The motion of billiards in ellipses

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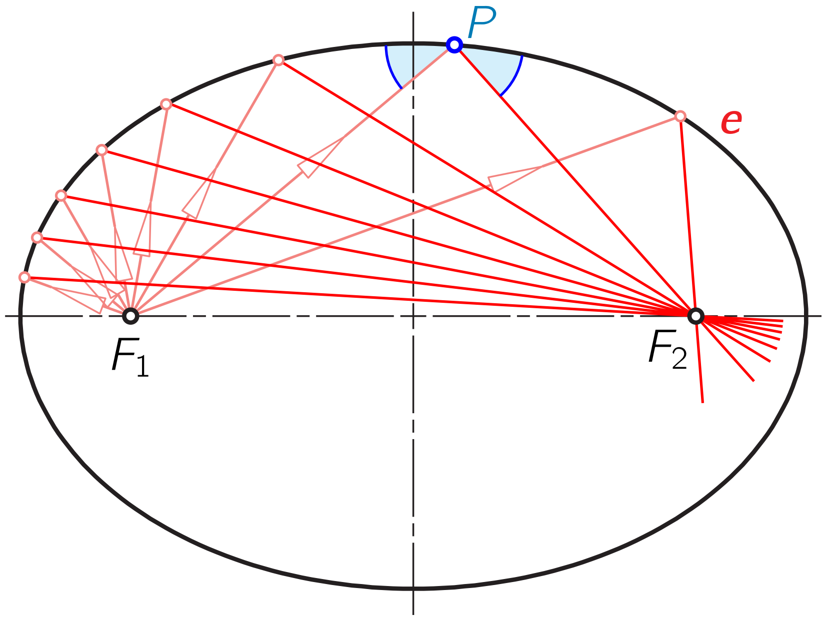
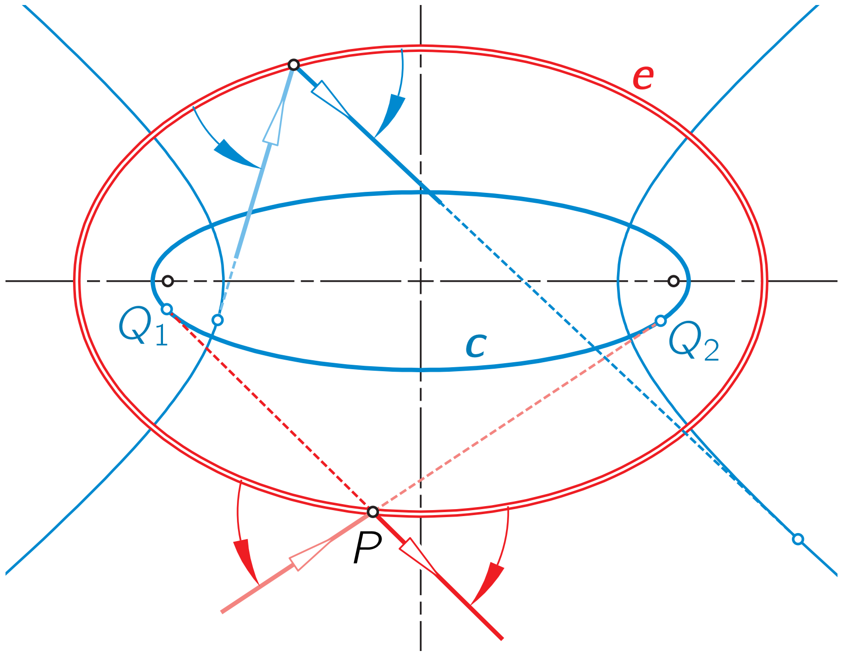


The optical property of ellipses is well known, and also the equivalence:

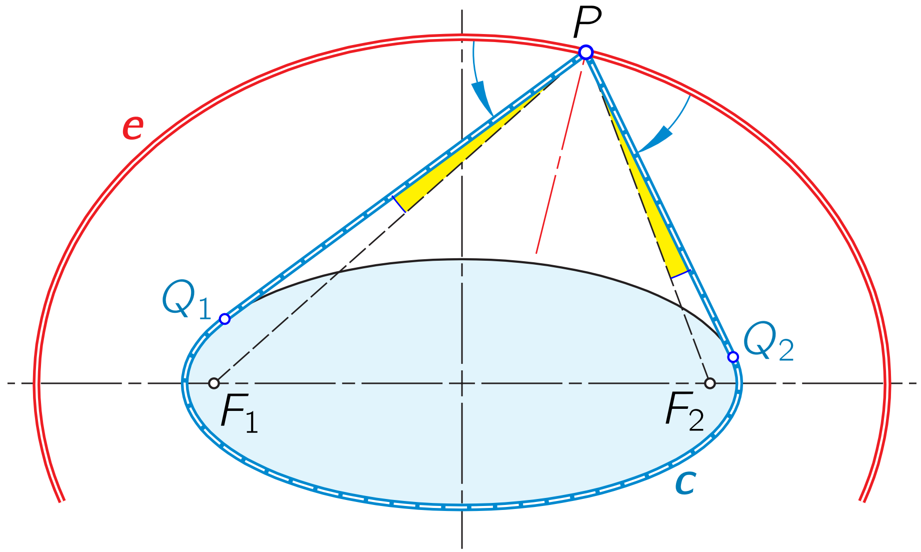
$$\text{equal angles} \iff \overline{F_1P} + \overline{F_2P} = \text{const.} \iff P \in e\,.$$

There is a generalization:

1. Metric properties of confocal conics



If any ray is reflected in a conic e then the incoming and the outgoing ray are tangent to the same confocal conic c, called **caustic**.

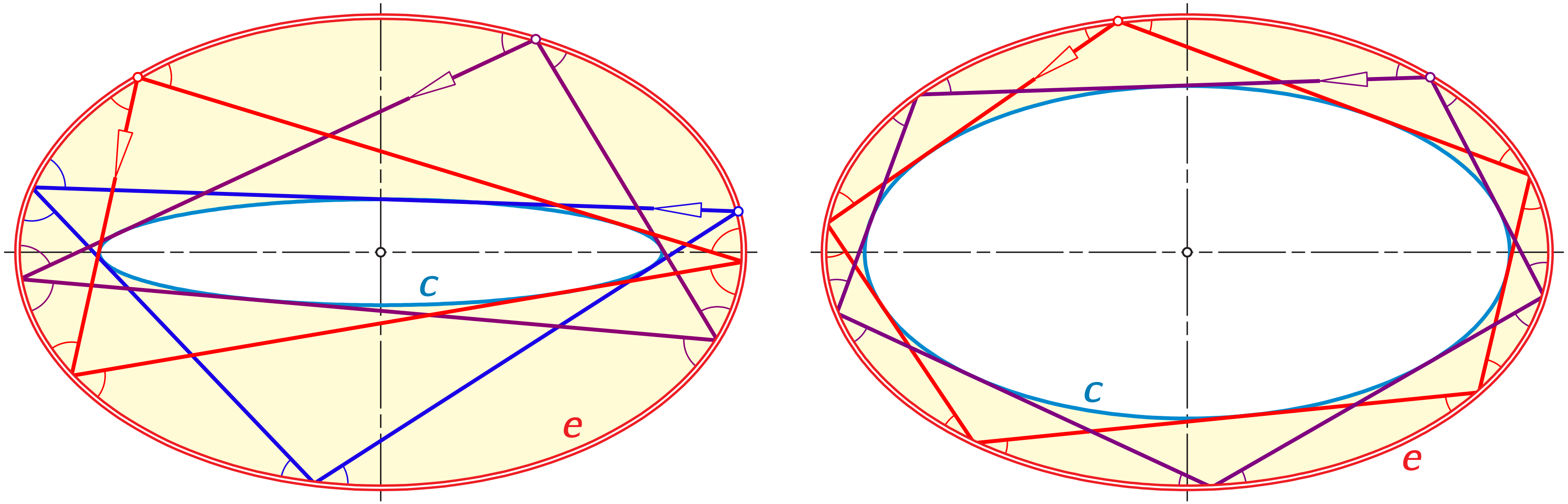


Charles **Graves** (1812-1899), bishop of Limerick and mathematician:

The locus of point P used to pull the string taut around c is a confocal ellipse e.

$$D_e := \overline{PQ_1} + \overline{PQ_2} - \overset{\frown}{Q_1Q_2} = \text{const}.$$

1. Metric properties of confocal conics



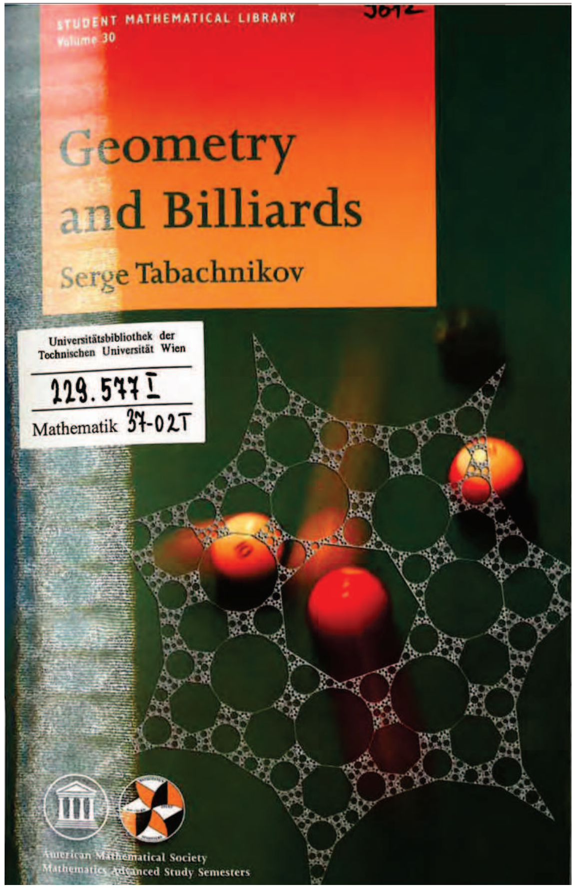
Billiards in an ellipse e are always tangent to a confocal ellipse or hyperbola.

If one billiard closes after N reflections, then all billiards close, independent of the initial point on c (**Poncelet porism**), and all these closed loops have the same length.

1. Metric properties of confocal conics

For centuries, billiards attracted the attention of mathematicians, beginning with J.-V. Poncelet and A. Cayley.

S. Tabachnikov: *Geometry and Billiards.*
American Mathematical Society, 2005



Recently, Dan Reznik revitalized the interest by computer animations showing the variation of periodic billiards. He identified 40 invariants, e.g., a constant sum of Cosines of interior angles.

Sum of square altitudes to N-periodic tangents is invariant

The two types of self-intersected 7-periodics in the Elliptic Billiard

1. Metric properties of confocal conics

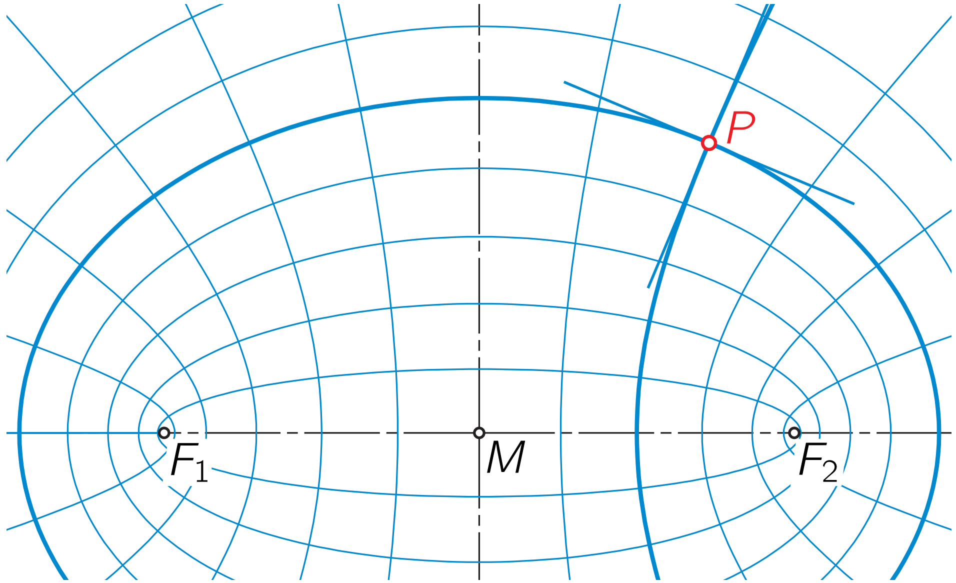
A family of confocal central conics

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} = 1,$$

$k \in \mathbb{R} \setminus \{-a^2, -b^2\}$ sends through each point P outside the axes one ellipse and one orthogonally intersecting hyperbola.

The parameters (k_e, k_h) define the *elliptic coordinates* of P with

$$-a^2 < k_h < -b^2 < k_e\,.$$



We specify the caustic c (semiaxes a_c, b_c) as $k = 0$ and the ellipse e with semiaxes a_e, b_e as $k = k_e \implies k_e = a_e^2 - a_c^2 = b_e^2 - b_c^2 > 0\,.$

1. Metric properties of confocal conics

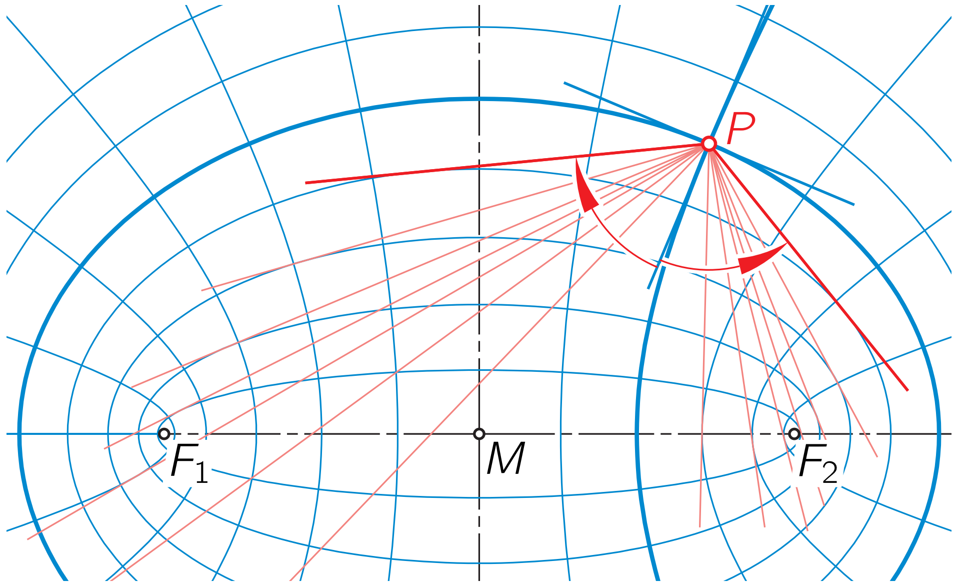
From given (k_e, k_h) follows

$$x^2 = \frac{(a^2+k_e)(a^2+k_h)}{d^2}, \qquad y^2 = -\frac{(b^2+k_e)(b^2+k_h)}{d^2}.$$

Conversely, $P = (a_e \cos t,\ b_e \sin t)$ with tangent vector

$$\mathbf{t}_e(t) = (-a_e \sin t,\ b_e \cos t)$$

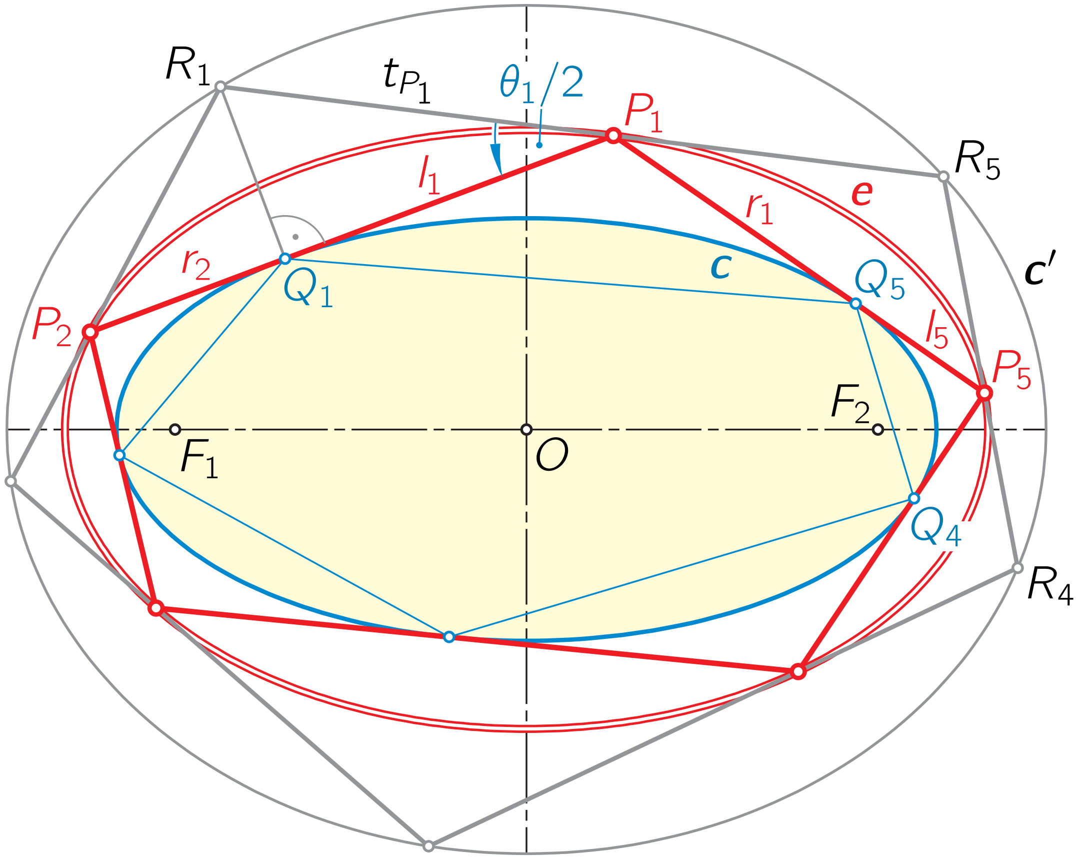
and $\mathbf{t}_c(t) = (-a_c \sin t,\ b_c \cos t)$ yield



$$k_h(t) = -\,(a_c^2 \sin^2 t + b_c^2 \cos^2 t) = -\|\mathbf{t}_c(t)\|^2 = -\|\mathbf{t}_e(t)\|^2 + k_e\,.$$

Points on confocal ellipses e and c with the same parameter t have the same coordinate k_h, i.e., they belong to the same confocal hyperbola.

1. Metric properties of confocal conics



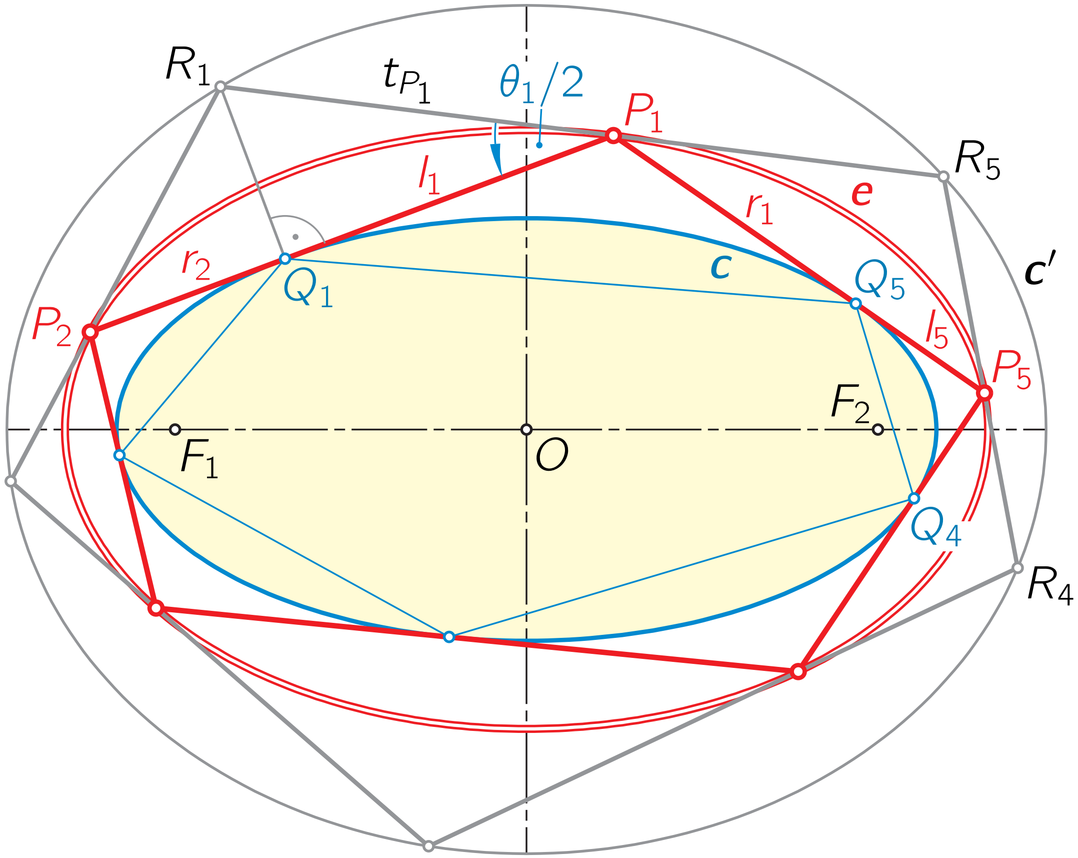
Given $P = (a_e \cos t,\ b_e \sin t)$, the tangents from P to c ($k = 0$) include the angle $\theta(t)$ where

$$\sin^2 \frac{\theta(t)}{2} = \frac{k_e}{\|\mathbf{t}_e(t)\|^2},$$

$$\tan \frac{\theta(t)}{2} = \pm \frac{\sqrt{k_e}}{\|\mathbf{t}_c(t)\|},$$

$$\cos\theta = 1 - \frac{2k_e}{\|\mathbf{t}_e(t)\|^2} = \frac{k_h(t) + k_e}{k_h(t) - k_e}.$$

1. Metric properties of confocal conics



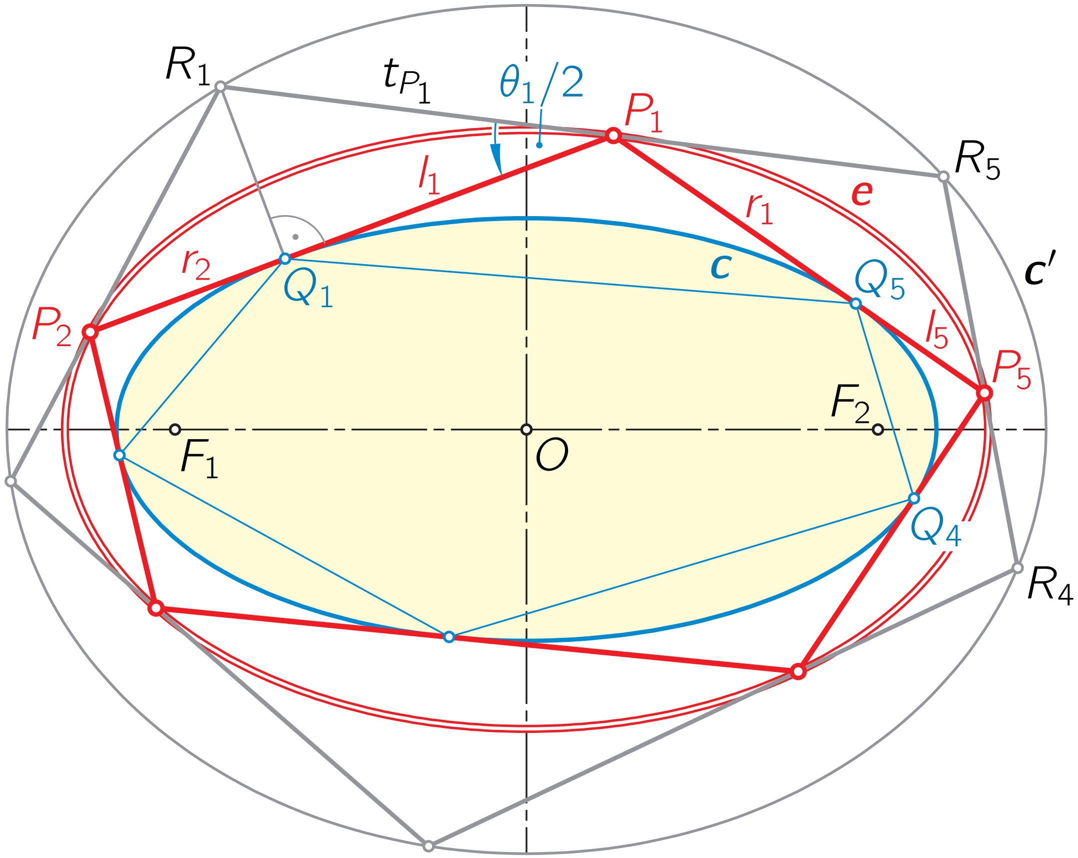
The signed distances of the side $t_Q = P_1P_2$ from the center O and the pole R_1 w.r.t. e have the constant product $-k_e$.

For $Q_1 = (a_c \cos t'_1,\ b_c \sin t'_1)$,

$$\overline{O t_Q} = \frac{-a_c b_c}{\|\mathbf{t}_c(t'_1)\|}$$

$$\overline{R t_Q} = \overline{R_1 Q_1} = \frac{k_e \|\mathbf{t}_c(t'_1)\|}{a_c b_c} \,.$$

1. Metric properties of confocal conics



The side P_1P_2 (parameters t_1, t_2) contacts the caustic c iff

$$\sin^2\frac{t_1-t_2}{2} = \frac{k_e}{a_e b_e}\left\|\mathbf{t}_e\left(\frac{t_1+t_2}{2}\right)\right\|^2.$$

If t_1' is the parameter of the tangency point $Q_1 \in c$, then

$$\tan t_1' = \frac{b_c a_e}{a_c b_e}\tan\frac{t_1+t_2}{2}.$$

Half angle substitution yields ...

1. Metric properties of confocal conics

The half-angle substitution $\tau_i := \tan\frac{t_i}{2}$ yields for P_1 and P_2 a symmetric biquadratic equation in projective coordinates on e, namely

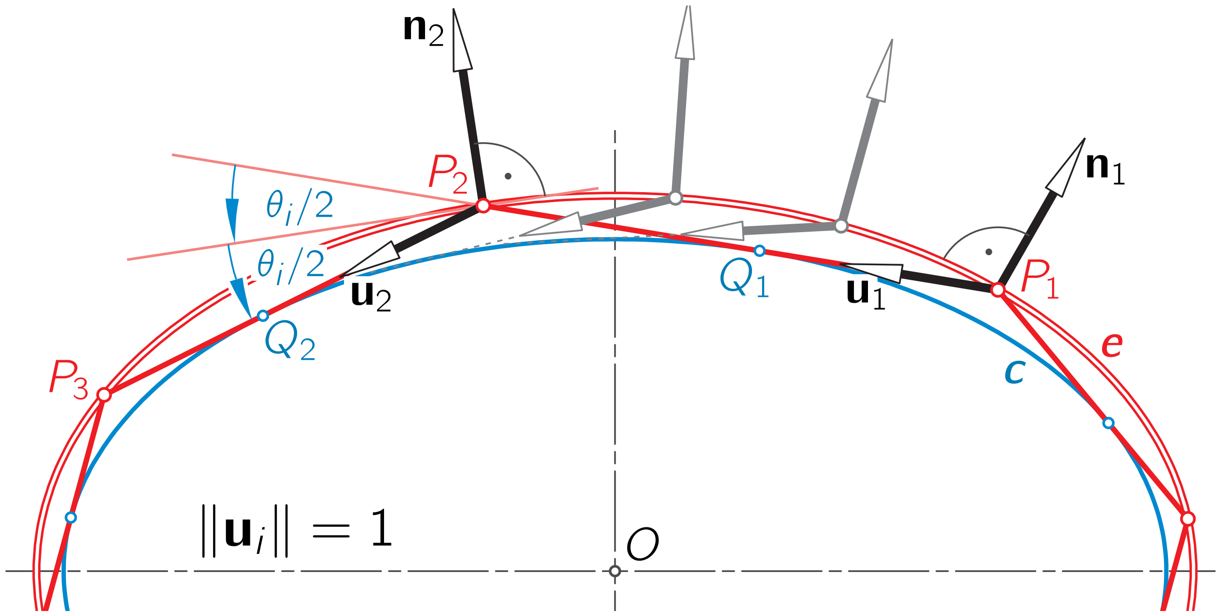
$$b_e^2 k_e \tau_1^2 \tau_2^2 - b_c^2 a_e^2 (\tau_1^2 + \tau_2^2) + 2(a_e^2 k_e + a_c^2 b_e^2)\tau_1 \tau_2 + b_e^2 k_e = 0,$$

which defines a 2-2-correspondence between consecute points P_1, P_2 of a billiard. The same holds after N iteration between P_1 and P_{N+1}.

We recall a classical algebraic argument for the Poncelet porism:

A 2-2-correspondence ($\neq$ id) has at most four fixed points. However, four fixed points are already known as contact points between e and the common (isotropic) tangents with the caustic c. Hence, if one N-sided billiard closes, then the correspondence is the identity, and each billiard inscribed in e with caustic c must close.

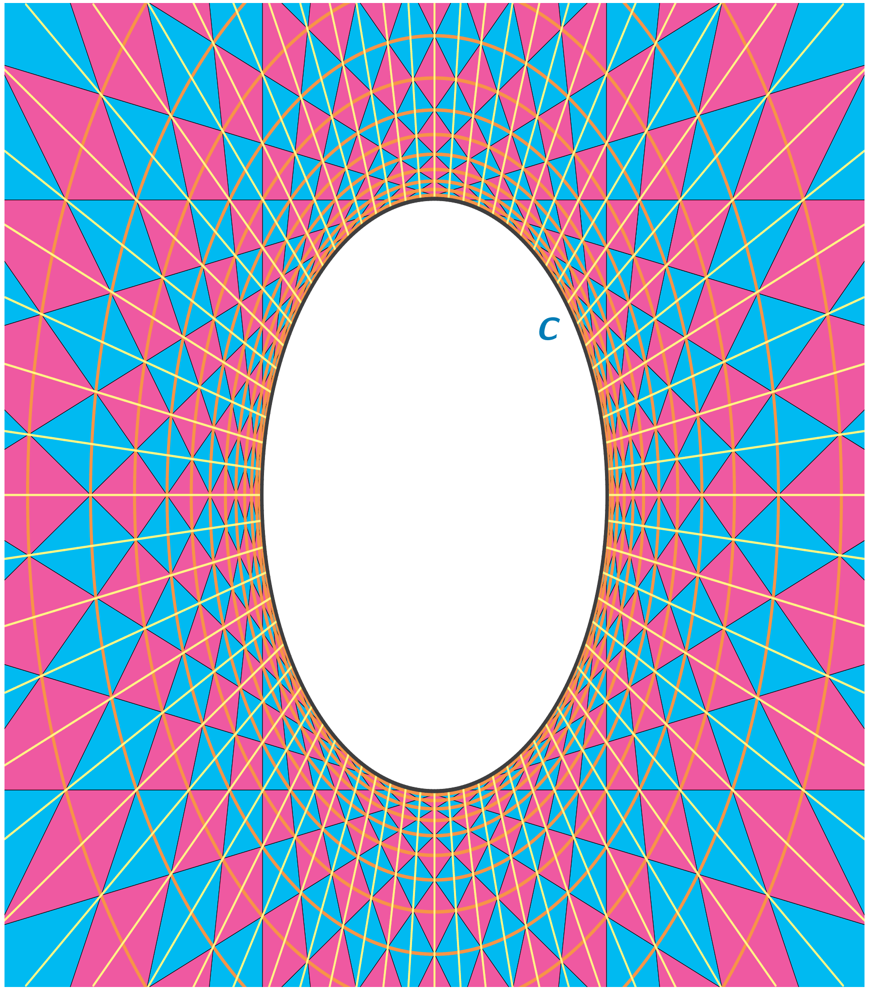
2. Confocal conics and billiards



S. Tabachnikov: the key result for the integrability of billiards is the **Joachimsthal integral** $J_e := -\langle \mathbf{u}_i, \mathbf{n}_i \rangle$ with $\mathbf{u}_i$ as unit vector of P_iP_{i+1} and $\mathbf{n}_i = (\cos t/a_e, \sin t/b_e)$ as a normal vector of e. This holds in all dimensions ($\mathbf{n}_i = \mathbf{A}\,\mathbf{p}_i = \mathbf{p}_i^*$). In the plane

$$J_e = -\langle \mathbf{u}_i,\, \mathbf{n}_i \rangle = -\cos\!\left(\frac{\pi}{2} + \frac{\theta_i}{2}\right)\|\mathbf{n}_i\| = \sin\frac{\theta_i}{2}\,\|\mathbf{n}_i\| = \frac{\sqrt{k_e}}{\|\mathbf{t}_e\|}\,\frac{\|\mathbf{t}_e\|}{a_e b_e} = \frac{\sqrt{k_e}}{a_e b_e}\,.$$

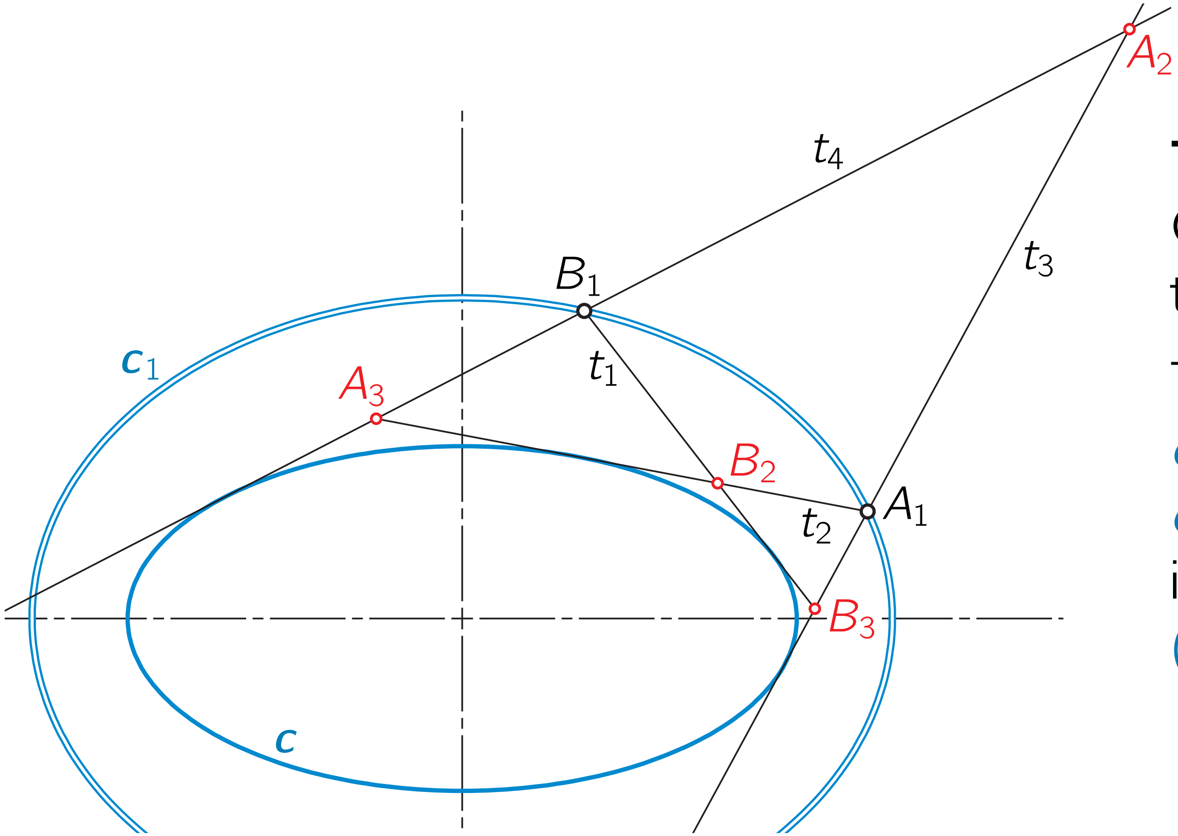
2. Confocal conics and billiards



The extended sides of a billiard intersect at points of confocal ellipses and hyperbolas and form a **Poncelet grid**.

affinely transformed 72-sided periodic billiard with associated Poncelet grid (G. Glaeser, B. Odehnal, H.S.: *The Universe of Conics*, 7 KB!)

2. Confocal conics and billiards



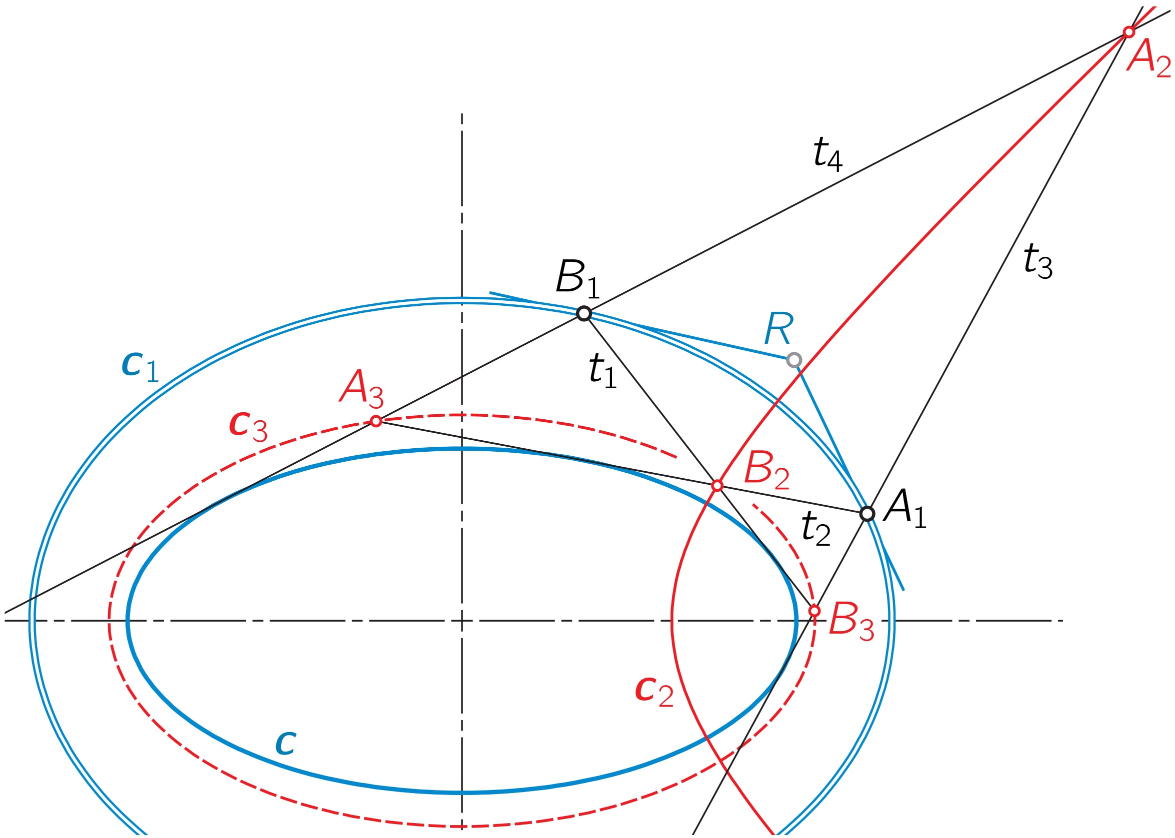
Theorem:
Given a quadrilateral $t_1, \ldots, t_4$ of tangents to c from $A_1, B_1 \in c_1$.

Then the range $\mathcal{R}_c$ spanned by c and c_1 contains conics c_2, c_3 passing through the remaining pairs of opposite vertices (A_2, B_2) and (A_3, B_3).

(**range** = 'dual pencil')

= summary of results from M. Chasles (1843), W. Böhm (1961), Izmestiev & Tabachnikov (2016), Akopyan & Bobenko (2017).

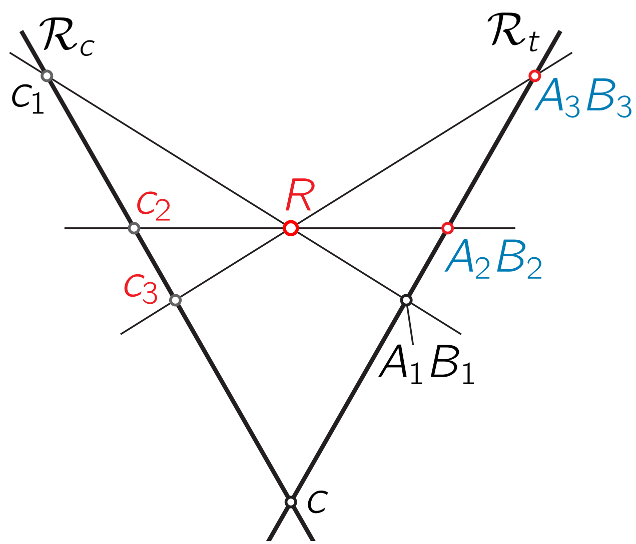
2. Confocal conics and billiards



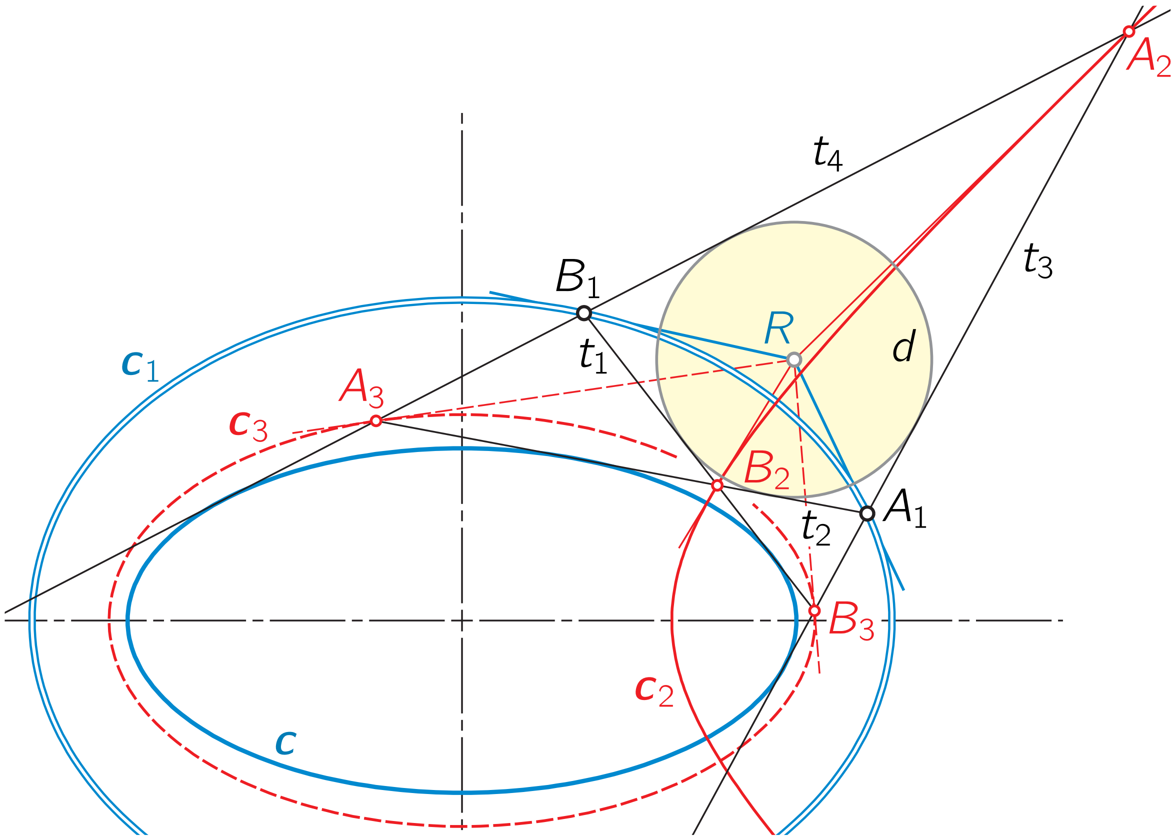
$\mathcal{R}_t := \{\text{conics tangent to } t_1, \ldots, t_4\}$

$\mathcal{R}_c \cap \mathcal{R}_t = \{c\} \implies$ they span a net $\mathcal{N}$ (2-parameter set).

In $\mathcal{N}$, the line elements of c_1 at A_1 and B_1 span a range which contains the rank-1 conic R.



2. Confocal conics and billiards



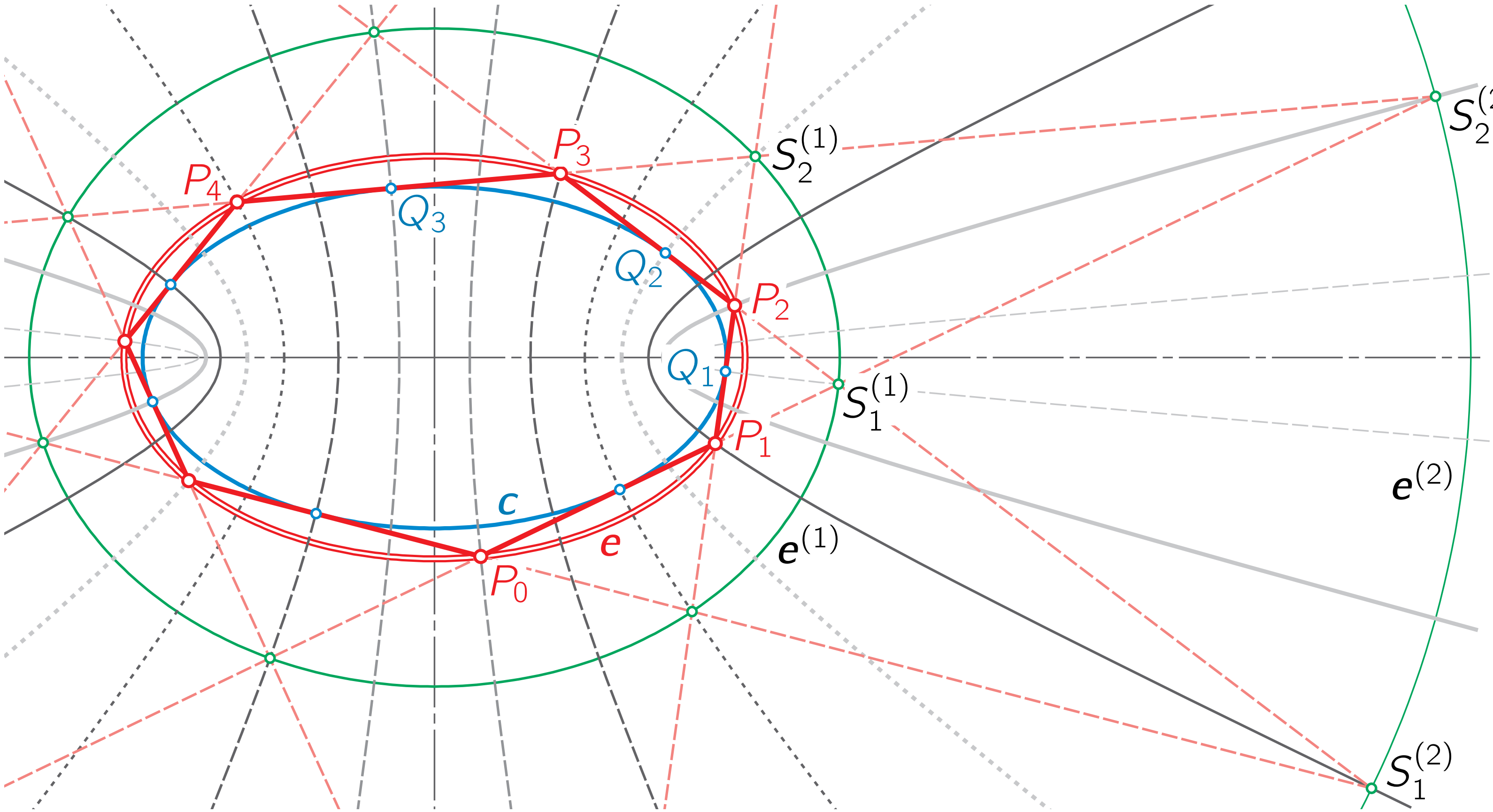
In $\mathcal{N}$, the line pencils A_i, B_i and the pencil R (2-fold) span a range which intersects $\mathcal{R}_c$ at c_i. The range contains conics sharing the line elements at A_i and B_i.

The tangents to c_i at A_i and B_i pass through R.

Confocal $c, c_1 \implies$ concyclic quadrilateral.

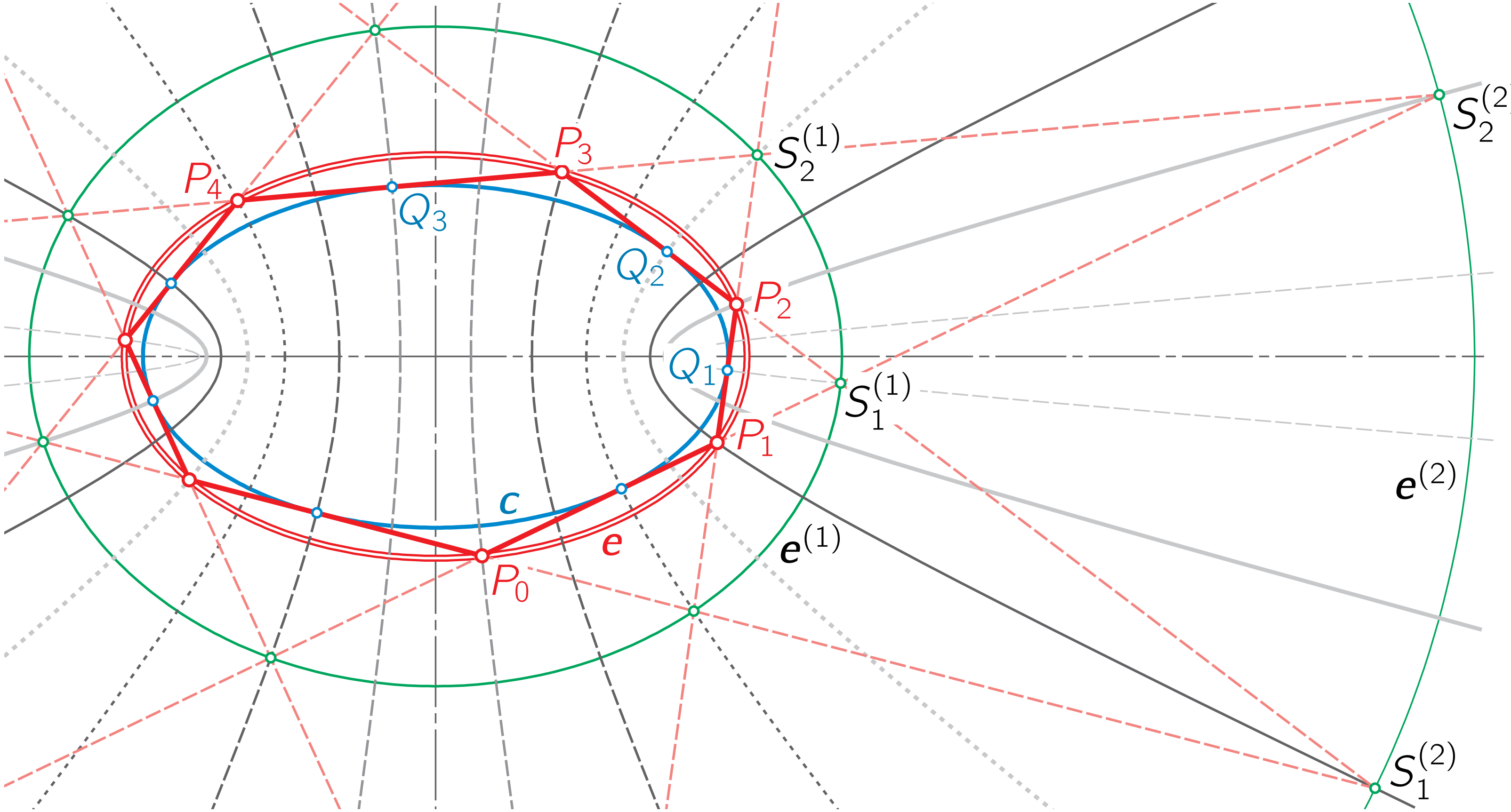
This holds also when $B_2 \in c$ $(t_1 = t_2)$.

2. Confocal conics and billiards



$S_2^{(1)} := [P_1, P_2] \cap [P_3, P_4]$ on the confocal hyperbola through Q_2,

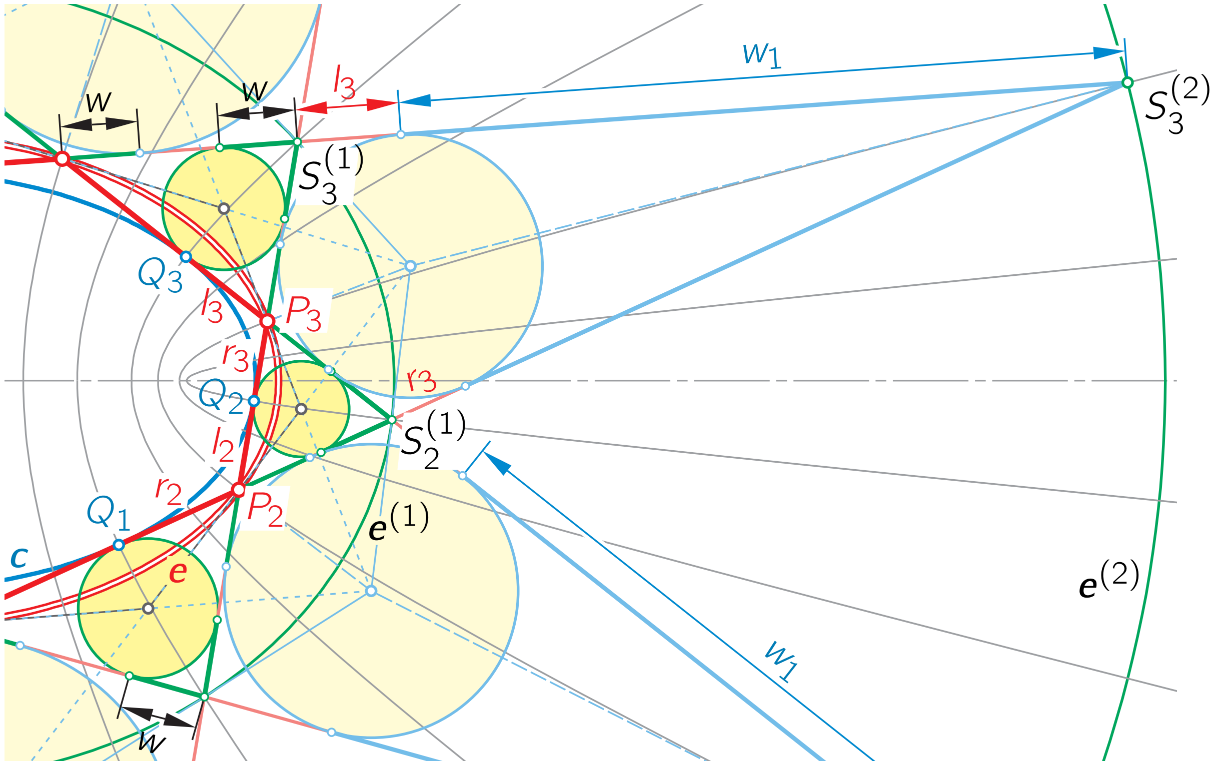
$S_2^{(2)} := [P_0, P_1] \cap [P_3, P_4] \in e^{(2)}$ on the confocal hyperbola through P_2.



$S_i^{(1)} := [P_{i-1}, P_i] \cap [P_{i+1}, P_{i+2}]$ on the confocal ellipse $e^{(1)}$ for all i,

and $e^{(1)}$ is invariant of the initial data, $k_{e|1} = k_e \left(\dfrac{2a_c b_c a_e b_e}{a_c^2 b_c^2 - k_e^2} \right)^2.$

2. Confocal conics and billiards



Surprisingly, the **distance *w*** between $S_2^{(1)}$ and the contact point with the incircle is invariant; the same for w_1.

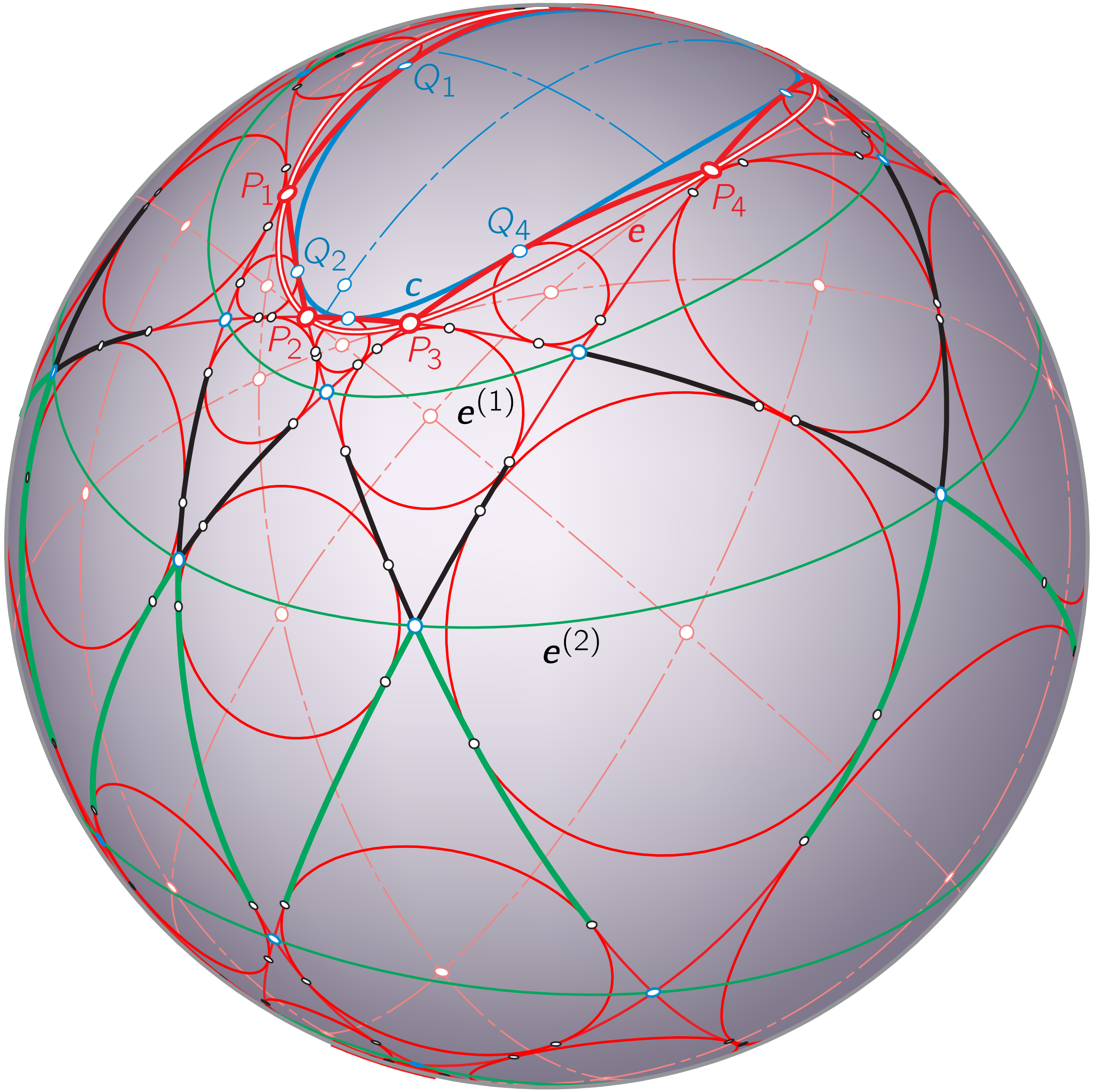
Graves's construction $\Longrightarrow$

$$D_e := \overline{Q_{i-1}P_i} + \overline{P_iQ_i} - \overset{\frown}{Q_{i-1}Q_i}$$

$$= r_i + l_i - \overset{\frown}{Q_{i-1}Q_i}$$

is constant.
Similarly, for $e^{(1)}$ follows

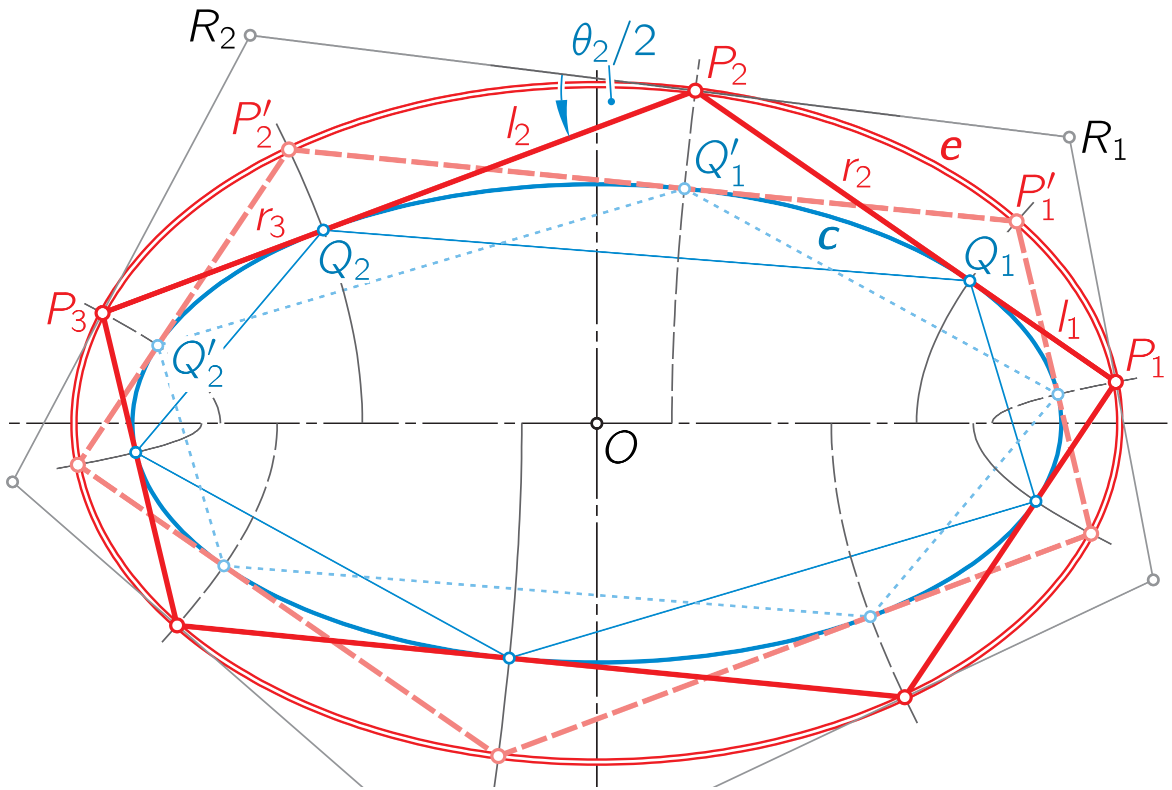
$$D_{e|1} := \overline{Q_1S_2^{(1)}} + \overline{S_2^{(1)}Q_3} - \overset{\frown}{Q_1Q_3} = (r_2+l_2+w) + (r_3+l_3+w) - \overset{\frown}{Q_1Q_3} = 2D_e + 2w.$$



The same invariants show up on the sphere. All circular arcs in black have the same length. The same 'in green'.

In the plane: $w = \dfrac{2a_e b_e \sqrt{k_e^3}}{a_c^2 b_c^2 - k_e^2}$

2. Confocal conics and billiards



For each billiard $P_0P_1P_2\dots$ in the ellipse e with caustic c, there exists a **conjugate** billiard $P_0'P_1'P_2'\dots$

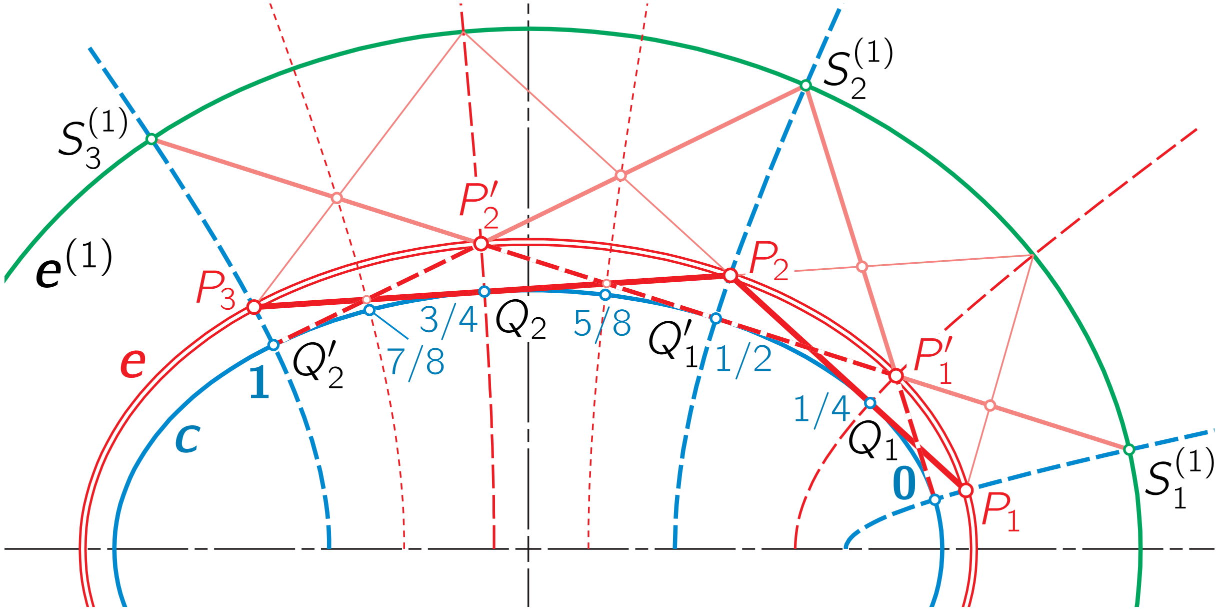
The axial scaling with $c \to e$,

$$\alpha\colon\ Q_i \mapsto P_i', \quad Q_{i-1}' \mapsto P_i\,,$$

transforms tangents P_1P_2 of c to tangents $P_1'P_2'$. This results from the symmetry between t_i and t_i' in the equation

$b_c a_e \cos t_i \cos t_i' + a_c b_e \sin t_i \sin t_i' = a_c b_c$, which expresses that P_i lies on the tangent to c at Q_i and P_i' on the tangent at Q_i' .

2. Confocal conics and billiards



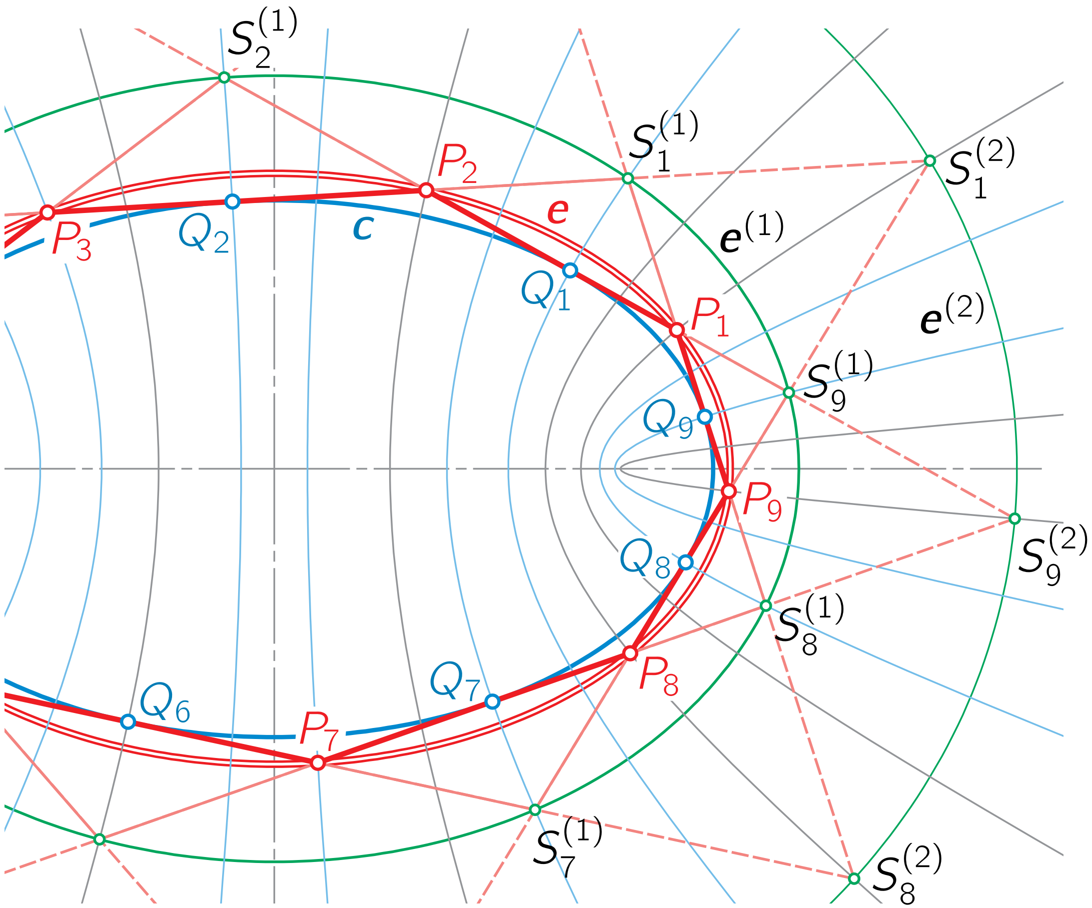
One might say, point P_i' is halfway from P_i to P_{i+1}.

In the sequence of parameters $t_1, t_1', t_2, t_2', \ldots$ for $P_1, P_1'(Q_1), P_2, P_2'(Q_2), \ldots$, the transition $P_i \mapsto P_i'$ means a shift $t_i \mapsto t_i'$.

I. Izmestiev, S. Tabachnikov (2017): There exists a **canonical parametrization** of e such that the billiard transformation $P_i \to P_{i+1}$ corresponds to a shift $u_i \to u_{i+1} = u_i + 2\,\Delta u$.

Above, an example of canonical parameters: $P_1 \sim 0$, $P_2 \sim \frac{1}{2}$, $P_3 \sim 1$.

3. Periodic N-sided billiards



The **turning number** $\boldsymbol{\tau}$ of a periodic billiard counts the loops around the center.

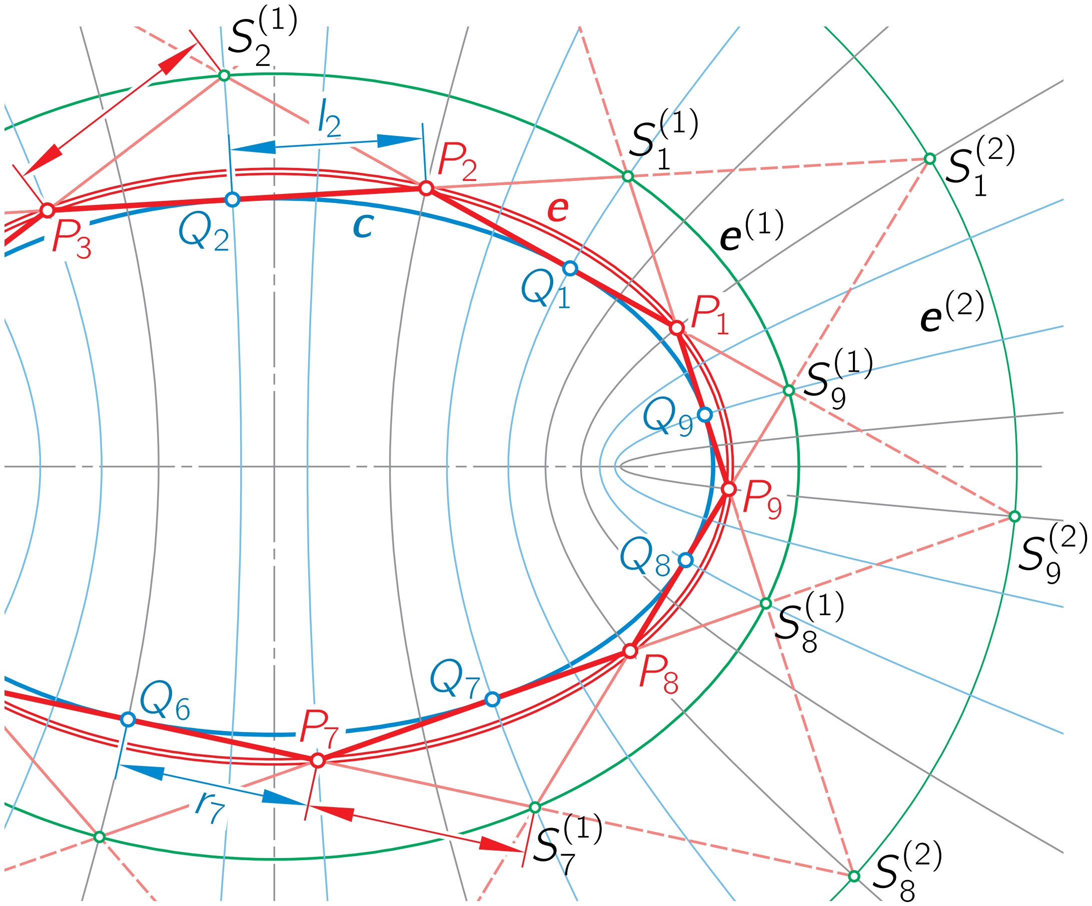
Theorem:

(i) A periodic billiard with even N and odd τ is centrally symmetric.

(ii) For odd N and odd τ, the billiard is centrally symmetric to the conjugate billiard.

(iii) For odd N and even τ the billiard conincides with the conjugate billiard.

3. Periodic N-sided billiards



From Graves' theorem

$$D_e = \overline{Q_{i-1}P_i} + \overline{P_iQ_i} - \overset{\frown}{Q_{i-1}Q_i}$$

follows for the perimeter of the N-sided billiard

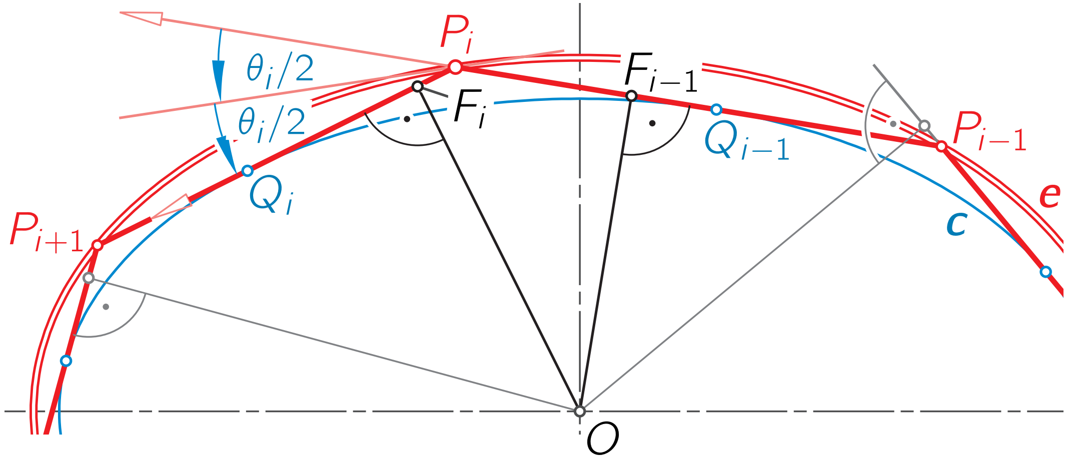
$$L_e = N \cdot D_e + \tau \cdot P_c$$

with P_c as perimeter of e.

Ivory's theorem implies for **odd** $N = 2n+1$: the length l_i equals symmetric l'_{i+n} of the conjugate billiard and r_{i+n+1} of the original one.

Theorem: $\sum l_i = \sum r_i = L_e/2.$

3. Periodic N-sided billiards



$$\overline{P_iF_i} + \overline{P_iF_{i-1}} = \frac{2a_eb_e}{\|\mathbf{t}_e(t_i)\|^2}\sqrt{k_e}$$

$$L_e = \sum_{i=1}^N \left(\overline{P_iF_i} + \overline{P_iF_{i-1}}\right) \Longrightarrow$$

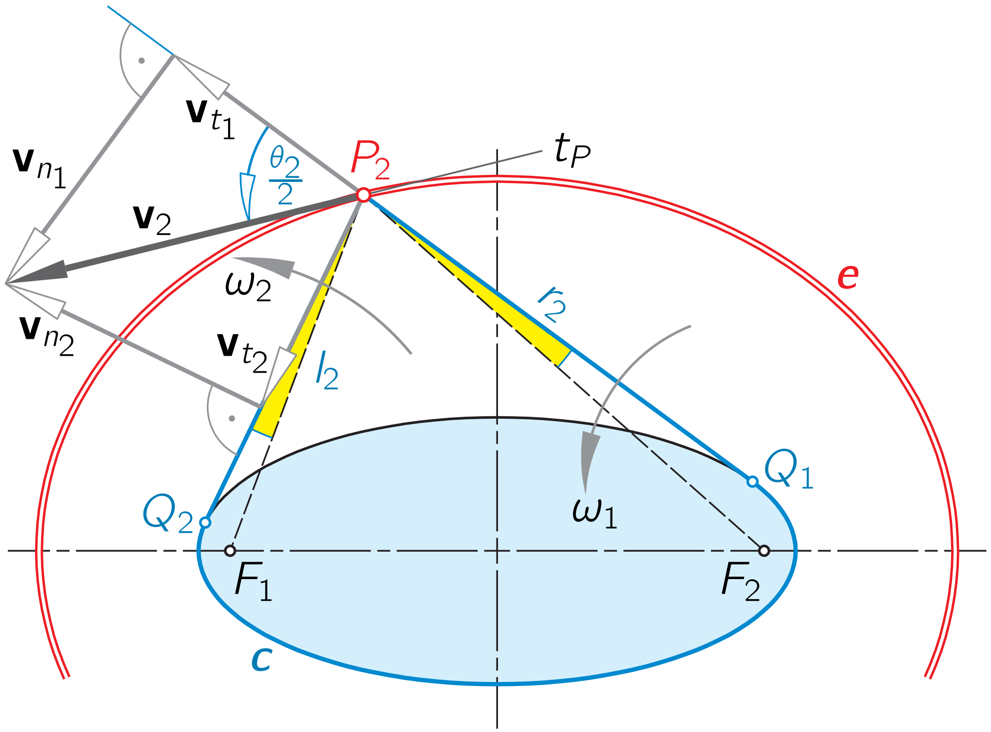
$$L_e = \frac{a_eb_e}{\sqrt{k_e}} \sum_{i=1}^N \frac{2k_e}{\|\mathbf{t}_e(t_i)\|^2}$$

$$= \frac{a_eb_e}{\sqrt{k_e}} \sum_{i=1}^N (1 - \cos\theta_i)$$

Theorem [Akopyan, Schwartz, Tabachnikov, Bialy] $\sum_{i=1}^N \cos\theta_i = N - \frac{\sqrt{k_e}}{a_eb_e} L_e$.

With $\sum_{i=1}^N \frac{1}{\|\mathbf{t}_e(t_i)\|^2}$ also $\sum_{i=1}^N \overline{Ot_P}^2$ and $\sum_{i=1}^N \kappa_e(t_i)^{2/3}$ are invariant.

4. A switch to Analysis



With a **billiard motion** we denote a variation of the billiard and the Poncelet grid induced by the variation of a single vertex.

According to Graves' construction, the $\mathbf{v}_2$ of P_2 can be decomposed as

$$\mathbf{v}_2 = \mathbf{v}_{t_1} + \mathbf{v}_{n_1} = \mathbf{v}_{t_2} + \mathbf{v}_{n_2},$$

where due to the constant length

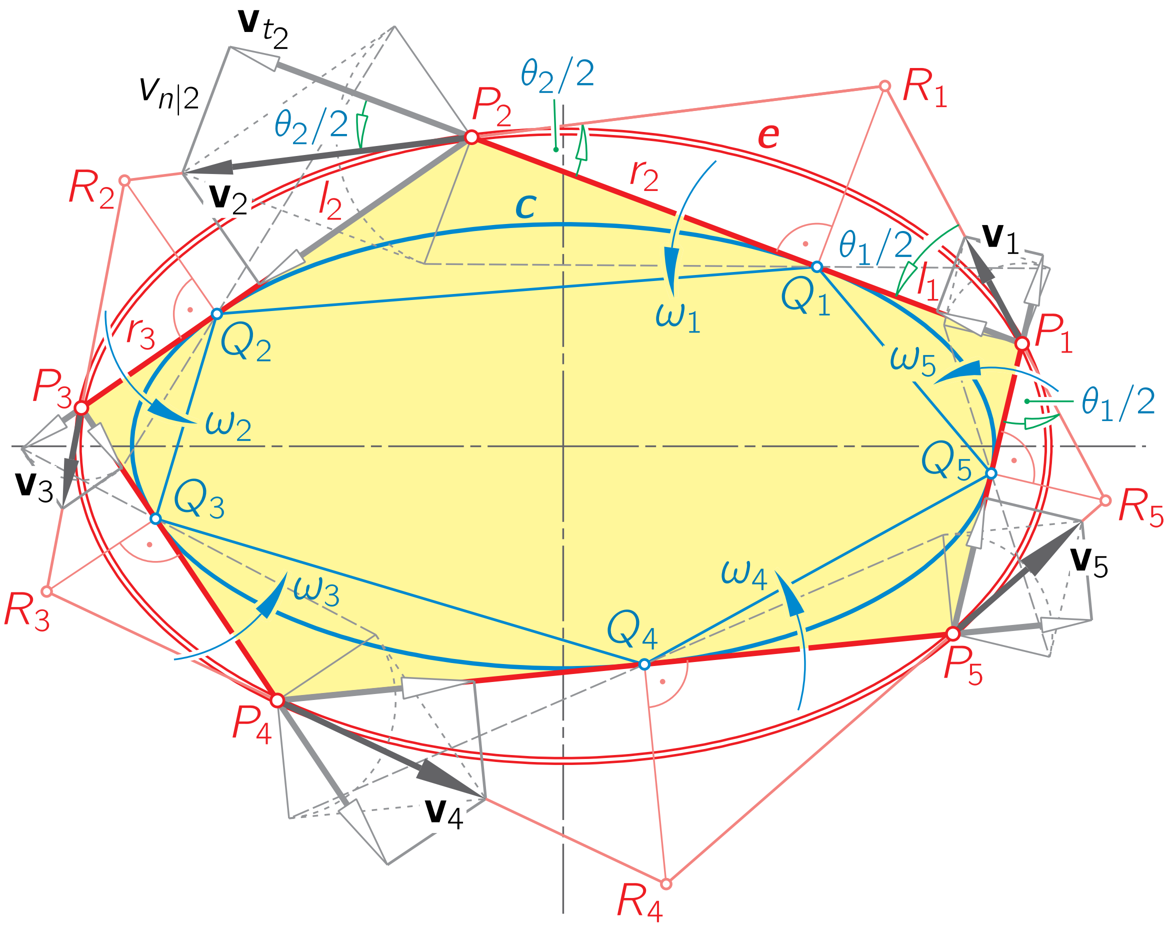
$\|\mathbf{v}_{t_2}\| = \|\mathbf{v}_{t_1}\|$ and $\|\mathbf{v}_{n_2}\| = \|\mathbf{v}_{n_1}\|$

$l_2\,\omega_2 = r_2\,\omega_1$, hence $\dfrac{\omega_1}{\omega_2} = \dfrac{l_2}{r_2}$.

If the billiard is periodic, then the product of all ratios l_i/r_i yields

$$\frac{l_1}{r_1}\cdot\frac{l_2}{r_2}\cdots\frac{l_N}{r_N} = \frac{\omega_N}{\omega_1}\cdot\frac{\omega_1}{\omega_2}\cdots\frac{\omega_{N-1}}{\omega_N} = 1, \quad \text{hence} \quad l_1 l_2 \dots l_N = r_1 r_2 \dots r_N\,.$$

4. A switch to Analysis



For periodic billiards, a given velocity vector $\mathbf{v}_2$ of any vertex defines all velocities.

In terms of the exterior angles $\theta_1, \ldots, \theta_N$ we obtain

$$\sin\frac{\theta_2}{2} = \frac{l_2\,\omega_2}{v_2} = \frac{r_2\,\omega_1}{v_2} \quad \text{and}$$

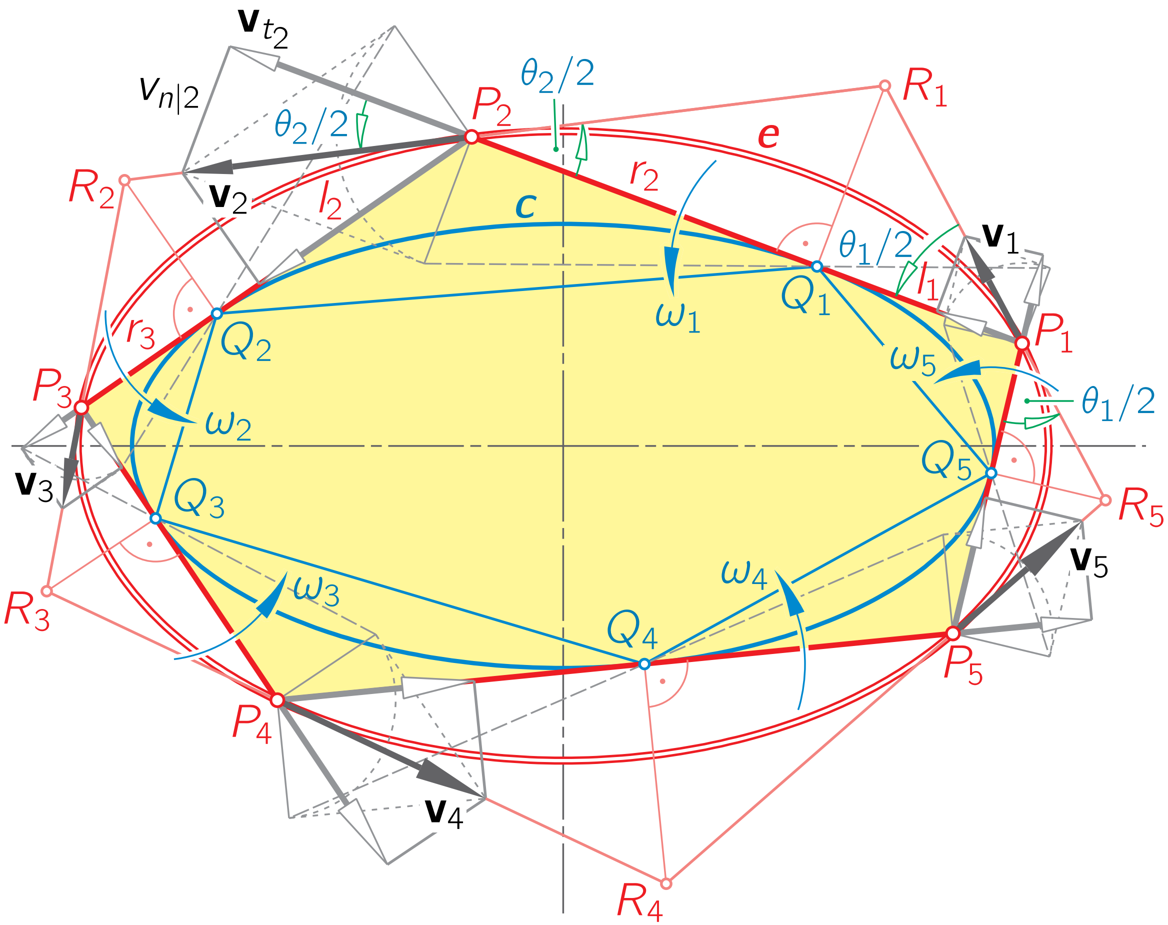
$$\cos\frac{\theta_2}{2} = \frac{v_{t|2}}{v_2}, \quad \text{where}$$

$v_2 := \|\mathbf{v}_2\|, \ \ v_{t|2} := \|\mathbf{v}_{t|2}\|.$

From $P_1P_2 \perp Q_1R_1$ follows

$$\overline{R_1Q_1} = l_1 \tan\frac{\theta_1}{2} = r_2 \tan\frac{\theta_2}{2}\,.$$

4. A switch to Analysis



After some manipulations follows

$$v_{t|1}\tan^2\frac{\theta_1}{2} = v_{t|2}\tan^2\frac{\theta_2}{2}$$

$$\cdots = v_{t|i}\,\frac{k_e}{\|\mathbf{t}_c(t_i)\|^2} =: C\,.$$

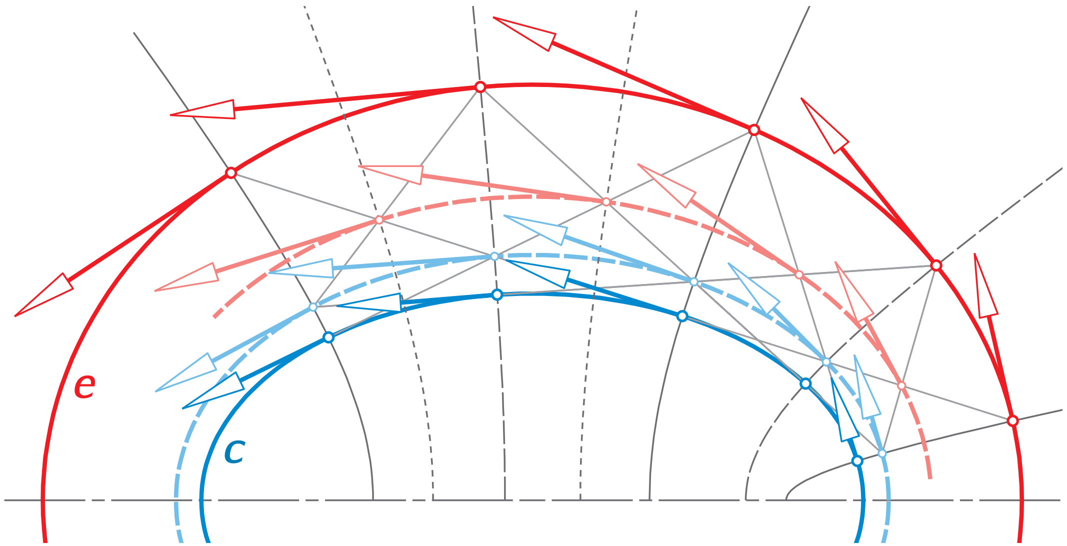
Instead of a free choice of v_2, we set $C = k_e$.

$$v_{t|i} = \|\mathbf{t}_c\|^2,\;\; v_i = \|\mathbf{t}_c\|\,\|\mathbf{t}_e\|,$$

$$v_{n|i} = v_i \sin\frac{\theta}{2} = \|\mathbf{t}_c\|\,\sqrt{k_e}$$

for all $t = t_i$ and for all confocal ellipses e.

4. A switch to Analysis



To each point

$$\mathbf{p} = (x, y) = (a_e \cos t,\ b_e \sin t)$$

we assign a velocity vector

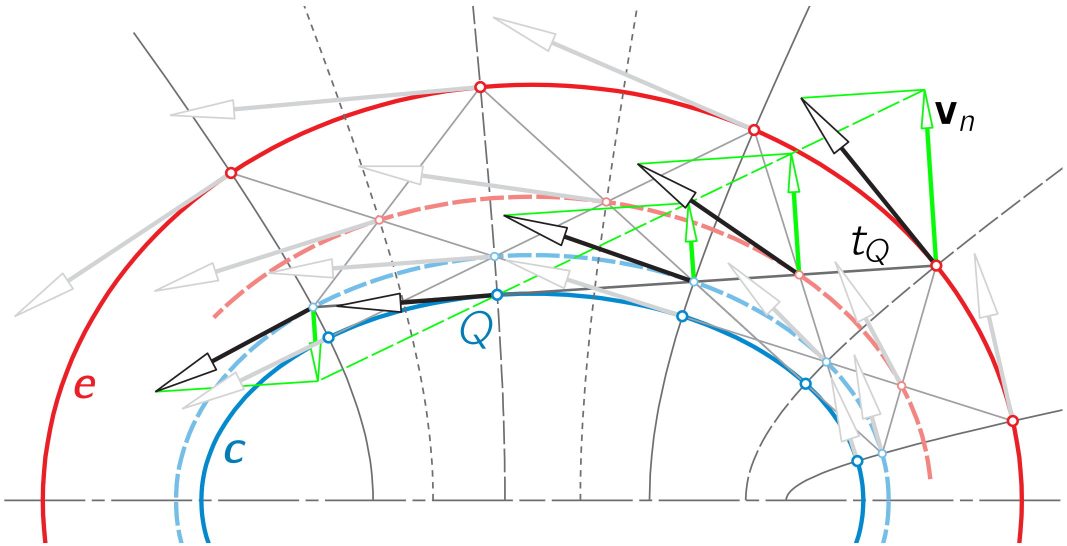
$$\mathbf{v} = \|\mathbf{t}_c\|\,\mathbf{t}_e = \sqrt{a_c^2 \sin^2 t + b_c^2 \cos^2 t}\left(-\frac{a_e y}{b_e},\ \frac{b_e x}{a_e}\right).$$

Theorem:

This vector field defines an infinitesimal motion which preserves confocal ellipses and permutes the confocal hyperbolas and the tangents of the caustic c.

The infinitesimal motion generates a one-parameter Liegroup Γ which carries out the billiard transformation along e and simultaneously that of the associated Poncelet grid.

4. A switch to Analysis



We set $\mathbf{v} = \dfrac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}u} = \dot{\mathbf{p}}$. Then, u is a canonical parameter of e and of Γ, i.e., $\gamma(u_2) \circ \gamma(u_1) = \gamma(u_1 + u_2)$.

$$\mathbf{v}(t) = \|\mathbf{t}_c(t)\|\,\mathbf{t}_e(t) = \dot{t}\,\mathbf{t}_e(t)$$

$$\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}u} = \sqrt{a_c^2 \sin^2 t + b_c^2 \cos^2 t}\,.$$

Proof: Γ permutes hyperbolas since $\dot{t}$ (and $\dot{k}_h$) is independent of e (and k_e).

The condition $b_c a_e \cos t \cos t' + a_c b_e \sin t \sin t' = a_c b_c$ is equivalent to the fact that $P = (a_e \cos t,\ b_e \sin t)$ lies on the tangent t_Q of $Q = (a_c \cos t' + b_c \sin t')$. Differentiation by u yields an identity. Hence, Γ permutes the tangents of c.

4. A switch to Analysis

In order to express the action of $\gamma(u) \in \Gamma$ on $(a_e \cos t,\ b_e \sin t)$, we integrate

$$\frac{\mathrm{d}t}{\mathrm{d}u} = \sqrt{a_c^2 \sin^2 t + b_c^2 \cos^2 t} = \sqrt{a_c^2 \sin^2 t + (a_c^2 - d^2)\cos^2 t} = a_c\sqrt{1 - m^2 \cos^2 t}$$

with $m := d/a_c < 1$ as numeric eccentricity of the caustic c. We substitute $\varphi := t - \frac{\pi}{2}$ and get under the initial condition $\varphi = 0$ for $u = 0$

$$\frac{\mathrm{d}\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}} = a_c\,\mathrm{d}u\,, \quad \text{hence} \quad a_c\,u(\varphi) = F(\varphi, m) = \int_0^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}$$

with $F(\varphi, m)$ as the elliptic integral of the first kind with the modulus m. This function shows the canonical coordinate u in terms of φ with the quarter period

$$K := a_c\,u\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}\,.$$

4. A switch to Analysis

For the sake of simplicity, we define $\widetilde{u}(\varphi) := a_c\, u(\varphi)$ as a new canonical coordinate.

The inverse function of $\widetilde{u} = F(\varphi, m)$, the Jacobian amplitude $\varphi = \mathrm{am}(\widetilde{u}\,)$ leads to the Jacobian elliptic functions,

$$\mathrm{sn}\,\widetilde{u} = \sin(\mathrm{am}(\widetilde{u}\,)) \quad \text{and} \quad \mathrm{cn}\,\widetilde{u} = \cos(\mathrm{am}(\widetilde{u}\,)),$$

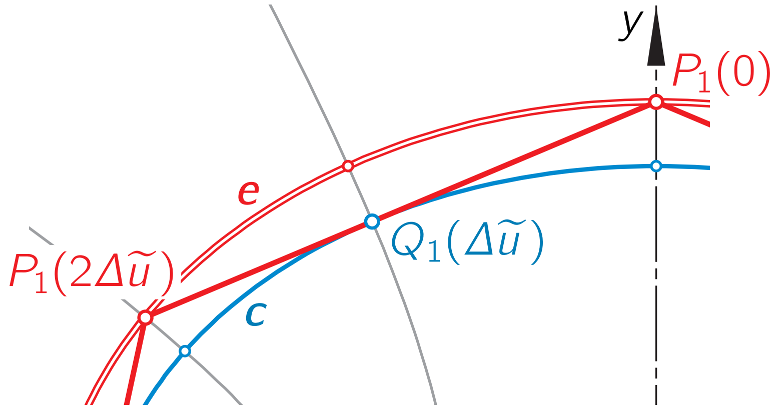
which can be extended in $\mathbb{R}$ to periodic functions with period $4K$.

4. A switch to Analysis

Theorem:
For the ellipse c with semiaxes (a_c, b_c) and eccentricity $d = \sqrt{a_c^2 - b_c^2}$ and all confocal ellipses e with semiaxes (a_e, b_e), the inscribed billiards with caustic c can be canonically parametrized as $(-a_e \operatorname{sn} \widetilde{u},\ b_e \operatorname{cn} \widetilde{u})$, using the Jacobian elliptic functions to the modulus $m = d/a_c$.

If $b_c = b_e \operatorname{cn}(\Delta\widetilde{u})$, then the vertices of the billiard in e have the canonical parameters $\widetilde{u} = (\widetilde{u}_0 + 2k\Delta\widetilde{u})$ for $k \in \mathbb{Z}$ and any given initial $\widetilde{u}_0$.

Conversely, we obtain an N-sided billiard with turning number τ, where $\gcd(N, \tau) = 1$, by the choice $\Delta\widetilde{u} = \frac{2\tau K}{N}$ with K as the complete elliptic integral of the first kind to the modulus m, provided that $b_e = b_c/\operatorname{cn}(\Delta\widetilde{u})$.



Schönbrunn Castle, Vienna

Thank you for your attention!

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