A NECESSARY FLEXIBILITY CONDITION OF A NONDEGENERATE SUSPENSION IN LOBACHEVSKY 3-SPACE

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Abstract

We show that some combination of the lengths of all edges of the equator of a flexible suspension in Lobachevsky 3-space is equal to zero (each length is taken either positive or negative in this combination).

Keywords

flexible polyhedron, Lobachevsky space, hyperbolic space, flexible suspension, Connelly method, equator of suspension, length of edge.

1 Introduction

A polyhedron (more precisely, a polyhedral surface) is said to be flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e., if every face remains congruent to itself during the flex.

In 1897 R. Bricard [1] described all flexible octahedra in Euclidean 3-space. The Bricard's octahedra were the first examples of flexible polyhedra (with self-intersections). BricardTs octahedra are special cases of Euclidean flexible suspensions. In 1974 R. Connelly [2] proved that some combination of the lengths of all edges of the equator of a flexible suspension in Euclidean 3-space is equal to zero (each length is taken either positive or negative in this combination). The method applied by R. Connelly, is to reduce the problem to the study of an analytic function of complex variable in neighborhoods of its singular points.

In 2001 S. N. Mikhalev [3] reproved the above-mentioned result of R. Connelly by algebraic methods. Moreover, S. N. Mikhalev proved that for every spatial quadrilateral formed by edges of a flexible suspension and containing its both poles there is a combination of the lengths (taken either positive or negative) of the edges of the quadrilateral, which is equal to zero.

The aim of this work is to prove a similar result for the equator of a flexible suspension in Lobachevsky 3-space, applying the method of Connelly [2].

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2 Formulating the main result

Let $\mathcal{K}$ be a simplicial complex. A *polyhedron* (a *polyhedral surface*) in Lobachevsky 3-space is a continuous map from $\mathcal{K}$ to $\mathbb{H}^3$, which sends every k-dimensional simplex of $\mathcal{K}$ into a subset of a k-dimensional plane of Lobachevsky space ($k \leq 2$). Images of topological 2-simplices are called faces, images of topological 1-simplices are called edges and images of topological 0-simplices are called vertices of the polyhedron. Note that in our definition an image of a simplex can be degenerate (for instance, a face can lie on a straight hyperbolic line, and an edge can be reduced to one point), and faces can intersect in their interior points. If $v_1, ..., v_W$ are the vertices of $\mathcal{K}$, and if $\mathcal{P} : \mathcal{K} \to \mathbb{H}^3$ is a polyhedron, then $\mathcal{P}$ is determined by W points $P_1, ..., P_W \in \mathbb{H}^3$, where $P_j \stackrel{\text{def}}{=} \mathcal{P}(v_j)$, $j = 1, ..., W$.

If $\mathcal{P} : \mathcal{K} \to \mathbb{H}^3$ and $\mathcal{Q} : \mathcal{K} \to \mathbb{H}^3$ are two polyhedra, then we say $\mathcal{P}$ and $\mathcal{Q}$ are *congruent* if there exists a motion $\mathcal{A} : \mathbb{H}^3 \to \mathbb{H}^3$ such that $\mathcal{Q} = \mathcal{A} \circ \mathcal{P}$ (i.e. the isometric mapping $\mathcal{A}$ sends every vertex of $\mathcal{P}$ into a corresponding vertex of $\mathcal{Q}$: $Q_j = \mathcal{A}(P_j)$, or in other words $\mathcal{Q}(v_j) = \mathcal{A}(\mathcal{P}(v_j))$, $j = 1, ..., W$). We say $\mathcal{P}$ and $\mathcal{Q}$ are *isometric* (*in the intrinsic metric*) if each edge of $\mathcal{P}$ has the same length as the corresponding edge of $\mathcal{Q}$, i.e. if $\langle v_j, v_k \rangle$ is a 1-simplex of $\mathcal{K}$ then $\mathrm{d}_{\mathbb{H}^3}(Q_j, Q_k) = \mathrm{d}_{\mathbb{H}^3}(P_j, P_k)$, where $\mathrm{d}_{\mathbb{H}^3}(\cdot, \cdot)$ stands for the distance in Lobachevsky space $\mathbb{H}^3$.

A polyhedron $\mathcal{P}$ is *flexible* if, for some continuous one parameter family of polyhedra $\mathcal{P}_t : \mathcal{K} \to \mathbb{H}^3$, $0 \leq t \leq 1$, the following three conditions hold true: (1) $\mathcal{P}_0 = \mathcal{P}$; (2) each $\mathcal{P}_t$ is isometric to $\mathcal{P}_0$; (3) some $\mathcal{P}_t$ is not congruent to $\mathcal{P}_0$.

Let $\mathcal{K}$ be defined as follows: $\mathcal{K}$ has vertices $v_0, v_1, ..., v_V, v_{V+1}$, where $v_1, ..., v_V$ form a cycle (v_j adjacent to v_{j+1}, $j = 1, ..., V-1$, and v_V adjacent to v_1), and v_0 and v_{V+1} are each adjacent to all of $v_1, ..., v_V$. Each polyhedron $\mathcal{P}$ based on $\mathcal{K}$ is called a *suspension*. Call $N \stackrel{\text{def}}{=} \mathcal{P}(v_0)$ the north pole, and $S \stackrel{\text{def}}{=} \mathcal{P}(v_{V+1})$ the south pole, and $P_j \stackrel{\text{def}}{=} \mathcal{P}(v_j)$, $j = 1, ..., V$ vertices of the *equator* $\mathcal{P}$.

Assume that a suspension $\mathcal{P}$ is flexible. If we suppose the segment NS to be an extra edge, then $\mathcal{P}$ becomes a set of V tetrahedra glued cyclically along their common edge NS. We call a suspension *nondegenerate* if none of these tetrahedra lies on a hyperbolic 2-plane. Note that a nondegenerate suspension $\mathcal{P}$ does not flex if the distance between N and S remains constant. Therefore, as in the Euclidean case [2] we assume that the length of NS is variable during the flex of $\mathcal{P}$. Examples of degenerate suspensions are a double covered cap — a suspension with coinciding poles (see Fig. 1), and a suspension with a wing — a suspension whose vertices N, S, P_{i-1}, and P_{i+1} lie on a straight line for some i (see Fig. 2). In this paper we will not study the degenerate flexible suspensions.

The main result of the paper is

Theorem 1 *Let $\mathcal{P}$ be a nondegenerate flexible suspension in Lobachevsky 3-space with the poles S and N, and with the vertices of the equator P_j, $j = 1, ..., V$. Then for some set of signs $\sigma_{j,j+1} \in \{+1, -1\}$, $j = 1, ..., V$, the combination of the lengths $e_{j,j+1}$ of all edges P_jP_{j+1} of the equator of $\mathcal{P}$ taken with the corresponding signs $\sigma_{j,j+1}$ is equal to*

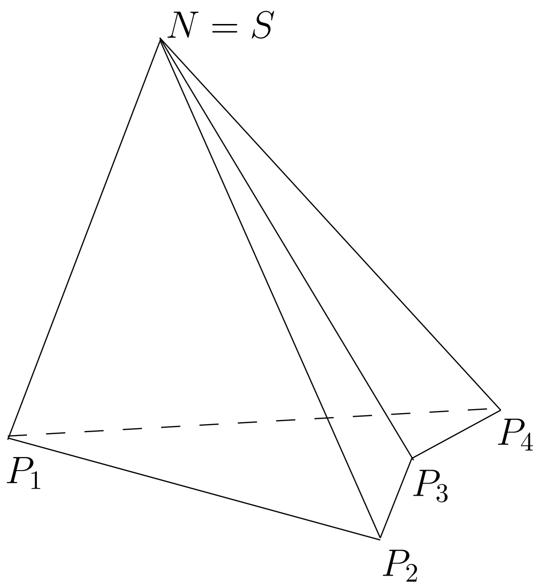

Figure 1: A double covered cap.

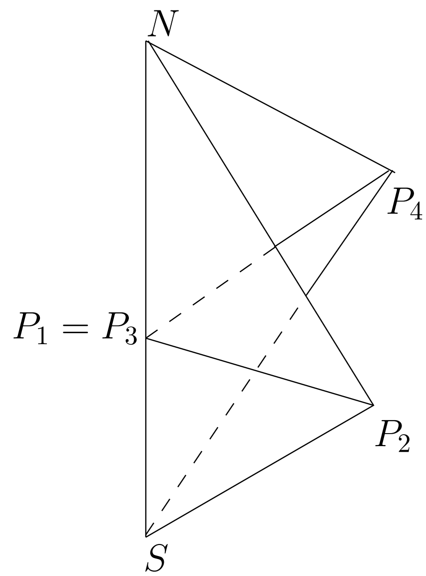

Figure 2: A suspension with a wing.

zero, i.e.

$$\sum_{j=1}^{V} \sigma_{j,j+1} e_{j,j+1} = 0. \tag{1}$$

(*Here and below, by definition, it is considered that* $P_{V+1} \stackrel{\text{def}}{=} P_1$, $P_V P_{V+1} \stackrel{\text{def}}{=} P_V P_1$, $\sigma_{V,V+1} \stackrel{\text{def}}{=} \sigma_{V,1}$, *and* $e_{V,V+1} \stackrel{\text{def}}{=} e_{V,1}$.)

3 Connelly's equation of flexibility of a suspension

R. Connelly in [2] obtained an equation of flexibility of a nondegenerate suspension in Euclidean 3-space. Following him, in this section we will obtain an equation of flexibility of a nondegenerate suspension in Lobachevsky 3-space.

Let us place a nondegenerate suspension $\mathcal{P}$ into the Poincaré upper half-space model [4] of Lobachevsky 3-space $\mathbb{H}^3$ in such a way that the poles N and S of $\mathcal{P}$ lie on the axis Oz of the Cartesian coordinate system of the Poincaré model (see Fig. 3). Let S has the coordinates $(0,0,z_S)$, N has the coordinates $(0,0,z_N)$, and P_j has the coordinates (x_j, y_j, z_j), $j = 1, ..., V$. Also we denote the length of the edge NP_j by e_j, and the length of SP_j by e'_j, $j = 1, ..., V$.

Consider a Euclidean orthogonal projection $\widetilde{\mathcal{P}}$ of $\mathcal{P}$ on the plane Oxy (see Fig. 4). Also $\widetilde{\mathcal{P}}$ is a hyperbolic projection of $\mathcal{P}$ on Oxy from the only point at infinity of $\mathbb{H}^3$ which does not lie on Oxy. This projection sends poles N and S of $\mathcal{P}$ to the origin O $(0,0)$ on the plane Oxy, P_j to the point $\widetilde{P}_j$ (x_j, y_j), edges NP_j and SP_j to the Euclidean segment $O\widetilde{P}_j$, and the egde $P_j P_{j+1}$ of the equator of $\mathcal{P}$ to the Euclidean segment $\widetilde{P}_j \widetilde{P}_{j+1}$, $j = 1, ..., V$ (here and below $\widetilde{P}_{V+1} \stackrel{\text{def}}{=} \widetilde{P}_1$, $x_{V+1} \stackrel{\text{def}}{=} x_1$, $y_{V+1} \stackrel{\text{def}}{=} y_1$, $z_{V+1} \stackrel{\text{def}}{=} z_1$).

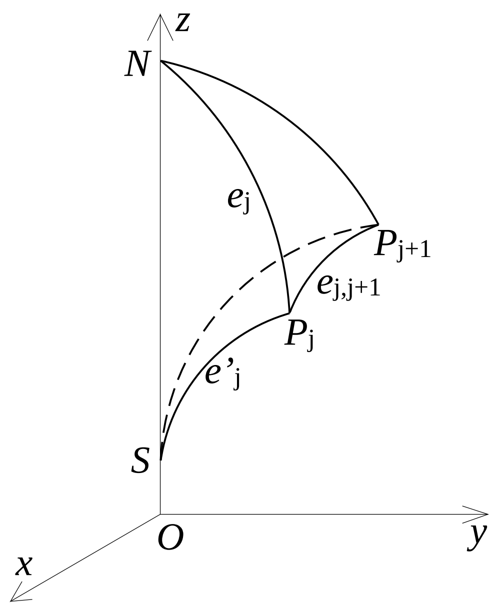

Figure 3: A fragment of the lateral surface of $\mathcal{P}$.

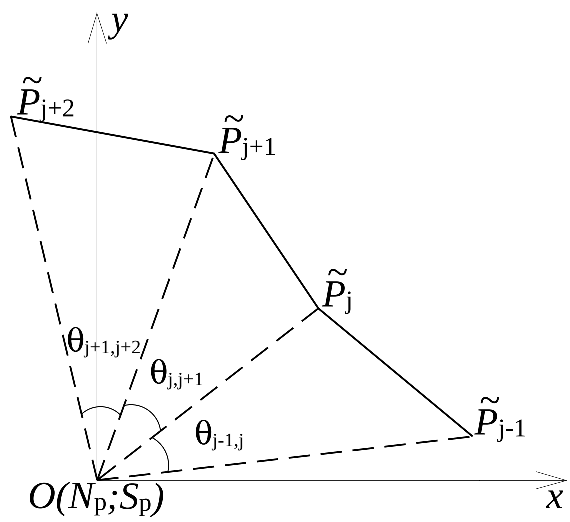

Figure 4: A projection of $\mathcal{P}$ on Oxy .

Polar coordinates (ρ_j, θ_j) of $\widetilde{P}_j$, $j = 1, ..., V$, are related to its Cartesian coordinates by the formulas (see Fig. 5):

$$\rho_j = \sqrt{x_j^2 + y_j^2}, \quad \sin\theta_j = \frac{y_j}{\rho_j} = \frac{y_j}{\sqrt{x_j^2 + y_j^2}}, \quad \cos\theta_j = \frac{x_j}{\rho_j} = \frac{x_j}{\sqrt{x_j^2 + y_j^2}}. \tag{2}$$

Note that by construction, the dihedral angle $\theta_{j,j+1}$ of the tetrahedron NSP_jP_{j+1} at the edge NS is equal to the flat angle $\angle \widetilde{P}_j O \widetilde{P}_{j+1}$, $j = 1, ..., V$, and

$$\theta_{j,j+1} = \theta_{j+1} - \theta_j. \tag{3}$$

Note as well that the value of $\theta_{j,j+1}$ can be negative. Applying the trigonometric ratio of the difference of two angles and (3), we get:

$$\cos\theta_{j,j+1} = \cos\theta_{j+1}\cos\theta_j + \sin\theta_{j+1}\sin\theta_j, \quad \sin\theta_{j,j+1} = \sin\theta_{j+1}\cos\theta_j - \cos\theta_{j+1}\sin\theta_j. \tag{4}$$

Taking into account (2) we reduce (4) to

$$\cos\theta_{j,j+1} = \frac{x_j x_{j+1} + y_j y_{j+1}}{\sqrt{x_{j+1}^2 + y_{j+1}^2}\sqrt{x_j^2 + y_j^2}}, \quad \sin\theta_{j,j+1} = \frac{x_j y_{j+1} - y_j x_{j+1}}{\sqrt{x_{j+1}^2 + y_{j+1}^2}\sqrt{x_j^2 + y_j^2}}.$$

Then, according to Euler's formula,

$$e^{i\theta_{j,j+1}} = \cos\theta_{j,j+1} + i\sin\theta_{j,j+1} = \frac{(x_j x_{j+1} + y_j y_{j+1}) + i(x_j y_{j+1} - y_j x_{j+1})}{\sqrt{x_{j+1}^2 + y_{j+1}^2}\sqrt{x_j^2 + y_j^2}}. \tag{5}$$

Following R. Connelly [2], we remark that the sum of the dihedral angles $\theta_{j,j+1}$ of all tetrahedra NSP_jP_{j+1}, $j = 1, ..., V$, at the edge NS is constant and a multiple of 2π (here

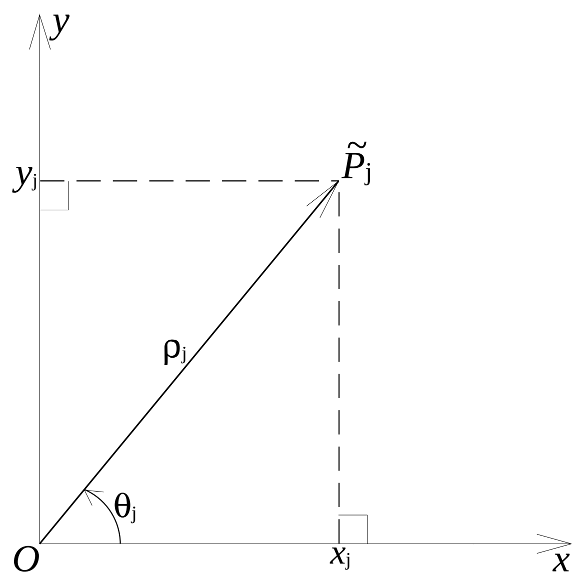

Figure 5: The coordinates of $\widetilde{P}_j$.

and below $\theta_{V,V+1} \stackrel{\text{def}}{=} \theta_{V,1}$, $\theta_{V+1} \stackrel{\text{def}}{=} \theta_1$, $\rho_{V+1} \stackrel{\text{def}}{=} \rho_1$), i.e.

$$\sum_{j=1}^{V} \theta_{j,j+1} = 2\pi m \quad \text{for some integer } m, \tag{6}$$

and remains so during the deformation of the suspension, when the values of the angles $\theta_{j,j+1}$, $j = 1, ..., V$, vary continuously.

We rewrite the equation of flexibility (6) in a convenient form:

$$\prod_{j=1}^{V} e^{i\theta_{j,j+1}} = 1. \tag{7}$$

Thus, taking into account (5), we see that coordinates of vertices of $\mathcal{P}$ are related as follows:

$$\prod_{j=1}^{V} \frac{(x_j x_{j+1} + y_j y_{j+1}) + i(x_j y_{j+1} - y_j x_{j+1})}{x_j^2 + y_j^2} = 1, \tag{8}$$

or in other notations

$$\prod_{j=1}^{V} F_{j,j+1} = \prod_{j=1}^{V} \frac{G_{j,j+1}}{\rho_j \rho_{j+1}} = \prod_{j=1}^{V} \frac{G_{j,j+1}}{\rho_j^2} = 1, \tag{9}$$

where $G_{j,m} = (x_j x_m + y_j y_m) + i(x_j y_m - y_j x_m)$,$F_{j,m} = \frac{G_{j,m}}{\rho_j \rho_m}$, $j, m = 1, ..., V$, and $G_{V,V+1} \stackrel{\text{def}}{=} G_{V,1}$, $F_{V,V+1} \stackrel{\text{def}}{=} F_{V,1}$.

When studying the deformation $\mathcal{P}_t$ of the suspension $\mathcal{P}$, all objects and values related to $\mathcal{P}_t$ naturally succeed from the notations for the corresponding entities related to $\mathcal{P}$. For example, the coordinate $x_j(t)$ of the point $P_j(t)$ of the deformation $\mathcal{P}_t$ corresponds to the coordinate x_j of the point P_j of the suspension $\mathcal{P}$, the dihedral angle $\theta_{j,j+1}(t)$ of the tetrahedron $N(t)S(t)P_j(t)P_{j+1}(t)$ at the edge $N(t)S(t)$ corresponds to the dihedral angle $\theta_{j,j+1}$ of the tetrahedron NSP_jP_{j+1} at the edge NS, etc.

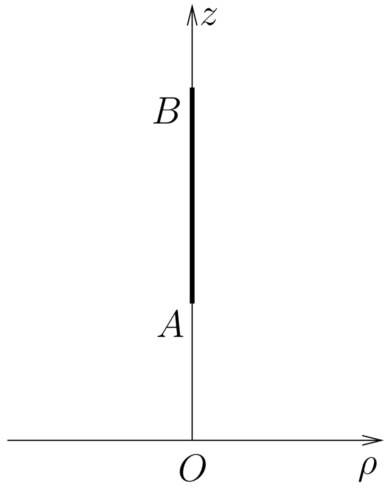

Figure 6: Points on a plane in the lemma 1.

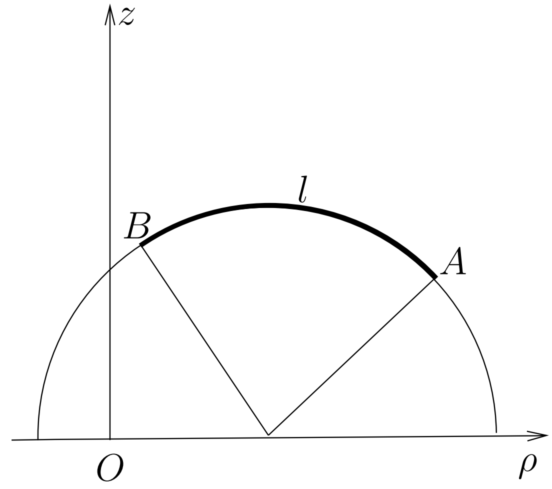

Figure 7: Points on a plane in the lemma 2.

4 The equation of flexibility of a suspension in terms of the lengths of its edges

In this section we are going to express the equation of flexibility of a suspension (8) in terms of the lengths of edges of $\mathcal{P}$. Recall that the lengths of the edges of $\mathcal{P}$ remain constant during the flex. To this purpose we need to demonstrate the truth of two following statements. The first of them can be verified by direct calculation (see also Fig. 6).

Lemma 1 *Given a Poincaré upper half-plane $\mathbb{H}^2$ with the coordinates (ρ, z) (i.e., with the metric given by the formula $ds^2 = \frac{d\rho^2+dz^2}{z^2}$). Then the distance between the points A (ρ_0, z_A) and B (ρ_0, z_B), having the same first coordinate ρ_0, is calculated by the formula*

$$\mathrm{d}_{\mathbb{H}^2}(A,B) = \left|\ln \frac{z_B}{z_A}\right|. \tag{10}$$

Lemma 2 *Given a Poincaré upper half-plane $\mathbb{H}^2$ with the coordinates (ρ, z) (i.e., with the metric given by the formula $ds^2 = \frac{d\rho^2+dz^2}{z^2}$). Then the distance $l \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(A,B)$ between the points A (ρ_A, z_A) and B (ρ_B, z_B) is related to their coordinates by the formula*

$$(\rho_B - \rho_A)^2 + z_A^2 + z_B^2 = 2 z_A z_B \cosh l. \tag{11}$$

Proof. According to the part (2) of the Corollary $A.5.8$ [5], the distance between the points with the coordinates (x,t) and (y,s) in the Poincaré upper half-space model $\mathbb{R}^n \times \mathbb{R}^+$ of Lobachevsky $(n+1)$-space $\mathbb{H}^{n+1}$ is calculated by the formula

$$\mathrm{d}_{\mathbb{H}^{n+1}}((x,t),(y,s)) = 2\,\mathrm{artanh}\left(\frac{\|x-y\|^2+(t-s)^2}{\|x-y\|^2+(t+s)^2}\right)^{1/2}, \tag{12}$$

where the symbol $\|\cdot\|$ stands for the standard Euclidean norm in $\mathbb{R}^n$.

By (12) the distance between the points A and B (see Fig. 7) is calculated by the formula

$$l = 2\,\mathrm{artanh}\left(\frac{(\rho_A-\rho_B)^2+(z_A-z_B)^2}{(\rho_A-\rho_B)^2+(z_A+z_B)^2}\right)^{1/2}, \tag{13}$$

where $n=1$, $(x,t)=(\rho_A,z_A)$ and $(y,s)=(\rho_B,z_B)$.

After a series of transformations of the formula (13) we get:

$$(\rho_A-\rho_B)^2\left(\cosh^2\frac{l}{2}-\sinh^2\frac{l}{2}\right)+(z_A^2+z_B^2)\left(\cosh^2\frac{l}{2}-\sinh^2\frac{l}{2}\right) = 2z_Az_B\left(\cosh^2\frac{l}{2}+\sinh^2\frac{l}{2}\right). \tag{14}$$

By two identities of hyperbolic geometry, $\cosh^2\frac{l}{2}-\sinh^2\frac{l}{2}=1$ and $\cosh l=\cosh^2\frac{l}{2}+\sinh^2\frac{l}{2}$, (14) reduces to (11). $\square$

Let us express $G_{j,j+1}$ and ρ_j^2 in terms of the length of edges of $\mathcal{P}$.

We assume that the coordinates of the south pole S are $(0,0,1)$. Let $t \stackrel{\text{def}}{=} e^{\mathrm{d}_{\mathbb{H}^3}(N,S)}$, where $\mathrm{d}_{\mathbb{H}^3}(N,S)$ is the distance between the poles N and S of $\mathcal{P}$. Without loss of generality, we assume that $z_N \geq z_S$. Then, by Lemma 1, the coordinates of N are $(0,0,t)$.

Applying Lemma 2 to the points S and P_j lying on the hyperbolic plane SNP_j, by the formula (11) we get:

$$\rho_j^2+z_j^2+1 = 2z_j\cosh e_j'. \tag{15}$$

Now we apply Lemma 2 to the vertices N and P_j:

$$\rho_j^2+z_j^2+t^2 = 2tz_j\cosh e_j. \tag{16}$$

Subtracting (15) from (16), under the assumption that $t\cosh e_j \neq \cosh e_j'$, we get:

$$z_j = \frac{t^2-1}{2(t\cosh e_j-\cosh e_j')}. \tag{17}$$

Also, taking into account (15) and (17), we obtain:

$$\rho_j^2 = 2z_j\cosh e_j' - z_j^2 - 1 = \frac{(t^2-1)\cosh e_j'}{(t\cosh e_j-\cosh e_j')} - \frac{(t^2-1)^2}{4(t\cosh e_j-\cosh e_j')^2} - 1. \tag{18}$$

Let $\rho_{j,j+1}$ denote the Euclidean distance between the points $\widetilde{P}_j$ and $\widetilde{P}_{j+1}$, $j=1,...,V$ (here and below $\rho_{V,V+1} \stackrel{\text{def}}{=} \rho_{V,1}$). Applying Lemma 2 to the vertices P_j and P_{j+1}, we get:

$$\rho_{j,j+1}^2 = 2z_jz_{j+1}\cosh e_{j,j+1} - z_j^2 - z_{j+1}^2. \tag{19}$$

By the Pythagorean theorem $\rho_{j,j+1}$ is related to the Cartesian coordinates of $\widetilde{P}_j$ and $\widetilde{P}_{j+1}$ by the formula

$$\rho_{j,j+1} = \sqrt{(x_{j+1}-x_j)^2+(y_{j+1}-y_j)^2}. \tag{20}$$

By (2) the equation (20) reduces to:

$$\rho_{j,j+1}^2 = (x_j^2+y_j^2)+(x_{j+1}^2+y_{j+1}^2)-2(x_jx_{j+1}+y_jy_{j+1}) = \rho_j^2+\rho_{j+1}^2-2(x_jx_{j+1}+y_jy_{j+1}).$$

Thus, taking into account (18) and (19), the expression $x_jx_{j+1}+y_jy_{j+1}$, which is a part of $G_{j,j+1}$ from (9), is related to the lengths of edges of $\mathcal{P}$ by the formula

$$x_jx_{j+1}+y_jy_{j+1}=\frac{\rho_j^2+\rho_{j+1}^2-\rho_{j,j+1}^2}{2}=z_j\cosh e'_j+z_{j+1}\cosh e'_{j+1}-z_jz_{j+1}\cosh e_{j,j+1}-1. \tag{21}$$

Substituting (17) in (21) we get:

$$x_jx_{j+1}+y_jy_{j+1}=\frac{1}{2}\Bigg(\frac{(t^2-1)\cosh e'_j}{(t\cosh e_j-\cosh e'_j)}+\frac{(t^2-1)\cosh e'_{j+1}}{(t\cosh e_{j+1}-\cosh e'_{j+1})}-$$

$$-\frac{(t^2-1)^2\cosh e_{j,j+1}}{2(t\cosh e_j-\cosh e'_j)(t\cosh e_{j+1}-\cosh e'_{j+1})}-2\Bigg). \tag{22}$$

Let us now express $x_jy_{j+1}-y_jx_{j+1}$, which is also a part of $G_{j,j+1}$, in terms of the length of edges of $\mathcal{P}$.

According to (5) we know that

$$\cos\theta_{j,j+1}=\frac{x_jx_{j+1}+y_jy_{j+1}}{\rho_j\rho_{j+1}}\quad\text{and}\quad\sin\theta_{j,j+1}=\frac{x_jy_{j+1}-y_jx_{j+1}}{\rho_j\rho_{j+1}}. \tag{23}$$

Note that by definition (2), $\rho_j>0$, $j=1,...,V$.

By the Pythagorean trigonometric identity, the formula

$$\sin\theta_{j,j+1}=\sigma_{j,j+1}\sqrt{1-\cos^2\theta_{j,j+1}} \tag{24}$$

holds true, where $\sigma_{j,j+1}=1$ if $\sin\theta_{j,j+1}\geq 0$, and $\sigma_{j,j+1}=-1$ if $\sin\theta_{j,j+1}<0$ (remind that $\theta_{j,j+1}$ is determined in (3)). Then (23) and (24) imply

$$x_jy_{j+1}-y_jx_{j+1}=\rho_j\rho_{j+1}\sin\theta_{j,j+1}=\sigma_{j,j+1}\rho_j\rho_{j+1}\sqrt{1-\cos^2\theta_{j,j+1}}=$$

$$=\sigma_{j,j+1}\rho_j\rho_{j+1}\sqrt{1-\frac{(x_jx_{j+1}+y_jy_{j+1})^2}{\rho_j^2\rho_{j+1}^2}}=\sigma_{j,j+1}\sqrt{\rho_j^2\rho_{j+1}^2-(x_jx_{j+1}+y_jy_{j+1})^2}. \tag{25}$$

Substituting (18) and (22) in (25) we get

$$x_jy_{j+1}-y_jx_{j+1}=\sigma_{j,j+1}\Bigg[\Bigg(\frac{(t^2-1)\cosh e'_j}{(t\cosh e_j-\cosh e'_j)}-\frac{(t^2-1)^2}{4(t\cosh e_j-\cosh e'_j)^2}-1\Bigg)\times$$

$$\times\Bigg(\frac{(t^2-1)\cosh e'_{j+1}}{(t\cosh e_{j+1}-\cosh e'_{j+1})}-\frac{(t^2-1)^2}{4(t\cosh e_{j+1}-\cosh e'_{j+1})^2}-1\Bigg)-\frac{1}{4}\Bigg(\frac{(t^2-1)\cosh e'_j}{(t\cosh e_j-\cosh e'_j)}+$$

$$+\frac{(t^2-1)\cosh e'_{j+1}}{(t\cosh e_{j+1}-\cosh e'_{j+1})}-\frac{(t^2-1)^2\cosh e_{j,j+1}}{2(t\cosh e_j-\cosh e'_j)(t\cosh e_{j+1}-\cosh e'_{j+1})}-2\Bigg)^2\Bigg]^{\frac{1}{2}}. \tag{26}$$

Substituting (18), (22), and (26) in (8) we obtain the equation of flexibility of a suspension in terms of the lengths of edges of $\mathcal{P}$.

5 Proof of the theorem

In order to prove the theorem 1 we shall study singular points of the equation of flexibility of a suspension.

Assume that a nondegenerate suspension $\mathcal{P}$ flexes. Then, as we have already mentioned in the section 2, the distance l_{NS} between the poles of $\mathcal{P}$ changes during the flex. Let $t \stackrel{\text{def}}{=} e^{l_{NS}}$ be the parameter of the flex of $\mathcal{P}$. The identity (9) holds true at every moment t of the flex, as the values of the expressions $F_{j,j+1}$, $G_{j,j+1}$, ρ_j^2, $j = 1, ..., V$, which make part (9), vary as t changes. Here the functions $G_{j,j+1}(t) = [x_j x_{j+1} + y_j y_{j+1}](t) + i[x_j y_{j+1} - y_j x_{j+1}](t)$ and $\rho_j^2(t)$, $j = 1, ..., V$, are determined in (18), (22) and (26).

Assume now that for some $j \in \{1, ..., V\}$ the dihedral angle $\theta_{j,j+1}(t)$ remains constant (the value of $\theta_{j,j+1}(t)$ can also be equal to zero) as t changes. In this case the length of the edge $N(t)S(t)$ of the tetrahedron $N(t)S(t)P_j(t)P_{j+1}(t)$ must be constant as well (all other edges of the tetrahedron are also the edges of $\mathcal{P}_t$, therefore there lengths are fixed), i.e. the value of t does not change. As we mentioned in the section 2, in this case $\mathcal{P}$ can not be flexible. Thus we have the contradiction. Therefore, the values of the angles $\theta_{j,j+1}(t)$, $j = 1, ..., V$, change continuously during the flex. Hence, there exists such an interval (t_1, t_2) that for all $t \in (t_1, t_2)$ it is true that $\theta_{j,j+1}(t) \neq 0$ for every $j \in \{1, ..., V\}$.

We extend both sides of the equation of flexibility (9) as functions in t on the whole complex plane $\mathbb{C}$. By the theorem on the uniqueness of the analytic function [6], the expression (9) remains valid.

Analytic functions $F_{j,j+1}(t)$, $j = 1, ..., V$, have a finite number of algebraic singular points. Without loss of generality we can assume that none of these points lies in the interval (t_1, t_2). For every $F_{j,j+1}(t)$, $j = 1, ..., V$, we choose a single-valued branch $(F_{j,j+1}(t), D)$, where $D \subset \mathbb{C}$ is an unbounded domain containing (t_1, t_2). Let $\mathcal{W} \subset D$ be a path connecting $t_0 \in (t_1, t_2)$ and ∞, such that t_0 is a unique real point of $\mathcal{W}$. Let us calculate the limit of $F_{j,j+1}(t)$ as $t \to \infty$ along $\mathcal{W}$.

Taking into account (18) we get

$$\lim_{t\to\infty} \frac{\rho_j^2(t)}{t^2} = \lim_{t\to\infty} \Big[\frac{1}{t^2}\Big(\frac{(t^2-1)\cosh e_j'}{(t\cosh e_j - \cosh e_j')} - \frac{(t^2-1)^2}{4(t\cosh e_j - \cosh e_j')^2} - 1\Big)\Big] = -\frac{1}{4\cosh^2 e_j}. \tag{27}$$

Similarly, from (22) we derive that

$$\lim_{t\to\infty} \frac{(x_j x_{j+1} + y_j y_{j+1})(t)}{t^2} = -\frac{\cosh e_{j,j+1}}{4\cosh e_j \cosh e_{j+1}}. \tag{28}$$

Also from (25) and taking into account (27) and (28) we have:

$$\lim_{t\to\infty} \frac{(x_j y_{j+1} - y_j x_{j+1})^2(t)}{t^4} = \lim_{t\to\infty} \left[\frac{\rho_j^2(t)\rho_{j+1}^2(t) - (x_j x_{j+1} + y_j y_{j+1})^2(t)}{t^4}\right] =$$

$$= \frac{1}{16\cosh^2 e_j \cosh^2 e_{j+1}} - \frac{\cosh^2 e_{j,j+1}}{16\cosh^2 e_j \cosh^2 e_{j+1}} = \frac{1-\cosh^2 e_{j,j+1}}{16\cosh^2 e_j \cosh^2 e_{j+1}}.$$

Hence,

$$\lim_{t\to\infty}\frac{(x_j y_{j+1}-y_j x_{j+1})(t)}{t^2}=i\sigma_{j,j+1}\frac{\sqrt{\cosh^2 e_{j,j+1}-1}}{4\cosh e_j\cosh e_{j+1}},\qquad(29)$$

where $\sigma_{j,j+1}\in\{+1,-1\}$ is determined by the single-valued branch $(F_{j,j+1}(t),D)$ and by the path $\mathcal{W}$.

By definition of $G_{j,j+1}(t)$ and according to (28) and (29), we get:

$$\lim_{t\to\infty}\frac{G_{j,j+1}(t)}{t^2}=-\frac{\cosh e_{j,j+1}+\sigma_{j,j+1}\sqrt{\cosh^2 e_{j,j+1}-1}}{4\cosh e_j\cosh e_{j+1}}.\qquad(30)$$

By (30) and (27), the limit of the left-hand side of (9) at $t\to\infty$

$$\lim_{t\to\infty}\prod_{j=1}^{V}F_{j,j+1}(t)=\lim_{t\to\infty}\prod_{j=1}^{V}\frac{F_{j,j+1}(t)/t^2}{\rho_j^2(t)/t^2}=\prod_{j=1}^{V}\left(\cosh e_{j,j+1}+\sigma_{j,j+1}\sqrt{\cosh^2 e_{j,j+1}-1}\right),$$

and (9) at $t\to\infty$ transforms to

$$\prod_{j=1}^{V}\left(\cosh e_{j,j+1}+\sigma_{j,j+1}\sqrt{\cosh^2 e_{j,j+1}-1}\right)=1.\qquad(31)$$

By the following trigonometric identity of hyperbolic geometry, $\cosh^2 x-\sinh^2 x=1$, and because $e_{j,j+1}>0$, we have

$$\sqrt{\cosh^2 e_{j,j+1}-1}=\sqrt{\sinh^2 e_{j,j+1}}=\sinh e_{j,j+1}.\qquad(32)$$

By (32) the equation (31) transforms to

$$\prod_{j=1}^{V}\left(\cosh e_{j,j+1}+\sigma_{j,j+1}\sinh e_{j,j+1}\right)=1.\qquad(33)$$

By $\cosh x=\frac{e^x+e^{-x}}{2}$ and $\sinh x=\frac{e^x-e^{-x}}{2}$, we have

$$\cosh e_{j,j+1}+\sigma_{j,j+1}\sinh e_{j,j+1}=\begin{cases}e^{e_{j,j+1}} & \text{-}\quad \sigma_{j,j+1}=1,\\ e^{-e_{j,j+1}} & \text{-}\quad \sigma_{j,j+1}=-1.\end{cases}=e^{\sigma_{j,j+1}e_{j,j+1}}.\qquad(34)$$

Substituting (34) in (33) and taking the logarithm of the resulting equation, we get (1) □.

The study of the behavior of the equation of flexibility (9) in neighborhoods of other singular points of the left-hand side of (9) did not give us interesting results: either we were obtaining trivial identities like $1=1$ (for instance, as $t\to\pm1$), or the limit of the left-hand side of the equation of flexibility was too complicated to distinguish interesting patterns there.

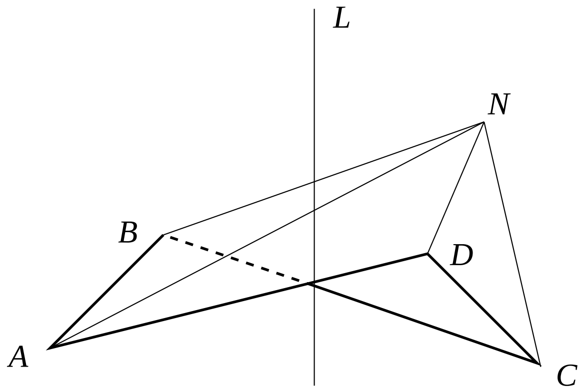

Figure 8: The construction of the Bricard-Stachel octahedron of type 1. Step 1.

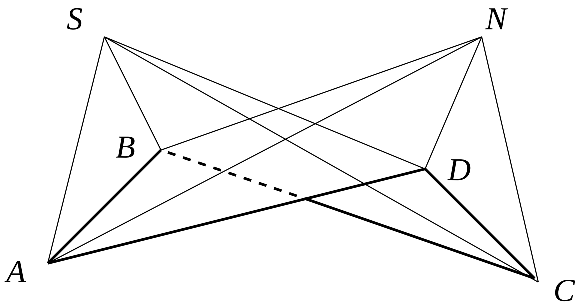

Figure 9: The construction of the Bricard-Stachel octahedron of type 1. Step 2.

6 Verification of the necessary flexibility condition of a nondegenerate suspension for the Bricard-Stachel octahedra in Lobachevsky 3-space

In 2002 H. Stachel [7] proved the flexibility of the analogues of the BricardTs octahedra in Lobachevsky 3-space. Let us verify the validity of the necessary flexibility condition of a nondegenerate suspension for the Bricard-Stachel octahedra in Lobachevsky 3-space.

We define an *octahedron* $\mathcal{O}$ as the suspension $NABCDS$ with the poles N and S, and with the vertices of the equator A, B, C, and D. Note that we can consider vertices A and C as the poles of $\mathcal{O}$ (in this case the quadrilateral $NDSB$ serves as the equator of $\mathcal{O}$). Also we can consider vertices B and D as the poles of $\mathcal{O}$ (in this case the quadrilateral $NCSA$ serves as the equator of $\mathcal{O}$).

6.1 Bricard-Stachel octahedra of types 1 and 2

The procedure of construction of the Bricard-Stachel octahedra of types 1 and 2 in Lobachevsky 3-space is the same as for the Bricard's octahedra of types 1 and 2 in Euclidean 3-space [7], [8].

Any *Bricard-Stachel octahedron of type 1* in $\mathbb{H}^3$ can be constructed in the following way. Consider a disk-homeomorphic piece-wise linear surface $\mathcal{S}$ in $\mathbb{H}^3$ composed of four triangles ABN, BCN, CDN, and DAN such that $d_{\mathbb{H}^3}(A,B) = d_{\mathbb{H}^3}(C,D)$ and $d_{\mathbb{H}^3}(B,C) = d_{\mathbb{H}^3}(D,A)$. It is known that a spatial quadrilateral $ABCD$ which opposite sides have the same lengths, is symmetric with respect to a line $\mathcal{L}$ passing through the middle points of its diagonals AC and BD (see Fig. 8; for a more precise analogy with the Euclidean case, in this Figure as well as in the following Figures we draw polyhedra in the Kleinian model of Lobachevsky space where lines and planes are intersections of Euclidean lines and planes with a fixed unit ball). Glue together $\mathcal{S}$ and its symmetric image with respect to L along $ABCD$. Denote by S the symmetric image of N under the symmetry with respect to L

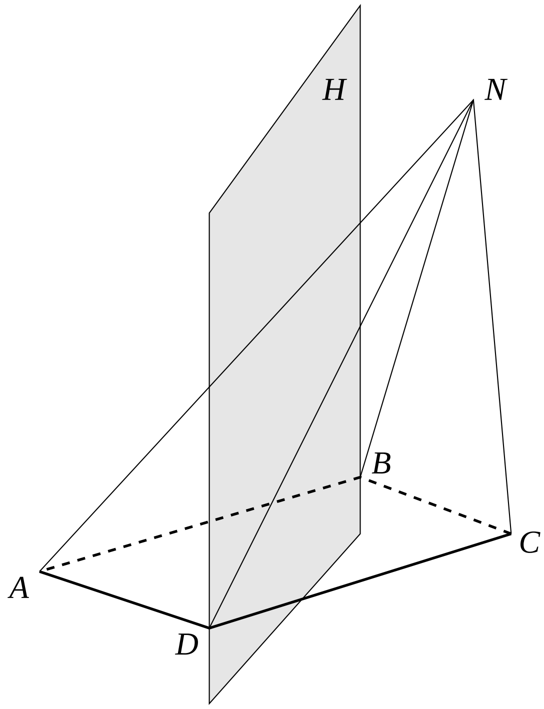

Figure 10: The construction of the Bricard-Stachel octahedron of type 2. Step 1.

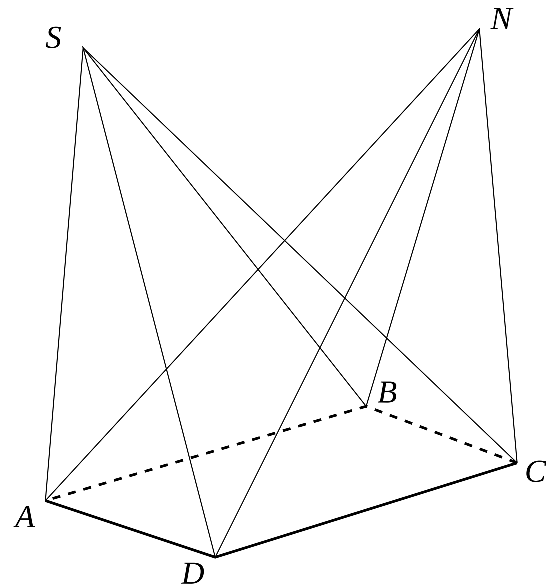

Figure 11: The construction of the Bricard-Stachel octahedron of type 2. Step 2.

(see Fig. 9). The resulting polyhedral surface $NABCDS$ with self-intersections is flexible (because $\mathcal{S}$ is flexible) and combinatorially it is an octahedron (according to the definition given above). We will call it a Bricard-Stachel octahedron of type 1. By construction it follows that $\mathrm{d}_{\mathbb{H}^3}(A,N) = \mathrm{d}_{\mathbb{H}^3}(C,S)$, $\mathrm{d}_{\mathbb{H}^3}(B,N) = \mathrm{d}_{\mathbb{H}^3}(D,S)$, $\mathrm{d}_{\mathbb{H}^3}(C,N) = \mathrm{d}_{\mathbb{H}^3}(A,S)$, and $\mathrm{d}_{\mathbb{H}^3}(D,N) = \mathrm{d}_{\mathbb{H}^3}(B,S)$.

Any *Bricard-Stachel octahedron of type 2* in $\mathbb{H}^3$ can be constructed as follows. Consider a disk-homeomorphic piece-wise linear surface $\mathcal{S}$ in $\mathbb{H}^3$ composed of four triangles ABN, BCN, CDN, and DAN such that $\mathrm{d}_{\mathbb{H}^3}(A,B) = \mathrm{d}_{\mathbb{H}^3}(B,C)$ and $\mathrm{d}_{\mathbb{H}^3}(C,D) = \mathrm{d}_{\mathbb{H}^3}(D,A)$. It is known that a spatial quadrilateral $ABCD$ which neighbor sides at the vertices B and D have the same lengths, is symmetric with respect to a plane H which dissects the dihedral angle between the half-planes ABD and CBD (see Fig. 10). Glue together $\mathcal{S}$ and its symmetric image with respect to H along $ABCD$. Denote by S the symmetric image of N under the symmetry with respect to H (see Fig. 9). The resulting polyhedral surface $NABCDS$ with self-intersections is flexible (because $\mathcal{S}$ is flexible) and combinatorially it is an octahedron. We will call it a Bricard-Stachel octahedron of type 2. By construction it follows that $\mathrm{d}_{\mathbb{H}^3}(A,N) = \mathrm{d}_{\mathbb{H}^3}(C,S)$, $\mathrm{d}_{\mathbb{H}^3}(C,N) = \mathrm{d}_{\mathbb{H}^3}(A,S)$, $\mathrm{d}_{\mathbb{H}^3}(B,N) = \mathrm{d}_{\mathbb{H}^3}(B,S)$, and $\mathrm{d}_{\mathbb{H}^3}(D,N) = \mathrm{d}_{\mathbb{H}^3}(D,S)$.

It remains to note that for every considered octahedron each of three its equators has two pairs of edges of the same lengths. Hence, the theorem 1 is valid for the Bricard-Stachel octahedra of types 1 and 2.

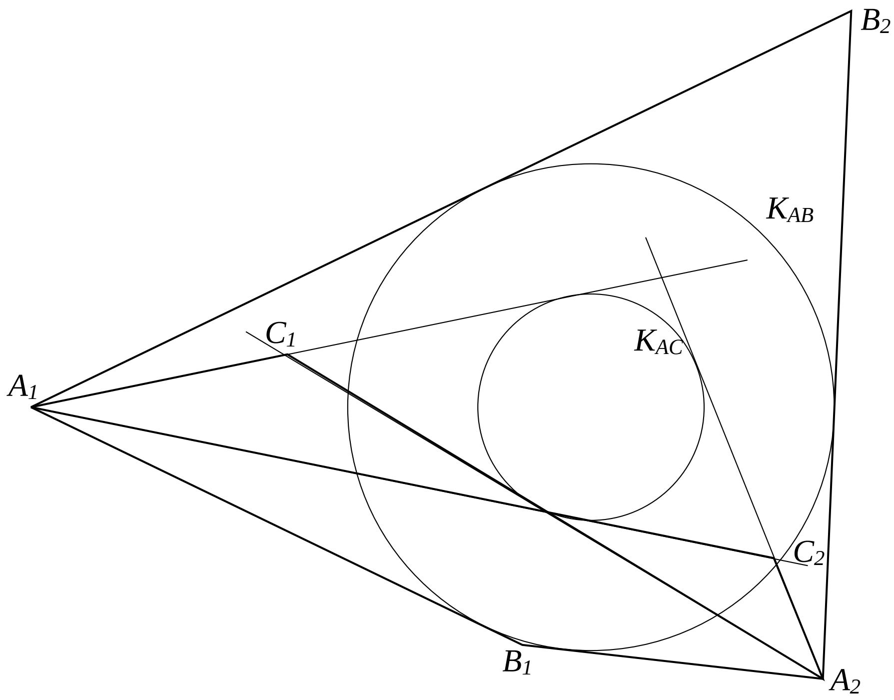

Figure 12: The construction of the Bricard-Stachel octahedron of type 3 based on circles. Step 1.

6.2 Bricard-Stachel octahedra of type 3

There are three subtypes of the Bricard-Stachel octahedra of type 3 in Lobachevsky space [7] which construction is based on circles, horocycles or hypercircles correspondingly. The procedure of construction is common for all subtypes of the Bricard-Stachel octahedra of type 3 and it is the same as for the Bricard's octahedra of type 3 in Euclidean space.

Any *Bricard-Stachel octahedron of type 3* in $\mathbb{H}^3$ can be constructed in the following way. Let K_{AC} and K_{AB} be two different circles (horocycles, hypercircles) in $\mathbb{H}^2$ with the common center M and let A_1, A_2 be two different finite points outside K_{AC} and K_{AB}. In addition, suppose that K_{AC}, K_{AB}, A_1 and A_2 are taken in such a way that the straight lines tangent to K_{AB} and passing through A_1 and A_2 intersect pairwise in finite points of $\mathbb{H}^2$ and form a quadrilateral $A_1B_1A_2B_2$ tangent to K_{AB}; moreover, that the straight lines tangent to K_{AC} and passing through A_1 and A_2 intersect pairwise in finite points of $\mathbb{H}^2$ and form a quadrilateral $A_1C_1A_2C_2$ tangent to K_{AC} (see Fig. 12; for clarity, we placed circles K_{AB} and K_{AC} so that their common center coincides with the center of the Kleinian model of Lobachevsky space. In this case K_{AB} and K_{AC} are Euclidean circles as well). A polyhedron $\mathcal{O}$ with the vertices A_i, B_j, C_k, with the edges A_iB_j, A_iC_k, B_jC_k, and with the faces $\triangle A_iB_jC_k$, $i, j, k \in \{1, 2\}$, is an octahedron in the sense of the definition given above (see Fig. 13). The following pairs of vertices can serve as the poles of $\mathcal{O}$: (A_1, A_2) with the corresponding equator $B_1C_1B_2C_2$, (B_1, B_2) with the equator $A_1C_1A_2C_2$, and (C_1, C_2) with the equator $A_1B_1A_2B_2$. Suppose in addition that $\mathcal{O}$ does not have symmetries. We will call such octahedron $\mathcal{O}$ a Bricard-Stachel octahedron of type 3.

According to H. Stachel [7], $\mathcal{O}$ flexes continuously in $\mathbb{H}^3$. Moreover, $\mathcal{O}$ admits two flat

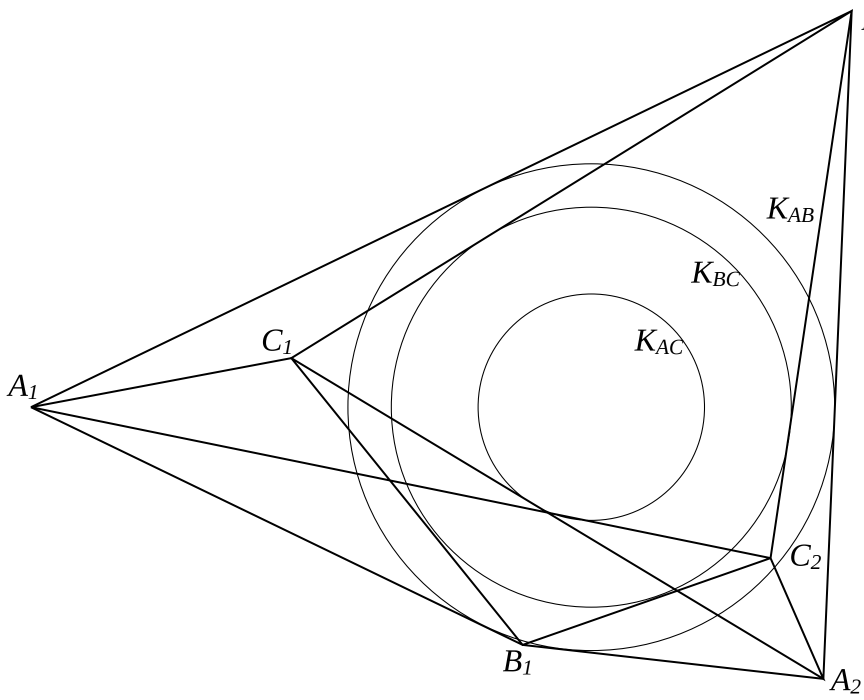

Figure 13: The construction of the Bricard-Stachel octahedron of type 3 based on circles. Step 2.

positions during the flex (we constructed $\mathcal{O}$ in one of its flat positions). Hence, for every equator of $\mathcal{O}$, $A_1B_1A_2B_2$, $B_1C_1B_2C_2$, and $A_1C_1A_2C_2$, all straight lines containing a side of the equator are tangent to some circle (horocycle, hypercircle) at least in one flat position of $\mathcal{O}$. Using this fact, we will prove that the theorem 1 is valid for the Bricard-Stachel octahedra of type 3. We have to consider three possible cases: when an equator of $\mathcal{O}$ is tangent to a circle, to a horocycle, or to a hypercircle in $\mathbb{H}^2$. Here we study the most common situation when any three vertices of an equator of a flexible octahedron in its flat position do not lie on a straight line.

6.2.1 An equator of a Bricard-Stachel octahedron of type 3 is tangent to a circle in $\mathbb{H}^2$

Let M be the center of the circle K_{AB} with the radius R in $\mathbb{H}^2$ and let all straight lines containing a side of the quadrilateral $A_1B_1A_2B_2$ are tangent to K_{AB}. Let us draw the segments MP_1, MP_2, MP_3, MP_4 connecting M with the straight lines A_1B_2, A_2B_2, A_2B_1, A_1B_1 and perpendicular to the corresponding lines. By construction, $\mathrm{d}_{\mathbb{H}^2}(M,P_1)=\mathrm{d}_{\mathbb{H}^2}(M,P_2)=\mathrm{d}_{\mathbb{H}^2}(M,P_3)=\mathrm{d}_{\mathbb{H}^2}(M,P_4)=R$.

By the Pythagorean theorem for Lobachevsky space [9] applied to $\triangle A_1MP_1$ and $\triangle A_1MP_4$, we obtain: $\cosh \mathrm{d}_{\mathbb{H}^2}(A_1,P_1)=\cosh \mathrm{d}_{\mathbb{H}^2}(A_1,P_4)=\cosh \mathrm{d}_{\mathbb{H}^2}(A_1,M)/\cosh R$. Then $a \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(A_1,P_1)=\mathrm{d}_{\mathbb{H}^2}(A_1,P_4)$. Similarly we get: $b \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(B_2,P_1)=\mathrm{d}_{\mathbb{H}^2}(B_2,P_2)$, $c \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(A_2,P_2)=\mathrm{d}_{\mathbb{H}^2}(A_2,P_3)$, and $d \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(B_1,P_3)=\mathrm{d}_{\mathbb{H}^2}(B_1,P_4)$.

If the circle K_{AB} is inscribed in the quadrilateral $A_1B_1A_2B_2$ (see Fig. 12), then $\mathrm{d}_{\mathbb{H}^2}(A_1,B_2)=a+b$, $\mathrm{d}_{\mathbb{H}^2}(A_2,B_2)=b+c$, $\mathrm{d}_{\mathbb{H}^2}(A_2,B_1)=c+d$, $\mathrm{d}_{\mathbb{H}^2}(A_1,B_1)=a+d$, and the identity

$$\mathrm{d}_{\mathbb{H}^2}(A_1, B_2) - \mathrm{d}_{\mathbb{H}^2}(A_2, B_2) + \mathrm{d}_{\mathbb{H}^2}(A_1, B_1) - \mathrm{d}_{\mathbb{H}^2}(A_1, B_1) = 0 \tag{35}$$

holds true.

If the circle K_{AB} is tangent to the quadrilateral $A_1B_1A_2B_2$ externally (this case corresponds to the quadrilateral $A_1C_1A_2C_2$ and to the circle K_{AC} in the Fig. 12), then $\mathrm{d}_{\mathbb{H}^2}(A_1, B_2) = a - b$, $\mathrm{d}_{\mathbb{H}^2}(A_2, B_2) = b + c$, $\mathrm{d}_{\mathbb{H}^2}(A_2, B_1) = c - d$, $\mathrm{d}_{\mathbb{H}^2}(A_1, B_1) = a + d$, and the identity

$$\mathrm{d}_{\mathbb{H}^2}(A_1, B_2) + \mathrm{d}_{\mathbb{H}^2}(A_2, B_2) - \mathrm{d}_{\mathbb{H}^2}(A_1, B_1) - \mathrm{d}_{\mathbb{H}^2}(A_1, B_1) = 0 \tag{36}$$

holds true.

By (35) and (36), the theorem 1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a circle in at least one of its flat positions.

6.2.2 An equator of a Bricard-Stachel octahedron of type 3 is tangent to a horocycle in $\mathbb{H}^2$

Let us consider the Poincaré upper half-plane model of Lobachevsky plane $\mathbb{H}^2$ with the coordinates (ρ, z) (i.e., with the metric given by the formula $ds^2 = \frac{d\rho^2 + dz^2}{z^2}$). Without loss of generality we can assume that the center of the horocycle tangent to the equator of a Bricard-Stachel octahedron $\mathcal{O}$ of type 3, coincides with the (unique) point ∞ at infinity of $\mathbb{H}^2$ which does not lie on the Euclidean line $z = 0$. We denote the family of such horocycles by $K = \{\rho = R | R > 0\}$. Let $K_R \in K$ and let $A_1 = (\rho_{A_1}, z_{A_1})$ and $A_2 = (\rho_{A_2}, z_{A_2})$ be two opposite vertices of $\mathcal{O}$, such that the straight line (in $\mathbb{H}^2$) passing through A_1 and A_2 is not tangent to K_R. All the vertices of $\mathcal{O}$ are located outside K_R, hence $z_{A_1} < R$ and $z_{A_2} < R$. We will construct all possible quadrangles tangent to K_R with the opposite vertices A_1 and A_2, i.e., all quadrangles that can serve as equators of $\mathcal{O}$. Then we will verify the validity of the theorem 1 for such quadrangles.

Let $T = (\rho_T, z_T)$ be a point in $\mathbb{H}^2$ and let Λ be a straight line in $\mathbb{H}^2$ passing through T which is realized in the Poincaré upper half-plane as the Euclidean demi-circle with the radius $\sqrt{(\rho_T - \rho_{T,\Lambda})^2 + z_T^2}$ and with the center $O_\Lambda^T = (\rho_{T,\Lambda}, 0)$. Then the angle $\varphi_T^\Lambda \overset{\mathrm{def}}{=} \angle T O_\Lambda^T \rho \in (0, \pi)$ determines uniquely a position of T on Λ.

Remark 1 *For every finite point $T = (\rho_T, z_T)$, $z_T < R$, there exist precisely two straight lines Λ_l^T and Λ_r^T tangent to the horocycle K_R and containing T. They are realized in the Poincaré upper half-plane as the Euclidean demi-circles with the radius R and with the centers $O_l^T = (\rho_{T,l}, 0)$ and $O_r^T = (\rho_{T,r}, 0)$, $\rho_{T,l} \leq \rho_T \leq \rho_{T,r}$. The angles $\varphi_T^l \overset{\mathrm{def}}{=} \angle T O_l^T \rho$ and $\varphi_T^r \overset{\mathrm{def}}{=} \angle T O_r^T \rho$ serve as the coordinates of T on Λ_l^T and Λ_r^T correspondingly. Then, by construction, we get: $\varphi_T^r = \pi - \varphi_T^l$. Hence,*

$$\cos \varphi_T^r = -\cos \varphi_T^l. \tag{37}$$

According to the remark 1, there are two straight lines, $\Lambda_l^{A_1}$, and $\Lambda_r^{A_1}$, passing through A_1 and tangent to K_R, which are realised in $\mathbb{H}^2$ as the Euclidean demi-circles with the radius R and with the centers $O_l^{A_1} = (\rho_{A_1,l}, 0)$, $O_r^{A_1} = (\rho_{A_1,r}, 0)$, $\rho_{A_1,l} \leq \rho_{A_1} \leq \rho_{A_1,r}$. The angles $\varphi_{A_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_l^{A_1} \rho$, $\varphi_{A_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_r^{A_1} \rho$ serve as the coordinates of A_1 on $\Lambda_l^{A_1}$ and $\Lambda_r^{A_1}$ correspondingly. Moreover,

$$\cos \varphi_{A_1}^{\Lambda_r^{A_1}} = -\cos \varphi_{A_1}^{\Lambda_l^{A_1}}. \tag{38}$$

Similarly, there are two straight lines, $\Lambda_l^{A_2}$, and $\Lambda_r^{A_2}$, passing through A_2 and tangent to K_R, which are realised in $\mathbb{H}^2$ as the Euclidean demi-circles with the radius R and with the centers $O_l^{A_2} = (\rho_{A_2,l}, 0)$, $O_r^{A_2} = (\rho_{A_2,r}, 0)$, $\rho_{A_2,l} \leq \rho_{A_2} \leq \rho_{A_2,r}$. The angles $\varphi_{A_2}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_l^{A_2} \rho$, $\varphi_{A_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_r^{A_2} \rho$ serve as the coordinates of A_2 on $\Lambda_l^{A_2}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{A_2}^{\Lambda_r^{A_2}} = -\cos \varphi_{A_2}^{\Lambda_l^{A_2}}. \tag{39}$$

Suppose that $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point B_1. Then the angles $\varphi_{B_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_1} \rho$, $\varphi_{B_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_2} \rho$ serve as the coordinates of B_1 on $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{B_1}^{\Lambda_l^{A_2}} = -\cos \varphi_{B_1}^{\Lambda_l^{A_1}}. \tag{40}$$

Also suppose that $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$ intersect at a point B_2. Then the angles $\varphi_{B_2}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_1} \rho$, $\varphi_{B_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_2} \rho$ serve as the coordinates of B_2 on $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{B_2}^{\Lambda_r^{A_2}} = -\cos \varphi_{B_2}^{\Lambda_r^{A_1}}. \tag{41}$$

Let the straight lines $\Lambda_r^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point C_1. Then the angles $\varphi_{C_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle C_1 O_r^{A_1} \rho$, $\varphi_{C_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle C_1 O_l^{A_2} \rho$ serve as the coordinates of C_1 on $\Lambda_r^{A_1}$ and $\Lambda_l^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{C_1}^{\Lambda_l^{A_2}} = -\cos \varphi_{C_1}^{\Lambda_r^{A_1}}. \tag{42}$$

Also, let the straight lines $\Lambda_l^{A_1}$ and $\Lambda_r^{A_2}$ intersect at a point C_2. Then the angles $\varphi_{C_2}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle C_2 O_l^{A_1} \rho$, $\varphi_{C_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle C_2 O_r^{A_2} \rho$ serve as the coordinates of C_2 on $\Lambda_l^{A_1}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos \varphi_{C_2}^{\Lambda_r^{A_2}} = -\cos \varphi_{C_2}^{\Lambda_l^{A_1}}. \tag{43}$$

By construction, the quadrangles $A_1B_1A_2B_2$ and $A_1C_1A_2C_2$ are tangent to K_R, and the points A_1, A_2 are opposite vertices of each of these quadrangles. In order to verify the validity of the theorem 1 for the flexible octahedra with the equator $A_1B_1A_2B_2$ or $A_1C_1A_2C_2$ we need to prove the following easy statement.

Lemma 3 *Given a Poincaré upper half-plane $\mathbb{H}^2$ with the coordinates (ρ, z) (i.e., with the metric given by the formula $ds^2 = \frac{d\rho^2 + dz^2}{z^2}$). Let A and B be points on the straight line Λ realized in $\mathbb{H}^2$ as the Euclidean demi-circle with the raduis R and with the center $O_\Lambda = (\rho_{O_\Lambda}, 0)$, and let the angles $\varphi_A \stackrel{\text{def}}{=} \angle A O_\Lambda \rho$, $\varphi_B \stackrel{\text{def}}{=} \angle B O_\Lambda \rho$ serve as the coordinates of A and*

B correspondingly on Λ. *Also we assume that* $0 < \varphi_A \leq \phi_B < \pi$. *Then the distance between* A *and* B *is calculated as follows:*

$$\mathrm{d}_{\mathbb{H}^2}(A,B) = \frac{1}{2}\ln\Big[\Big(\frac{1+\cos\varphi_A}{1+\cos\varphi_B}\Big)\Big(\frac{1-\cos\varphi_B}{1-\cos\varphi_A}\Big)\Big]. \tag{44}$$

Proof. The hyperbolic segment Λ_{AB} connecting the points A and B is specified parametrically by the formulas $\Lambda_{AB}(t) : (\rho(\varphi), z(\varphi))$, $\varphi \in [\varphi_A, \varphi_B]$, where $\rho(\varphi) = \rho_{O_\Lambda} + R\cos\varphi$, $z(\varphi) = R\sin\varphi$, $A = \Lambda_{AB}(\varphi_A)$, $B = \Lambda_{AB}(\varphi_B)$. The direct calculation shows that the lengths of Λ_{AB} is equal to the right-hand side of (44). $\square$

By Lemma 3, the lengths of the edges of the quadrilateral $A_1B_1A_2B_2$ are calculated as follows:

$$\mathrm{d}_{\mathbb{H}^2}(A_1,B_1) = \frac{1}{2}\ln\left[\left(\frac{1+\cos\varphi_{A_1}^{\Lambda_l^{A_1}}}{1+\cos\varphi_{B_1}^{\Lambda_l^{A_1}}}\right)\left(\frac{1-\cos\varphi_{B_1}^{\Lambda_l^{A_1}}}{1-\cos\varphi_{A_1}^{\Lambda_l^{A_1}}}\right)\right], \tag{45}$$

$$\mathrm{d}_{\mathbb{H}^2}(A_2,B_1) = \frac{1}{2}\ln\left[\left(\frac{1+\cos\varphi_{A_2}^{\Lambda_l^{A_2}}}{1+\cos\varphi_{B_1}^{\Lambda_l^{A_2}}}\right)\left(\frac{1-\cos\varphi_{B_1}^{\Lambda_l^{A_2}}}{1-\cos\varphi_{A_2}^{\Lambda_l^{A_2}}}\right)\right], \tag{46}$$

$$\mathrm{d}_{\mathbb{H}^2}(B_2,A_1) = \frac{1}{2}\ln\left[\left(\frac{1+\cos\varphi_{B_2}^{\Lambda_r^{A_1}}}{1+\cos\varphi_{A_1}^{\Lambda_r^{A_1}}}\right)\left(\frac{1-\cos\varphi_{A_1}^{\Lambda_r^{A_1}}}{1-\cos\varphi_{B_2}^{\Lambda_r^{A_1}}}\right)\right], \tag{47}$$

$$\mathrm{d}_{\mathbb{H}^2}(B_2,A_2) = \frac{1}{2}\ln\left[\left(\frac{1+\cos\varphi_{B_2}^{\Lambda_r^{A_2}}}{1+\cos\varphi_{A_2}^{\Lambda_r^{A_2}}}\right)\left(\frac{1-\cos\varphi_{A_2}^{\Lambda_r^{A_2}}}{1-\cos\varphi_{B_2}^{\Lambda_r^{A_2}}}\right)\right]. \tag{48}$$

Then, by (38)—(41), we get:

$$\mathrm{d}_{\mathbb{H}^2}(A_1,B_1) + \mathrm{d}_{\mathbb{H}^2}(A_2,B_1) - \mathrm{d}_{\mathbb{H}^2}(B_2,A_1) - \mathrm{d}_{\mathbb{H}^2}(B_2,A_2) = 0. \tag{49}$$

By Lemma 3, the lengths of the edges of the quadrilateral $A_1C_1A_2C_2$ are calculated as follows:

$$\mathrm{d}_{\mathbb{H}^2}(C_1,A_1) = \frac{1}{2}\ln\left[\left(\frac{1+\cos\varphi_{C_1}^{\Lambda_r^{A_1}}}{1+\cos\varphi_{A_1}^{\Lambda_r^{A_1}}}\right)\left(\frac{1-\cos\varphi_{A_1}^{\Lambda_r^{A_1}}}{1-\cos\varphi_{C_1}^{\Lambda_r^{A_1}}}\right)\right], \tag{50}$$

$$\mathrm{d}_{\mathbb{H}^2}(C_2,A_1) = \frac{1}{2}\ln\left[\left(\frac{1+\cos\varphi_{C_2}^{\Lambda_l^{A_1}}}{1+\cos\varphi_{A_1}^{\Lambda_l^{A_1}}}\right)\left(\frac{1-\cos\varphi_{A_1}^{\Lambda_l^{A_1}}}{1-\cos\varphi_{C_2}^{\Lambda_l^{A_1}}}\right)\right], \tag{51}$$

$$\mathrm{d}_{\mathbb{H}^2}(A_2,C_1) = \frac{1}{2}\ln\left[\left(\frac{1+\cos\varphi_{A_2}^{\Lambda_l^{A_2}}}{1+\cos\varphi_{C_1}^{\Lambda_l^{A_2}}}\right)\left(\frac{1-\cos\varphi_{C_1}^{\Lambda_l^{A_2}}}{1-\cos\varphi_{A_2}^{\Lambda_l^{A_2}}}\right)\right], \tag{52}$$

$$\mathrm{d}_{\mathbb{H}^2}(A_2,C_2) = \frac{1}{2}\ln\left[\left(\frac{1+\cos\varphi_{A_2}^{\Lambda_r^{A_2}}}{1+\cos\varphi_{C_2}^{\Lambda_r^{A_2}}}\right)\left(\frac{1-\cos\varphi_{C_2}^{\Lambda_r^{A_2}}}{1-\cos\varphi_{A_2}^{\Lambda_r^{A_2}}}\right)\right]. \tag{53}$$

By (38), (39), (42), and (43), it is easy to verify that

$$\mathrm{d}_{\mathbb{H}^2}(C_2,A_1) + \mathrm{d}_{\mathbb{H}^2}(C_1,A_1) - \mathrm{d}_{\mathbb{H}^2}(A_2,C_1) - \mathrm{d}_{\mathbb{H}^2}(A_2,C_2) = 0. \tag{54}$$

According to (49) and (54), the theorem 1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a horocycle in at least one of its flat positions.

6.2.3 An equator of a Bricard-Stachel octahedron of type 3 is tangent to a hypercircle in $\mathbb{H}^2$

Let us consider the Poincaré upper half-plane model of Lobachevsky plane $\mathbb{H}^2$ with the coordinates (ρ, z) (i.e., with the metric given by the formula $ds^2 = \frac{d\rho^2+dz^2}{z^2}$). Without loss of generality we can assume that the hypercircle tangent to the equator of a Bricard-Stachel octahedron $\mathcal{O}$ of type 3, passes through the (unique) point ∞ at infinity of $\mathbb{H}^2$ which does not lie on the Euclidean line $z = 0$, and through the point $O = (0, 0)$ at infinity of $\mathbb{H}^2$. Every such hypercircle is specified by the equation $z = \rho \tan\alpha$ for some $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. By the symmetry of $\mathbb{H}^2$ with respect to the straight line $\rho = 0$, it is sufficient to consider the family of hypercircles $K = \{z = \rho\tan\alpha | \alpha \in (0, \frac{\pi}{2})\}$. Let $K_\alpha \in K$. We will construct all possible quadrangles tangent to K_α such that none of their vertices belongs to K_α, i.e., all quadrangles that can serve as equators of $\mathcal{O}$. Then we will verify the validity of the theorem 1 for such quadrangles.

Let us study the quadrangles based on the straight lines $\Lambda_l^{A_1}$, $\Lambda_r^{A_1}$, $\Lambda_l^{A_2}$, $\Lambda_r^{A_2}$, tangent to K_α, which are realised in $\mathbb{H}^2$ as the Euclidean demi-circles with the centers $O_l^{A_1} = (\rho_{A_1,l}, 0)$, $O_r^{A_1} = (\rho_{A_1,r}, 0)$, $O_l^{A_2} = (\rho_{A_2,l}, 0)$, $O_r^{A_2} = (\rho_{A_2,r}, 0)$. Also, let $\Lambda_l^{A_1}$ and $\Lambda_r^{A_1}$ intersect at a point A_1, $\Lambda_l^{A_2}$ and $\Lambda_r^{A_2}$ intersect at a point A_2. Assume that A_1 and A_2 are two opposite vertices of $\mathcal{O}$, and that the inequalities $0 < \rho_{A_1,l} < \rho_{A_1,r}$, $0 < \rho_{A_2,l} < \rho_{A_2,r}$ hold true.

Remark 2 *Let $T = (\rho_T, z_T)$ be a point in $\mathbb{H}^2$, which serves as the intersection of straight lines Λ_l^T and Λ_r^T tangent to a hypercircle K_α, and let Λ_l^T and Λ_r^T are realised in $\mathbb{H}^2$ as the Euclidean demi-circles with the centers $O_l^T = (\rho_{T,l}, 0)$, $O_r^T = (\rho_{T,r}, 0)$ $(\rho_{T,l} < \rho_{T,r})$. Then, by Remark 1, the angles $\varphi_T^l \overset{\text{def}}{=} \angle TO_l^T\rho$ and $\phi_T^r \overset{\text{def}}{=} \angle TO_r^T\rho$ determine uniquely the positions of T on Λ_l^T and Λ_r^T correspondingly. Moreover,*

$$\cos\varphi_T^l = \frac{\rho_{T,r}}{\rho_{T,l}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \textit{and} \quad \cos\varphi_T^r = \frac{\rho_{T,l}}{\rho_{T,r}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \quad (55)$$

Proof. Λ_l^T and Λ_r^T are tangent to K_α. Hence, the radii R_l and R_r of the demi-circles realizing Λ_l^T and Λ_r^T in $\mathbb{H}^2$ are determined by the formulas

$$R_l = \rho_{T,l}\sin\alpha \quad \text{and} \quad R_r = \rho_{T,r}\sin\alpha. \quad (56)$$

Let T_∞ be a point with coordinates $(\rho_T, 0)$. Applying the Euclidean Pythagorean theorem to $\triangle TT_\infty O_r^T$ and simplifying the obtained expression, we get:

$$\rho_T^2 + z_T^2 = 2\rho_T\rho_{T,l} - \rho_{T,l}^2\cos^2\alpha. \quad (57)$$

Similarly, from $\triangle TT_\infty O_l^T$ we get that

$$\rho_T^2 + z_T^2 = 2\rho_T\rho_{T,r} - \rho_{T,r}^2\cos^2\alpha. \quad (58)$$

Subtracting (57) from (58), we easily deduce:

$$\rho_T = \frac{\rho_{T,r} + \rho_{T,l}}{2}\cos^2\alpha. \quad (59)$$

From the definitions of the cosines of φ_T^l and φ_T^r ($\cos\varphi_T^l = (\rho_T - \rho_{T,l})/R_l$ and $\cos\varphi_T^r = (\rho_T - \rho_{T,r})/R_r$), taking into account (56) and (59), we obtain (55). $\square$

By Remark 2, the angles $\varphi_{A_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_l^{A_1}\rho$ and $\varphi_{A_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_r^{A_1}\rho$ determine uniquely the positions of A_1 on $\Lambda_l^{A_1}$ and $\Lambda_r^{A_1}$ correspondingly. Moreover,

$$\cos\varphi_{A_1}^{\Lambda_l^{A_1}} = \frac{\rho_{A_1,r}}{\rho_{A_1,l}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{A_1}^{\Lambda_r^{A_1}} = \frac{\rho_{A_1,l}}{\rho_{A_1,r}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{60}$$

Similarly, the angles $\varphi_{A_2}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_l^{A_2}\rho$ and $\varphi_{A_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_r^{A_2}\rho$ serve as the coordinates of A_2 on $\Lambda_l^{A_2}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{A_2}^{\Lambda_l^{A_2}} = \frac{\rho_{A_2,r}}{\rho_{A_2,l}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{A_2}^{\Lambda_r^{A_2}} = \frac{\rho_{A_2,l}}{\rho_{A_2,r}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{61}$$

Suppose that the straight lines $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point B_1. Then the angles $\varphi_{B_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_1}\rho$ and $\varphi_{B_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_2}\rho$ serve as the coordinates of B_1 on $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{B_1}^{\Lambda_l^{A_1}} = \frac{\rho_{A_2,l}}{\rho_{A_1,l}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{B_1}^{\Lambda_l^{A_2}} = \frac{\rho_{A_1,l}}{\rho_{A_2,l}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{62}$$

Suppose also that $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$ intersect at a point B_2. Then the angles $\varphi_{B_2}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_1}\rho$ and $\varphi_{B_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_2}\rho$ serve as the coordinates of B_2 on $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{B_2}^{\Lambda_r^{A_1}} = \frac{\rho_{A_2,r}}{\rho_{A_1,r}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{B_2}^{\Lambda_r^{A_2}} = \frac{\rho_{A_1,r}}{\rho_{A_2,r}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{63}$$

Suppose that $\Lambda_r^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point C_1. Then the angles $\varphi_{C_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle C_1 O_r^{A_1}\rho$ and $\varphi_{C_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle C_1 O_l^{A_2}\rho$ serve as the coordinates of C_1 on $\Lambda_r^{A_1}$ and $\Lambda_l^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{C_1}^{\Lambda_l^{A_2}} = \frac{\rho_{A_1,r}}{\rho_{A_2,l}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{C_1}^{\Lambda_r^{A_1}} = \frac{\rho_{A_2,l}}{\rho_{A_1,r}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{64}$$

Suppose also that $\Lambda_l^{A_1}$ and $\Lambda_r^{A_2}$ intersect at a point C_2. Then the angles $\varphi_{C_2}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle C_2 O_l^{A_1}\rho$ and $\varphi_{C_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle C_2 O_r^{A_2}\rho$ serve as the coordinates of C_2 on $\Lambda_l^{A_1}$ and $\Lambda_r^{A_2}$ correspondingly. Moreover,

$$\cos\varphi_{C_2}^{\Lambda_l^{A_1}} = \frac{\rho_{A_2,r}}{\rho_{A_1,l}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2} \quad \text{and} \quad \cos\varphi_{C_2}^{\Lambda_r^{A_2}} = \frac{\rho_{A_1,l}}{\rho_{A_2,r}}\frac{\cos^2\alpha}{2\sin\alpha} - \frac{1}{2\sin\alpha} - \frac{\sin\alpha}{2}. \tag{65}$$

As in the case of the quadrangles tangent to a horocycle in $\mathbb{H}^2$, the lengths of the edges of $A_1B_1A_2B_2$ are expressed in (45)—(48), and the lengths of the edges of $A_1C_1A_2C_2$ are calculated in (50)—(53). Taking into account (60)—(65), it is easy to state the validity of (49) and (54).

According to (49) and (54), the theorem 1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a hypercircle in at least one of its flat positions.

The case when three vertices of an equator of a flexible octahedron in its flat position lie on a straight line, is similar. The case when all four vertices of an equator lie on a straight line, is trivial.

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References

[1] *Bricard R.* Mémoire sur la théorie de l'octaèdre articulé// J. Math. Pures Appl. 1897. V. 3. P. 113-150.

[2] *Connelly R.* An Attack on Rigidity// Preprint, Cornell University, 1974. There is a Russian translation in the book: *A.N. Kolmogorov and S.P. Novikov (eds.)* Investigations in the metric theory of surfaces// Moscow: Mir. 1980. P. 164–209.

[3] *Mikhalev S. N.* Several necessary metric conditions of flexibility of suspensions// Moscow Univ. Math. Bull. 2001. V. 56. N. 3. P. 14–20.

[4] *Anderson J. W.* Hyperbolic Geometry//Springer, 2005.

[5] *Benedetti R., Petronio C.* Lectures on hyperbolic geometry// Springer, 2003.

[6] *Bitsadze A. V.* Fundamentals of the theory of analytic functions of a complex variable// Moscow: Nauka, 1984. (in Russian)

[7] *Stachel H.* Flexible octahedra in the hyperbolic space// In the book: Non-Euclidean geometries. Janos Bolyai memorial volume (Eds. A. Prekopa et al.). New York: Springer. Mathematics and its Applications (Springer) 2006. V. 581. P. 209-225.

[8] *Alexandrov V.* The Dehn invariants of the Bricard octahedra// Journal of Geometry 2010. V. 99. N. 1-2. P. 1-13.

[9] *Alekseevskij D. V., Vinberg Eh. B., and Solodovnikov A. S.* Geometry II: Spaces of Constant Curvature// Encycl. Math. Sci. 1993. V. 29. P. 1–138.

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