ELSEVIER

Computer Aided Geometric Design 15 (1998) 223–249

COMPUTER AIDED GEOMETRIC DESIGN

A Laguerre geometric approach to rational offsets

Martin Peternell, Helmut Pottmann *

Institut für Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria

Received May 1996; revised April 1997

Abstract

Laguerre geometry provides a simple approach to the design of rational curves and surfaces with rational offsets. These so-called PH curves and PN surfaces can be constructed from arbitrary rational curves or surfaces with help of a geometric transformation which describes a change between two models of Laguerre geometry. Closely related to that is their optical interpretation as anticaustics of arbitrary rational curves/surfaces for parallel illumination. A theorem on rational parametrizations for envelopes of natural quadrics leads to algorithms for the computation of rational parametrizations of surfaces; those include canal surfaces with rational spine curve and rational radius function, offsets of rational ruled surfaces or quadrics, and surfaces generated by peripheral milling with a cylindrical or conical cutter. Laguerre geometry is also useful for the construction of PN surfaces with rational principal curvature lines. New families of such principal PN surfaces are determined. © 1998 Elsevier Science B.V.

Keywords: Laguerre geometry; NC milling; geometrical optics; rational curve; rational surface; offset; rational offset; Pythagorean-hodograph curve; principal patch; principal curvature line

Introduction

In this paper, we continue the discussion of the role of Laguerre geometry in geometric design which has been started in (Pottmann and Peternell, 1997). We refer to that paper concerning the fundamentals of euclidean Laguerre geometry and its different models.

The focus of this paper is on offset curves and surfaces, which arise in various applications including NC milling, path planning for rapid prototyping, font design and geometric tolerancing. It is well known that rational curves or surfaces do in

* Corresponding author. E-mail: pottmann@geometric.tuwien.ac.at.

PII S0167-8396(97)00024-1

general not possess rational offsets. In order to comply with current industry standards, offset curves or surfaces therefore have to be approximated in rational B-spline form. As an alternative approach to offsets, Farouki and Sakkalis (1990) proposed to use only those curves in the design process that do possess rational offsets. Farouki and Sakkalis introduced the so-called *Pythagorean-hodograph (PH) curves*, which are polynomial curves $x(t) = (x_1(t), x_2(t))$ with polynomial parametric speed $\sigma(t) = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$. Among other remarkable properties, these curves possess rational offsets $x_d(t) = x(t) + dn(t)$, where $n(t)$ is a field of unit normal vectors of the progenitor curve $x(t)$. Recent work on PH curves and their generalizations to the full class of rational curves with rational offsets showed that these curves are well suited for practical design purposes (Ait Haddou and Biard, 1995; Albrecht and Farouki, 1996; Farouki, 1992, 1994; Farouki and Neff, 1995; Farouki et al., 1994; Fiorot and Gensane, 1994; Lü, 1994, 1995; Pottmann, 1995a, 1995b).

Using the dual representation, it has been shown by Pottmann (1995a) that all rational curves and surfaces with rational offsets can be described explicitly and constructed via the dual representation. We denote them briefly by *PH curves* and *PN surfaces* in the sequel, where PN stands for *Pythagorean Normal vector field.*

We will now present a new geometric approach to PN surfaces based on Laguerre geometry. A geometric transformation (representing the change between two different models of Laguerre geometry) maps any rational curve/surface onto a PH curve/PN surface. Any PH curve/PN surface may be obtained in this way. This serves as a simple framework for the solution of several design problems with PH curves or PN surfaces such as surface design with parabolic Dupin cyclides.

Recently, unexpected results on offsets of special classes of surfaces could be derived (Lü, 1996; Lü and Pottmann, 1996; Peternell and Pottmann, 1996, 1997; Pottmann, Lü and Ravani, 1996). As a generalization, we use Laguerre geometry to prove a theorem on rational envelopes of natural quadrics. As applications, this results yields algorithms for the computation of rational parametrizations of special classes of surfaces including canal surfaces with rational spine curve and rational radius function, offsets of rational ruled surfaces or quadrics, and surfaces generated by peripheral milling with a cylindrical or conical cutter.

The new approach to PN surfaces enables us to determine classes of PN surfaces all whose principal curvature lines are rational. For example, all PN surfaces with planar rational lines of curvature can be determined explicitly.

1. Rational curves and surfaces with rational offsets

1.1. PH curves and PN surfaces in the cyclographic model

A curve in $\mathbb{R}^2$ is a *PH curve* if and only if it possesses a rational parametric representation $x(t) = (x_1(t), x_2(t))$ with rational parametric speed $\sigma(t) = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$. Then, the field of unit normal vectors $n(t) = (-\dot{x}_2, \dot{x}_1)/\sigma(t)$ and all offsets $x_d(t) = x(t) + dn(t)$ are rational.

Orienting the tangents of x with n, we obtain as their image in the cyclographic model a rational one-parameter family of γ-planes (we exclude the trivial special case of a straight line x),

$$E = (-n_1 x_1 - n_2 x_2, n_1, n_2, 1). \tag{1}$$

The envelope is a *rational γ-developable*. It is a γ-cone for a circle x, otherwise the tangent surface of a rational γ-curve $p(t)$, whose normal projection $p'(t) = (p_1(t), p_2(t))$ onto $\mathbb{R}^2$ is the evolute of x. Because of its constant inclination angle $\gamma = \pi/4$ against $\mathbb{R}^2$, i.e., $\dot{p}_3^2 = \dot{p}_1^2 + \dot{p}_2^2$, the third coordinate function $p_3(t)$ equals the arclength $s(t)$ of $p'(t)$, counted from some starting point $t = t_0$. Therefore, p' is a *rational curve with a rational arc length parameter function*. Conversely, having such a curve (p_1, p_2), the graph curve (p_1, p_2, s) is a rational γ-curve. Let us summarize these geometric relations (Pottmann, 1995a). To avoid lengthy formulations, we do not worry about the trivial PH curves, circles and lines, and the trivial special PH curves with rational arc length, namely the straight lines.

Theorem 1.1. *PH curves are the cyclographic images of rational γ-curves in $\mathbb{R}^3$. Their evolutes, the orthogonal projections of γ-curves onto the reference plane, are exactly the planar rational curves with rational arc length parameter function.*

We recall the 'anticaustic map' $(\Lambda^*)^{-1}$, which is discussed in (Pottmann and Peternell, 1997). Let $a: x_n = 0$ be a fixed hyperplane in $\mathbb{R}^n$. We consider light rays perpendicular to a. The anticaustic map $(\Lambda^*)^{-1}$ with respect to a maps a hyperplane $y = (y_0, \ldots, y_n)$ to a hyperplane

$$z := (\Lambda^*)^{-1}(y) = \left(y_0, \ldots, y_{n-1}, \frac{y_n^2 - y_1^2 - \cdots - y_{n-1}^2}{2y_n}\right). \tag{2}$$

Let y be tangent hyperplanes of a hypersurface m', then z are tangent hyperplanes of an anticaustic x with respect to the mirror m' and light rays perpendicular to a. Further, if y_i are rational functions, then z possesses a rational unit normal vector.

Intersecting the γ-developable determined by a γ-curve p with a γ-plane Γ, we obtain a rational curve m. Viewing its projection m' as mirror and considering light rays normal to $g := \Gamma \cap \mathbb{R}^2$, p' is the caustic and x is an anticaustic. Conversely we see that any anticaustic for parallel light rays to a rational mirror m' is a rational PH curve. In coordinates, let g be the or. line with coordinates $(0, 0, 1)$ and m' determined by the tangent lines $y(t) = (y_0, y_1, y_2)(t)$. Since m' is rational, y_i are rational functions in t. With formula (2), an anticaustic x possesses tangent lines $z(t)$ and their unit normals are rational. This gives a short proof of results by Fiorot and Gensane (1994).

Theorem 1.2. *PH curves are exactly the anticaustics of rational curves for parallel illumination; the corresponding caustics form the family of rational curves with rational arc length parameter function.*

Example 1.1. Let m be a *parabola* in a γ-plane Γ, see Example 2.2 in (Pottmann and Peternell, 1997). Then the pencil P of planes through the ideal line of Γ is a component of the reducible γ-developable of class 4 through m. It counts with multiplicity 2, if the axis of the parabola is a γ line, i.e., m is a pe circle. Then, the cyclographic image different from $\Gamma \cap \mathbb{R}^2$ is a cycle (see Example 2.1 in (Pottmann and Peternell, 1997)). The optical interpretation in Theorem 1.2 then just gives the elementary property that light rays parallel to the axis of a parabolic mirror are reflected through a fixed point (focal point = projection of the vertex of the γ-cone through m). If m is not a pe circle, P has multiplicity 1 and the irreducible part of the γ-developable is of class 3, i.e., the tangent surface of a cubic γ-curve p. Its projection p' onto $\mathbb{R}^2$ is known to be a *Tschirnhaus cubic* (Tschirnhaus, 1690; Wunderlich, 1973); it is a polynomial PH curve (Farouki and Sakkalis, 1990). Theorem 1.2 yields the known result that *the caustic of a parabola for parallel illumination (but not parallel to the axis) is a Tschirnhaus cubic* and the anticaustics (involutes) are so-called *Tschirnhaus quartics*. The latter are PH curves used in (Pottmann, 1994) and (Ait Haddou and Biard, 1995) for curve design.

Moreover, we can take a parabola m not in a γ-plane. The γ-developable through m is irreducible of class 4. According to Theorem 1.1, we find: *the anticaustics of a parabola for refraction of parallel light rays are PH curves of class* 4. The caustics are rational sextics with rational arc length parameter function.

The most convenient representation of PH curves is the *dual Bézier form* (Pottmann, 1995a). The same holds for rational γ-developables. We start with a rational representation of a circle,

$$(n_1, n_2) = \frac{1}{a^2 + b^2} (2ab, a^2 - b^2),$$

with polynomials $a(t), b(t)$. Eq. (1) with $x_1 = x_2 \equiv 0$,

$$E = \big(0, 2abf, (a^2 - b^2)f, (a^2 + b^2)f\big),$$

is a homogeneous dual representation of the γ-cone $\Gamma(O)$ with vertex at the origin O, degree elevated with the polynomial $f(t)$. Expanding the polynomial coordinate functions (degree m) in the Bernstein basis over a real interval, the coefficients determine the Bézier planes B_i and frame planes $F_i = B_i + B_{i+1}$ of the cone (for the dual representation of developable surfaces, see (Pottmann and Farin, 1995)). They pass through O and intersect each plane $x_3 =$ const in the control lines of a circle segment. Changing the first coordinate function e_0 from 0 to an arbitrary polynomial of degree m is equivalent to performing arbitrary translations to the control planes of the cone and yields a representation of a rational γ-developable.

Theorem 1.3. *The dual Bézier representation of a rational γ-developable is obtained by performing arbitrary translations to the Bézier and frame planes of a γ-cone.*

This has several applications. An explicit dual representation of PH curves is

$$e = \big(g, 2abf, (a^2 - b^2)f\big), \tag{3}$$

with arbitrary polynomials a, b, f, g. The intersection of the control planes with the reference plane $\mathbb{R}^2$ yields the property that the control lines of the dual representation of a

PH curve segment x are parallel to the corresponding lines in the dual control structure of a circular arc. Intersection with $x_3 = d$ shows, that the offset x_d of x has Bézier and frame lines which arise from those of x and a circle c of radius d via Minkowski addition (leaving the normals unchanged and adding the signed distances from the center of the circle; see (Pottmann, 1995b)). Furthermore, intersection with a γ-plane and projecting the lines into $\mathbb{R}^2$ gives the dual control structure of a mirror for parallel illumination. The construction of the control points of the edge of regression p with the formula in (Pottmann, 1994) yields the control points of a rational Bézier curve with rational arc length parameter function. Of course, one may treat piecewise rational γ-developables and PH curves in a completely analogous way.

It is an advantage of the present approach that the transition to rational surfaces with rational offsets is straightforward. A surface in $\mathbb{R}^3$ is called a *PN surface* if and only if it possesses a rational parametric representation $x(u,v) = (x_1(u,v), \dots, x_3(u,v))$ with rational unit normal vectors $n(u,v)$; then all offsets $x_d = x + dn$ are rational. Excluding the trivial planes x, we orient the tangent planes of x with n and obtain in the cyclographic model a rational two-parameter family of γ-hyperplanes,

$$E(u,v) = (-n_1x_1 - n_2x_2 - n_3x_3, n_1, n_2, n_3, 1)(u,v). \tag{4}$$

The envelope of $E(u,v)$ is a *rational γ-hypersurface* T, which contains a 2-parameter family of generating γ-lines $g = E \cap E_u \cap E_v$. The singular set s (γ-surface) of T is not necessarily rational, but the generating lines $g(u,v)$ depend rationally on (u,v). This implies that the intersection of T with a γ-hyperplane Γ is a rational surface m. Again we view its projection m' as mirror and consider light rays perpendicular to $\varepsilon := \Gamma \cap \mathbb{R}^3$. Then, s' is the caustic and x is an anticaustic. Conversely, the anticaustic map (2) shows that any anticaustic for parallel light rays to a rational mirror surface m' is a rational PN surface. Further, $x \cup \varepsilon$ is the cyclographic image of m.

It is important to note the following special case. It may happen that $m = T(s) \cap \Gamma$ is just a rational *curve*. Its projection m' in $\mathbb{R}^3$ is interpreted as 2-parameter set of tangent planes, namely the union of tangent planes passing through each tangent line of m'. We may also perform the reflection of light rays perpendicular to ε. The anticaustic x is a rational canal surface with the constant tangent plane ε. Again, $x \cup \varepsilon$ is the cyclographic image of m. We will later return to this case, when we construct rational parametrizations of canal surfaces. The spatial extension of the result of Fiorot and Gensane (1994) reads as follows.

Theorem 1.4. *PN surfaces are exactly the anticaustics of rational curves or surfaces for parallel light rays. An anticaustic to a rational mirror curve is a rational canal surface which is in line contact with a plane orthogonal to the light rays.*

The discussion of the dual control structure of PN surfaces is completely analogous to the curve case (Pottmann, 1995a) and can be omitted. For later use, we just present the dual representation of PN surfaces,

$$e = \left(g, 2acf, 2bcf, (a^2 + b^2 - c^2)f\right), \tag{5}$$

with polynomials a, b, c, f, g of one or two variables.

The present geometric setting is also well suited for the study of *rational curves* $p'(t)$ *in* $\mathbb{R}^3$ *with rational arc length* $s(t)$. The curve p' is projection of a rational γ-curve $p(t) = (p_1(t), p_2(t), p_3(t), s(t))$. An explicit representation of all curves $p(t)$ may be obtained as follows. The tangents of a γ-curve intersect the ideal plane in a rational curve $U(t) \subset \Omega$. Using homogeneous coordinates $(x_0, \dots, x_4)$ in 4-space, Ω is represented as $x_0 = 0$; $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$. With the explicit form of rational spherical curves $x(t) \subset \mathbb{R}^3$ from (Dietz, Hoschek and Jüttler, 1993), we get

$$U(t) = \big(0, 2X_0X_1 - 2X_2X_3, 2X_1X_3 + 2X_0X_2, \\ X_1^2 + X_2^2 - X_0^2 - X_3^2, X_1^2 + X_2^2 + X_0^2 + X_3^2\big), \tag{6}$$

with relatively prime polynomials $X_i(t)$. The osculating planes $V(t): x_0 = 0$, $v_1x_1 + \dots + v_4x_4 = 0$ of the curve U are spanned by $U(t)$ and the derivative points $\dot{U}(t), \ddot{U}(t)$. Setting $u = (u_1, \dots, u_4)$ and $v = (v_1, \dots, v_4)$, the vector product in $\mathbb{R}^4$ yields

$$v(t) = u(t) \times \dot{u}(t) \times \ddot{u}(t). \tag{7}$$

The osculating hyperplanes of the γ-curve $p(t)$ intersect the ideal plane in the osculating planes $V(t)$. Therefore, the osculating hyperplanes $H(t)$ of p possess the homogeneous plane coordinates

$$H(t) = \big(h(t), v_1(t), \dots, v_4(t)\big), \tag{8}$$

with an arbitrary rational function $h(t)$. Intersecting $H(t)$ with the first three derivative hyperplanes yields $p(t)$. Hence, the a homogeneous representation $P(t) = (P_0(t), \dots, P_4(t))$ of rational γ-curves is

$$P(t) = H(t) \times \dot{H}(t) \times \ddot{H}(t) \times H^{(3)}(t). \tag{9}$$

So far, we did not consider the special case where the curve p lies in a hyperplane H, such that it cannot be obtained with (9). Then, the ideal plane V of H must be pe, i.e., it intersects Ω in a real conic U. The (euclidean) unit tangent vectors $n(t)$ of the curve $p(t)$ form the constant angle $\gamma = \pi/4$ with $\mathbb{R}^3$, therefore their projection $n'(t)$ has constant length $\sqrt{2}/2$. On the other hand, the vectors $n(t)$ lie in H and thus the projection $n'(t)$ describes a planar section of the sphere $n^2 = 1/2$, i.e., a circle. The axis a of the circle is normal to the plane $\varepsilon := H \cap \mathbb{R}^3$. We see, that p' is a curve whose tangents form a constant angle with the axis a, or in other words, p is a *rational curve of constant slope*. From the discussion on planar curves $(p_1(t), p_2(t))$ with rational arc length $s(t)$, we see that p' has the form $(p_1, p_2, \lambda s)$ with a real constant $\lambda \neq 0$; the arc length of p' is $\sqrt{1 + \lambda^2}s$.

Theorem 1.5. *Any rational space curve* $p(t) = (p_1(t), p_2(t), p_3(t))$ *with a rational arc length parameter function* $s = p_4(t)$ *is either a rational curve of constant slope or it is representable in the form* (9) *with* $p_i = P_i/P_0$ *and* (6), (7) *and* (8).

As discussed in (Pottmann and Peternell, 1997), the cyclographic image of a γ-curve p is a principal curvature strip, consisting of a curve $x(t)$, the intersection of the tangent surface $T(p)$ with $\mathbb{R}^3$, and a tangent plane (or surface normal $n(t)$) at each point. The normals are the tangents of p' and therefore form a developable surface. Any surface in

$\mathbb{R}^3$ containing $x(t)$ and possessing the normals $n(t)$ along $x(t)$ possesses the curve x as principal curvature line. In our case, the curve

$$x(t) = p(t) - \frac{p_4(t)}{\dot{p}_4(t)}\,\dot{p}(t), \tag{10}$$

as well as the unit normal field

$$n(t) = \frac{1}{\dot{p}_4(t)}\,\dot{p}(t), \tag{11}$$

are rational. Hence, also the offset strips with curve $x(t) + dn(t)$ and normals $n(t)$ are rational. Thus, we have found a spatial analogue of Theorem 1.1.

Theorem 1.6. *The cyclographic images of rational γ-curves in $\mathbb{R}^4$ are rational principal curvature strips, with strip curve* (10) *and rational unit normal field* (11); *this implies rationality of their offsets. The normals of the strip are the tangents of a rational curve $p'(t)$ with rational arc length parameter function.*

The osculating planes $Y(t)$ of the strip curve x are the intersections $H(t) \cap \mathbb{R}^3$, and therefore they possess the homogeneous coordinates

$$Y(t) = \big(h(t), v_1(t), \ldots, v_3(t)\big). \tag{12}$$

These planes are parallel to the osculating planes of the rational spherical curve

$$c(t) = \frac{1}{u_4(t)}\big(u_1(t), u_2(t), u_3(t)\big), \tag{13}$$

with u_i as in Eq. (6). The set of osculating planes of a rational curve form its dual representation; selecting an interval (segment), one can express it with Bézier and frame planes. This allows us to state the following generalization of Theorem 1.3.

Theorem 1.7. *The dual Bézier representation of a nonplanar strip curve to a rational principal curvature strip with rational offsets is obtained by performing arbitrary translations to the Bézier and frame planes in the dual representation of a nonplanar rational spherical curve $c(t)$. The strip normals are parallel to the connections of $c(t)$ with the center of the sphere.*

A curve of constant slope p' determines as 'involutes' planar PH curves as strip curves. Their dual Bézier representation is generated as described in the discussion following Theorem 1.3.

It may be an interesting project to develop an algorithm which allows one to design a surface from a set of curves which are supposed to be principal curvature lines of the final surface. We have shown here which curves are admissible for this kind of shape preserving interpolation.

Remark 1.1. An example for rational space curves with rational arc length are the *polynomial PH space curves* in (Farouki and Sakkalis, 1994). The cubics among these curves yield just planar strip curves, namely Tschirnhausen quartics (see Example 1.1). The spatial involutes of PH quintics lead to principal curvature strips with rational strip curves of degree 9.

1.2. Using the Blaschke cylinder and the isotropic model

In the following, we study PH curves $x(t) \subset \mathbb{R}^2$ and PN surfaces $x(u,v) \subset \mathbb{R}^3$ in the Blaschke model and the isotropic model of euclidean Laguerre plane or space, respectively. Concerning fundamentals of euclidean Laguerre space, models and notations we refer to (Pottmann and Peternell, 1997). In the cyclographic model, we have the representations (1) and (4). Applying the Blaschke map (duality) δ, we get a rational curve or 2-surface in the Blaschke cylinder Δ in $\mathbb{R}^3$ or $\mathbb{R}^4$, respectively. Laguerre transformations appear as projective maps in the Blaschke model and projective maps preserve the rationality of a curve/surface. This proves the following result.

Theorem 1.8. *Laguerre transformations in $\mathbb{R}^2$ map PH curves (viewed as sets of oriented tangents) onto PH curves. Laguerre transformations in $\mathbb{R}^3$ map PN surfaces (as sets of oriented tangent planes) onto PN surfaces.*

Remark 1.1. PH curves/PN surfaces are also invariant under Möbius transformations in 2- or 3-space, respectively. For the simple proof of this fact, it is sufficient to show invariance under inversions. Together with the result above, we can state that *planar PH curves and PN surfaces are invariant under Lie transformations.* For details on Lie geometry, we refer the reader to Blaschke (1929) or Cecil (1992). Moreover, we can define PH curves in $\mathbb{R}^n$ as rational curves with rational parametric speed. Then, one can prove that *Möbius transformations in $\mathbb{R}^n$ map PH curves onto PH curves.* In particular we can use this in $\mathbb{R}^3$ to map planar PH curves onto *spherical PH curves* (see also (Wagner and Ravani, 1996)). These curves possess *rational geodesic offsets.*

The stereographic projection σ defines a bijective map between rational curves/surfaces in Δ and rational curves/surfaces in the isotropic model I^2/I^3. This gives us a simple construction of PH curves/PN surfaces.

Theorem 1.9. *Let Λ^{-1} be the geometric transformation which describes the change from the isotropic model of 2- or 3-dimensional Laguerre space to the standard model. Λ^{-1} maps a rational curve in I^2 onto a planar PH curve. The Λ^{-1}-image of a rational curve or surface in I^3 is a developable or nondevelopable PN surface, respectively. Any PH curve/PN surface may be obtained in this way.*

A degenerate case should be mentioned. An isotropic line corresponds to a pencil of parallel oriented lines/planes in design space. A rational cylinder in I^3 with isotropic generators belongs to a 2-parameter set of planes that touch a rational curve at infinity. Moreover, if the rational surface $\Psi \subset I^3$ possesses a real curve along which it is touched by an isotropic cylinder (occluding contour for projection in isotropic direction), the corresponding PN surface contains a real curve at infinity.

As discussed in (Pottmann and Peternell, 1997), the mapping $(\Lambda^*)^{-1}$ from the dual isotropic model to the standard model is a geometric transformation that maps a curve/surface in I^2/I^3 onto an anticaustic for parallel illumination. Therefore, Theorem 1.9 is dual to the interpretation of PH curves/PN surfaces as anticaustics in Theorems 1.2 and 1.4.

Let us recall that Λ maps an or. hyperplane e in $\mathbb{R}^n$ with normalized coordinates $(e_0, \ldots, e_n)$, $e_1^2 + \cdots + e_n^2 = 1$, onto the point

$$\Lambda(e) = \frac{1}{1 - e_n}(e_1, \ldots, e_{n-1}, e_0). \tag{14}$$

The preimage of a point $y = (y_1, \ldots, y_n) \in I^n$ is a hyperplane z in $\mathbb{R}^n$

$$\begin{aligned} z &:= \Lambda^{-1}(y) \\ &= \frac{1}{y_1^2 + \cdots + y_{n-1}^2 + 1}\left(2y_n, 2y_1, \ldots, 2y_{n-1}, y_1^2 + \cdots + y_{n-1}^2 - 1\right). \end{aligned} \tag{15}$$

Let y_i be rational functions, it follows that the unit normals of z are rational. With (14) it follows that the Λ-image of a PH curve (3) is

$$y = \left(\frac{a}{b}, \frac{g}{2b^2 f}\right). \tag{16}$$

Let us now consider those PH curves which are constructed in the sense of Theorem 1.3 from a quadratic circle representation (including a possible degree elevation). This means that we insert linear polynomials a, b. After a linear fractional reparameterization which leaves the degree unchanged, these may be chosen as $a = t$, $b = 1$. Then (16) shows that in the isotropic model we get the graph of the function $h = g/2f$. For a PH curve of class m, f and g have to be polynomials of degree $m - 2$ and m, respectively.

Theorem 1.10. *A rational PH curve of class m, derived from a dual quadratic representation of a circular arc via degree elevation and translations of the control lines, appears in the isotropic model as graph of a rational function with numerator degree m and denominator degree $m - 2$.*

The simplest example of such a PH curve is a circle $(m = 2)$, which is mapped onto the graph of a quadratic polynomial (see Pottmann and Peternell, 1997). For $m = 3$ we get the Tschirnhausen quartics from Example 1.1.

Analogously, a PN surface (5) appears in the isotropic model as

$$y = \left(\frac{a}{c}, \frac{b}{c}, \frac{g}{2c^2 f}\right). \tag{17}$$

Nondevelopable surfaces derived from a quadratic spherical representation (via reparameterization simplified to $a = u$, $b = v$, $c = 1$) are mapped to graphs of rational functions in I^3.

Example 1.2. Let us consider as surface $y(u, v) \subset I^3$ the graph of a quadratic polynomial. From (Pottmann and Peternell, 1997) we know already that a plane or paraboloid of revolution (isotropic sphere) y corresponds to a sphere in design space. Otherwise, we obtain a well-known class of PN surfaces, namely *parabolic Dupin cyclides*. For a discussion of this important special case, see (Peternell and Pottmann, 1996), where it has also been shown how to derive a modeling scheme with cyclides from bivariate quadratic spline algorithms.

As PN surfaces, parabolic Dupin cyclides may also be generated with help of the optical interpretation in Theorem 1.4 and to do so, we have to convert them to the dual isotropic model. Transforming y under the polarity of an isotropic sphere, yields in general a paraboloid y^*; a special case occurs for a parabolic cylinder y, whose points are mapped onto the planes tangent to a parabola with y_3-parallel axis. Using Theorem 1.4, *we recognize parabolic Dupin cyclides as anticaustics of paraboloids or parabolas for light rays parallel to the axis of the mirror curve or surface* (Wunderlich, 1948).

The presented results can be used for *modeling with PH curves and PN surfaces based on the geometric transformation Λ. Λ preserves the order of geometric contact*, but one has to take into account that we are working with the dual representation in design space. This means that an inflection of a curve in $\mathbb{R}^2$ appears as a cusp in the isotropic model. Parabolic points of a surface in $\mathbb{R}^3$ correspond to singularities of the Gauss map and therefore to singularities in the isotropic model. Conversely, having a regular curve/surface in the isotropic model, then the preimage in design space is regular as line/plane set, but not necessarily as point set. Note also that in Laguerre geometry a curve/surface and its offsets are equivalent, since they are related by Laguerre transformations. But clearly, the offsets of a regular curve/surface need not be regular.

2. PH curves

The oriented tangents of a planar PH curve $x(t)$ appear in the Blaschke model as rational curve $X(t)$ on the cylinder Λ (see Fig. 1). Transforming the Bézier points B_i (homogeneous coordinates $(b_{i,0}, \dots, b_{i,3})$) of a segment of X with the duality δ^{-1}, we obtain the Bézier planes $u_i = (b_{i,3}, b_{i,1}, b_{i,2}, b_{i,0})$ of a γ-developable. Intersecting the developable and the planes u_i with the reference plane $x_3 = 0$ yields the curve x and the Bézier lines $b_i = (b_{i,3}, b_{i,1}, b_{i,2})$ in its dual representation.

As an example for the transfer between the standard model and the Blaschke model we consider a *PH curve of class* 4. The curve X on Δ is a rational quartic. We pick a segment of it with Bézier points $B_0, \dots, B_4$. Let $\alpha_0 = B_0B_1B_2$ and $\alpha_1 = B_2B_3B_4$ be the osculating planes of X at the end points B_0 and B_4. The middle Bézier point B_2 lies on the intersection line $a = \alpha_0 \cap \alpha_1$ (Fig. 1). Assume that the osculating planes are not tangent to Δ. Then, they intersect Δ in ellipses K_0, K_1. They have contact of order 2 with X and therefore are the Blaschke images of the oriented osculating circles k_0, k_1 at the endpoints x_0, x_1 of the corresponding segment of x. Transforming a back to the cyclographic model, we obtain the line connecting the vertices of the γ-cones $\zeta(k_0)$ and $\zeta(k_1)$. This line intersects the reference plane in the center s of similarity of the two osculating cycles k_0, k_1. Since a passes through B_2, b_2 contains s (Fig. 2). Thus, we have

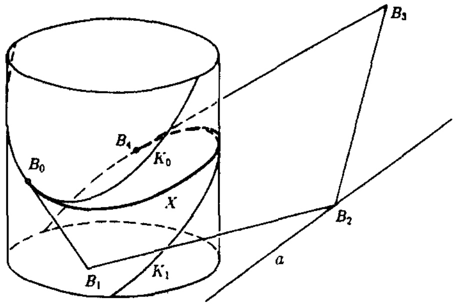

Fig. 1. Osculating planes of a rational curve on a quadratic cylinder.

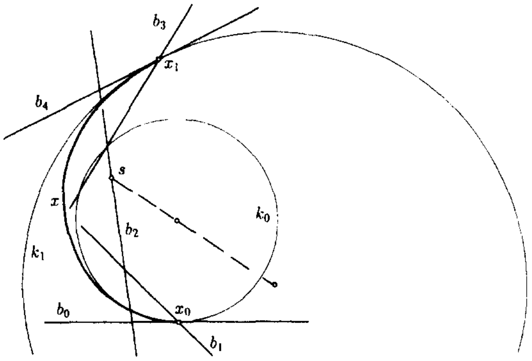

Fig. 2. Two osculating circles of a PH curve of class 4.

found a result that has been proved in a different way in (Pottmann, 1995b), where it forms the basis of the construction of G^2 PH splines of class 4.

Proposition 2.1. *The middle Bézier line b_2 of the dual representation of a PH curve segment x of class 4 passes through the center of similarity s of the oriented osculating circles k_0, k_1 at its end points.*

Rational curves $X(t) \subset \Delta$ may be written as

$$(X_0, \ldots, X_3) = \left(p_3^2 + p_0^2, p_3^2 - p_0^2, -2p_3p_0, 2(p_3p_1 - p_2p_0)\right), \tag{18}$$

with polynomials $p_i(t)$. Let p_i be coordinate functions in a projective 3-space $\mathbb{P}^3$. Formula (18) may be interpreted as quadratic mapping $\sigma : \mathbb{P}^3 \rightarrow \Delta$, which is called 'generalized stereographic projection' (Dietz, 1995). Note that the preimage of each point $X \in \Delta$ is a line of a parabolic net in $\mathbb{P}^3$. Mapping a curve segment $p \subset \mathbb{P}^3$ together with its Bézier points P_i and frame points Q_i onto Δ, we get X with an intrinsic control structure. It consists of control points $S_i = \sigma(P_i)$ and frame points $T_i = \sigma(Q_i)$. The image of the projective control polygon of p (cf. (Pottmann and Farin, 1995)), is a curved control polygon consisting of elliptic or straight line segments S_iS_{i+1} that carry T_i (Fig. 3).

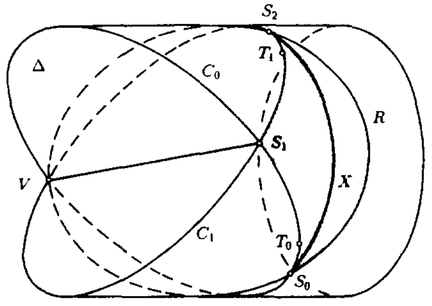

Fig. 3. Intrinsic control structure of a rational curve on a quadratic cylinder.

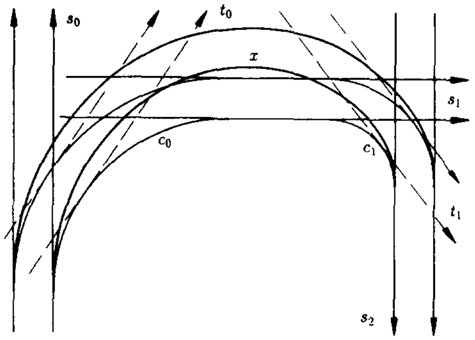

Fig. 4. Laguerre geometric control structures of a PH curve and an offset.

The intrinsic control structure is useful for interactive design, since control and frame points may be chosen interactively. However, if two of the three points S_i, S_{i+1}, T_i lie on the same generator, all three must be collinear. To compute p and X, choose $P_0 \in \sigma^{-1}(S_0)$ and compute the remaining Bézier points $P_i \in \sigma^{-1}(S_i)$ and frame points $Q_i \in \sigma^{-1}(T_i)$ by maintaining collinearity of P_i, P_{i+1}, Q_i.

This curve design technique, which works on other quadrics as well, has been used in (Wallner and Pottmann, 1997) for the construction of rational blending surfaces between quadrics. Here, we interpret it as Laguerre geometric control structure and transform it to the standard model. We obtain a sequence of or. control lines s_i and frame lines t_i of the *Laguerre geometric control structure of the PH curve* x. They may be chosen arbitrarily. However, if two of the three or. lines s_i, t_i, s_{i+1} are parallel, also the third must be chosen parallel to them. Otherwise, s_i, t_i, s_{i+1} are tangent to a cycle c_i and the segment with end tangents s_i, s_{i+1} and interior tangent t_i corresponds to a segment of the control polygon of X. The cycles c_i (maybe degenerate to parts of a pencil of parallel or. lines) form the *complete Laguerre geometric control structure of* x (Fig. 4).

Theorem 2.2. *Any PH curve* x *possesses a Laguerre geometric control structure, consisting of oriented control lines* $s_0, \ldots, s_m$ *and oriented frame lines* $t_0, \ldots, t_{m-1}$. *The lines may be chosen arbitrarily; however, if two of the three or. lines* s_i, t_i, s_{i+1} *are par-*

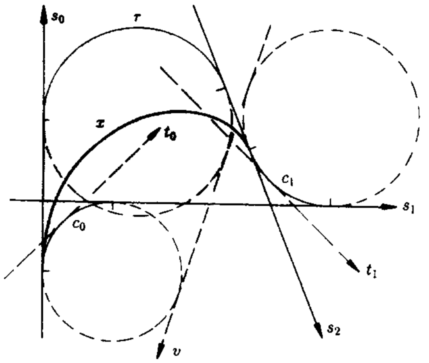

Fig. 5. Control structure of a PH curve of class 4 and convex hull property.

allel, also the third must be parallel to them. The connection between the curve x and the control structure $C(x)$ is invariant under Laguerre transformations. Particularly, the offset of x at distance d is controlled by the offset of $C(x)$ at distance d.

Some well-known properties of the Bézier polygon of p may be translated into properties of $C(x)$. For example, the cycle c_0 touches s_0 at an end point of x and the other end point is determined by c_{m-1} and s_m (Fig. 5).

The most interesting case for applications is $m = 2$ and concerns PH curves x of class 4. The corresponding curve X on the Blaschke cylinder Δ is a rational quartic with intrinsic control points S_0, S_1, S_2. We assume that the points S_i lie on different generators. The control conic arcs C_i between S_i and S_{i+1} $(i = 0, 1)$ intersect at S_1 and a second point, say V. The point V is a double point of the quartic X (parameterized over $\mathbb{R}$) and V is also the vertex of a quadratic cone Γ through X. The cone is determined by the generators VS_0 and VS_2, the tangent planes VS_0S_1 and VS_1S_2 along them and the tangent plane VT_0T_1. Therefore, we can formulate a *convex hull property* (illustrated in Fig. 3) and a *variation diminishing property* of the intrinsic control polygon as in the plane; but straight lines are replaced by conics on Δ passing through V.

Converting to the standard model, we have three or. control lines s_0, s_1, s_2 and 2 or. frame lines t_0, t_1. They determine the control cycles c_0, c_1. Apart from s_1, the 2 cycles c_0, c_1 have a second common or. tangent v (corresponding to the point V). To formulate a convex hull and variation diminishing property, we have to consider the cycles touching v. The curve then lies in the region bounded by $s_0, c_0, s_1, c_1, s_2, r$, where the cycle r is tangent to s_0, s_2, v (Fig. 5). The variation diminishing property reads: *Consider a cycle c tangent to v. Then, the number of or. lines in the complete Laguerre geometric control structure of a PH curve x of class* 4, *which touch c, is greater or equal than the number of or. tangents of x that are tangent to c.*

Example 2.1. Let x be a sufficiently smooth curve in $\mathbb{R}^2$ and p not an inflection point of x. In a neighborhood of p one may represent x by a dual normalized parameterization

$$x(t) = \left(f(t), \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right).$$

The point p shall correspond to the parameter value $t = 0$, that means $p = x(0) \cap \dot{x}(0)$. We want to determine a rational planar PH curve y of class m, which has maximum order of contact with x in p. We restrict y to be contained in the family of PH curves, which are derived from a quadratic representation of a circular arc. With Theorem 1.10 and formula (16) one obtains

$$\Lambda(x) = \left(t, \frac{f(t)}{2}\right) \quad \text{and} \quad \Lambda(y) = \left(t, \frac{g(t)}{2}\right) = \left(t, \frac{e(t)}{2d(t)}\right),$$

with polynomials e of degree m and d of degree $m-2$. One has to determine a rational function $g(t)$ such that the condition $(f-g)(0) = \mathrm{O}(t^{2m-1})$ holds. Usually $g(t)$ is called a *Padé approximant* of f of order $(m, m-2)$. An introduction to this topic is given by Brezinski (1993). This condition results in a linear homogeneous system of $2m-1$ equations

$$f(0) = g(0),\ f'(0) = g'(0), \ldots, f^{(2m-2)}(0) = g^{(2m-2)}(0),$$

for the $2m$ coefficients e_i, $i = 0, \ldots, m$ and d_i, $i = 0, \ldots, m-2$, of the polynomials e and d. This system has at least one nontrivial solution. Applying Λ^{-1} yields in a dual representation of y.

Example 2.2. Let x be a sufficiently smooth curve in $\mathbb{R}^2$ and p, q two distinct points in x, which define a segment $c \subset x$. Assume c does not contain inflection points. Then c can be represented by a dual normalized parameterization as given in Example 2.1

$$c(t) = \left(f(t), \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right).$$

The points p and q shall correspond to parameter values $t_p = 0$ and $t_q = 1$. We want to determine a rational planar PH curve y of class m, which has maximal order of contact with x in p and q. Assume $m = 4$, then x and y should be in contact of order 2 in p and q. Further we require y to be contained in the same family described in Example 2.1. Using analogous arguments one has to determine a rational function $g(t)$, satisfying

$$\begin{aligned} &f(0) = g(0), && f'(0) = g'(0), && f''(0) = g''(0), \\ &f(1) = g(1), && f'(1) = g'(1), && f''(1) = g''(1). \end{aligned}$$

Above conditions lead to a homogeneous system of 6 linear equations for 8 coefficients e_i, d_i of the polynomials e and d with $g = e/d$. This system has at least a one parameter set of nontrivial solutions. To obtain a unique curve one may require $f(t_0) = g(t_0)$ for a $t_0 \neq 0, 1$. Applying Λ^{-1} results in a dual parameterization of the PH curve y which has order of contact 2 in p and q and interpolates a further tangent line $c(t_0)$.

Example 2.3. Let (p_1, t_1) and (p_2, t_2) be two oriented line elements and let a be a cycle in $\mathbb{R}^2$. A rational planar PH curve x of class 3 should be determined, which is tangent to t_i in p_i and touches a. Problems of this kind have been studied by Hoffmann and Peters (1995). We may assume that x has a dual parameterization

$$x(t) = \left(\frac{e_0 + e_1 t + e_2 t^2 + e_3 t^3}{d_0 + d_1 t}, \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right).$$

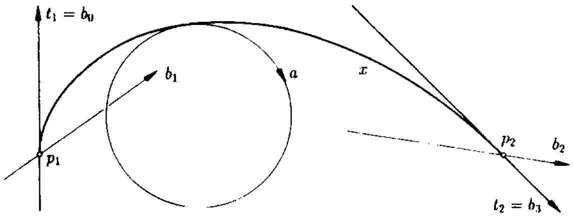

Fig. 6. PH curve of class 3 passing through two line elements and tangent to a cycle.

One may choose a coordinate system in $\mathbb{R}^2$ such that $a = (0, -r_a, r_a)$ with r_a as radius of the cycle a. That implies that Λ maps a to a line parallel to the y_1 axis in I^2. The line elements (p_i, t_i) are mapped to line elements (q_i, s_i) where $q_i = \Lambda(t_i)$ and s_i is the tangent of the parabola $\Lambda(p_i)$ in q_i. The interpolation problem in I^2 reads as follows: determine a graph of a rational function e/d with degrees 3 and 1, which interpolates the two given line elements (q_i, s_i) and is tangent to $\Lambda(a)$ anywhere. This results in a system of polynomial equations, which has in general 4 solutions. A real solution is displayed in Fig. 6.

3. PN surfaces

3.1. Rational parametrizations for envelopes of natural quadrics

In CAD, spheres, and cones and cylinders of revolution are often called *natural quadrics*. We define here *real rational one-parameter sets of natural quadrics* as follows. A one-parameter set $c(t)$ of cycles with a rational center curve $m(t)$ and a rational radius function $r(t)$ is called rational. Given two rational sets $c_1(t), c_2(t)$, we may form the common tangent cone $\Delta(t)$ to the cycles $c_1(t), c_2(t)$ for each t. We assume that $\Delta(t)$ is real for all $t \in \mathbb{R}$. Otherwise, we would reparametrize each interval $[a, b]$, in which this is true, by a quadratic function $t = (a + bs^2)/(1 + s^2)$ and then $\Delta(s)$ is real for all $s \in \mathbb{R}$. Note that $\Delta(t)$ may be a cylinder or a line, as a pencil of planes. For isolated parameter values t_i, the degeneracy to a surface element is allowed, which occurs for tangent cycles $c_1(t_i), c_2(t_i)$.

A one-parameter set $c(t)$ of cycles possesses a canal surface as envelope. This canal surface is also the envelope of a rational set of cones of revolution. The canal surface and the set of cones is real, if $\dot{m}(t)^2 - \dot{r}(t)^2 \geqslant 0$ for all t, where equality shall hold only for isolated parameter values. Only those sets of cycles are called real rational sets.

With the cyclographic map, we may define *a real rational one-parameter set of natural quadrics $\Delta(t)$ as cyclographic image of a rational ruled surface $\Phi = e(t)$ in $\mathbb{R}^4$ with only hyperbolic generators $e(t)$, except for isolated parabolic generators.* Real rational sets of cycles are cyclographic images of rational curves without elliptic and with only isolated parabolic tangents. Only these are discussed in the sequel. The cones of revolution $\Delta(t)$ occur as cyclographic images of the generators $e(t)$.

The image of the entire surface is the envelope of the set of cones $\Delta(t)$. In general, $\Delta(t)$ is tangent to the envelope in points of a rational quartic curve obtained by $\Delta(t) \cap \dot{\Delta}(t)$.

We omit the discussion of all cases here, but note that the quartic can be reducible, for instance contain a conic and two, not necessarily real generating lines of $\Delta(t)$. Thus, the envelope is reducible and contains for instance a surface generated by a one parameter family of conics and additionally two developable surfaces. In view of applications, we require that the envelope contains a surface, which is regular as set of tangent planes. This means that the tangent planes and its first derivatives are linearly independent. The technique we use is based on the dual representation and the parametrizations only works for the part of the envelope, which possesses a two parameter set of tangent planes. It is clear that the possibly occurring developables possess tangent planes contained in this two-parameter family.

Theorem 3.1. *The envelope of a real rational one-parameter set of natural quadrics is a PN surface.*

Proof. It is sufficient to consider the set $\Delta(t)$ generated by two rational cycle sets $c_i(t)$ with centers $(m_{i,1}(t), m_{i,2}(t), m_{i,3}(t))$ and radius $r_i(t)$, $i = 1, 2$. The cycles, as sets of oriented tangent planes, appear in the isotropic model as isotropic spheres (see (Pottmann and Peternell, 1997)),

$$\Psi_i(t)\colon\ 2y_3 + (y_1^2 + y_2^2)(r_i + m_{i,3}) + 2y_1 m_{i,1} + 2y_2 m_{i,2} + r_i - m_{i,3} = 0. \tag{19}$$

For each t, the two surfaces $\Psi_i(t)$ intersect in an isotropic Möbius circle $d(t)$, the image of $\Delta(t)$ in I^3. Its projection onto $y_3 = 0$ is

$$d'(t)\colon\ (y_1^2 + y_2^2)(R + M_3) + 2y_1 M_1 + 2y_2 M_2 + R - M_3 = 0, \tag{20}$$

with $M_j := m_{1,j} - m_{2,j}$ and $R := r_1 - r_2$. For $R + M_3 \neq 0$, $d'(t)$ is a euclidean circle with center

$$(n_1, n_2)(t) = \frac{-1}{R + M_3}(M_1, M_2). \tag{21}$$

The radius function $\rho(t)$ is not rational, however we have

$$\rho^2 = \frac{1}{(R + M_3)^2}\left(M_1^2 + M_2^2 + M_3^2 - R^2\right). \tag{22}$$

Due to our assumptions on a real rational set $\Delta(t)$, we have $\rho^2 \geqslant 0$, i.e., real circles only. The surface $d(t)$ contains a one parameter family of isotropic circles. Our task is to parametrize this surface rationally. We first construct two rational functions $\rho_1(t)$ and $\rho_2(t)$, satisfying

$$\rho_1^2(t) + \rho_2^2(t) = \rho^2(t). \tag{23}$$

This is done by factorization over the complex field, and is equivalent to determine all roots of the nonnegative polynomial ρ^2, which is of even degree. We note that in several applications there is not only a numerical solution of the factorization, but ρ_1 and ρ_2 can be calculated explicitly. With a solution of (23), we form the planar rational curve

$$f(t) = (n_1 + \rho_1, n_2 + \rho_2), \tag{24}$$

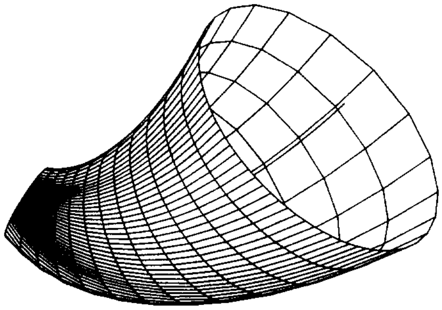

Fig. 7. Canal surface with cubic spine curve and radius function; the displayed curves are rational parameter lines.

whose points lie on the circles $d'(t)$. For fixed t, let the normals of the diameters of the circle $d'(t)$ be parametrized as $c(u) = (u, 1)$. To get all diameters, the domain for u has to be the projective line $\mathbb{P}^1$; appropriate sampling techniques for $\mathbb{P}^1$ are described in (DeRose, 1991). We reflect $f(t)$ at all diameters and get a rational parametrization of the projection of the circle surface $d(t)$ as

$$(z_1, z_2)(t, u) = f(t) + 2\,\frac{(n(t) - f(t)) \cdot c(u)}{c(u) \cdot c(u)}\,c(u). \tag{25}$$

We insert $y_i = z_i(t, u)$, $i = 1, 2$, in (19) and compute $y_3 = z_3(t, u)$. Thus, we have found a rational parametrization $z(t, u)$ of the envelope in the isotropic model. The special case $R + M_3 \equiv 0$ is much simpler, since the rational one-parameter set of lines $d'(t)$ is easily parametrized in rational form and again we obtain a rational parametrization for the Λ-image of the envelope. By Theorem 1.9, the envelope of $\Delta(t)$ is a PN surface. □

As a first application of the theorem, we get a proof for Theorem 2.1 in (Pottmann and Peternell, 1997), which states the *rationality of canal surfaces with rational spine curve and rational radius function*. In particular, we have shown that pipe surfaces (canal surfaces with constant radius) to a rational spine curve are always rational. Another proof of this result may be found in (Lü and Pottmann, 1996). It seems to be an interesting topic for future research to derive conditions under which the degree of the canal surface parametrization is reduced. The important special case of a polynomial cubic spine curve and radius function yields a representation of degree five in t and two in u (Fig. 7). Details on the parametrization and on an appropriate calculation can be found in (Peternell and Pottmann, 1997).

Furthermore, we may consider a set $\Delta(t)$ with generating lines, interpreted as pencils of planes. Due to Theorem 3.1 they can be parametrized by inserting radius functions $r_i = 0$. We obtain rational PN parametrizations of rational ruled surfaces. However, for a developable surface, we parametrize just the edge of regression l as set of tangent planes. The offsets are pipe surfaces with spine curve l. Otherwise, we obtain the following result which has been derived in a different way in (Pottmann et al., 1996).

Corollary 3.1. *The offsets of nondevelopable rational ruled surfaces are rational.*

Applications of this result lie in NC milling. With a spherical cutter one can use rational cutter paths to generate a rational ruled surface. For example, the curves on the offsets to the generators of the progenitor surface are rational quartics and may be chosen as cutter paths (for details, see (Pottmann et al., 1996)). Clearly, the offsets of a ruled surface x at distances d, $-d$ are envelopes of a cylinder of revolution with radius d, whose axis traces out the progenitor surface x. Hence, offsets of rational ruled surfaces are generated during peripheral milling with a cylindrical cutter.

Assume that we are leading a conical or cylindrical cutter under a *rational motion*. For a rational motion, there exists a parameter t such that the paths of all points in the moving space are rational in t. Hence, a moving cone or cylinder of revolution generates a real rational set $\Delta(t)$, and we recognize the envelope (surface generated with peripheral milling) as rational.

Corollary 3.2. *The envelope of a cone or cylinder of revolution under a rational motion is a PN surface.*

Note that as envelope of a moving cylinder, whose axis runs on a developable surface x, we get the pipe surface around the cuspidal edge of x and not the offset of x. More generally, one can show that any moving developable surface has a rational envelope (Jüttler and Wagner, 1996). Note, that for a moving cylinder, Corollary 3.1 is more general than Corollary 3.2: only special rational ruled surfaces can be generated by a line under a rational motion, namely those whose unit vectors of the generators describe a rational spherical curve.

3.2. Hypercyclides

Let H be a hyperplane in $\mathbb{R}^4$ and $s \subset H$ a quadric in it. Then, the cyclographic image $c(s)$ of s is called a *hypercyclide*. A brief discussion of these surfaces is given in (Blaschke, 1910). To generate $c(s)$ one forms the γ-hypersurface $T(s)$, which is the envelope of the two parameter set of γ-hyperplanes, passing through the tangent planes of s. Then, the hypercyclide $c(s)$ is the intersection $T(s) \cap \mathbb{R}^3$.

Let Ω be the quadric defined by $x_0 = 0$, $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$. The hypersurface $T(s)$ can be interpreted as envelope of common tangent hyperplanes of a pencil of dual hyperquadrics $\lambda s + \mu\Omega$ in $\mathbb{R}^4$, where s and Ω are considered as sets of tangent hyperplanes. They are singular hypersurfaces in this pencil.

From classical geometry we know that the intersection surface of a pencil of hyperquadrics in $\mathbb{R}^4$ is rational, and usually called a del Pezzo surface. But a del Pezzo surface is dual to the hypersurface $T(s)$, such that also $T(s)$ possesses a rational two parameter set of tangent hyperplanes. This proves the following result.

Proposition 3.1. *Hypercyclides are rational surfaces.*

Hypercyclides are natural generalizations of Dupin cyclides and contain interesting surfaces such as the offsets of quadrics. Those occur if H is parallel to $\mathbb{R}^3$. The cyclographic image of a quadric in any euclidean hyperplane H (see Pottmann and Peternell, 1997) is related to the offset of a quadric by a Laguerre transformation.

To obtain rational parametrizations of hypercyclides one can follow ideas of classical geometry, which provides parametrizations of a del Pezzo surface. We will construct rational parametrizations of these surfaces, if s is regular (i.e., nondevelopable) in another way. We consider the case where H containing s is a euclidean hyperplane. Here it is sufficient to study offsets of regular quadrics. Their rationality is a recent result by Lü (1996). We will present a new geometric proof based on Theorem 3.1.

Theorem 3.2. *All regular quadrics are PN surfaces.*

Proof. We show that any regular quadric x is the envelope of a real rational one-parameter set of natural quadrics. For quadrics of revolution this follows from the interpretation as canal surfaces. Ruled quadrics are subsumed in Corollary 3.1. Otherwise, we note that the quadric is the envelope of a one-parameter set of cones of revolution. The vertices of the cones lie on a focal curve of x. Real cones correspond to the focal conic f that intersects the quadric orthogonally at its umbilic points. For an ellipsoid, f is a hyperbola (Fig. 8) and the ideal points of f correspond to two circumscribed cylinders of revolution. A hyperboloid of two sheets possesses a focal ellipse f and an elliptic paraboloid has a focal parabola f.

Let $m_1(t)$ be an improper rational parametrization over $\mathbb{R}$ of a segment of f, outside x and bounded by umbilics (Fig. 8). The envelope of the corresponding cones $\Delta(t)$ is the entire quadric x. Furthermore, pick a fixed circumscribed cone or cylinder Δ_0. Its axis is intersected by the axes of $\Delta(t)$ at points $m_2(t)$. Since $m_1(t) \to m_2(t)$ is a projectivity, also $m_2(t)$ is rational. Each of the two cones $\Delta(t)$ and Δ_0 touches x along a conic. At the two intersection points of these conics, $\Delta(t)$ and Δ_0 are tangent to each other and therefore they possess a common inscribed sphere $c_2(t)$. Its radius $r_2(t)$ is rational in t, since the spheres $c_2(t)$ generate Δ_0. Finally, we have generated $\Delta(t)$ according to our definition of a real rational set with the sets $c_1(t) = (m_1(t), r_1(t) = 0)$ and $c_2(t) = (m_2(t), r_2(t))$. □

Example 3.1. Let Φ be an ellipsoid

$$\Psi(x_1, x_2, x_3)\colon \ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1,$$

where x_i are cartesian coordinates in $\mathbb{R}^3$ and $a \geqslant b \geqslant c > 0$ are constant. First we assume $a > b > c$, which implies that one focal conic is a hyperbola f defined by

$$f\colon \ \frac{x_1^2}{a^2 - b^2} - \frac{x_3^2}{b^2 - c^2} = 1 \quad \text{and} \quad x_2 = 0.$$

If $a = b$ or $b = c$, f degenerates to a segment of a line and Φ is an ellipsoid of revolution. We discuss this later. To obtain low degree representations, we apply a rotation of the frame with axis x_2. The new x_3 axis shall be parallel to the normal to Φ at one umbilic point. With respect to this coordinate system and by the substitution $\alpha = \sqrt{b^2 - c^2}$, $\beta = \sqrt{a^2 - c^2}$, $\gamma = \sqrt{a^2 - b^2}$ a rational parametrization of an appropriate segment of f is

$$f = m_1(t) = \frac{1}{b(t^4 - 1)}\left(\alpha\gamma(t^4 + 1), 0, ac(t^4 + 1) + 2b^2t^2\right).$$

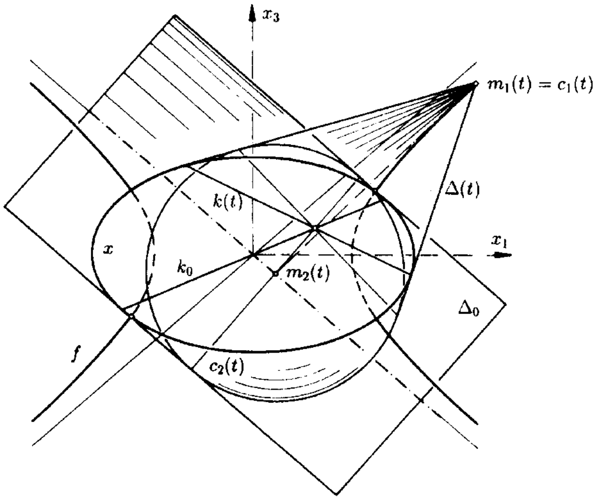

Fig. 8. Ellipsoid as envelope of a rational set of cones of revolution.

It defines the vertices of the cones $\Delta(t)$ and also the cycles $c_1(t)$ in accordance with Theorems 3.1 and 3.2. Let Δ_0 be a cylinder of revolution, tangent to Φ along k_0 (see Fig. 8). The intersection points of the tangent lines of f with the axis of Δ_0 are the centers

$$m_2(t) = \frac{(t^2-1)}{b(t^2+1)}(\alpha\gamma, 0, ac-b^2)$$

of the cycles $c_2(t)$ with constant radius $r_2 = b$. Applying Λ one finds according to formulae (21) and (22)

$$(n_1, n_2) = \left(\frac{-\alpha\gamma t^2}{act^2+b^2}, 0\right),$$

$$\rho^2 = \frac{b^2t^2}{(act^2+b^2)^2}\left(ac(t^4+1)+t^2(a^2+c^2)\right),$$

$$(\rho_1, \rho_2) = \frac{bt}{(act^2+b^2)}\left(\sqrt{ac}(t^2+1), (a-c)t\right).$$

With formula (25) one constructs a rational parametrization of Φ as set of tangent planes, in general of degrees 6 and 2 in t and u. It is not difficult to see that ellipsoids of revolution Φ can be treated in the same way. For instance, let $a = b$. A dual rational parametrization of the one-sided offset Φ_d at distance d is

$$\phi_0 = -(1+u^2)\left(a(ct^4+2at^2+c)+d(at^4+2ct^2+a)\right),$$

$$\phi_1 = -2t\left(\sqrt{ac}(t^2+1)(u^2-1)+2tu(a-c)\right),$$

$$\phi_2 = 2t\big(-\sqrt{ac}(t^2+1)u + t(u^2-1)(a-c)\big),$$
$$\phi_3 = a(1+u^2)(t^4-1).$$

Intersecting the planes $\phi_0 x_0 + \cdots + \phi_3 x_3 = 0$ with the derivative planes with respect to t and u leads to a point representation of Φ_d

$$x_0 = -(1+u^2)(ct^4+2at^2+c)(at^4+2ct^2+a),$$
$$x_1 = -2t\big(d(ct^4+2at^2+c)+a(at^4+2ct^2+a)\big)$$
$$\big((u^2-1)\sqrt{ac}(t^2+1)+2ut(a-c)\big),$$
$$x_2 = 2t\big(d(ct^4+2at^2+c)+a(at^4+2ct^2+a)\big)$$
$$\big(t(a-c)(u^2-1)-2\sqrt{ac}u(t^2+1)\big),$$
$$x_3 = -(1+u^2)(t^4-1)\big(ad(ct^4+2at^2+c)+c^2(at^4+2ct^2+a)\big).$$

The t-lines on Φ are rational quartics, containing an isolated double point. The t-lines on Φ_d are rational curves of degree 8.

4. Principal PN surfaces

The extensive study of Dupin cyclides in geometric design was initiated by the dissertation of Martin (1982), in which he studied *principal patches*. These are rational patches whose parameter lines are principal curvature lines. Curvature lines are sometimes used as cutter paths and thus they are interesting on the offsets of the designed surface. Therefore, we will now study *principal PN surfaces*. These surfaces and all their offsets are principal surfaces. Dupin cyclides are a special case of principal PN surfaces; we will derive a much larger class of such surfaces.

Let $x(u,v)$ be a surface and $y(u,v)$ the corresponding surface in the isotropic model. We can map a curve c on x to a curve d on y in the following way: consider the set of oriented tangent planes of x along c and map it with Λ to I^3. It follows easily from well known properties of principal curvature lines (Blaschke, 1929), that *principal curvature lines of x are mapped onto isotropic principal curvature lines of y*. The isotropic principal curvature directions at a surface point y are those conjugate directions, whose orthogonal projection onto $y_3 = 0$ is orthogonal. In view of Theorem 1.9, our approach is based on the following fact.

Theorem 4.1. *Any principal PN surface appears in the isotropic model of Laguerre space as a rational surface with rational isotropic curvature lines.*

In I^3 we have to construct rational surfaces $y(u,v)$, such that the parameter lines are isotopically orthogonal,

$$\langle y_u, y_v\rangle_i = 0, \tag{26}$$

where the isotropic inner product of two vectors $a, b \in \mathbb{R}^3$ is $\langle a,b\rangle_i := a_1b_1 + a_2b_2$. Furthermore, the parameter lines have to be conjugate,

$$\det(y_u, y_v, y_{uv}) = 0. \tag{27}$$

The determination of all principal PN surfaces is still a difficult problem. One could try to construct all orthogonal nets of rational curves in the plane first and then solve the partial differential equation (27) for the third component $y_3(u,v)$. Even the first part seems to be unsolved so far.

However, we can determine all *PN surfaces whose parameter lines are planar principal curvature lines*. Surfaces with planar principal curvature lines have been studied in the classical literature (see (Blaschke, 1929, pp. 278–283)). We use here the following known facts: The Gaussian spherical image of a planar principal curvature line is a circle. If the surface possesses only planar curvature lines, their Gaussian images form an orthogonal net of circles on the unit sphere Σ. This can be the set of circles through 2 fixed points A, B and their orthogonal trajectories (whose planes pass through the polar line of AB with respect to Σ); we refer to it as type 1. The second type is the limit case, where the circles pass through a fixed point $A \in \Sigma$ and touch orthogonal tangents there. The two cases are now studied separately. Note that the class of surfaces to be constructed is invariant under Laguerre transformations, since a Laguerre transformation induces a Möbius transformation in the Gaussian image and the latter preserves orthogonal nets of circles.

Type 1. We apply a Laguerre transformation and choose an appropriate coordinate system to get $A = (0,0,1)$ and $B = (0,0,-1)$. The circular net is now the usual 'geographic net' on the unit sphere. Then the principal curvature lines appear in the isotropic model as curve net whose projection onto $y_3 = 0$ is the set of circles with center $(0,0,0)$ plus the pencil of lines through the origin. We do not yet consider the rational parametrization $y(u,v)$ of the desired surfaces and use the polar coordinate representation

$$y(r,\phi) = (r\cos\phi, r\sin\phi, z(r,\phi)).$$

Its parameter lines possess the derived projection onto $y_3 = 0$. They are conjugate iff

$$\det(y_r, y_\phi, y_{r\phi}) = 0 \iff z_\phi - r z_{r\phi} = 0.$$

The solution of this equation is

$$z(r,\phi) = g(r) + r f(\phi),$$

with two arbitrary C^2 functions $f(\phi), g(r)$. We see that the surface $y(r,\phi)$ is the 'sum' of the surface of revolution Ψ_1 to $z = g(r)$ and the cone Ψ_2 with $z = r f(\phi)$. For a rational parametrization $y(u,v)$ the two components Ψ_1, Ψ_2 must be rational and thus we get

$$y(u,v) = \left(r(v)\,\frac{2a(u)b(u)}{a^2(u)+b^2(u)},\, r(v)\,\frac{a^2(u)-b^2(u)}{a^2(u)+b^2(u)},\, g(v) + r(v)f(u) \right), \tag{28}$$

with polynomials a, b and rational functions r, f, g. Application of Λ^{-1} (15) yields the nonnormalized dual representation of the desired surfaces,

$$\begin{aligned} e(r,\phi) &= \big(2rf(\phi) + 2g(r), 2r\cos\phi, 2r\sin\phi, r^2 - 1\big), \\ e(u,v) &= \bigg(2r(v)f(u) + 2g(v), 2r(v)\frac{2a(u)b(u)}{a^2(u)+b^2(u)}, \\ &\qquad 2r(v)\,\frac{a^2(u)-b^2(u)}{a^2(u)+b^2(u)}, r^2(v) - 1\bigg). \end{aligned} \tag{29}$$

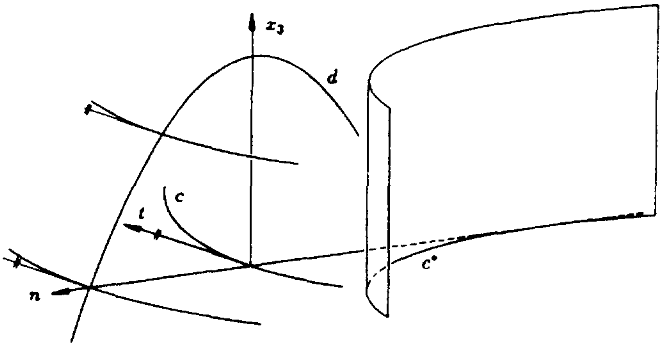

Fig. 9. Kinematic generation of a profile surface.

Surface $\Phi_1 = \Lambda^{-1}(\Psi_1)$ $(f \equiv 0)$ is a surface of revolution, obtained by rotating a PH curve, e.g., in the plane $x_1 = 0$, around the x_3-axis. The surface Φ_2 degenerates to a PH curve c in the plane $x_3 = 0$. The general solution Φ is obtained by Minkowski addition of Φ_1 and Φ_2; thereby one picks all pairs of parallel or. tangent planes of the two surfaces and constructs a new tangent plane by keeping their normal and adding their distances from the origin. This shows immediately that Φ is a *profile surface* with the following kinematical generation: choose a planar PH curve c and consider the motion of its Frenet frame (t, n, x_3). A PH curve d in the normal plane of the moving frame then generates the surface Φ. Note that the positions of the moving PH curve d and the trajectories of its points form the principal curvature lines on Φ; the offsets of Φ are generated by the offsets of d. The motion of the Frenet frame is rational in the sense that all point paths are rational in the curve parameter u of c (rational Frenet frame motions have been studied by (Wagner and Ravani, 1996)). The motion can also be generated as rolling of a plane along a cylinder which has a rational curve c^* with rational arc length parameter function (evolute of c) as cross section (Fig. 9). Note that we have applied a Laguerre transformation to get the present solution and therefore the type 1 surfaces can be described as follows.

Theorem 4.2. *Type* 1 *of PN surfaces whose parameter lines are planar rational curvature lines, are Laguerre transforms of profile surfaces, kinematically generated during the Frenet frame motion of a planar PH curve c by another PH curve which lies in the normal plane of c.*

In the simplest example, both PH curves are circles. The kinematic generation yields a torus and the Laguerre transformation a (nonparabolic) Dupin cyclide Φ. A surface generated by Tschirnhausen quartics is shown in Fig. 10.

Type 2. Again we set $A = (0, 0, 1)$ and obtain an orthogonal net of straight lines as projection of the isotropic curvature line net of y. Inserting the ansatz $y(r, s) = (r, s, z(r, s))$ into the conjugacy condition (27) we find $z = f(r) + g(s)$ and thus $y(u, v)$ is a *rational translational surface*,

$$y(u, v) = \big(a(u), b(v), c(u) + d(v)\big), \tag{30}$$

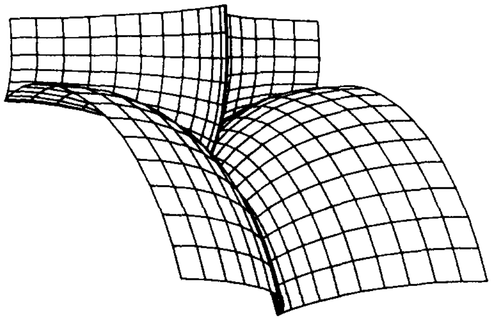

Fig. 10. PN surface with planar rational principal curvature lines.

with rational functions a, b, c, d. Conversion to the standard model can be done with (15), and delivers the nonnormalized dual form

$$e(u,v) = \big(2c(u) + 2d(v), 2a(u), 2b(v), a^2(u) + b^2(v) - 1\big). \tag{31}$$

For the geometric interpretation, we first transform y into the dual isotropic model and find again a surface y^* of type (30). The desired surface is then anticaustic of y^* for y_3-parallel illumination.

Theorem 4.3. *Type 2 PN surfaces whose parameter lines are planar rational curvature lines, are Laguerre transforms of anticaustics of rational translational surfaces. The translational surfaces are generated by translating planar rational curves in orthogonal planes along each other and the light direction is parallel to these planes.*

A simple example is found with

$$a = u, \quad b = v, \quad c = c_0 u^2, \quad d = d_0 v^2$$

and yields paraboloids y and y^* and a parabolic Dupin cyclide as PN surface (cf. Example 1.2).

If one wants to model a surface with principal PN patches, the surfaces from Theorems 4.2 and 4.3 offer more flexibility than Dupin cyclides, but still possess the limitations which arise from the very special spherical image. Therefore, it is desirable to derive a larger class of principal PN surfaces. We will now determine all *principal PN surfaces with one family of planar principal curvature lines whose Gaussian image is a family of circles through a fixed point A*. We set $A = (0, 0, 1)$. Then the projections of the corresponding family of isotropic curvature lines are straight lines. Their orthogonal trajectories form a family of offset curves and must be rational. Hence, the projection of the complete isotropic curvature line net of y consists of an *offset family of PH curves and their common normals*. As before, it is simpler to use the rational parameterization later, and thus we write y as

$$y(d,s) = \big(e(s) + (d-s)e'(s), f(s) + (d-s)f'(s), z(d,s)\big), \tag{32}$$

with a planar curve $(e(s), f(s))$ parametrized with respect to arc length s. This curve is the common evolute of the offset curve family. The condition on $z(s,d)$ to get a conjugate net of parameter lines is

$$z_s = (d-s)z_{ds},$$

with the solution

$$z(d,s) = g(d) + h(s) + (d-s)h'(s). \tag{33}$$

With $z = g(d)$ we obtain a *profile surface* Ψ_1, generated by a curve in a y_3-parallel plane that rolls on the cylinder with cross section (e, f); in accordance with type 1 surfaces above, Ψ_1 is a surface of revolution when the cylinder degenerates to a line. The second component Ψ_2 of the solution to $g \equiv 0$ is a *developable surface* with cuspidal edge $(e(s), f(s), h(s))$.

For a rational solution, the profile surface is generated by an arbitrary rational curve in the normal plane of a planar PH curve c during the Frenet frame motion along c. Ψ_2 is rational exactly for a rational cuspidal edge. An explicit representation $y(u,v)$ for the general solution is found by inserting the representation formula for rational curves $(e(u), f(u))$ with rational arc length parameter function $s(u)$ (see (Pottmann, 1995a)) and rational functions $d(v), g(v), h(u)$ with (33) into (32).

The transformation Λ^{-1} maps the surface Ψ_2 into a canal surface Ψ_2 which is formed by spheres that touch a fixed plane; the spheres are the Λ^{-1} images of the tangent planes of Ψ_2. It is harder to develop a simple geometric construction of the other component Φ_1 in the general solution Φ. It is however easy to see from the isotropic model, that the rational developables of constant slope, which touch Φ along its planar principal curvature lines (planar PH curves), are Laguerre transforms of each other. Thus, the surface may be generated as *envelope of a rational developable of constant slope during an appropriate Laguerre motion* (curve in the Lie group of Laguerre transformations). For further details and connections to kinematics we refer to (Pottmann and Wagner, 1997).

Conclusion and future research

We have shown that classical Laguerre geometry provides a theoretical framework in which several geometric design problems can be elegantly treated. Here, we focused on special rational curves and surfaces and on algebraic properties. However, it can be expected that Laguerre differential geometry (Blaschke, 1929) may be applicable as well, for example in variational curve and surface design. Furthermore, we plan to investigate whether the other classical sphere geometries, in particular Möbius geometry, are similarly useful in CAGD.

Acknowledgements

This research has been supported by the Austrian Science Foundation through project P09790.

References

Ait Haddou, R. and Biard, L. (1995), G^2 approximation of an offset curve by Tschirnhausen quartics, in: Daehlen, M., Lyche, T. and Schumaker, L.L., eds., *Mathematical Methods for Curves and Surfaces*, Vanderbilt University Press, 1–10.

Albrecht, G. and Farouki, R.T. (1996), Construction of C^2 Pythagorean-hodograph interpolating splines by the homotopy method, Advances in Comp. Math., to appear.

Aumann, G. (1995), Curvature continuous connections of cones and cylinders, Computer-Aided Design 27, 293–310.

Blaschke, W. (1910), Untersuchungen über die Geometrie der Speere in der Euklidischen Ebene, Monatshefte für Mathematik und Physik 21, 3–60.

Blaschke, W. (1929), *Vorlesungen über Differentialgeometrie III*, Springer, Berlin.

Brezinski, C. (1993), An Introduction to Padé Approximations, in: Laurent, P.J., Le Méhauté, A. and Schumaker, L.L., eds., *Curves and Surfaces II*, A K Peters, Boston, 1–10.

Cecil, T.E. (1992), *Lie Sphere Geometry*, Springer, Berlin.

DeRose, T.D. (1991), Rational Bézier curves and surfaces on projective domains, in: Farin, G., ed., *NURBS for Curve and Surface Design*, SIAM, Philadelphia, 35–45.

Dietz, R. (1995), Rationale Bézier-Kurven und Bézier-Flächenstücke auf Quadriken, dissertation, TH Darmstadt.

Dietz, R., Hoschek, J. and Jüttler, B. (1993), An algebraic approach to curves and surfaces on the sphere and on other quadrics, Computer Aided Geometric Design 10, 211–229.

Farouki, R.T. (1992), Pythagorean-hodograph curves in practical use, in: Barnhill, R.E., ed., *Geometry Processing for Design and Manufacturing*, SIAM, Philadelphia, 3–33.

Farouki, R.T. (1994), The conformal map $z \mapsto z^2$ of the hodograph plane, Computer Aided Geometric Design 11, 363–390.

Farouki, R.T. and Neff, C.A. (1995), Hermite interpolation by Pythagorean-hodograph quintics, Math. Comp. 64, 1589–1609.

Farouki, R.T. and Sakkalis, T. (1990), Pythagorean hodographs, IBM J. Res. Develop. 34, 736–752.

Farouki, R.T. and Sakkalis, T. (1994), Pythagorean-hodograph space curves, Advances in Comp. Math. 2, 41–66.

Farouki, R.T. et al. (1994), Offset curves in layered manufacturing, ASME PED, Vol. 68-2, Manufacturing Science and Engineering Book No. G0930B, 557–568.

Fiorot, J.C. and Gensane, T. (1994), Characterization of the set of rational parametric curves with rational offsets, in: Laurent, P.J., Le Méhauté, A. and Schumaker, L.L., eds., *Curves and Surfaces in Geometric Design*, A K Peters, Wellesley, MA, 153–160.

Hoffmann, Ch. M. and Peters, J. (1995), Geometric Constraints for CAGD, in: M. Daehlen, T. Lyche and Schumaker, L.L., eds., *Mathematical Methods in CAGD III*, 1–16.

Jüttler, B. and Wagner, M. (1996), Computer aided design with spatial rational B-spline motions, ASME J. of Mechanical Design 118, 193–201.

Lü, W. (1994), Rationality of the offsets to algebraic curves and surfaces, Applied Mathematics 9 (Ser. B), 265–278.

Lü, W. (1995), Offset-rational parametric plane curves, Computer Aided Geometric Design 12, 601–616.

Lü, W. (1996), Rational parameterization of quadrics and their offsets, Computing 57, 135–147.

Lü, W. and Pottmann, H. (1996), Pipe surfaces with rational spine curve are rational, Computer Aided Geometric Design 13, 621–628.

Martin, R.R. (1982), Principal patches for computational geometry, PhD Thesis, Cambridge University.

Peternell, M. and Pottmann, H. (1996), Designing rational surfaces with rational offsets, in: Fontanella, F., Jetter, K. and Laurent, P.J., eds., *Advanced Topics in Multivariate Approximation*, World Scientific, 275–286.

Peternell, M. and Pottmann, H. (1997), Computing rational parametrizations of canal surfaces, J. Symbolic Computation 23, 255–266.

Pottmann, H. (1994), Applications of the dual Bézier representation of rational curves and surfaces, in: Laurent, P.J., Le Méhauté, A. and Schumaker, L.L., eds., *Curves and Surfaces in Geometric Design*, A K Peters, Wellesley, MA, 377–384.

Pottmann, H. (1995a), Rational curves and surfaces with rational offsets, Computer Aided Geometric Design 12, 175–192.

Pottmann, H. (1995b), Curve design with rational Pythagorean-hodograph curves, Advances in Comp. Math. 3, 147–170.

Pottmann, H. and Farin, G. (1995), Developable rational Bézier and B-spline surfaces, Computer Aided Geometric Design 12, 513–531.

Pottmann, H., Lü, W. and Ravani, B. (1996), Rational ruled surfaces and their offsets, Graphical Models and Image Processing 58, 544–552.

Pottmann, H., and Peternell, M. (1997), Applications of Laguerre geometry in CAGD, Computer Aided Geometric Design, to appear.

Pottmann, H., and Wagner, M. (1997), Principal Surfaces, in: T.N.T. Goodman (ed.), *The Mathematics of Surfaces VII*, Information Geometers Ltd., to appear.

Tschirnhaus, W. von (1690), Methodus curvas determinandi, quae formantur a radiis reflectis, quorum incidentes ut paralleli considerantur, Acta Eruditorum, Febr. 1690, 68–75.

Wagner, M. and Ravani, B. (1996), Computer aided design of robot trajectories using rational Frenet–Serret motions, in: J. Lenarcic and V. Parenti-Castelli (eds.) *Recent Advances in Robot Kinematics*, Kluwer Academic Publishers, Dordrecht, 151–158.

Wallner, J. and Pottmann, H. (1997), Rational blending surfaces between quadrics, Computer Aided Geometric Design, to appear.

Wunderlich, W. (1948), Spiegelung am elliptischen Paraboloid, Monatsh. Math. 52, 13–37.

Wunderlich, W. (1973), Algebraische Böschungslinien dritter und vierter Ordnung, Sitzungsberichte der Österreichischen Akademie der Wissenschaften 181, 353–376.