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TU Wien

Differential Geometry Seminar

TU Wien, May 6, 2020

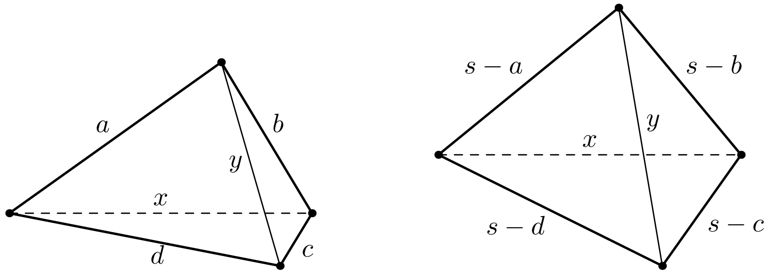
Regge symmetries: part one

Let Δ be a tetrahedron in $\mathbb{R}^3$, $\mathbb{S}^3$, or $\mathbb{H}^3$ with edge lengths a, b, c, d, x, y as shown on the left. Put

$$s = \frac{a+b+c+d}{2}.$$

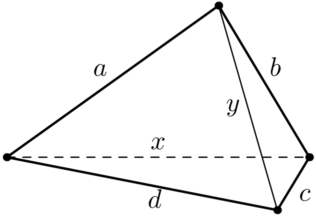
Then there is $\overline{\Delta}$ with edge lengths $s-a, s-b, s-c, s-d, x, y$, and:

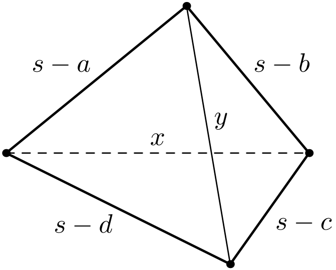
- Tetrahedra Δ and $\overline{\Delta}$ have equal volumes.



Regge symmetries: part two

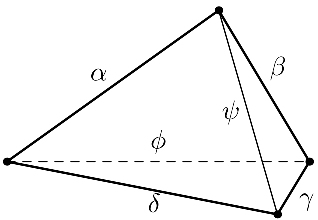
- The dihedral angles at the x-edges in Δ and $\bar{\Delta}$ are equal. The same holds for the angles at the y-edges.

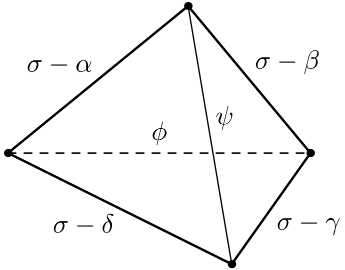




Regge symmetries: part two

- The dihedral angles at the *x*-edges in Δ and $\bar{\Delta}$ are equal. The same holds for the angles at the *y*-edges.
- If α, β, γ, δ are the dihedral angles at the edges *a*, *b*, *c*, *d* of Δ, then the angles at the edges $s - a$, $s - b$, $s - c$, $s - d$ of $\bar{\Delta}$ are equal to $\sigma - \alpha$, $\sigma - \beta$, $\sigma - \gamma$, $\sigma - \delta$, where $\sigma = \dfrac{\alpha + \beta + \gamma + \delta}{2}$.





Where does it come from

- Ponzano, Regge, *Semiclassical limit of Racah coefficients*, 1968.
- Roberts, *Classical 6j-symbol and the tetrahedron*, 1999.
- Taylor, Woodward, *6j-symbols for* $U_q(\mathfrak{sl}_2)$ *and non-Euclidean tetrahedra*, 2005.

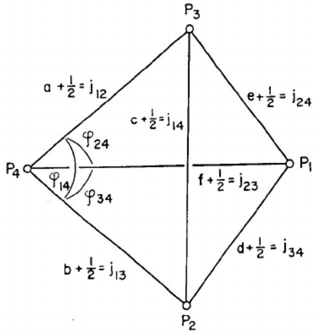


Fig. 1. Three-dimensional representation of the 6j-symbol $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$.

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$6j$-symbols are related to angular momenta in quantum mechanics.

Their asymptotics produces volumes and angles in a tetrahedron.

Symmetries of $6j$-symbols $\Rightarrow$ Regge symmetries for tetrahedra.

Scissors congruence

Regge symmetries imply that Δ and $\overline{\Delta}$ have equal Dehn invariants

$$\sum_i \ell_i \otimes \theta_i.$$

In $\mathbb{R}^3$ this implies that Δ and $\overline{\Delta}$ are scissors congruent,
that is can be glued from the same set of pieces.

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It is known in the hyperbolic case (makes use of ideal polyhedra):

- Mohanty, *The Regge symmetry is a scissors congruence in hyperbolic space*, 2003.

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Potentially, by cut-and-paste one could also prove the angle relations...

In this talk, I will present an elementary proof of Regge symmetries:

- Akopyan, Izmestiev, *The Regge symmetry, confocal conics, and the Schläfli formula*, 2019.

Volumes of Euclidean tetrahedra

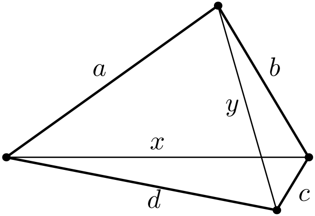
Brute-force computation of the Cayley-Menger determinant:

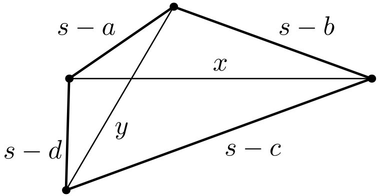
$$\mathrm{vol}(\Delta)^2 = \frac{1}{288}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & y^2 \\ 1 & a^2 & 0 & x^2 & d^2 \\ 1 & b^2 & x^2 & 0 & c^2 \\ 1 & y^2 & d^2 & c^2 & 0 \end{vmatrix}$$

$$= \frac{1}{288}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (s-a)^2 & (s-b)^2 & y^2 \\ 1 & (s-a)^2 & 0 & x^2 & (s-d)^2 \\ 1 & (s-b)^2 & x^2 & 0 & (s-c)^2 \\ 1 & y^2 & (s-d)^2 & (s-c)^2 & 0 \end{vmatrix} = \mathrm{vol}(\overline{\Delta})^2$$

The degenerate case

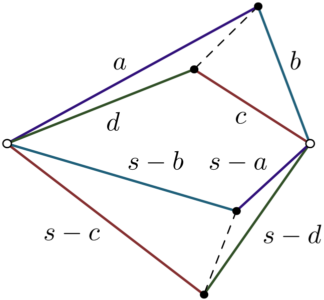
If (a, b, c, d)-quadrilateral and $(s - a, s - b, s - c, s - d)$-quadrilateral have equal x-diagonals, then they have equal y-diagonals.
(Volumes vanish, angles $\in \{0, \pi\}$.)





The degenerate case: proof

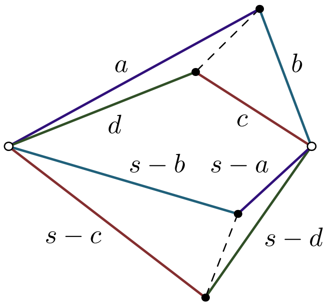
That is, the dashed segments have equal length.



The degenerate case: proof

$$a + b = (s - c) + (s - d),\ c + d = (s - a) + (s - b),$$
$$a - b = (s - b) - (s - a),\ c - d = (s - d) - (s - c)$$

Get two ellipses and two hyperbolas with the same foci.



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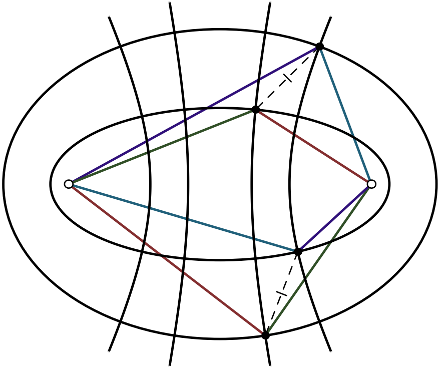
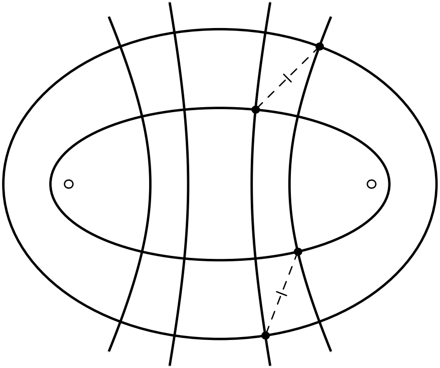


The degenerate case: proof

But this is the Ivory theorem:
a curved quadrilateral between two confocal ellipses and hyperbolas has diagonals of equal lengths.



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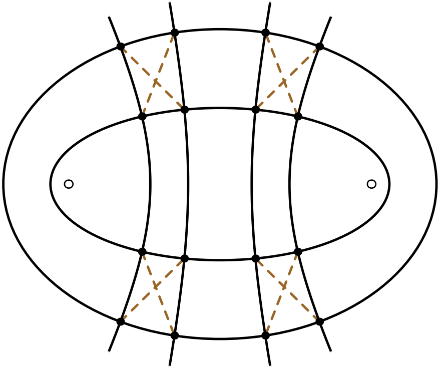
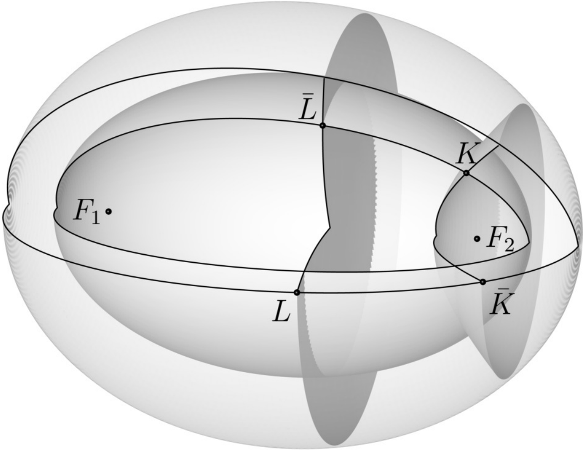


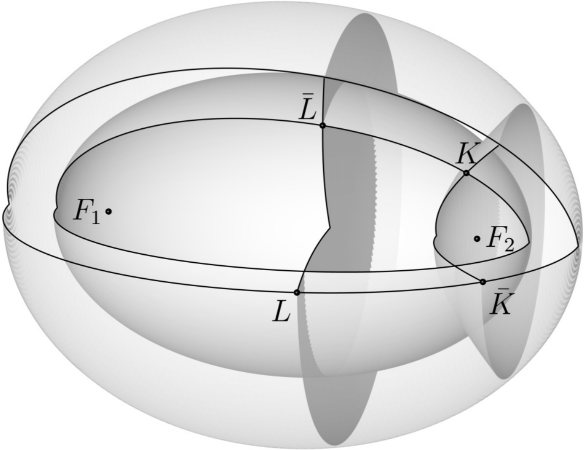
Dihedral angles at the *x*- and *y*-edges

Ivory's theorem holds for confocal quadrics in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$.
The confocal family can degenerate to prolate spheroids, hyperboloids of rotation, planes through the axis. Thus one has $KL = \overline{KL}$.



Dihedral angles at the x- and y-edges

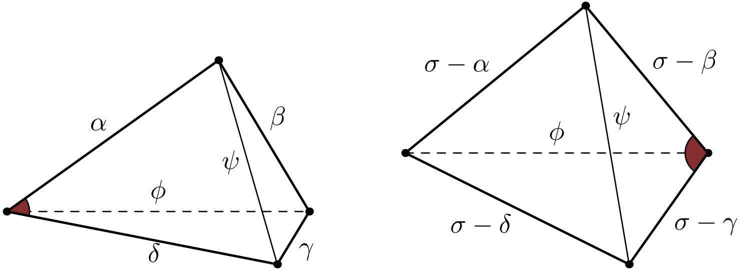
Thus if $\Delta = F_1 F_2 KL$ with $|F_1 F_2| = x$, then $F_1 F_2 \overline{KL} = \overline{\Delta}$.
The angles at x in Δ and $\overline{\Delta}$ are equal by construction.
Similarly, the angles at the y-edges are equal.



Other dihedral angles

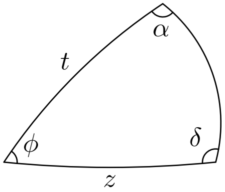
Relations between other dihedral angles are equivalent to:

- $\alpha + \delta = \overline{\beta} + \overline{\gamma},\ \alpha - \delta = \overline{\delta} - \overline{\alpha}$ etc.



In particular, the solid angles in the figure are equal.

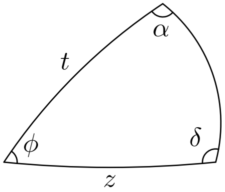
On the area of a spherical triangle



For a triangle with fixed angle ϕ:

$$\text{Area} \quad \text{is a function of} \quad \tan\frac{z}{2}\tan\frac{t}{2}$$

On the area of a spherical triangle



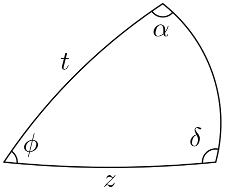
For a triangle with fixed angle ϕ:

$$\text{Area} \quad \text{is a function of} \quad \tan\frac{z}{2}\tan\frac{t}{2}$$

The actual formula is:

$$\tan\frac{\text{Area}}{2} = \frac{\tan\frac{z}{2}\tan\frac{t}{2}\sin\phi}{1+\tan\frac{z}{2}\tan\frac{t}{2}\cos\phi}$$

Sums and differences of angles

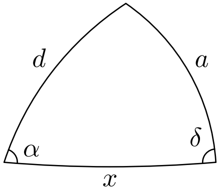


For a triangle with fixed angle ϕ:

$$\alpha + \delta \quad \text{is a function of} \quad \tan\frac{z}{2}\tan\frac{t}{2}$$

$$\alpha - \delta \quad \text{is a function of} \quad \frac{\tan\frac{z}{2}}{\tan\frac{t}{2}}$$

Dual version

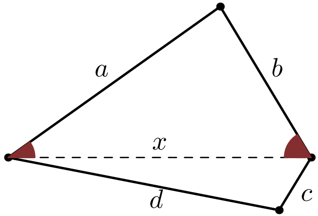


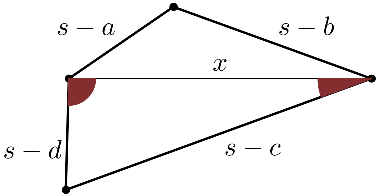
For a $\mathbb{R}^2, \mathbb{S}^2$ or $\mathbb{H}^2$ triangle with fixed side x:

$$a + d \quad \text{is a function of} \quad \tan\frac{\alpha}{2}\tan\frac{\delta}{2}$$

$$a - d \quad \text{is a function of} \quad \frac{\tan\frac{\alpha}{2}}{\tan\frac{\delta}{2}}$$

Proof of the angle relations





$$
\begin{aligned}
\measuredangle \cdot \reflectbox{$\measuredangle$} &= \overline{\scalebox{1}[-1]{$\measuredangle$}} \cdot \overline{\rotatebox[origin=c]{180}{$\measuredangle$}} \\
\measuredangle / \reflectbox{$\measuredangle$} &= \overline{\reflectbox{$\measuredangle$}} / \overline{\measuredangle} \\
\scalebox{1}[-1]{$\measuredangle$} \cdot \rotatebox[origin=c]{180}{$\measuredangle$} &= \overline{\measuredangle} \cdot \overline{\reflectbox{$\measuredangle$}} \\
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\end{aligned}
\quad \Rightarrow \quad
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\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\alpha + \delta &= \overline{\beta} + \overline{\gamma} \\
\alpha - \delta &= \overline{\delta} - \overline{\alpha}
\end{aligned}
$$

(Here $\measuredangle$ etc. denote the tangent of the corresponding half-angle.)

Proof of $\mathrm{vol}(\Delta) = \mathrm{vol}(\overline{\Delta})$ in $\mathbb{S}^3$ and $\mathbb{H}^3$

Theorem (Schläfli)

For every smooth deformation of a tetrahedron in $\mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$ one has

$$2K \cdot d\,\mathrm{vol} = \sum_{i=1}^{6} \ell_i d\theta_i.$$

$K \in \{0, 1, -1\}$*: the curvature,* ℓ_i*: edge lengths,* θ_i*: dihedral angles.*

During any deformation of Δ the volume derivatives are equal:

$$\frac{d}{dt}\mathrm{vol}(\Delta_t) = \frac{1}{2K}(a\dot{\alpha} + b\dot{\beta} + c\dot{\gamma} + d\dot{\delta} + x\dot{\phi} + y\dot{\psi})$$

$$= \frac{1}{2K}\Big((s-a)(\dot{\sigma} - \dot{\alpha}) + \cdots + (s-d)(\dot{\sigma} - \dot{\delta}) + x\dot{\phi} + y\dot{\psi}\Big) = \frac{d}{dt}\mathrm{vol}(\overline{\Delta_t})$$

If Δ flattens in the end, then $\overline{\Delta}$ flattens as well $\Rightarrow \mathrm{vol}(\Delta) = \mathrm{vol}(\overline{\Delta})$.