Discrete conformality and ideal cone-manifolds

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Smooth conformality

Definition

*Let M be a smooth manifold and g_0, g_1 be two Riemannian metrics on M. They are called **pointwise conformally equivalent** if*

$$g_0 = e^{2u} g_1$$

for a smooth function u on M.

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A **conformal structure** *on M is a pointwise conformal equivalence class of Riemannian metrics on M.*

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A smooth map $F : (M_1, g_1) \to (M_2, g_2)$ *between two Riemannian manifolds is called* **conformal** *if it is a local diffeomorphism and*

$$F^* g_2 = e^{2u} g_1$$

for a smooth function u *on* M_1*.*

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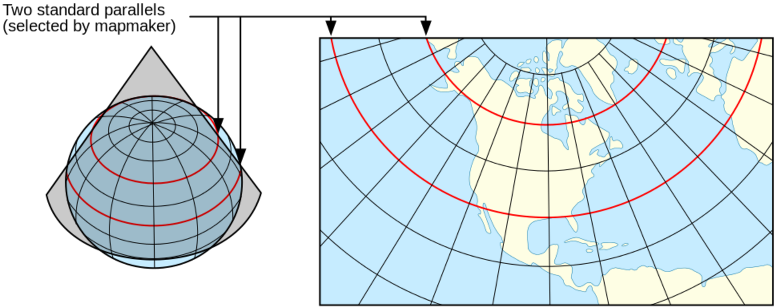

Smooth conformality: uniformization

Theorem (Uniformization Theorem)

Let (S, g) be a closed orientable Riemannian surface. Then it is pointwise conformally equivalent to a unique metric of constant Gaussian curvature equal to -1, 0 or 1 (in case of 0 curvature the uniqueness is up to scaling). The sign is equal to the sign of the Euler characteristic of S.

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One can attempt to understand Riemannian metrics on S as follows:

- Understand the Teichmüller space
- Understand diffeomorphisms
- Understand the conformal classes

Smooth conformality: prescribing Gaussian curvature

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Theorem (The Gauss-Bonnet theorem)

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The Gauss-Bonnet condition:

- $\chi(S) > 0$: K is positive somewhere;
- $\chi(S) = 0$: K changes sign unless $K \equiv 0$;
- $\chi(S) < 0$: K is negative somewhere.

Smooth conformality: prescribing Gaussian curvature

Theorem (Berger, 1971)

Let $\chi(S) < 0$, g_0 be a Riemannian metric on S and K be a non-positive smooth function on S such that $K \not\equiv 0$. Then there exists a unique metric g on S in the conformal class of g_0 with $K = K_g$.

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Theorem (Kazdan–Warner, 1974)

Let $\chi(S) = 0$, g_0 be a Riemannian metric on S and K be a smooth function on S such that K changes sign and $\int_S K dA_0 < 0$. Then there exists a metric g on S in the conformal class of g_0 with $K = K_g$.

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Theorem (Kazdan–Warner, 1975)

Let g_0 be a Riemannian metric on S and K be a smooth function on S satisfying the Gauss-Bonnet condition. Then there exists a metric g on S with $K = K_g$ and a conformal diffeomorphism $(S, g_0) \to (S, g)$.

Discrete conformality: cone-metrics

Definition

A ***Euclidean (resp. hyperbolic or spherical) cone-metric*** d *on a surface* S *is locally isometric to the Euclidean plane* $\mathbb{E}^2$ *(resp. the hyperbolic plane* $\mathbb{H}^2$ *or the standard sphere* $\mathbb{S}^2$*) except finitely many points called* ***conical or singular points****. At a conical point* v *the metric* d *is locally isometric to a cone with angle* $\lambda_v(d) \neq 2\pi$*.*

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The number $\kappa_v(d) := 2\pi - \lambda_v(d)$ is called the **curvature** of v.
Let $V \subset S$ be a set of marked points. We say that d is a cone-metric on (S, V) if the set of conical points of d is a subset of V.

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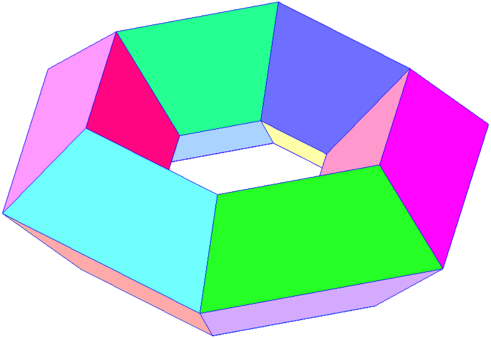
Example

The induced metric on the boundary of a convex polytope in $\mathbb{E}^3$ is a Euclidean cone-metric. Moreover, it is ***convex****, i.e., for all conical points we have $\kappa_v(d) > 0$.*

Discrete conformality: cone-metrics



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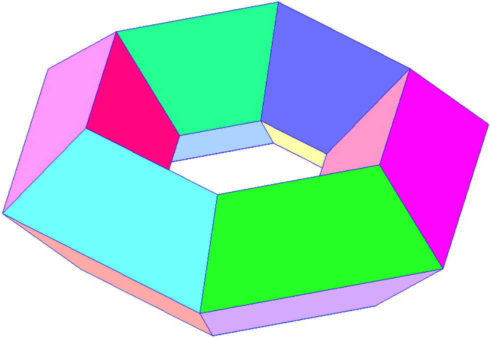

Theorem

For each cone metric d on (S, V) there exists a geodesic triangulation of (S, V, d) with vertices at V.

Discrete conformality

Definition

Two Euclidean cone-metrics d, d' on (S, V) with a common triangulation $\mathcal{T}$ are discretely conformally equivalent with respect to $\mathcal{T}$ if there exists a function $u : V \to \mathbb{R}$ such that for every edge e with the ends v_1 and v_2 we have

$$l(e, d) = e^{u(v_1)+u(v_2)} l(e, d').$$

Discrete conformality: Delaunay triangulations

Definition

Let d be a cone-metric on (S, V). A decomposition of (S, d, V) into geodesic polygons with vertices at V is called ***Delaunay*** *if each polygon can be inscribed in a circle and all vertices of V except the vertices of the polygon lie outside the circle.*

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A ***Delaunay triangulation*** *of (S, d, V) is any triangulation refining the Delaunay decomposition.*

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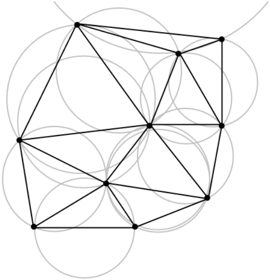
A Delaunay decomposition always exists and is unique.



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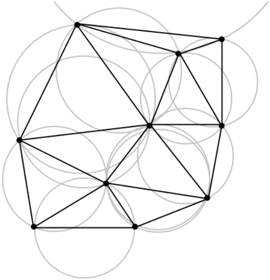


Discrete conformality

Definition

Two Euclidean cone-metrics d and d' on (S, V) are discretely conformally equivalent if there is a sequence of metrics with triangulations $(d_1, \mathcal{T}_1), \ldots, (d_m, \mathcal{T}_m)$ on (S, V) such that

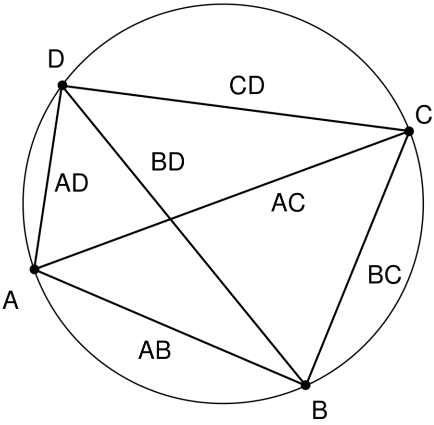
(i) $d_1 = d$, $d_m = d'$;

(ii) $\mathcal{T}_i$ is Delaunay for d_i;

(iii) either $\mathcal{T}_i = \mathcal{T}_{i+1}$ and d_i, d_{i+1} are discretely conformally equivalent with respect to $\mathcal{T}_i$;

(iv) or $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, but $d_i = d_{i+1}$ and $\mathcal{T}_i$, $\mathcal{T}_{i+1}$ are two different Delaunay triangulations of the same metric.

Discrete conformality: discussion



Ptolemy's relation
$ABCD$ is cyclic iff
$AC \cdot BD = AB \cdot CD + BC \cdot AD$.

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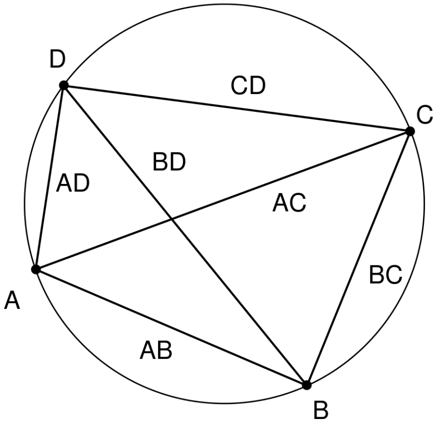

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If $\mathcal{T}_1$, $\mathcal{T}_2$ be two triangulations of (S, V, d) distinct by a diagonal flip in a quadrilateral Q and d' is d.c.e. to d with respect to $\mathcal{T}_1$, then d' is d.c.e to d with respect to $\mathcal{T}_2$ iff Q is cyclic.

Discrete conformality: curvature

Theorem (The Gauss–Bonnet theorem for Euclidean cone-metrics)

$$\sum_{v \in V} \kappa_v(d) = 2\pi\chi(S).$$

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Theorem (Gu–Luo–Sun–Wu, 2018)

Let $\widetilde{\kappa} : V \to (-\infty; 2\pi)$ *be a function such that* $\sum_{v \in V} \widetilde{\kappa}_v(d) = 2\pi\chi(S)$*. Then in every class of discrete conformal class of Euclidean cone-metrics there exists a unique up to scaling metric* d *with* $\kappa_v(d) = \widetilde{\kappa}(v)$ *for each* $v \in V$*. Moreover, there exists an algorithm to construct* d*.*

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Corollary (Discrete uniformization on a torus)

Each Euclidean cone-metric on a torus is discretely conformally equivalent to a unique up to scaling Euclidean metric.

Discrete conformality: discussion

Theorem

There exists an algorithm to decide if two cone-metrics on (S, V) are discretely conformally equivalent.

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Theorem (Smooth uniformization)

Let S be the (open) disk and g be a Riemannian metric on the closure of S. Then (S, g) is conformally diffeomorphic to the (open) unit disk in $\mathbb{E}^2$.

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There exists an algorithm to decide if two cone-metrics on (S, V) are discretely conformally equivalent.

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Theorem

There exists an algorithm, which computes the uniformization map from Theorem above.

Discrete conformality: the hyperbolic case

Definition

Two hyperbolic cone-metrics d and d' on (S, V) are discretely conformally equivalent if there is a sequence of metrics with triangulations $(d_1, \mathcal{T}_1), \ldots, (d_m, \mathcal{T}_m)$ on (S, V) such that

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$$\sinh \frac{l(e, d_i)}{2} = e^{u(v_1)+u(v_2)} \sinh \frac{l(e, d_{i+1})}{2};$$

(iv) or $\mathcal{T}_i \neq \mathcal{T}_{i+1}$*, but* $d_i = d_{i+1}$ *and* $\mathcal{T}_i$, $\mathcal{T}_{i+1}$ *are two different Delaunay triangulations of the same metric.*

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$$\sum_{v \in V} \kappa_v(d) = 2\pi\chi(S) + \mathrm{area}(S, d).$$

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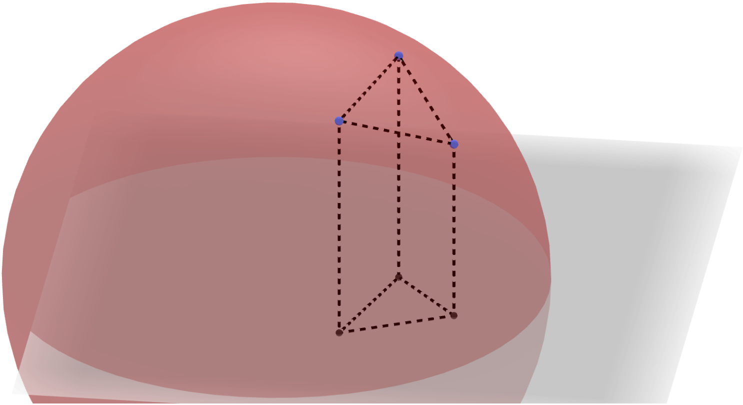
Corollary (Discrete uniformization with genus ≥ 2)

Each hyperbolic cone-metric on S_g, $g \geq 2$, is discretely conformally equivalent to a unique hyperbolic metric.

Approach: Ideal prisms



- An **ideal prism** is a prism in $\mathbb{H}^3$ with two triangular faces, all vertices of the upper face are ideal, all lateral edges are orthogonal to the lower face. It is uniquely determined by the lower face.

Approach: Ideal Fuchsian cone-manifolds

- - S_g is a closed surface of genus $g \geq 2$;
  - V is a set of marked points;
  - d is a hyperbolic cone-metric on (S_g, V);
  - $\mathcal{T}$ be a geodesic triangulation of (S_g, V, d).

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- Take each triangle T of $\mathcal{T}$ and construct the ideal prism with T as the lower boundary. Glue all these prisms according to $\mathcal{T}$. We obtain a hyperbolic cone-3-manifold. It is called an *ideal Fuchsian cone-manifold* $P_{\downarrow}(d, \mathcal{T})$. It is homeomorphic to $S_g \times [0; 1]$ minus points at the upper boundary.

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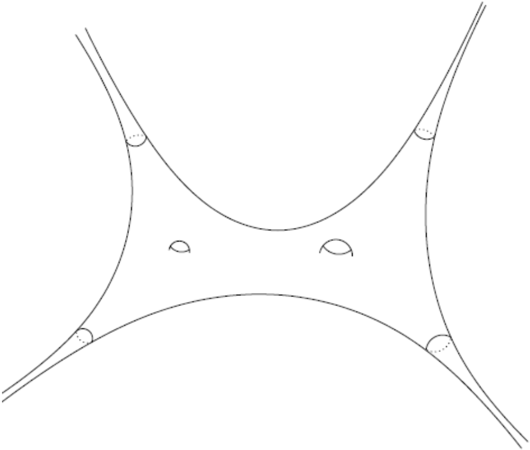
- If $\mathcal{T}$ is Delaunay, then it is denoted just as $P_{\downarrow}(d)$ and it is convex.
- The upper boundary of $P_{\downarrow}(d, \mathcal{T})$ is glued from ideal hyperbolic triangles. The obtained metric on the upper boundary is called a *hyperbolic cusp-metric* on S_g. In the case of $P_{\downarrow}(d)$ we denote it by $\tilde{d}$.

Approach: Hyperbolic cusp-metric



On a hyperbolic cusp-metric $\tilde{d}$ marked points are at infinite distance from each other.

Approach: Ideal Fuchsian cone-manifolds

Theorem

Let d, d' be two hyperbolic cone-metrics on (S_g, V). Then they are discretely conformally equivalent iff $\tilde{d}$ is isometric to $\tilde{d}'$.

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- Thus, to study a discrete conformal class one can study convex ideal Fuchsian cone-manifolds with a fixed upper boundary metric.

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- *Discrete uniformization.* We have a hyperbolic cone-metric d and we try to uniformize it. Construct $P_\downarrow(d)$, obtain $\tilde{d}$. To uniformize d means to deform $P_\downarrow(d)$, while preserving the upper boundary, so that cone-singularities dissolve.

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- $\tilde{d}$ is a hyperbolic cusp-metric on (S_g, V). Let $H(\tilde{d})$ be the set of convex ideal Fuchsian cone-manifolds having $\tilde{d}$ at its upper boundary.

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Theorem

$H(\tilde{d})$ can be identified with $\mathbb{R}^V$ with the help of hyperbolic decorations.

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- *Corollary*: uniqueness in the discrete uniformization.

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- *Corollary*: critical points of S in $\mathbb{R}^V$ correspond to metrics in the given discrete conformal class without cone-singularities.
- *Fact*: S is strictly concave over $\mathbb{R}^V$.
- *Corollary*: uniqueness in the discrete uniformization.
- *Task*: Prove existence.

Approach: the discrete curvature

- Let P be an ideal Fuchsian cone-manifold. Define the *discrete curvature* $S(P)$ as the sum of its total discrete scalar curvature and total discrete mean curvature of its boundary.
- Consider S as a function over $H(\tilde{d}) = \mathbb{R}^V$.
- *Fact*: $\frac{\partial S}{\partial h_v} = \kappa_v$. *Proof*: The Schläfli formula.
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- Follow the gradient flow of S!
  Prove that it stays in a compact convex subset of $\mathbb{R}^V$.

Step aside: isometric realizations

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Let M be a compact 3-manifold with boundary and d be a metric on ∂M. Does there exist a constant curvature metric on M such that the induced metric on ∂M is d? Is it unique?

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For every convex Euclidean (resp. hyperbolic) cone-metric d on S^2 there exists a unique convex polyhedron $P \subset \mathbb{E}^3$ (resp. $P \subset \mathbb{H}^3$) such that (S^2, d) is isometric to the boundary of P.

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Theorem (Rivin, 1994; Springborn 2020)

For every hyperbolic cusp-metric d on S^2 there exists a unique ideal convex polyhedron $P \subset \mathbb{H}^3$ such that (S^2, d) is isometric to the boundary of P.

Step aside: geometrization

- Our case: the manifold is $S_g \times [0;1]$, d is a hyperbolic cusp metric on $S_g \times \{1\}$. Additional condition: $S_g \times \{0\}$ is geodesic. It turns out that this is equivalent to the discrete uniformization.

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- In 70s Thurston proposed an analogue of the unformization for 3-manifolds: each 3-manifold can be decomposed canonically into pieces and each piece can be endowed with one of 8 canonical geometries. Among all of them hyperbolic manifolds are the most mysterious.
- Hyperbolic structures on closed 3-manifolds are unique. On manifolds with convex boundary they should be determined by the boundary metric.

The end

Thank you!

Roman Prosanov