbf{Fd\bar{3}m}$ induces the hyperbolic plane group **2**$\star$**2,3**. That means, the reflection lines R_0R_2, R_2B has two corners: $\frac{\pi}{2}$ at $R_0\sim B$ and $\frac{\pi}{3}$ at R_2. Furthermore, the halfturn in $\mathbf{Fd\bar{3}m}$ induces a plane halfturn, about R_1, as denoted by **2** before the star $\star$ opening the *boundary component* of the *factor surface (orbifold)* $\mathbb{H}^2/(\mathbf{2\star2,3})$. Thus we have obtained the hyperbolic plane group pair

$$\star\mathbf{2,4,6/2\star2,3}\quad \text{of } D \text{ surface.} \tag{3}$$

We remark that besides the generators of $\mathbf{Pn\bar{3}m}$ and $\star$**2,4,6**, also in general, the relations (by Poincaré algorithm [16]) refer to each other:

$$\begin{aligned} \mathbf{Pn\bar{3}m} \;>\; \big(\mathbf{r}_1,\mathbf{r}_2,\mathbf{m} \text{ —— } 1=\mathbf{r}_1^2=\mathbf{r}_2^2=\\ =\mathbf{m}^2=(\mathbf{r}_1\mathbf{r}_2)^2=(\mathbf{m}\mathbf{r}_1)^4=(\mathbf{m}\mathbf{r}_2)^6\big)=\star\mathbf{2,4,6}, \end{aligned} \tag{4}$$

according to the fact that the action of $\mathbf{Pn\bar{3}m}$ on the D-surface induces just the action of $\star$**2,4,6**. This also guarantees that the D surface does not have *self-intersections*. The symbol $>$ means that $\mathbf{Pn\bar{3}m}$ has a further relation, namely [15]:

$$1=(\mathbf{m}\mathbf{r}_1\mathbf{r}_2)^6. \tag{5}$$

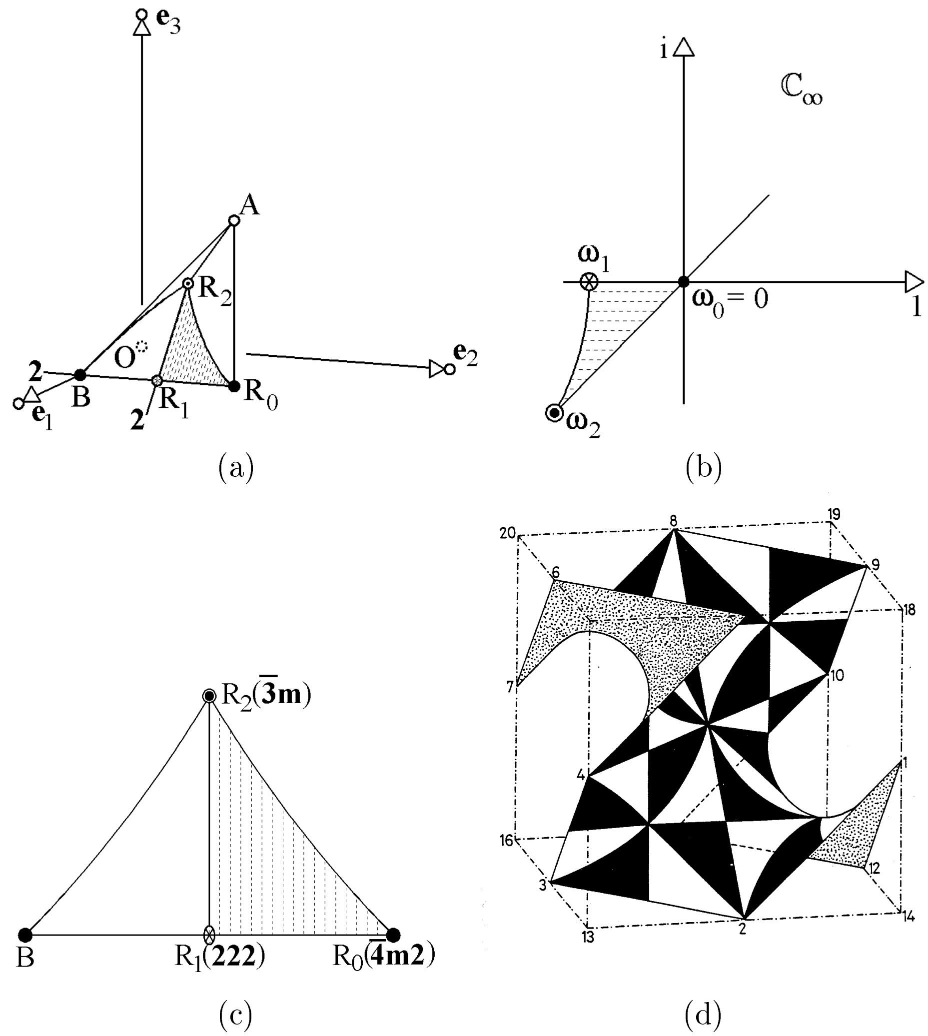


FIGURE 1. H.A.SCHWARZ' *D* (diamond) surface: (a) its space group pair **Pn$\bar{3}$m/ Fd$\bar{3}$m** by fundamental domains, (b) its complex domain $\omega_0\omega_1\omega_2$, (c) its symbolic hyperbolic surface group pair **⋆2,4,6/ 2⋆2,3** (d) some global drawing of *D* surface by Alan L. MACKAY [14], glued together from tetrahedral saddle surface pieces. *Warning:* Such a piece is not a hyperbolic paraboloid (as a usual ruled (straight line) surface, called saddle simply).

In Fig. 1d we see a fragment of our *D* surface by A.L. MACKAY from [14] which is very inspiring, although the black and white coloring is not traditional. In general, the side preserving subgroups **Fd$\bar{3}$m** $>$ **2⋆2,3** are presented as keeping the colours. This convention is not followed here. Congruent tetrahedral saddle surface pieces are glued together in a cube. Then the continuation is more obvious.

2. About the classical theory

In this sketch we follow the nicely motivated books [7, 8] of Kommerell brothers, although the newer books, e.g. [1], [11] are more complete in some details. J.L. Lagrange (1760/61) posed the following

Problem: *Let $\mathcal{C}$ be a closed curve and $z = f(x, y)$ be a surface incident to $\mathcal{C}$ such that the surface area*

$$S = \iint_T \sqrt{1 + f_x^2 + f_y^2}\, dx \wedge dy \tag{6}$$

is minimal over a parameter domain T whose border curve defines $\mathcal{C}$. □

Lagrange himself, by his *variatonal method*, found the partial differential equation

$$(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0 \tag{7}$$

for a function $\mathbb{R}^2 \supset T \ni (x, y) \mapsto f(x, y) \in \mathbb{R}$ (smooth enough). Here e.g. $f_x := \frac{\partial f}{\partial x}$, $f_{xy} = f_{yx} := \frac{\partial^2 f}{\partial x \partial y}$ denote the partial derivatives as usual. Meusnier (1776) observed that (7) expresses

$$\kappa_1 + \kappa_2 = 0 = \frac{1}{R_1} + \frac{1}{R_2} \tag{8}$$

for the main (principal) curvatures κ_i and their radii $R_i = \frac{1}{\kappa_i}$, $(i = 1, 2)$.

In Gauss' surface setting

$$\mathbb{R}^2 \supset U \ni (u, v) \mapsto \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \in \mathbb{E}^3 \tag{9}$$

the equation of minimal surfaces

$$0 = \frac{1}{R_1} + \frac{1}{R_2} = \frac{EN - 2FM + GL}{EG - F^2} \tag{10}$$

holds for the first and second fundamental values

$$E := \langle \mathbf{r}_u, \mathbf{r}_u \rangle, \quad F := \langle \mathbf{r}_u, \mathbf{r}_v \rangle, \quad G := \langle \mathbf{r}_v, \mathbf{r}_v \rangle;$$
$$L := \langle \mathbf{r}_{uu}, \mathbf{m} \rangle, \quad M := \langle \mathbf{r}_{uv}, \mathbf{m} \rangle, \quad N := \langle \mathbf{r}_{vv}, \mathbf{m} \rangle.$$

Here $\mathbf{m} := \mathbf{r}_u \times \mathbf{r}_v / D$ with $D := \sqrt{EG - F^2} = |\mathbf{r}_u \times \mathbf{r}_v|$ denotes the surface normal unit vector

$$\mathbf{m}(u, v) =: (a(u, v), b(u, v), c(u, v)) \tag{11}$$

and its coordinates.

G. Monge (1784) found the ingenious idea: *Try to solve the equation (10) by choosing $E = 0 = G$ through new complex parameters*

$$u = u_1 + iu_2, v = v_1 + iv_2, \text{ both in } \mathbb{C}. \tag{12}$$

Then the first fundamental form becomes

$$(ds)^2 = 2F du dv, \text{ with } F \neq 0, \tag{13}$$

and the following consequences hold:

$$0 = M := \langle \mathbf{r}_{uv}, \mathbf{m} \rangle, 0 = E_v := 2\langle \mathbf{r}_{uv}, \mathbf{r}_u \rangle, 0 = G_u := 2\langle \mathbf{r}_{uv}, \mathbf{r}_v \rangle \tag{14}$$

$$\Longrightarrow \quad \mathbf{r}_{uv} = \mathbf{0}. \tag{15}$$

Now $\mathbf{r}_{uv} = \mathbf{0}$ and $0 = E = G$ imply for $\mathbf{r}(x(u,v), y(u,v), z(u,v))$:

$$x = U_1(u) + V_1(v), \quad y = U_2(u) + V_2(v), \quad z = U_3(u) + V_3(v),$$

$$\text{and} \quad U_1'^2 + U_2'^2 + U_3'^2 = 0 = V_1'^2 + V_2'^2 + V_3'^2 \tag{16}$$

$$\text{with} \quad U_1' := \frac{d}{du} U_1, \quad V_1' := \frac{d}{dv} V_1, \quad \text{etc.} \ldots$$

Let us insert from [7, 8] some preliminary

Remarks on minimal curves:
1. Consider a *complex cone*

$$0 = x^2 + y^2 + z^2 = (x + iy)(x - iy) + z^2 \tag{17}$$

by parameters $(u, v) \in \mathbb{C}^2$ in

$$\text{a)} \ \frac{x + iy}{-z} = \frac{z}{x - iy} =: u; \quad \text{b)} \ \frac{x - iy}{-z} = \frac{z}{x + iy} =: v. \tag{18}$$

These describe so-called *minimal straight line families* of the cone (17). Each straight line has a complex variable w, e.g. for u-family (18a)

$$x = \frac{w}{2}(1 - u^2), \quad y = \frac{iw}{2}(1 + u^2), \quad z = wu. \tag{19}$$

2. Analogously, the *complex unit sphere* equation

$$0 = x^2 + y^2 + z^2 - 1 = (x + iy)(x - iy) + (z + 1)(z - 1) \tag{20}$$

leads to two minimal straight line families, parametrized by u and v, respectively:

$$\text{a)} \ \frac{x + iy}{1 - z} = \frac{1 + z}{x - iy} = u \quad \text{b)} \ \frac{x - iy}{1 - z} = \frac{1 + z}{x + iy} = v, \tag{21}$$

and to a complex parametrization of the complex unit sphere:

$$x = \frac{u + v}{1 + uv}, \quad y = \frac{i(v - u)}{1 + uv}, \quad z = \frac{uv - 1}{1 + uv} \tag{22}$$

with some extra conditions on $(x, y, z) = (0, 0, 1)$. We obtain *real points* of the unit sphere iff

$$v = \overline{u}, \text{ moreover } (0,0,1) \text{ for } u = \infty, \ \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}. \tag{23}$$

Then we get the *real unit sphere* by one complex parameter

$$\begin{aligned} u = u_1 + iu_2: \qquad & a = \frac{u+\overline{u}}{1+u\overline{u}} = \frac{2u_1}{1+u_1^2+u_2^2}, \\ & b = \frac{i(\overline{u}-u)}{1+u\overline{u}} = \frac{2u_2}{1+u_1^2+u_2^2} \\ & c = \frac{u\overline{u}-1}{1+u\overline{u}} = \frac{u_1^2+u_2^2-1}{1+u_1^2+u_2^2}, \\ u = \infty \quad \mapsto \quad & (0,0,1). \end{aligned} \tag{24}$$

This is the stereographic projection of $\mathbb{C}_\infty$, as $(a, b, 0) = (u_1, u_2)$ plane onto the unit sphere from $(0, 0, 1)$.

3. A *minimal curve* is defined as a complex curve

$$\mathbb{C} \supset I \ni u \mapsto \mathbf{r}(u) \in \mathbb{C}^3 \tag{25}$$

whose any tangent $\frac{d}{du}\mathbf{r} = (\frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du}) := (x', y', z')$ satisfies

$$x'^2 + y'^2 + z'^2 = 0, \text{ i.e., } 0 = (ds)^2 := (dx)^2 + (dy)^2 + (dz)^2. \tag{26}$$

As above by (19) with an 'arbitrary' complex function $w = \Phi(u)$, we get the *general equation of a minimal curve*

$$x = \frac{1}{2}\int (1-u^2)\Phi(u)du; \quad y = \frac{i}{2}\int (1+u^2)\Phi(u)du; \quad z = \int u\Phi(u)du \tag{27}$$

by integrating along a curve in a simply connected domain from any starting point.□

4. We notice at (26) that the sign of coordinates in (27) can independently be chosen. This will be important also later on!

Putting together the formulas in (16), (26), (27) we get the *complex minimal surface*

$$\begin{aligned} & \mathbb{C}^2 \supset U \ni (u,v) \mapsto \mathbf{r}(u,v) \in \mathbb{C}^3, \text{ defined by} \\ & x = \frac{1}{2}\int (1-u^2)\Phi(u)du + \frac{1}{2}\int (1-v^2)\Psi(v)dv, \\ & y = \frac{i}{2}\int (1+u^2)\Phi(u)du - \frac{i}{2}\int (1+v^2)\Psi(v)dv, \\ & z = \int u\Phi(u)du + \int v\Psi(v)dv. \end{aligned} \tag{28}$$

This will provide the *real minimal surface* equation iff u and $v = \overline{u}$ are conjugate, and for $\Phi(u)$ and $\Psi(v)$ holds

$$\Psi(\overline{u}) = \overline{\Phi(u)}, \text{ i.e., they are conjugate functions.} \tag{29}$$

(The formal generalization by a further smooth bijection $u \mapsto \Omega(u)$ of $\mathbb{C}$ is not essential.)

Equivalently we get the MONGE-ENNEPER-WEIERSTRASS (1866, briefly: 'MEW') formulas for the real minimal surfaces:

$$\begin{aligned} x &= Re \int (1-u^2)\Phi(u)du; \\ y &= Re \int i(1+u^2)\Phi(u)du \\ z &= Re \int 2u\Phi(u)du \end{aligned} \tag{30}$$

taken the real parts of the integrals up to a sign, independently for each coordinate (see Rem. 2.4). Here we integrate on any complex curve (with real parameter)

$$\mathbb{R} \supset I \ni t \mapsto u(t) = u_1(t) + iu_2(t) \in U \subset \mathbb{C} \tag{31}$$

in a simply connected domain U of $\mathbb{C}$, where the starting point is fixed. The endpoint $u = u_1 + iu_2$ provides us the two real parameters (u_1, u_2) of the minimal surface (30). □

The other important data can be read off (28), first again for a complex minimal surface, then for a real one:

$$(ds)^2 = (1+uv)^2\Phi(u)\Psi(v)dudv; \tag{32}$$

the surface normal unit vector $\mathbf{m}(a, b, c)$ is

$$\begin{aligned} &\mathbf{m} = \mathbf{r}_u \times \mathbf{r}_v / D; \\ &a = \frac{u+v}{1+uv}, \quad b = \frac{i(v-u)}{1+uv}, \quad c = \frac{uv-1}{1+uv.} \end{aligned} \tag{33}$$

For the spherical mapping, which associates any (u, v)-point with its unit normal vector, we have the image arc length quadrat

$$\begin{aligned} &(ds_0)^2 = (da)^2 + (db)^2 + (dc)^2 = \frac{4dudv}{(1+uv)^2}, \\ \text{and} \quad &L = -\Phi(u), \quad M = 0, \quad N = -\Psi(v). \end{aligned} \tag{34}$$

The Gauss curvature is

$$K = \kappa_1\kappa_2 = \frac{1}{R_1R_2} = \frac{-4}{(1+uv)^4\Phi(u)\Psi(v)},$$

$$\text{i.e.} \qquad R_1 = -R_2 = \frac{1}{2}(1+uv)^2\sqrt{\Phi(u)\Psi(v)} \tag{35}$$

$$(ds_0)^2 = -K(ds)^2 \quad \text{with} \quad \Psi(\overline{u}) = \overline{\Phi(u)}.$$

In general, it holds the

Theorem: *A minimal surface is conform (locally similar) with its spherical image by the surface normals.* □

This is a characteristic theorem for the minimal surfaces (and for the spheres).

3. Associate minimal surfaces

The last statement can be strengthened by the following

Theorem 3.1: *Isometric minimal surfaces S and S_α can be equally orientied in the space $\mathbb{E}^3$ so that they have parallel surface normals in the corresponding (equally parametrized) points. Then S_α has the functions $\Phi_\alpha(u)$ and $\Psi_\alpha(v)$ such that*

$$\Phi_\alpha(u) = e^{i\alpha}\Phi(u), \quad \Psi_\alpha(v) = e^{-i\alpha}\Psi(v). \tag{36}$$

Here α is the angular constant by Bonnet *(1853), it measures the angle of corresponding line elements $d\mathbf{r}$ and $d\mathbf{r}_\alpha$ by*

$$\cos\alpha = \frac{\langle d\mathbf{r}, d\mathbf{r}_\alpha\rangle}{|d\mathbf{r}||d\mathbf{r}_\alpha|}, \quad 0 \le \alpha \le \pi \tag{37}$$

□

The above S and S_α are called *associate minimal surfaces. Conjugate minimal surfaces* are defined by $\alpha = \frac{\pi}{2}$. We recall the

Problem: *(*Björling-Bonnet-Schwarz*, 1864) Construct a minimal surface to a given curve and to a normal vector field given along the curve.*

Bonnet and independently H.A. Schwarz found the basic

Theorem 3.2: *The above problem has a unique solution for a smooth curve and for its smooth normal vector field.* □

Consequences: *1. Any incident straight line is a halfturn axis of a minimal surface (see the introductory D surface).*

2. Any plane, which is perpendicular to a minimal surface in each intersection point, is a symmetry plane of it.

3. In a reflection point or at a rotation centre of order at least 3, both main curvatures have to be zero (flat point of the minimal surface)

4. A minimal surface is uniquely determined (locally)

a) by a geodesic curve,

b) by an asymptotic curve,

c) by a main curvature curve.

The most famous real minimal surfaces are the catenoid as revolution surface about the z axis

$$\sqrt{x^2+y^2} = a(e^{z/2a} + e^{-z/2a}), \tag{38}$$

and the helicoid as screw ruled (straight line) surface along the z axis

$$x = y \tan \frac{z}{2a}. \tag{39}$$

The catenoid and the helicoid are conjugate minimal surfaces for the same real parameter $a > 0$.

4. Construction of minimal surfaces by MEW formulas

We turn back to our D minimal surface in Fig. 1, and we shall construct it by Monge-Enneper-Weierstrass (briefly MEW) formulas (30). The function $\Phi(u)$ has to be produced, first, by the surface normal

$$\mathbf{m}(a(u), b(u), c(u)) \quad \text{with} \quad u \in U \subset \mathbb{C}, \tag{40}$$

second, by its symmetries, thus by the flat points of zero curvature (see formulas (24), (32), (35), (30)).

In the points $R_0\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $R_1\left(\frac{1}{2}, \frac{1}{4}, 0\right)$, $R_2\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ we can choose the surface normals by (24):

$$\begin{aligned} \mathbf{m}_0(0,0,-1) \quad &\text{to} \quad \omega_0 = 0, \\ \mathbf{m}_1\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) \quad &\text{to} \quad \omega_1 = -(\sqrt{2}-1), \\ \mathbf{m}_2\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \quad &\text{to} \quad \omega_2 = -\frac{1}{2}(\sqrt{3}-1)(1+i), \end{aligned} \tag{41}$$

respectively as Fig. 1b shows. By the symmetries $\bar{\mathbf{3}}\mathbf{m}$ at $R_2(\omega_2)$, the Gauss curvature K has to be 0, thus it holds

$$\Phi[\omega_2 = -\frac{1}{2}(\sqrt{3}-1)(1+i)] = \infty \quad \text{as a limit.} \tag{42}$$

We look at (24) that the parameter transform

$$u \mapsto u' = -1/\overline{u}, \quad 0 \longleftrightarrow \infty \quad \text{yields } (a,b,c) \mapsto (-a,-b,-c) \tag{43}$$

and the point reflection of $\mathbf{Pn\bar{3}m}$ in R_2

$$\begin{aligned} \mathbf{R}_2: \quad & X(x,y,z) \mapsto X^\star(-x+\frac{1}{2}, -y+\frac{1}{2}, -z+\frac{1}{2}) \text{ produces} \\ & R_2^\star[-1/\overline{\omega}_2 = \frac{1}{2}(\sqrt{3}+1)(1+i)] = R_2\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) \\ & R_1^\star(-1/\overline{\omega}_1) = \left(0,\frac{1}{4},\frac{1}{2}\right), \quad R_0^\star(\infty) = (0,0,1). \end{aligned} \tag{44}$$

We can also check that, with ω_2 from (41),

$$\omega_2,\ i\omega_2,\ -\omega_2,\ -i\omega_2,\ \frac{-1}{\overline{\omega}_2},\ \frac{-i}{\overline{\omega}_2},\ \frac{1}{\overline{\omega}_2},\ \frac{i}{\overline{\omega}_2,}, \text{ i.e., the roots of} \tag{45}$$

$$(u^4-\omega_2^4)\left[u^4-(\frac{-1}{\overline{\omega}_2})^4\right] = u^8+14u^4+1 \text{ all make} \quad \Phi \to \infty.$$

We could continue these observations which indicate, how to choose other equivalent domains $\omega_0^\star\omega_1^\star\omega_2^\star \subset \mathbb{C}_\infty$ for describing our minimal surface, in general. Our essential (and may be new)

Theorem 4.1: *The MEW function $\Phi(u)$ satisfies the functional equation*

$$\Phi\left(\frac{-1}{u}\right) = u^4\Phi(u) \tag{46}$$

if a point reflection belongs to the symmetry group of the minimal surface (the point coincides with the surface).

The proof comes from (43) by the integral transform

$$u' = \frac{-1}{\overline{u}}, \quad du' = \left(\frac{1}{\overline{u}}\right)^2 \cdot d\overline{u}, \quad d\overline{u}' = \left(\frac{1}{u}\right)^2 du \tag{47}$$

in (28) with $v = \overline{u}$ and (29). □

Then we can 'easily' check a well used

Theorem 4.2: *The MEW function for the D minimal surface is (with freedom of sign of the root, see at (30))*

$$\Phi_D(u) = -k_D/\sqrt{u^8+14u^4+1} \tag{48}$$

where $k_D \approx i \cdot 0.593208$ □

The normalized constant k_D is computed by J.SZIRMAI. To this the MEW formulas (30) have to be integrated, first from 0 to $\omega_1 = -(\sqrt{2}-1)$ for the zero z-difference of R_0 and R_1, then from 0 to ω_2, say, for z-difference $\frac{1}{4}$ of points R_0 and R_2.

I saw the SCHWARZ formula (48) at A.L. MACKAY and H. TERRONES [17] in their lecture in Bielefeld (1990) in equivalent form. I would accept any information about the constant k_D with thanks, also for the later constants. The accuracy is questionable yet.

5. THE CUBIC P SURFACE AND THE G (GYROID) SURFACE

The cubic P surface of H.A. SCHWARZ will be conjugate to the D surface (Th. 3.1), the G (gyroid) surface of A. SCHOEN [20] will be associate to both ones.

In Fig. 2a we see the P surface patch spanned by points

$$R_0\left(\frac{1}{2}, 0, \frac{1}{4}\right),\ R_1\left(\frac{1}{2}, y_1, y_1\right),\ R_2\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right). \tag{49}$$

This patch will be continued by the space group $\mathbf{Im\bar{3}m}$ of $\mathbb{E}^3$ (No. 229 by [5]), generated by

$$\begin{aligned} \mathbf{Im\bar{3}m}: \quad & \mathbf{m}_1 : (x, y, z) \mapsto (-x+1, y, z), \\ & \mathbf{m}_2 : (x, y, z) \mapsto (x, z, y), \\ & \mathbf{r} : (x, y, z) \mapsto (-y+\frac{1}{2}, -x+\frac{1}{2}, -z+\frac{1}{2}), \end{aligned} \tag{50}$$

i.e., we have two plane reflections **(m)** and a halfturn **(2)** about R_0R_2. The halfturn $\mathbf{r}$ changes the sides of P surface, both reflections preserve them. The index two subgroup of side preserving transforms in $\mathbf{Im\bar{3}m}$ will be

$$\begin{aligned} & \mathbf{Pm\bar{3}m} \text{ (No. 221), generated by} \\ & \mathbf{m}_1 \text{ and } \mathbf{m}_2 \text{ above and by their } \mathbf{r}\text{-conjugates}, \end{aligned} \tag{51}$$

i.e., in the side planes of tetrahedron $OABC$. In Fig. 2c we see the induced hyperbolic group pair $\star$**2,4,6**/$\star$**2,2,2,3**. The first group is as at the D surface, the second one is generated by 4 line reflections by dihedral angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{3}$ as listed in the symbol. Again, the presentations refer to each other

$$\begin{aligned} \mathbf{Im\bar{3}m} \quad > \quad & (\mathbf{m}_1, \mathbf{m}_2, \mathbf{r} —— 1 = \mathbf{m}_1^2 = \mathbf{m}_2^2 = \\ & = \mathbf{r}^2 = (\mathbf{m}_1\mathbf{m}_2)^2 = (\mathbf{m}_1\mathbf{r})^4 = (\mathbf{m}_2\mathbf{r})^6) =: \star\mathbf{2{,}4{,}6} \end{aligned} \tag{52}$$

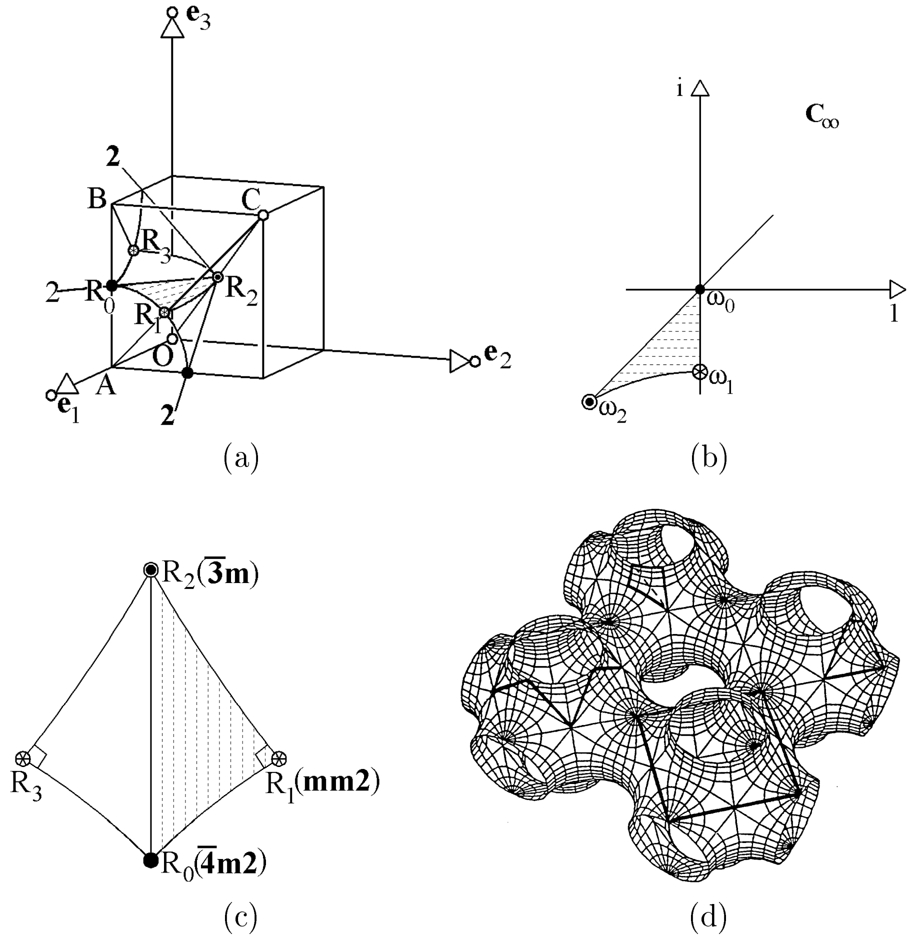


FIGURE 2. H. A. SCHWARZ' cubic P surface: (a) its space group pair $\mathbf{Im\bar{3}m/Pm\bar{3}m}$, (b) its complex domain $\omega_0\omega_1\omega_2$, (c) its hyperbolic surface group pair $\star$**2,4,6**/$\star$**2,2,2,3** in CONWAY's notation, (d) A global picture of cubic P minimal surface by H. KARCHER and K. POLTHIER from [6].

proving the lack of selfintersection. $\mathbf{Im\bar{3}m}$ has one relation more [15]:

$$1 = (\mathbf{m}_1\mathbf{r}\mathbf{m}_2\mathbf{r})^4. \tag{53}$$

Now, we have to adjust the patch as indicated in Fig. 2b by the unit surface normals. Then by the complex parameters, as at (41) before, we have

$$\begin{aligned}
&\omega_0 = 0 \mapsto \mathbf{m}_0(0,0,-1) \text{ to } R_0,\\
&\omega_1 = -i(\sqrt{2}-1) \mapsto \mathbf{m}_1(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) \text{ to } R_1,\\
&\omega_2 = -\frac{1}{2}(\sqrt{3}-1)(1+i) \mapsto \mathbf{m}_2(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}) \text{ to } R_2.
\end{aligned} \tag{54}$$

Now, by Th. 4.1 and arguments there, we formulate

Theorem 5.1: *The MEW function for the cubic P minimal surface is*

$$\Phi_P(u) = -k_P/\sqrt{u^8+14u^4+1} \text{ where } -k_P = 0.463711 \tag{55}$$

$$\text{i.e.} \quad \Phi_P(u) \sim i\Phi_D, \text{ then } y_1 = 0.175091\ldots \text{ to } R_1\left(\frac{1}{2}, y_1, y_1\right). \quad \square$$

The sign $\sim$ means proportionality by a positive constant.

See also a global picture of cubic P surface in Fig. 2d from [6] where other (non-cubic) P-surfaces with fewer symmetries are drawn as well.

Remark: Side reversing transforms might be not required, in general (see Fig. 4b). Then one space group characterizes the situation (not uniquely). For instance **Pmmm** (No. 47 by [5] from the orthorhombic crystal system) allows many minimal surfaces. Then a brick of 6 plane reflections, as a fundamental domain, may contain many hyberbolic rectangular hexagons, each of them as generating surface piece, perpendicular to the brick sides. Further point reflection, reversing the sides of the surface and introducing the pair **Immm/Pmmm**, makes the surface more stabil.

In Figure 3a there is described A. Schoen's G surface, indicated by its fundamental patch which is spanned by points

$$R_0\left(\frac{1}{2}, 0, -\frac{1}{4}\right),\ R_2(0,0,0),\ R_1\left(\frac{1}{4}, y_1, -\frac{1}{2}+y_1\right),\ R_3\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right). \tag{56}$$

This patch will be extended by the space group (see also Fig. 5b) **Ia$\bar{3}$d** (No. 230 by [5]), generated only by two transforms

$$\begin{aligned}
\mathbf{Ia\bar{3}d}: \quad &\mathbf{z}: (x,y,z) \mapsto (y+\frac{1}{2}, -x+\frac{1}{2}, -z-\frac{1}{2}) \text{ at } R_0\\
&\mathbf{r}: (x,y,z) \mapsto (-x+\frac{1}{2}, z+\frac{1}{2}, y-\frac{1}{2}) \text{ at } R_1.
\end{aligned} \tag{57}$$

Thus, we have a rotatory reflection $\mathbf{z}$, reversing the space orientation and the sides of G surface. This is why the two labirinths, bounded by

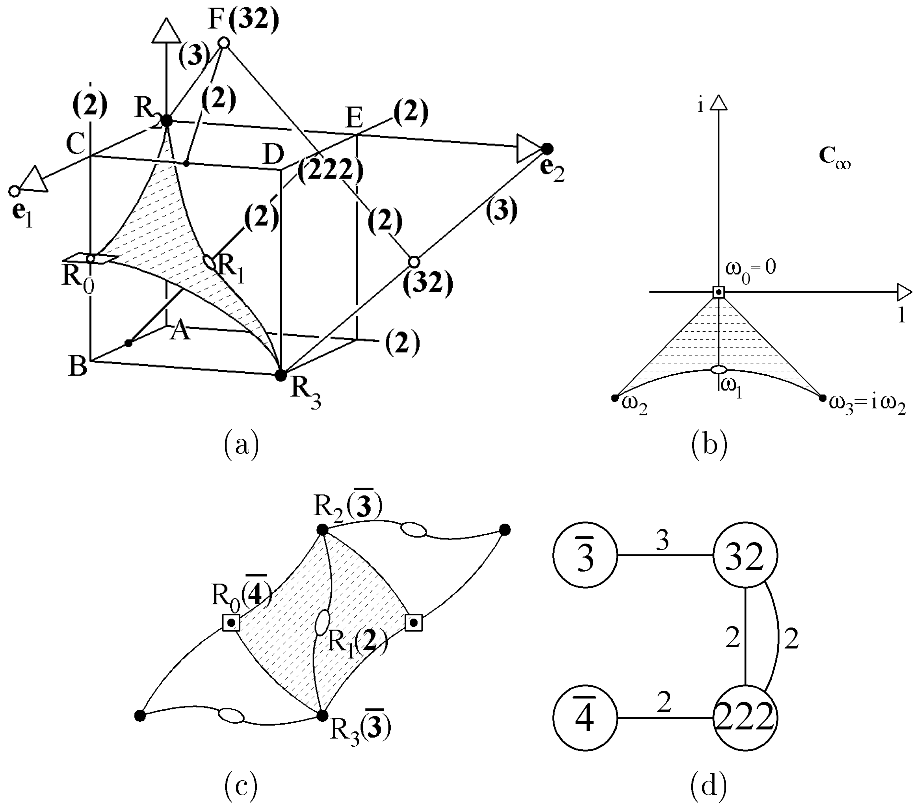


FIGURE 3. A. SCHOEN's G (gyroid) surface: (a) its space group pair $\mathbf{Ia\bar{3}d/I4_132}$, (b) its complex domain $\omega_0\omega_2\omega_1\omega_3$, (c) its hypberbolic surface group pair **2,4,6/2,2,2,3** in CONWAY's notation, the second group **2,2,2,3** preserves the 'colours'. The rotatory reflection ($\bar{\mathbf{4}}$) at R_0, reversing the sides of G surface, induces a 4-rotation on the surface, which changes the colours. The halfturn (**2**) at R_1 keeps the sides (so the colours), its axis is perpendicular to the G surface. (d) A simplified orbifold diagram refers to the orbit types of $\mathbf{Ia\bar{3}d}$, constructed by its generators and fundamental domain $ABCDER_2F$ [3, 21].

the surface, will be *enantiomorphic*, which was a new phenomenon in the topic. The author observed that $\mathbf{Ia\bar{3}d}$ has a *minimal presentation*

$$\begin{aligned}\mathbf{Ia\bar{3}d} \;&:=\; (\mathbf{r},\mathbf{z} \text{ --- } 1=\mathbf{r}^2=\mathbf{z}^4=(\mathbf{zr})^6=(\mathbf{z}^{-1}\mathbf{r}\mathbf{z}^2\mathbf{r}\mathbf{z}\mathbf{r})^2)\\ &> \;(\mathbf{r},\mathbf{z} \text{ --- } 1=\mathbf{r}^2=\mathbf{z}^4=(\mathbf{zr})^6)=\mathbf{2,4,6} \qquad (58)\end{aligned}$$

according to the hyperbolic rotation group above. To derive this we can construct a fundamental domain for $\mathbf{Ia\bar{3}d}$ as the half cube R_2ABCDE and the pyramid FR_2CDE together (Fig. 5). Then the face pairing isometries as generators, by the POINCARÉ algorithm for relations [16], provide a presentation for $\mathbf{Ia\bar{3}d}$. Then we express the other generators by $\mathbf{z}$ and $\mathbf{r}$ and get the simple presentation in (58).

The side preserving transforms in $\mathbf{Ia\bar{3}d}$ form again a subgroup of index two. Now this is the space group $\mathbf{I4_132}$ (No. 214 in [5]), generated by three halfturns

$$\begin{aligned} \mathbf{I4_132}: \quad \mathbf{r} &: (x,y,z) \mapsto (-x+\frac{1}{2}, z+\frac{1}{2}, y-\frac{1}{2}) \text{ above at } R_1, \\ \mathbf{r}_0 &: (x,y,z) \mapsto (-x+1, -y, z) = \mathbf{z}^2 \text{ about } BC, \\ \mathbf{r}^\star &: (x,y,z) \mapsto (-z+\frac{1}{2}, -y+\frac{1}{2}, -x+\frac{1}{2}) = \mathbf{z}^{-1}\mathbf{r}\mathbf{z}. \end{aligned} \tag{59}$$

We can 'easily' present $\mathbf{I4_132}$ as a subgroup of $\mathbf{Ia\bar{3}d}$

$$\begin{aligned} \mathbf{I4_132} \;=\; & (\mathbf{r}, \mathbf{r}_0, \mathbf{r}^\star \text{ —— } 1 = \mathbf{r}^2 = \mathbf{r}_0^2 = \\ & = \mathbf{r}^{\star 2} = (\mathbf{r}_0\mathbf{r}^\star\mathbf{r})^3 = (\mathbf{r}^\star\mathbf{r}_0\mathbf{r}^\star\mathbf{r})^2 = (\mathbf{r}_0\mathbf{r}\mathbf{r}_0\mathbf{r}\mathbf{r}_0\mathbf{r}^\star)^2). \end{aligned} \tag{60}$$

The presentation of **2,2,2,3** is a 'part' of (60) as our Fig. 3c shows. Namely the last two relations are missing.

Again, we adjust the patch $R_0R_2R_1R_3$ by the MEW theory in Fig. 3b. The surface normals involve

$$\begin{aligned} &\omega_0 = 0 \mapsto \mathbf{m}_0(0,0,-1) \text{ to } R_0, \\ &\omega_1 = -i(\sqrt{2}-1) \mapsto \mathbf{m}_1\left(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \text{ to } R_1, \\ &\omega_2 = -\frac{1}{2}(\sqrt{3}-1)(1+i) \mapsto \mathbf{m}_2\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \text{ to } R_2, \\ &\omega_3 = i\omega_2 \mapsto \mathbf{m}_3\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ to } R_3. \end{aligned} \tag{61}$$

Analogously as earlier, we can formulate

Theorem 5.2: *The MEW function for the G (gyroid) surface is (with freedom of sign of the root, see at (30))*

$$\begin{aligned} &\Phi_G(u) = -k_G/\sqrt{u^8+14u^4+1}, \textit{ where} \\ &-k_G = 0.463711 - i\cdot 0.593208, \textit{ and} \\ &y_1 = 0.175091 \textit{ to } R_1 \textit{ in formula (56).} \end{aligned} \tag{62}$$

The G surface is associate to the D and P surfaces.

6. A strategy for classifying all TPMS's, some comments

Alan Schoen [20] was brave enough to overview the classical examples of H. A. Schwarz and his students, and he found also new examples by a systematic method. Hermann Karcher [6] approached to the topic by interpreting Schoen's work with contemporary mathematics, refreshing the classical MEW theory. He collaborated with his colleagues and extended the topic to spherical $\mathbb{S}^3$ and hyperbolic $\mathbb{H}^3$ spaces [10, 18] as well. They combined these with modern discrete computations and computer graphics [19]. All these have been reported e.g. in (the preprint of) [9] where other references are listed, too.

The author met this problem, first, in a lecture of Alan L. Mackay [14] who wrote also other papers, partly with his doctorand H. Terrones [17]. As a 'lucky work', at the change of 1999/2000 my colleague Henrik Farkas (Dept. of Chemical Physics of BUTE) asked me for refereeing two related papers of Paul Gandy and Jacek Klinowski (Dept. of Chemistry, University of Cambridge, they do not know this story yet). They computed and pictured the P and G surfaces by the theory of elliptic integrals. (I hope their work will be published soon with my suggestions). Then I had to refresh my knowledge about the classical theory, crystallography non-euclidean geometry and topology as reported here.

We discussed the topic in our geometry seminars. Jenö Szirmai helped me in some computations. Meanwhile I get the prepint [9] in the framework of our collaboration with TU Berlin.

I am very grateful to all colleagues mentioned above, and to the organizers of this conference mainly to Helmut Pottmann and Hellmuth Stachel for the kind invitation in the framework of our contract between TU's Vienna and Budapest.

Since we are far from a complete classification of TPMS's (without selfintersection and with equivalent two labyrinths), I am optimistic enough to suggest a **Strategy** for this problem:

1. The *principle of classification* is how to form a concept of equivalence. As expressed in this report, a TPMS is characterized by the complete symmetry group of its self isometries, and by the subgroups of side preserving isometries. We may assume that side reversing transform occurs, to guarantee the congruence of the two labyrinths, bounded by TPMS, and so a balance for its stability. Triply periodicity just means the existence of three independent translations. Thus we have a space group pair for each TPMS, say Γ/Γ', where Γ means the complete isometry group, and Γ' is the side preserving subgroup of index two.

a). Since the classification of space groups are by some *geometric equivalence* — called namely, affine *equivariance* — in general, we chose first the group of topological transforms of $\mathbb{E}^3$ to select the so called *maximal classes* of TPMS's, each representing a *family* (see [2] for analogous problem).

For instance, P type mimal surfaces, as to group pairs $\mathbf{Im\bar{3}m}/\mathbf{Pm\bar{3}m}$ (Th. 5.1) and **Immm/Pmmm** (in Remark of Sect. 5), respectively, belong to the same family, and will be represented by the cubic P surface to the first mentioned group pair. Intuitively, the fundamental brick to **Pmmm** with the rectangular hexagon surface, now with a side reversing point reflection in the centre of the brick which extends **Pmmm** to **Immm**, can topologically be deformed to a cube with a rectangular 'regular' hexagon surface, with maximal self-symmetries ($\mathbf{\bar{3}m}$), thus we get $\mathbf{Im\bar{3}m}/\mathbf{Pm\bar{3}m}$. The hyperbolic surface group pair was first $\mathbf{2{\star}2{,}2{,}2/{\star}2{,}2{,}2{,}2{,}2{,}2}$ and then it became to $\mathbf{{\star}2{,}4{,}6/{\star}2{,}2{,}2{,}3}$.

b). Now we are ready to formulate

Definition 6.1: *Two minimal surfaces F_1, F_2 with space group pairs Γ_1/Γ_1' and Γ_2/Γ_2', respectively, belong to the same family, iff there is a topological transform φ of $\mathbb{E}^3$, mapping the first surface onto the second one and the following subgroup relations hold*

$$\varphi^{-1}\Gamma_1\varphi < \Gamma_2, \quad \varphi^{-1}\Gamma_1'\varphi < \Gamma_2'. \tag{63}$$

By words: φ deforms the actions of Γ_1/Γ_1' on F_1 onto the actions of $\Gamma_1^\varphi := \varphi^{-1}\Gamma_1\varphi$ and $\Gamma_1'^\varphi := \varphi^{-1}\Gamma_1'\varphi$ on F_2 so that the subgroup relations in (63) stand, i.e., the two surfaces are homeomorphic but the second one has more symmetries. If there exist φ above, such that equalities stand in (63), then F_1 and F_2 are called equivariant, and they belong to the same (equivariance) class. □

A TPMS of maximal space group pair always exists, e.g., if the automorphism group of the surface can be realised by isometries. Such a realization can be proved (!!!?). Then the 'symmetry breakings' of such a representative surface $F(\Gamma/\Gamma')$ can be produced.

Werner Fischer and Elke Koch [4] indicated some analogous things for cubic space group pairs and surfaces containing certain straight lines. They found new TPMS's and made a mistake as well, being mentioned later.

2. To look for representative surfaces of maximal space group pairs, we consider a 'simple fundamental domain' of each space group, first, for those with higher order point groups [5], e.g., in cubic and hexagonal systems. Then we insert a surface into a fundamental domain,

taking in mind the face paring and the simple requirements in Sect. 2–3. This hyperbolic surface has an induced hyperbolic plane group whose factor group will be the original space group with corresponding generators and more relations. This also holds for the subgroups of index two. This last criterion might exclude many space group pairs. Reflection planes and axes of halfturns always help. If further symmetry occurs, then we examine this richer space group pair with smaller (easier) fundamental domains.
a). We have finitely many combinatorial possibilities for a surface contour in a fundamental domain (i.e., in an orbit space or orbifold $\mathbb{E}^3/\Gamma$ [3, 21]) of a space group Γ. Moreover, such a contour — by its side pairing transforms — must determine a hyperbolic surface group (see e.g. [13], such groups have already been classified by A. M. Macbeath) with a simply connected fundamental surface domain.
b). Any TPMS may have only finitely many super space group pairs which can be read of [5], step by step, not easily. For instance, the cubic P surface with $\mathbf{Im\bar{3}m/Pm\bar{3}m}$ is maximal (as Fig. 2a,c illustrate it), but surprisingly it is not unique, as the surface $C(P)$ shows [4] with H^2 group pair $\star\mathbf{446}/\star\mathbf{2434}$.

3. To a combinatorial contour in a fundamental domain of a maximal space group (consider its pair as well), we determine the fitting minimal surface by the symmetry criteria (as in Sect. 3).
a). The MEW theory gives one possibility by the given surface normals, thus the complex parameters in given points, as we illustrated in this survey. This program needs computer of course, but nowadays it is obvious and hopeful [6].
b). The method of discrete minimal surface gives a newer possibility of realization [19, 9]. This is related to the general J.A.F. Plateau problem (as Lagrange posed, especially, for smooth curve in Sect. 2). The method is illustrated in Fig. 4a. We recall a

Lemma *(from [K-P96]; Balancing Condition). The formula*

$$\frac{\partial}{\partial P}\text{Area (of triangulation)} := \sum_{i=1}^{\substack{\#\text{ neigh-}\\ \text{bours of } P}} (\text{ctg}\alpha_i + \text{ctg}\beta_i)(\mathbf{p} - \mathbf{q}_i) = \mathbf{0} \tag{64}$$

has to be fulfilled by every discrete minimal surface in every point $P(\mathbf{p})$ (meaning that the surface tension balances at P, i.e., the area gradient vanishes in every P, as for a soap film.) □

The point system must be fitted to an appropriate boundary, with increasing number of points.

I think that this method has a good chance in the future (see the nice illustrations of [9], we show only Fig. 4b of them with kind permission of the authors).

Remarks: Our strategy seems to be useful with a lot of technical problems. We mention the following.

1. The criterion of the presentation of space group pair and its hyperbolic plane group pair *excludes e.g. the existence of TPMS by* $\mathbf{Ia\bar{3}}/\mathbf{Pa\bar{3}}$. The space group $\mathbf{Ia\bar{3}}$ (No. 206 in [5]) is presented in Fig. 5a by generators

$$\begin{aligned}\mathbf{Ia\bar{3}}: \quad & \mathbf{r}: AA_1C \mapsto AA_2C \text{ (about } AC),\\ & \mathbf{z}: CA_1A_{12} \mapsto CA_{12}A_2 \text{ (at } A),\\ & \mathbf{h}: AA_1A_{12}A_2 \mapsto A_1AA_2A_{12} \ (B_1\overline{B}_1). \end{aligned} \tag{65}$$

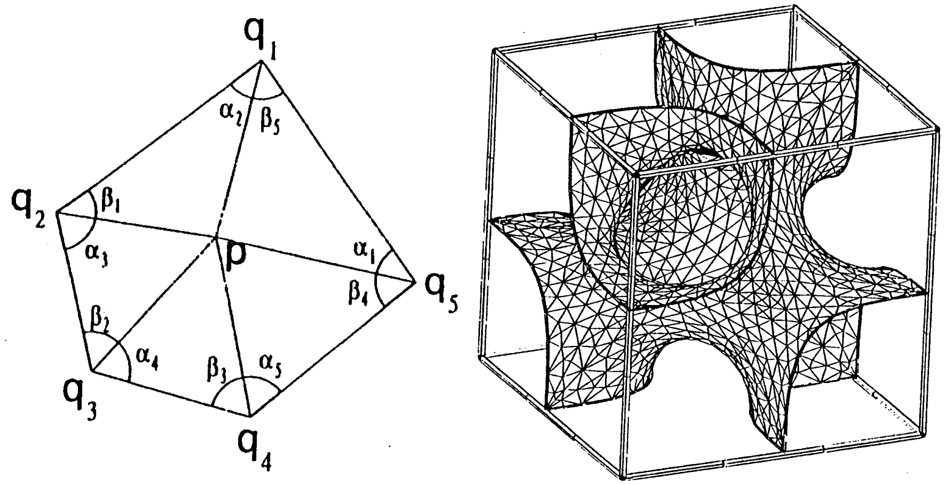
Going around the edge equivalence classes we get the relations by the Poincaré algorithm [16] (see also the hyperbolic generalization $3 \to p$ in [22]):

$$\mathbf{Ia\bar{3}} := (\mathbf{r}, \mathbf{z}, \mathbf{h} \text{ --- } 1 = \mathbf{r}^3 = \mathbf{z}^2\mathbf{r}^{-1} = \mathbf{h}^2 = \mathbf{rhzhz}^{-1}\,\mathbf{hr}^{-1}\mathbf{h}). \tag{66}$$

The subgroup $\mathbf{Pa\bar{3}}$ has a double fundamental domain obtained by the halfturn $\mathbf{h}$ from that of $\mathbf{Ia\bar{3}}$. Besides $\mathbf{r}$ and $\mathbf{z}$ their $\mathbf{h}$-conjugates in $\mathbf{Ia\bar{3}}$, i.e., $\overline{\mathbf{r}} = \mathbf{hrh}$
($\overline{\mathbf{r}}: A_1AD \mapsto A_1A_{12}D4$) and $\mathbf{z} = \mathbf{hzh}$ ($\overline{\mathbf{z}}: DAA_2 \mapsto DA_2A_{12}$)) will be the generators of $\mathbf{Pa\bar{3}}$. We get the presentation:

$$\begin{aligned}&\mathbf{Pa\bar{3}} \text{ (No. 205, in [5])}\\ &= (\mathbf{r}, \mathbf{z}, \overline{\mathbf{r}}, \overline{\mathbf{z}} \text{ --- } 1 = \mathbf{r}^3 = \mathbf{z}^2\mathbf{r}^{-1} = \overline{\mathbf{r}}^3 = \overline{\mathbf{z}}^2\overline{\mathbf{r}}^{-1} = \mathbf{r}\overline{\mathbf{z}}\mathbf{z}^{-1}\overline{\mathbf{r}}^{-1}).\end{aligned} \tag{67}$$

Now, think a simply connected surface in the interior of the bipyramid, so that the halfturn $\mathbf{h}$ transforms it onto, moreover, the surface boundary follows the $\mathbf{Ia\bar{3}}$-paired faces of the bipyramid, similarly as it is indicated by a dotted line in Fig. 5a. We can do this for a while, e.g. $B_1\,X\,B_2$, but after that the surface pairing of $\mathbf{Ia\bar{3}}$ involves more than two boundary lines meeting $\overline{B}_1$ (say, then also $\overline{B}_2$, etc.) on the faces of the bipyramid. Thus we get selfintersection e.g. in $\overline{B}_1$. Other starting (broken) line from B_1, say to Y on AC leads to similar contradiction in few steps. A curve from B_1 to A on the surface of the bipyramid can not be orthogonal to AC, since both angles B_1AC and DAC less than rectangle. Since the rotational order of A is 3 (bigger than 2), this would be a necessary condition for a TPMS through A. We conclude to our



(a) (b)

FIGURE 4. (a) Neighbourhood around a point on a triangulated discrete minimal surface. (b) The F-Rd surface of A. SCHOEN by [9] to the space group $\mathbf{Fm\bar{3}m}$ (No. 225) with non-equivalent labyrinths.

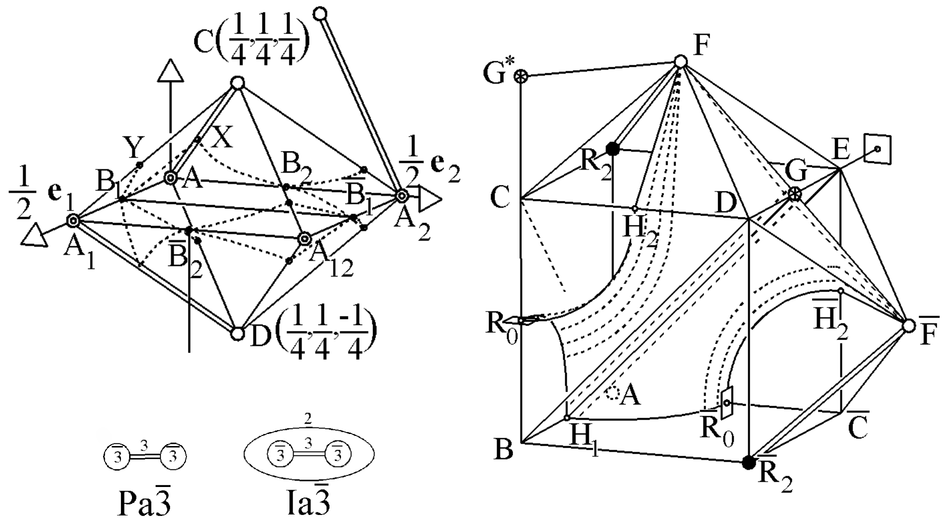
Theorem 6.1: *The space group pair* $\mathbf{Ia\bar{3}/Pa\bar{3}}$ *does not permit any TPMS.* □

This contradicts to [4], however, the authors 'modelled the surface' in their Fig. 5(?). They confess: 'The existence of minimal surfaces with the described properties is very probable but the mathematical proof has still to be done.' This is not possible anymore!
2. I can prove that TPMS to $\mathbf{Ia\bar{3}d/Ia\bar{3}}$ of FISCHER and KOCH [4] does exist, indeed. Here only a
Sketch of the proof. will be described: In Fig. 5b there are pictured the former fundamental domain $ABCDER_2F$ of $\mathbf{Ia\bar{3}d}$ as in Fig. 3a. But we double this domain now by the halfturn $\mathbf{r}$ about H_1G to get a fundamental domain, now for $\mathbf{Ia\bar{3}}$, after appropriate face pairing.

Then we place the surface through the halfturn axes of $\mathbf{Ia\bar{3}d}$, as follows

$$\begin{aligned} &H_1G \text{ for } \mathbf{r}, \quad GF \text{ for } \mathbf{r}_1, \quad FH_2 \text{ for } \mathbf{r}_2 \\ &\text{then through the arcs to } \mathbf{z} : R_0H_1 \mapsto R_0H_2. \end{aligned} \tag{68}$$

These transforms all reverse the side of the surface. Then we apply the hafturn $\mathbf{r}$ to get a domain for side preserving hyperbolic group and a new side pairing for the double polygon and double polyhedron, consequently.



(a) (b)

FIGURE 5. (a) Fundamental domains for space group $\mathbf{Ia\bar{3}}$: the pyramid $CAA_1A_{12}A_2$, and its pair $\mathbf{Pa\bar{3}}$: the bipyramid $CAA_1A_{12}A_2D$ which can not contain a TPMS (without self-intersection), contradicting to [4]. (b) A new possibility of TPMS by $\mathbf{Ia\bar{3}d/Ia\bar{3}}$ as indicated in [4] and interpreted by the author. Very probably it was discovered also by S. LIDIN [12] (1990).

Next we give the exact presentation of $\mathbf{Ia\bar{3}d}$ and its subgroup, now $\mathbf{Ia\bar{3}}$, of index two, as promised. The group $\mathbf{Ia\bar{3}d}$ (No. 230) is generated in Fig. 5b by

$$\begin{aligned}
\mathbf{Ia\bar{3}d}: \quad & \mathbf{r} : ABDE \mapsto BAED \text{ (about } H_1G) \\
& \mathbf{z}_1 : R_0BH_1AC \to R_0CH_2DB \text{ (at } R_0) \\
& \mathbf{z}_2 : R_2CA \mapsto R_2AE \text{ (at } R_2) \\
& \mathbf{r}_2 : FCD \mapsto FDC \; (FH_2) \\
& \mathbf{t} : R_2CF \mapsto R_2EF \; (R_2F) \\
& \mathbf{r}_1 : FDE \mapsto FED \; (GF)
\end{aligned} \tag{69}$$

Now

$$\mathbf{r},\ \mathbf{z}_1 := \mathbf{z},\ \mathbf{r}_2 = \mathbf{z}^{-1}\mathbf{r}\mathbf{z},\ \mathbf{z}_2 = \mathbf{z}\mathbf{r},\ \mathbf{t} = \mathbf{z_2}^2,\ \mathbf{r}_1 = \mathbf{r}_2\mathbf{t} = (\mathbf{z}^{-1}\mathbf{r}\mathbf{z})(\mathbf{z}\mathbf{r})^2 \tag{70}$$

can be defined and all relations can be derived by the angular conditions of H. POINCARÉ [16] Thus, we get a minimal presentation in (58), indeed. But now, according to the surface $H_1GFH_2R_0$ we can derive a hyperbolic plane group **4⋆2,3** as follows

$$\mathbf{Ia\bar{3}d} \quad > \quad (\mathbf{z}, \mathbf{r}, \mathbf{r}_1, \mathbf{r}_2 \text{ --- } 1 = \mathbf{z}^4 = \mathbf{r}^2 = \mathbf{r}_1^2 = \\ = \mathbf{r}_2^2 = \mathbf{z}^{-1}\mathbf{r}\mathbf{z}\mathbf{r}_2 = (\mathbf{r}_2\mathbf{r}_1)^3 = (\mathbf{r}_1\mathbf{r})^2) =: \mathbf{4{\star}2,3} \text{ (Fig. 5b)} \quad (71)$$

That means, the side reversing rotatory reflection **z** induces a 4-rotation in $\mathbb{H}^2$. This closes the boundary component (after ⋆), where dihedral corners $\frac{\pi}{2}$, $\frac{\pi}{3}$ appear. The group $\mathbf{Ia\bar{3}}$ (No. 206) is generated by Fig. 5b

$$\begin{aligned} \mathbf{Ia\bar{3}}: \quad & \mathbf{z}_2 := \mathbf{z}_1\mathbf{r} : R_2CBA \mapsto R_2A\overline{C}E \text{ (at } R_2) \\ & \mathbf{t} : R_2CF \mapsto R_2EF \text{ (about } R_2F) \\ & \mathbf{h} := \mathbf{r}_1\mathbf{r} : DFE\overline{F} \mapsto D\overline{F}EF \ (DE) \\ & \mathbf{s} := \mathbf{r}_2\mathbf{r} : FCD \mapsto \overline{F}E\overline{C} \text{ (screw motion)} \\ & \overline{\mathbf{z}}_2 := \mathbf{r}\mathbf{z}_2\mathbf{r} : \overline{R}_2\overline{C}AB \mapsto \overline{R}_2BCD \text{ (at } \overline{R}_2) \\ & \overline{\mathbf{t}} := \mathbf{r}\mathbf{t}\mathbf{r} : \overline{R}_2\bar{C}\bar{F} \mapsto \overline{R}_2D\overline{F} \ (\overline{R}_2\overline{F}). \end{aligned} \quad (72)$$

Now with

$$\mathbf{z} := \mathbf{z}_2 \text{ and } \mathbf{h},\ \mathbf{t} = \mathbf{z}^2,\ \mathbf{s} = \mathbf{t}\mathbf{h} = \mathbf{z}^2\mathbf{h},\ \overline{\mathbf{z}}_2 = \mathbf{z}\mathbf{h}\mathbf{z}^{-2},\ \overline{\mathbf{t}} = \mathbf{z}\mathbf{h}\mathbf{z}^{-1}\mathbf{h}\mathbf{z}^{-2} \quad (73)$$

we get just the presentation in (66).

For side preserving hyperbolic group we get the surface group with symbol: **2,2,3**⊗ by Fig. 5b [13]. That means, the corresponding orbifold is over the projective plane with cone points as the rotation orders show. The realization of minimal surface $\mathbf{Ia\bar{3}d/Ia\bar{3}}$ can be solved by MEW-theory again by careful computations, analogously as before or by discrete method (J. Szirmai obtained $k = 0.463711 + i \cdot 0.593208$ for the $\Phi(u)$ analogous to (48), but with opposite sign in the z-coordinate). It turns out that this surface is associate to P, D, G surfaces. In [9] there is mentioned and cited [12] that S. Lidin (with his colleagues) has found an associate surface to above ones by numerical method. The name 'lidinoid' has to be changed to FKL surface or, simply to $\mathbf{Ia\bar{3}d/Ia\bar{3}}$ surface. Or to $\mathbf{Ia\bar{3}d/I\bar{4}3d}$ surface as follows?

3. It is very probable now that *the minimal surface to space group pair* $\mathbf{Ia\bar{3}d/I\bar{4}3d}$, discovered and modelled (not proved) by W. Fischer and E. Koch, Fig. 1 in [4], does indeed exist and belong to the above associate company (This is the case with $\mathbf{I4_132/P4_132}$, too.) Our method in Fig. 5b seems to be effective. 'Only' the $\mathbf{Ia\bar{3}d/I\bar{4}3d}$ pair has to be modelled now by another plane intersection through $F\overline{F}H_1$ for the fundamental domain of $\mathbf{Ia\bar{3}d}$, with convenient side face pairing. Then $\mathbf{I\bar{4}3d}$ will have the former double polyhedron as fundamental domain with other face pairing as before. The surface will have the

frame, as indicated in [4] by WYCKOFF positions [5]

$$G(\mathbf{222}) - \mathbf{2} - \overline{F}(\mathbf{32}) - \overline{H}_2(\mathbf{2}) \smile R_2(\overline{\mathbf{3}}) \smile H_1(\mathbf{2}) - G \qquad (74)$$

for $\mathbf{Ia\bar{3}d}$ and a hyperbolic surface $\mathbf{6{\star}2,3}$. The halfturn about H_1G results the side preserving surface group $\mathbf{2,3,3}\otimes$ similarly as before. Then comes the realization by MEW theory or by discrete minimal surface.

We intend to deal with these problems with my colleages A. BÖLCSKEI and J. SZIRMAI and others. I expect for results in the collaboration with geometers in Berlin and Vienna and with colleagues interested in chemistry, crystallography, physics and other sciences.

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