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Towards a Proof of the Chern Conjecture for Isoparametric Hypersurfaces in Spheres

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Abstract: We present the framework and a short history of the Chern conjecture for isoparametric hypersurfaces in spheres and its generalizations. Main results will be presented and we summarize the progress for this topic.

1 Introduction

The Chern conjecture for isoparametric hypersurfaces in spheres can be stated as follows:

Let M be a closed, minimally immersed hypersurface of the $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$ with constant scalar curvature. Then M is isoparametric.

It was originally proposed in a less strong version by Chern in [11] and Chern, do Carmo and Kobayashi in [12], in 1968 and 1970 respectively. So far, no proof for the conjecture has been found, although partial results exist in particular for low dimensions and with additional conditions for the curvature functions of M. We will give an overview of these results and discuss several possible generalizations of the Chern conjecture.

Its original version relates to the following theorem, first proved by Simons [21]:

Theorem 1.1. *Let $M \subset \mathbb{S}^{n+1}$ be a closed, minimally immersed hypersurface and S the squared norm of its second fundamental form. Then*

$$\int_M (S-n)S \geq 0.$$

In particular, for $S \leq n$ one has either $S = 0$ or $S = n$ identically on M.

Note that since M is minimally immersed S is constant if and only if the scalar curvature κ is constant. In this case it follows that $S = 0$ or $S \geq n$, which led Chern to propose the following

Conjecture 1.2. *Consider closed minimal hypersurfaces $M \subset \mathbb{S}^{n+1}$ with constant scalar curvature κ. Then for each n the set of all possible values for κ (or equivalently S) is discrete.*

The only known examples for minimal hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}$ are isoparametric, i.e. all of their principal curvature functions are constant. From the classification of isoparametric hypersurfaces in spheres, given by Münzner in [17], one obtains that S equals $(g-1)n$, where g is the number of pairwise distinct principal curvatures and can only take the values 1, 2, 3, 4 or 6, which establishes the conjecture in this case. Based on this, Verstraelen, Montiel, Ros and Urbano first formulated the stronger version of the conjecture given initially (see [22]). Most of the later results refer to this version.

2 Preliminaries

In the following we present the natural framework in order to attack the problem for the case $n = 4$, since the recent results are given for this dimension and it is obvious how to generalize (or restrict) the equations given below.

Let M^4 be a 4-dimensional hypersurface in a unit sphere $\mathbb{S}^5(1)$. We choose a local orthonormal frame field $\{e_1, \ldots, e_5\}$ in $\mathbb{S}^5(1)$, so that restricted to M^4, $e_1, \ldots, e_4$ are tangent to M^4. Let $\omega_1, \ldots, \omega_5$ denote the dual co-frame field in $\mathbb{S}^5(1)$. We use the following convention for the indices: A, B, C, D range from 1 to 5 and i, j, k from 1 to 4. The structure equations of $\mathbb{S}^5(1)$ as a hypersurface of the Euclidean space $\mathbb{R}^6$ are given by

$$
\begin{aligned}
d\,\omega_A &= -\sum_B \omega_{AB}\wedge\omega_B, \;\; \omega_{AB}+\omega_{BA}=0,\\
d\,\omega_{AB} &= -\sum_C \omega_{AC}\wedge\omega_{CB} + \frac{1}{2}\sum_{C,D}\bar{R}_{ABCD}\omega_C\wedge\omega_D,
\end{aligned}
$$

where $\bar{R}$ is the Riemannian curvature tensor

$$\bar{R}_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}.$$

The contractions $\bar{R}_{AC} = \sum\limits_B \bar{R}_{ABCB}$ and $\bar{R} = \sum\limits_{A,B} \bar{R}_{ABAB}$ are the Ricci curvature tensor and the scalar curvature of $\mathbb{S}^5(1)$, respectively. Next, we restrict all the tensors to M^4. First of all, since $\omega_5 = 0$ on M^4, $\sum\limits_i \omega_{5i}\wedge\omega_i = d\omega_5 = 0$. By Cartan's lemma we can write

$$\omega_{5i} = \sum_j h_{ij}\omega_i, \;\; h_{ij} = h_{ji}. \tag{1}$$

Here $h = \sum\limits_{i,j} h_{ij}\omega_i\omega_j$ denotes the second fundamental form of M^4 and the principal curvatures λ_i are the eigenvalues of the matrix (h_{ij}). Furthermore the mean curvature is given by $H = \frac{1}{4}\sum\limits_i h_{ii} = \frac{1}{4}\sum\limits_i \lambda_i$ and $K = \det(h_{ij}) = \prod\limits_i \lambda_i$ is the Gauß-Kronecker curvature. On M^4 we have

$$
\begin{aligned}
d\omega_i &= -\sum_j \omega_{ij}\wedge\omega_j, \quad \omega_{ij}+\omega_{ji}=0,\\
d\omega_{ij} &= -\sum_k \omega_{ik}\wedge\omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l,
\end{aligned}
$$

where R is the Riemannian curvature tensor on M^4 with components satisfying

$$0 \;=\; R_{ijkl} + R_{ijlk}.$$

These structure equations imply the following integrability condition (Gauß equation):

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}).$$

For the scalar curvature we have

$$\kappa = 12 + 16H^2 - S,$$

where $S = \sum_{i,j} h_{ij}^2$ is the squared norm of h.

If we consider minimal hypersurfaces, the Ricci curvature and scalar curvature are given by, respectively,

$$R_{ij} = 3\delta_{ij} - \sum_k h_{ik}h_{jk}, \tag{2}$$

$$\kappa = 12 - S. \tag{3}$$

It follows from (3) that κ is constant if and only if S is constant. The covariant derivative ∇h with components h_{ijk} is given by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{jk}\omega_{ik} + \sum_k h_{ik}\omega_{jk}. \tag{4}$$

Then the exterior derivative of (2) together with the structure equations yields the following Codazzi equation

$$h_{ijk} = h_{ikj} = h_{jik}. \tag{5}$$

For any fixed point on M^4, we can choose a local orthonormal frame $\{e_1, \ldots, e_4\}$, such that

$$h_{ij} = \lambda_i \delta_{ij}.$$

We define the symmetric functions f_3 and f_4 on M^4 as follows:

$$f_3 := \sum_{i,j,k} h_{ij}h_{jk}h_{ki} = \sum_i \lambda_i^3, \quad f_4 := \sum_{i,j,k,l} h_{ij}h_{jk}h_{kl}h_{li} = \sum_i \lambda_i^4. \tag{6}$$

3 Results

The trivial case is given for $n = 2$. Here (under the premises of the conjecture)

$$\lambda_1 + \lambda_2 = 0, \tag{7}$$

$$\lambda_1^2 + \lambda_2^2 = const \tag{8}$$

and we have $\lambda_1 = -\lambda_2 = const$.

The first partial result was achieved by Peng and Terng, who gave further constraints for the possible values of S:

Theorem 3.1 (PENG, TERNG 1983 [18]). *For every $n \geq 3$ there exists a maximal $C(n)$ with the following property: Let $M \subset \mathbb{S}^{n+1}$ be a closed minimal hypersurface with constant $S > n$. Then it follows that $S \geq n + C(n)$ and one has $C(3) = 3$, $C(n) \geq \frac{1}{12n}$.*

Since for isoparametric hypersurfaces the next highest possible value for S is $2n$, they in particular proposed the following

Conjecture 3.2. *$C(n) \geq n$.*

The originally shown inequality has since been improved considerably by Yang and Cheng ([23],[24],[25]) to $C(n) \geq \frac{26}{61}n - \frac{16}{61} > \frac{1}{3}n$ and, under the additional assumption that the sum of cubes of the principal curvatures f_3 is constant, $C(n) \geq \frac{13}{15}n - \frac{4}{5} \geq \frac{2}{3}n$.

The lowest dimension for which the Chern conjecture is non-trivial is $n = 3$. In this case, a more general theorem has been proven:

Theorem 3.3 (ALMEIDA, BRITO 1990 [3]; CHANG 1993 [7]). *Let $M \subset \mathbb{S}^4$ be a closed hypersurface with constant mean curvature H and constant scalar curvature κ. Then M is isoparametric.*

Almeida and Brito initially showed this in [3] under the additional assumption that κ is non-negative. The approach of this proof has since been used to show a number of other results (see below), and can be sketched as follows: Let Y be the set of points where all principal curvatures are distinct. It is easy to see that it is sufficient to proof that the principal curvatures are constant on Y. One defines a three-form ψ on Y depending on the principal curvature directions, which satisfies $d\psi = F\mathrm{vol}$ for a non-negative function F. Using Stokes' theorem and an estimate on the boundary of Y, one obtains $F = 0$ from which the claim follows directly.

Chang then completed the proof in [7] by showing that κ is non-negative under these assumptions. He proved this separately for manifolds with three everywhere distinct principal curvatures and those where two principal curvatures coincide in a point, in the former case generalizing a proof earlier given by Peng and Terng in [19] for minimal hypersurfaces.

Instead of manifolds in low dimensions, one can also consider those with a certain number g of pairwise different principal curvatures. Again, $g = 3$ is the first non-trivial case, and one has the following result:

Theorem 3.4 (Chang 1994 [9])**.** *Let $M \subset \mathbb{S}^{n+1}$ be a closed hypersurface with constant mean and scalar curvatures which has exactly three pairwise distinct principal curvatures in every point. Then M is isoparametric.*

For the case $n = 4$ a partial result has recently been proven under the additional assumption that M is a Willmore hypersurface, i.e. a critical point of the Willmore functional $W(M) := \int_M \rho^n$ with $\rho^2 = S - nH^2$. For minimal hypersurfaces with constant scalar curvature in spheres this has been shown by Li in [14] to be equivalent to $f_3 = 0$. This is the case for most minimal isoparametric hypersurfaces of $\mathbb{S}^5$, which motivates the assumption. One has:

Theorem 3.5 (Lusala, Scherfner, Sousa Jr. 2005 [16])**.** *Let $M \subset \mathbb{S}^5$ be a closed minimal Willmore hypersurface with constant non-negative scalar curvature. Then M is isoparametric.*

The proof follows essentially the same approach as that of [3].

In fact, [16] claims that this is true even in the case of negative scalar curvature. However, the proof as given there contains an incorrect step; namely, an integral estimate is made to show that on the set Y of points with four distinct principal curvatures one has $\kappa = 0$. For this it is claimed that a certain integral term goes to zero in the limit, which is not generally the case.

4 Generalizations

One obvious generalization is that on non-closed manifolds, i.e. a local version of the conjecture. This has in particular been proposed by Bryant for the case $n = 3$:

Conjecture 4.1. *Let $M \subset \mathbb{S}^4$ be a minimal hypersurface with constant scalar curvature. Then M is isoparametric.*

The following is a result of the proof of the (global) Chern conjecture in this case:

Theorem 4.2 (CHANG 1993 [8]). *Let $M \subset \mathbb{S}^4$ be a minimal Hypersurface with constant scalar curvature such that there is a point $p \in M$ in which two principal curvatures coincide. Then M is isoparametric.*

Another possible generalization is that on hypersurfaces of constant mean curvature. It has already been mentioned that the proofs for the cases $n = 3$ and $g = 3$ remain valid under this more general assumption. One also has the following inequalities in analogy to Theorem 1.1:

Theorem 4.3 (ALENCAR, DO CARMO 1994 [1]). *There exist continous positive functions B_n with $B_n(0) = n$ and the following property:*

Let $M \subset \mathbb{S}^{n+1}$ be a closed hypersurface with constant mean curvature H. If

$$\widetilde{S} := \sum_i (\lambda_i - H)^2 \leq B_n(H),$$

then it follows that $\widetilde{S} = 0$ or $\widetilde{S} = B_n(H)$ identically on M.

Theorem 4.4 (HOU 1997 [13]). *Let $M \subset \mathbb{S}^{n+1}$ be a closed hypersurface with constant mean curvature. If $S < 2\sqrt{n-1}$, then M is a hypersphere.*

Note that in both cases there exist isoparametric hypersurfaces for which the upper bounds are assumed, such that the inequalities are sharp.

A number of the results mentioned in section 3 can also be generalized to hypersurfaces in Riemannian manifolds R^{n+1} of constant curvature $c \leq 0$; for details see the second table in section 5 (note that it is sufficient to consider $c \in \{-1, 0\}$).

One can also more generally ask what can be said about manifolds with other combinations of constant curvature functions. For this one defines the r-th mean curvature (or mean curvature of order r) σ_r as

$$\sigma_r := \binom{n}{r}^{-1} \sum_{i_1 < \ldots < i_r} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_r},$$

that is up to a factor as the r-th elementary symmetric polynomial of the principal curvatures. Note that σ_1 equals the mean curvature H, σ_2 equals the scalar curvature κ up to additive and multiplicative constants ($\kappa = n(n-1)(1+\sigma_2)$) and σ_n equals the Gauß-Kronecker curvature K.

The Chern conjecture can now be stated as follows: If $\sigma_1 = 0$ and σ_2 are constant, then so are all other σ_r. This suggests the question for which other combinations of indices such a statement is true. For hypersurfaces immersed in certain manifolds there exists the following remarkable result:

Theorem 4.5 (Bivens 1983 [6])**.** *Let M be a compact hypersurface in $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$ or the open half-sphere $\mathbb{S}^{n+1}_+$. If for some $1 \leq r < n$ the two mean curvature functions σ_r and σ_{r+1} are constant, then M is a geodesic hypersphere (and thus isoparametric).*

For hypersurfaces in $\mathbb{S}^4$, Almeida and Brito proved the following using a similar approach to that in [3]:

Theorem 4.6 (Almeida, Brito 1997 [4])**.** *Let $M \subset \mathbb{S}^4$ be a closed hypersurface with mean curvature $H = \sigma_1$, scalar curvature $\kappa = 12(1+\sigma_2)$ and Gauß-Kronecker curvature $K = \sigma_3$.*

If κ and K (or equivalently σ_2 and σ_3) are constant with $\kappa \geq 0$ ($\sigma_2 \geq -1$), then M is isoparametric.

If H und $K \neq 0$ (or equivalently σ_1 and $\sigma_3 \neq 0$) are constant with $HK^{-1} \geq -1$ ($\sigma_1\sigma_3^{-1} \geq -1$), then M is also isoparametric.

In [5] Almeida, Brito and Sousa recently claimed that this is the case even without assuming the inequalities. However, the proof given there uses the same incorrect argument as [16] (see above).

Lusala and Oliveira showed in [15] that if H and $K = 0$ are constant, H is also zero. In this case there exist non-isoparametric examples ([2], see also [20]).

5 Summary

The table given below recapitulates the dimensions and additional conditions for which the Chern conjecture and its generalizations for hypersurfaces of constant mean curvature and non-closed hypersurfaces have been proven.

In the following, let $\widetilde{S}$ and B_n be defined as in theorem 4.3 and let g be the number of pairwise distinct principal curvatures as a function on M.

n	Chern Conjecture	Chern Conjecture ($H \neq 0$)	Chern Conjecture (locally)
2	Yes	Yes	Yes
3	Yes, [19], [8]	Yes, [3], [7]	If $S \leq 3$, [12] or $g = 2$ in p, [8]
4	If $f_3 \equiv 0$, $S \leq 12$, [16] or f_3 const., $S < \frac{20}{3}$, [25] or $S < \frac{372}{61}$, [25] or $g \equiv 3$, [9]	If $\widetilde{S} \leq B_4(H)$, [1] or $S \leq 2\sqrt{3}$, [13] or $g \equiv 3$, [9] or $M \subset \mathbb{S}^5_+$, [6]	If $S \leq 4$, [12]
> 4	If f_3 const., $S < \frac{28}{15}n - \frac{4}{5}$, [25] or $S < \frac{97}{61}n - \frac{16}{61}$, [25] or $g \equiv 3$,[9]	If $\widetilde{S} \leq B_n(H)$, [1] or $S \leq 2\sqrt{n-1}$, [13] or $g \equiv 3$, [9] or $M \subset \mathbb{S}^{n+1}_+$, [6]	If $S \leq n$, [12]

For closed hypersurfaces in a Riemannian manifold R^{n+1} of constant curvature c which have constant mean and scalar curvature, the equivalent of the Chern conjecture can be proved in the following cases:

n	$c=0$	$R^{n+1}=\mathbb{R}^{n+1}$	$c=-1$	$R^{n+1}=\mathbb{H}^{n+1}$
2	Yes*	Yes*	Yes*	Yes*
3	Yes** (analogous to [10])	Yes** ([6],[10])	If $S \leq 9H^2-6$ ([3])	Yes ([6])
>3	If $g \equiv 3$ (analogous to [9])	Yes ([6])	If $g \equiv 3$ (analogous to [9])	Yes ([6])

$*$: Also true locally $**$: Also true for complete hypersurfaces.

Finally, the following table gives the results that are known for closed hypersurfaces in $\mathbb{S}^4$ with two constant mean curvature functions.

Constant σ_r		Result
σ_1	σ_2	isoparametric ([3],[7])
σ_1	$\sigma_3 \neq 0$	isoparametric if $\sigma_1\sigma_3^{-1} \geq -1$ ([4])
$\sigma_1 \neq 0$	$\sigma_3 = 0$	does not occur ([15])
$\sigma_1 = 0$	$\sigma_3 = 0$	non-isoparametric examples ([2], [20])
σ_2	σ_3	isoparametric if $\sigma_2 \geq -1$ ([4])

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