Conchoid surfaces of quadrics

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Abstract

The conchoid surface F_d of a surface F with respect to a fixed reference point O is a surface obtained by increasing the distance function with respect to O by a constant d. This contribution studies *conchoid surfaces of quadrics* in Euclidean $\mathbb{R}^3$ and shows that these surfaces admit real rational parameterizations. We present an algorithm to compute these parameterizations and discuss several configurations of the position of O with respect to F where the computation simplifies significantly.

Keywords: quadric, pencil of quadrics, del Pezzo surface, rational conchoid surface, rational polar representation, focal conic.

1. Introduction

The construction of a conchoid curve to a given curve dates back to the ancient Greeks. An example is the well known *conchoid of Nikomedes*, being the conchoid of a line, see Figure 1(a). This curve has been discovered while studying the problem of angle trisection.

Consider a plane curve $C \subset \mathbb{R}^2$ and a fixed *reference point* $O \in \mathbb{R}^2$. The conchoid curve C_d of C with respect to O at distance d consists of those points Q in the lines OP for $P \in C$, for which $\operatorname{dist}(PQ) = d$ holds. Generally speaking, the conchoid construction is *non-rational*, since at any line OP there are typically two points Q_1, Q_2 with $\operatorname{dist}(PQ_1) = \operatorname{dist}(PQ_2) = d$. These points are the intersections of OP with a circle of radius d, centered at P. Thus the conchoid curves C_d of rational planar curves C are typically non-rational.

Consider a conic C, its conchoid curves C_d are only rational for very particular choices of the reference point O, namely if $O \in C$ or O coincides with one of C's focal points. More details on conchoid curves can be found in text books on algebraic curves, for instance Wieleitner (1908); Kunz (2000); Gibson (1998). Algebraic attributes of conchoid curves and surfaces have been studied recently by Sendra and Sendra (2008, 2010), and by Albano and Roggero (2010).

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The construction of a conchoid surface F_d to a given surface $F \subset \mathbb{R}^3$ with respect to a fixed reference point O at distance d follows analogous lines as the construction of conchoid curves. Figure 1(b) displays the conchoid surface of a plane F. The conchoid surface F_d of a surface F is defined by

$$F_d = \{Q \in OP \text{ with } P \in F, \text{ and } \operatorname{dist}(QP) = d\}. \tag{1}$$

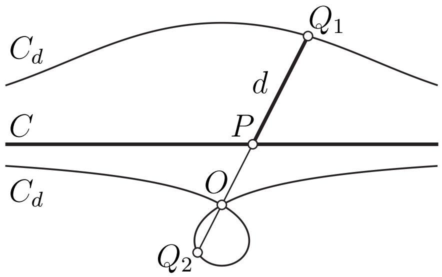
Since the construction is non-rational, the conchoid surfaces F_d of rational input surfaces F are typically non-rational and do not admit rational parameterizations. It has been shown in Peternell et al. (2012) and Peternell et al. (2011) that the conchoid surfaces F_d of spheres and rational ruled surfaces $F \subset \mathbb{R}^3$ always admit real rational parameterizations, independent of the choice of the reference point O. In both cases the parameterizations are improper.

Contribution. This article extends the class of real rational surfaces whose conchoid surfaces admit real rational parameterizations. In particular we show that the conchoid surfaces F_d of quadrics $F \subset \mathbb{R}^3$ admit real rational parameterizations, independent on the choice of the reference point O. These surfaces F_d are not bi-rationally equivalent to the projective plane and their invariants typically do not vanish, but they admit improper real rational parameterizations. The existence of these special parameterizations is proofed in Section 2, and Section 3 constructs these parameterizations in all details for regular quadrics $F \subset \mathbb{R}^3$ and general position of the reference point O with respect to F. We provide an explicit parameterization of bi-degree at most (6,2) for a quadric F which directly leads to real rational parameterizations of its conchoid surfaces F_d of bi-degree at most (12,4). Section 4 discusses singular quadrics and presents simplifications of the construction in case that $O \in F$ or that O is contained in a focal conic of F. Finally the paper concludes with a numerical example in Section 5.

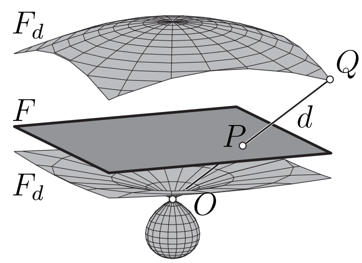
Remark. The geometric objects studied are surfaces F in real Euclidean space $\mathbb{R}^3$ admitting real rational parameterizations $\mathbf{f}(u, v) \in \mathbb{R}^3$, where (u, v) are coordinates in $\mathbb{R}^2$. Additionally the projective extension $\mathbb{P}^3$ of $\mathbb{R}^3$ is considered. This makes a main difference to traditional algebraic geometry, where surfaces are studied in a projective space over an algebraically closed field. For instance when speaking about rational surfaces Castelnuovo's theorem says that if a surface admits a rational parameterization it is bi-rationally equivalent to the projective plane over an algebraically closed field. This theorem is no longer valid if one considers real surfaces. There exist real surfaces admitting real rational parameterizations being improper. In this case the inverse map from the surface to the parameter domain is not rational since any point on the surface corresponds to two or more points in the parameter domain.

1.1. Conchoid surfaces

To fix the notation, points in $\mathbb{R}^3$ or $\mathbb{R}^4$ are identified with their Cartesian coordinate vectors $\mathbf{x} = (x, y, z)$ or $\mathbf{x} = (x, y, z, w)$. The scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is denoted by



(a) Conchoid C_d of a line C in $\mathbb{R}^2$.



(b) Conchoid F_d of a plane F in $\mathbb{R}^3$.

Figure 1: Conchoid of a line and a plane.

$\mathbf{x} \cdot \mathbf{y}$, and the same symbol denotes the product of a matrix and a vector in $\mathbb{R}^3$ or $\mathbb{R}^4$. The Euclidean norm of a vector $\mathbf{x}$ is defined by $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$.

Real surfaces $F \subset \mathbb{R}^3$ are represented by parameterizations $\mathbf{f}(u, v) = (f_1, f_2, f_3)(u, v)$, where (u, v) are coordinates in $\mathbb{R}^2$. The conchoid construction relies on the choice of a reference point $O \in \mathbb{R}^3$. Applying a translation we can always assume that $O = (0, 0, 0)$ is the origin of the chosen coordinate system in $\mathbb{R}^3$. This choice is made throughout the whole article. The conchoid surface F_d of F at distance d admits the parameterization

$$\mathbf{f}_d(u, v) = \mathbf{f}(u, v) \pm d \frac{\mathbf{f}(u, v)}{\|\mathbf{f}(u, v)\|}. \tag{2}$$

Locally F_d consists of two sheets for both signs of d. For algebraic input surfaces F, their conchoid surfaces F_d are algebraic as well. Consider a rationally parameterizable surface $F \subset \mathbb{R}^3$, the parameterization (2) is typically non-rational because of the dependency on the norm $\|\mathbf{f}(u, v)\|$. This is also evident geometrically since for any point $P \in F$ there exist typically two points $Q_1, Q_2 \in OP$ with $\mathrm{dist}(PQ_1) = \mathrm{dist}(PQ_2) = d$.

Because of the dependency of the conchoid construction on the reference point O, a polar representation of the surface $F \subset \mathbb{R}^3$ involving a unit direction vector and a distance from O is convenient. Spherical coordinates (ρ, φ, θ) in $\mathbb{R}^3$ are of this kind, but for our purposes we do not need to specify the angles φ and θ. In detail we define

Definition 1. *Consider a surface $F \subset \mathbb{R}^3$ represented by $\mathbf{f}(u, v)$. The parameterization*

$$\begin{aligned} \mathbf{f}(u, v) &= \rho(u, v)\mathbf{k}(u, v), \text{ with} \\ \rho(u, v) &= \|\mathbf{f}(u, v)\|, \text{ and} \\ \mathbf{k}(u, v) &= \frac{1}{\|\mathbf{f}(u, v)\|}\mathbf{f}(u, v), \end{aligned} \tag{3}$$

is called a polar representation *of F. The scalar valued function $\rho(u, v)$ is called its* radius function, *and the vector valued function $\mathbf{k}(u, v) \in S^2$ is called its* spherical part.

Any surface $F \subset \mathbb{R}^3$ admits a polar representation (3), at least locally. These parameterizations are well adapted for the conchoid construction, since the conchoid surfaces F_d of a surface F admit the polar representations

$$\mathbf{f}_d(u,v) = (\rho(u,v) \pm d)\mathbf{k}(u,v). \tag{4}$$

Moreover, the 'base surface' F and its conchoid surfaces F_d are in correspondence with respect to the identical spherical part $\mathbf{k}(u,v)$ of their polar representations.

The main goal is to show that quadrics in $\mathbb{R}^3$ have conchoid surfaces F_d which admit real rational parameterizations. In fact we will construct polar representations $\mathbf{f}(u,v) = \rho(u,v)\mathbf{k}(u,v)$ of quadrics F, where $\rho(u,v)$ is a rational function and $\mathbf{k}(u,v)$ is a rational (improper) parameterization of the unit sphere S^2. These parameterizations are denoted as *rational polar representations*. Finally, their conchoid surfaces F_d admit rational polar representations (4) as well.

Theoretically it might be possible that there exist rational surfaces $F \subset \mathbb{R}^3$ whose conchoid surfaces F_d admit real rational parameterizations $\mathbf{f}_d(u,v)$, not corresponding to $\mathbf{f}(u,v)$ by coincident spherical parts $\mathbf{k}(u,v)$. The following definition shall rule out these cases.

Definition 2. *A surface F is called* rational conchoid surface *with respect to the reference point $O = (0,0,0)$, if F admits a rational polar representation $\rho(u,v)\mathbf{k}(u,v)$, with a rational radius function $\rho(u,v)$ denoting the distance function from O to F and a rational parameterization $\mathbf{k}(u,v)$ of S^2.*

Remark. A rational conchoid surface $F \subset \mathbb{R}^3$ is not necessarily a rational surface in the sense that it is bi-rationally equivalent to the projective plane. It might be parameterized by a possibly improper rational polar representation $\mathbf{f}(u,v) = \rho(u,v)\mathbf{k}(u,v)$, where $\mathbf{k}(u,v)$ is a possibly improper rational parameterization of S^2. Thus F is often denoted as *unirational*.

1.2. The cone model

The construction of rational conchoid surfaces in the sense of Definition 2 is related to unirational two-dimensional surfaces in a three-dimensional cone in $\mathbb{R}^4$. Consider Euclidean $\mathbb{R}^4$ with coordinates x, y, z and w and let $\mathbb{R}^3$ be embedded in $\mathbb{R}^4$ by $w = 0$. Consider the quadratic three-dimensional cone

$$D : x^2 + y^2 + z^2 - w^2 = 0 \subset \mathbb{R}^4. \tag{5}$$

The correspondence between points in D and points in $\mathbb{R}^3$ is realized by the orthogonal projection π, see Figure 2,

$$\begin{aligned} \pi : D \subset \mathbb{R}^4 &\;\to\; \mathbb{R}^3, \\ (x,y,z,w) &\;\mapsto\; \mathbf{x} = (x,y,z), \text{ with } w = \pm\|\mathbf{x}\|. \end{aligned} \tag{6}$$

Theorem 3. *The rational conchoid surfaces $F \subset \mathbb{R}^3$ are in correspondence to those two-dimensional surfaces $\Phi \subset D : x^2 + y^2 + z^2 - w^2 = 0 \subset \mathbb{R}^4$, admitting rational parameterizations $\varphi(u, v)$.*

Proof: Let $\Phi \subset D$ be a uni-rational surface, admitting the real rational parameterization $\varphi(u, v) = (\varphi_1, \ldots, \varphi_4)(u, v)$. Consequently, the orthogonal projection $\pi(\Phi)$ is a rational conchoid surface F with rational radius function $\rho(u, v) = \varphi_4(u, v)$ and rational spherical part

$$\mathbf{k}(u, v) = \frac{1}{\varphi_4(u, v)}(\varphi_1(u, v), \varphi_2(u, v), \varphi_3(u, v)).$$

Let $F \subset \mathbb{R}^3$ be a rational conchoid surface with rational polar representation $\mathbf{f}(u, v) = \rho(u, v)\mathbf{k}(u, v)$ with $\|\mathbf{k}\| = 1$. Then there is a uni-rational surface $\Phi \subset D$ with $\pi(\Phi) = F$, and a rational parameterization of Φ reads

$$\varphi(u, v) = \rho(u, v)(k_1(u, v), k_2(u, v), k_3(u, v), 1).$$

$\square$

Note that $\varphi = \rho(k_1, k_2, k_3, -1)$ is also be a possible rational representation of Φ satisfying same requirements. Any point $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ has two pre-images $P^+ = (x, y, z, \|\mathbf{x}\|)$ and $P^- = (x, y, z, -\|\mathbf{x}\|)$ with respect to the projection π.

1.3. Admissible rational mappings

We study mappings that preserve the rationality of the polar representation of a surface. Consider the map $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\sigma(\mathbf{x}) = \mathbf{x}' = \frac{r(\mathbf{x})}{s(\mathbf{x})} R \cdot \mathbf{x}, \text{ with } R \in \mathbb{R}^{3\times 3}, \text{ and } R^T \cdot R = I = \mathrm{diag}(1, 1, 1), \tag{7}$$

and relatively prime polynomials $r(\mathbf{x})$ and $s(\mathbf{x})$. Consequently the norm of $\mathbf{x}'$ is

$$\|\mathbf{x}'\| = \sqrt{\mathbf{x}' \cdot \mathbf{x}'} = \frac{r(\mathbf{x})}{s(\mathbf{x})}\|\mathbf{x}\|.$$

Thus the rational map (7) preserves rational polar representations. It can be decomposed into a rotation $\mathbf{x} \mapsto R \cdot \mathbf{x}$ around a line through O and a 'scaling' $\mathbf{x} \mapsto f(\mathbf{x})\,\mathbf{x}$ with a rational function $f(\mathbf{x})$, fixing all lines through O. We note that a translation $\mathbf{x}' = \mathbf{x} + \mathbf{c}$, with a constant vector $\mathbf{c} \in \mathbb{R}^3$, typically does not preserve rationality of a parameterization.

If one chooses $s(\mathbf{x}) = s_0 + s_1 x + s_2 y + s_3 z$ as a linear polynomial, $r \in \mathbb{R}$ and $R = I$, then the rational map (7) becomes a *perspective collineation.* This is a projective linear map which fixes the point O and keeps the axis plane $r - s(\mathbf{x}) = 0$ point-wise fixed. The plane $s(\mathbf{x}) = 0$ contains points with improper image points, the plane $-r + s_1 x + s_2 y + s_3 z = 0$ is called vanishing plane and consists of points with improper pre-images. In Section 2.1 these perspective collineations together with rotations around lines through O are used to transform a quadric to a particular normal form.

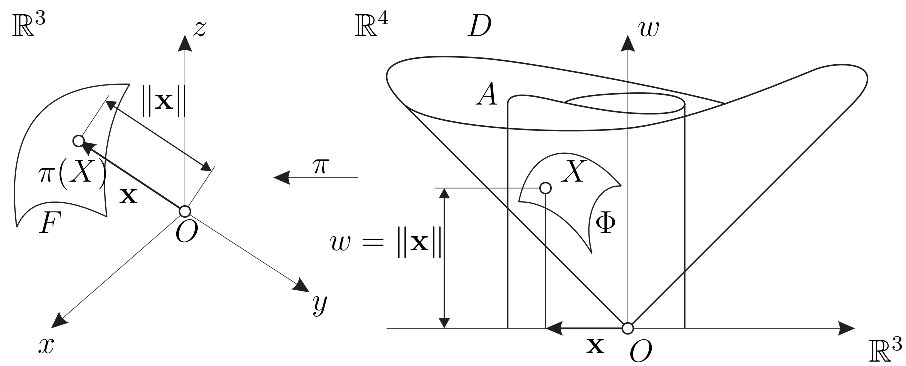


Figure 2: The cone model.

Corollary 4. *Any rational map of the form* (7) *preserves rational polar representations with respect to the reference point* $O = (0,0,0)$. *Choosing* $r \in \mathbb{R}$ *and a linear polynomial* $s(\mathbf{x})$ *these mappings are rotations about lines through* O *combined with perspective collineations with center* O.

2. Rational polar representation of quadrics – Theory

A quadric $F \subset \mathbb{R}^3$ is the zero set of a quadratic equation in x, y and z. In the following F denotes both, the quadric as well as its defining polynomial $F(x,y,z) = 0$, since it should be clear from the context whether F denotes a surface or a polynomial. We assume that the polynomial F has real coefficients and that the quadric F has more than one real point.

Quadrics $F \subset \mathbb{R}^3$ and conics $c \subset \mathbb{R}^2$ admit rational parameterizations. The conchoid curves c_d of conics $c \subset \mathbb{R}^2$ with respect to an arbitrary reference point O are typically non-rational curves. These curves c_d are rational if and only if $O \in c$ or if O is a focal point of c. In the first case c_d is an irreducible rational curve, in the second case c_d consists of two rational components.

Consider a regular quadric $F \in \mathbb{R}^3$ and a fixed reference point $O \notin F$, chosen as origin $(0,0,0)$. We prove that F admits a real rational polar representation $\mathbf{f}(u,v) = \rho(u,v)\mathbf{k}(u,v)$. Consequently, F is a rational conchoid surface in the sense of Definition 2. The explicit construction is carried out in several steps and leads to an explicit polar representation of F. An outline of the construction reads as follows:

- Apply admissible transformations to represent a quadric F by a normal form (Section 2.1).
- Compute the associated pencil of quadrics in $\mathbb{R}^4$. Its base locus Φ carries a rational one-parameter family of real conics $L(u)$ (Section 2.2).

Theoretically these two steps already prove the existence of real rational polar representations of quadrics and there are also techniques to compute those. Nevertheless, in order to find low degree parameterizations which can be computed symbolically we have to investigate the geometry of the conics $L(u)$ in more detail.

- Conics $L(u) \subset \Phi$ are transformed to circles $\overline{C}(u)$ in S^2 (Section 3.1). Explicit parameterizations of $\overline{C}(u) \subset S^2$ are provided in Section 3.2.
- Rational parameterizations of Φ and their corresponding rational polar representations of F are derived in Section 3.3.

By assuming regularity of F and $O \notin F$ we have excluded some special cases, whose treatment is actually significantly simpler than the generic case. Singular quadrics and special positions of the reference point O with respect to F are discussed in Section 4.

2.1. Transformation to normal form

A quadric F in $\mathbb{R}^3$ is given by

$$F(x, y, z) = X^T \cdot M \cdot X = 0, \text{ with } M^T = M \in \mathbb{R}^{4\times 4}, \text{ and } X = (1, x, y, z) \in \mathbb{R}^4, \tag{8}$$

where M is a regular symmetric 4×4 matrix with real entries and X is a column vector. Let $O = (0, 0, 0) \notin F$ be the reference point for the conchoid construction.

The aim is to apply admissible transformations and coordinate transformations such that the image quadric is represented by a diagonal matrix. We perform this in two steps. First we apply a perspective collineation κ with center $O \notin F$ according to Corollary 4, that O becomes the center of $\kappa(F) = F'$. Assume that M has entries $m_{ij} \in \mathbb{R}$, with $i, j = 1, \ldots, 4$, then this transformation reads

$$\kappa : \mathbf{x}' = \frac{1}{s(\mathbf{x})}\mathbf{x}, \text{ with } s(\mathbf{x}) = m_{11} + m_{12}x + m_{13}y + m_{14}z. \tag{9}$$

The polar plane δ of O with respect to F is given by $s(\mathbf{x}) = 0$. Let $\omega = \mathbb{P}^3 \setminus \mathbb{R}^3$ be the ideal plane of the projective space $\mathbb{P}^3$ extending $\mathbb{R}^3$. Then $\kappa(\delta) = \omega$, and κ maps F to the quadric

$$F' : \frac{1}{m_{11}} + \mathbf{x}'^T \cdot M' \cdot \mathbf{x}' = 0,$$

with a symmetric 3×3 matrix M'. The equation of F' does no longer contain linear terms in x', y' and z'. Further we may assume that $m_{11} = \pm 1$. Depending on the position of O and $\omega = \kappa(\delta)$ with respect to F' one distinguishes different affine types of F'. Since $O \notin F$ and thus ω is not a tangent plane of F', the image quadric F' is never a paraboloid.

- If O is inside of F', the intersection $F' \cap \omega$ does not contain real points. Thus F' is an *ellipsoid.*

- Otherwise if O is outside F', the intersection $F' \cap \omega$ is a conic containing real points. Thus F' is a *hyperboloid*, either *of one sheet* or *of two sheets*.

In a second step we apply a coordinate transformation where the new coordinate axes are chosen as eigenvectors of M'. This can be considered as rotation fixing O. Thus F' is represented by a diagonal matrix, and reads

$$F' : X^T \cdot \mathrm{diag}(\pm 1, \pm a^2, \pm b^2, \pm c^2) \cdot X = 0. \tag{10}$$

If all signs in (10) are positive, F' is a quadric without real points. Otherwise in case of strict inequalities between a, b and c and with a proper re-ordering of the coordinate axes the different combinations of signs imply the normal forms of Table 1.

Ellipsoid	$F' : -1 + a^2x^2 + b^2y^2 + c^2z^2 = 0$
Hyperboloid of two sheets	$F' : 1 - a^2x^2 + b^2y^2 + c^2z^2 = 0$
Hyperboloid of one sheet	$F' : -1 - a^2x^2 + b^2y^2 + c^2z^2 = 0$

Table 1: Normal forms of quadrics.

If there are two coincident eigenvalues, say $b = c$, F' is a rotational quadric with x as axis. If all three eigenvalues coincide, F' is a sphere, centered at O. The first case is obtained when O lies on a focal conic of F, and the second case appears when O is a focal point of a rotational quadric F. Both cases are studied in detail in Section 4.3. Since a hyperboloid of one sheet (last line of Table 1) is a real ruled quadric, the algorithm from Peternell et al. (2011) applies to this case, too.

2.2. Quadric pencil and base locus

Consider a quadric represented by one of the normal forms listed in Table 1. For simplicity it shall be denoted again by F instead of F'. We show that F contains a one-parameter family of conics admitting a real rational polar representation. Let $A \subset \mathbb{R}^4$ be the three-dimensional quadratic cylinder through F whose generating lines are parallel to the w-axis. Thus $A(x, y, z, w) = F(x, y, z)$=0. Let

$$\mathcal{B}(\alpha, \beta) = \alpha A + \beta D \subset \mathbb{R}^4, \text{ with } (\alpha, \beta) \in \mathbb{R}^2 \setminus (0, 0), \tag{11}$$

be the pencil of quadrics in $\mathbb{R}^4$ spanned by the cylinder A and the cone D from equation (5), illustrated in Figure 3(a).

A pencil of quadrics $\mathcal{B} \subset \mathbb{R}^4$ contains up to five singular quadrics. If one of these singular quadrics, say B, is a cone over a real ruled quadric, the cone B contains two one-parameter families of real planes corresponding to the two families of generating lines of the ruled quadric. Moreover, these families of planes are determined by linear equations with rational coefficients. We have a look at the three different normal forms of F, and show that in these cases the pencil of quadrics $\mathcal{B}$ contains a cone over a ruled quadric.

- Given an ellipsoid $F : -1 + a^2x^2 + b^2y^2 + c^2z^2 = 0$ in $\mathbb{R}^3$, with $0 < a^2 < b^2 < c^2$. The respective pencil of quadrics $\mathcal{B}$ contains the cylinder

$$B : -1 - (b^2 - a^2)x^2 + (c^2 - b^2)z^2 + b^2w^2 = 0 \tag{12}$$

  over the ruled two-dimensional quadric $B \cap (y = 0)$.

- Given a hyperboloid of two sheets $F : 1-a^2x^2+b^2y^2+c^2z^2 = 0$ in $\mathbb{R}^3$, with $0 < b^2 < c^2$. The respective pencil of quadrics contains the cylinder

$$B : 1 - (c^2 + a^2)x^2 - (c^2 - b^2)y^2 + c^2w^2 = 0$$

  over the ruled two-dimensional quadric $B \cap (z = 0)$.

- Given the hyperboloid of one sheet $F : -1 - a^2x^2 + b^2y^2 + c^2z^2 = 0$ in $\mathbb{R}^3$, with $0 < b^2 < c^2$. The corresponding cylinder A is already a cylinder over the ruled quadric F. The respective pencil of quadrics contains two further cylinders

$$\begin{aligned} B_1 \ &: -1 + (a^2 + b^2)y^2 + (a^2 + c^2)z^2 - a^2w^2 = 0, \text{ and} \\ B_2 \ &: -1 - (a^2 + b^2)x^2 + (c^2 - b^2)z^2 + b^2w^2 = 0, \end{aligned}$$

  over ruled quadrics $B_1 \cap x = 0$ and $B_2 \cap y = 0$, respectively.

Let B be a cylinder over a two-dimensional ruled quadric and let $\psi(u), u \in \mathbb{R}$, be one of its one-parameter families of generating planes. Consider the intersection surface $\Phi = A \cap D$, the base locus of the pencil of quadrics $\mathcal{B}$. Since $\Phi = B \cap D$, Φ contains a rational one parameter family of conics $L(u) = D \cap \psi(u)$, compare equation (14). Not all of these conics might contain real points, but there is at least a subset $L(s), s \in I \subset \mathbb{R}$ of conics containing real points; compare equation (20) in Section 3.2. It has been proved in Peternell (1997); Schicho (1998) that such a family of conics always admits a real rational parameterization. This proves that $\Phi \subset D$ is a uni-rational surface.

According to Theorem 3, a real rational parameterization of Φ represents a real rational polar representation of the quadric F. This proves

Theorem 5. *A quadric $F \subset \mathbb{R}^3$ is a rational conchoid surface independent of the position of the reference point O and the distance d.*

Remark. The two-dimensional surface Φ is a so called *del Pezzo surface* of degree four, and it is known that it admits real rational parameterizations, even a proper one over an algebraically closed field. For detailed information about del Pezzo surfaces see for example Griffiths and Harris (1978); Manin (1974); Schicho (1998).

The problem of computing real rational parameterizations of the intersection Φ of two quadrics in $\mathbb{R}^4$ has already been studied in Aigner et al. (2009). We give a brief outline of that method. Consider a point $P \in \Phi$. The projection of Φ from P to a three-dimensional space is a cubic surface, say Ψ. The cubic surface Ψ contains at least one real line, say g. Consider the one parameter family of planes $\varepsilon(t)$ through g. The intersection $\varepsilon(t) \cap \Psi$ consists of g and a curve $k(t)$ of degree two, typically a conic. A real rational parameterization of this family of curves $k(t)$ on Ψ is lifted back to a real rational parameterization of Φ.

Section 3 provides a detailed description of a another method for the construction of a real rational parameterization of the intersection Φ of two quadrics in $\mathbb{R}^4$. The construction is possibly a bit more involved compared to that of Aigner et al. (2009). But finally we are able to provide a *symbolic solution* and the resulting parameterization is of bi-degree $(6, 2)$.

3. Rational polar representation of quadrics – Details

This section provides a detailed construction of a polar representation of a quadric $F \subset \mathbb{R}^3$ with respect to a reference point O, chosen as origin. We have already explained how to transform F to a normal form with respect to O, given in Table 1. The last section lists the singular quadrics of the corresponding quadric pencil $\mathcal{B} \subset \mathbb{R}^4$, which are cylinders over a two-dimensional ruled quadric. The explicit construction is similar in all three cases and shall be performed exemplarily for an ellipsoid $F \subset \mathbb{R}^3$.

Consider the ellipsoid $F : -1 + a^2x^2 + b^2y^2 + c^2z^2 = 0$, with $0 < a^2 < b^2 < c^2$. The corresponding pencil of quadrics $\mathcal{B} \subset \mathbb{R}^4$ contains the singular quadric B, given by (12). Let $\Phi = B \cap D$. A rational parameterization of B reads

$$\mathbf{b}(u, v_1, v_2) = \mathbf{e}_0(u) + v_1\mathbf{e}_1(u) + v_2\mathbf{e}_2, \tag{13}$$

where $\mathbf{e}_0(u) + v_1\mathbf{e}_1(u)$ parameterizes a ruled quadric being contained in $y = 0$, and $\mathbf{e}_2$ is a direction vector of the generating lines of the cylinder B. Using the abbreviations $\beta = \sqrt{b^2 - a^2}$ and $\gamma = \sqrt{c^2 - b^2}$, the vector functions $\mathbf{e}_0, \mathbf{e}_1$ and $\mathbf{e}_2$ read

$$\mathbf{e}_0 = \left(-\frac{u}{\beta}, 0, -\frac{u}{\gamma}, \frac{1}{b}\right), \ \mathbf{e}_1 = \left(-\frac{u^2+1}{u\beta}, 0, -\frac{u^2-1}{u\gamma}, \frac{2}{b}\right), \text{ and } \mathbf{e}_2 = (0, 1, 0, 0)\,.$$

The generating planes $\psi(u)$ of B are spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. The intersection surface Φ carries a family of conics $L(u) = D \cap \psi(u)$. Inserting the parameterization (13) into $D : x^2 + y^2 + z^2 - w^2 = 0$ yields an implicit representation of $L(u)$, a quadratic equation in v_1 and v_2 whose coefficients are polynomials in u.

In order to obtain an equation (14) of $L(u)$ without linear terms in v_1, we choose the directrix curve $\mathbf{e}_0(u)+\lambda(u)\mathbf{e}_1(u)$ instead of $\mathbf{e}_0(u)$ of B in (13), with

$$\lambda(u)=\frac{(a^2b^2(u^2+1)-c^2b^2(u^2-1)-2a^2c^2)u^2}{a^2b^2(u^2+1)^2-c^2b^2(u^2-1)^2-4a^2c^2u^2}.$$

Consequently, the family of conics $L(u)\subset\Phi$ is represented by

$$L(u):l_0(u)+l_1(u)v_1^2+l_2(u)v_2^2=0, \tag{14}$$

whose coefficients are the polynomials

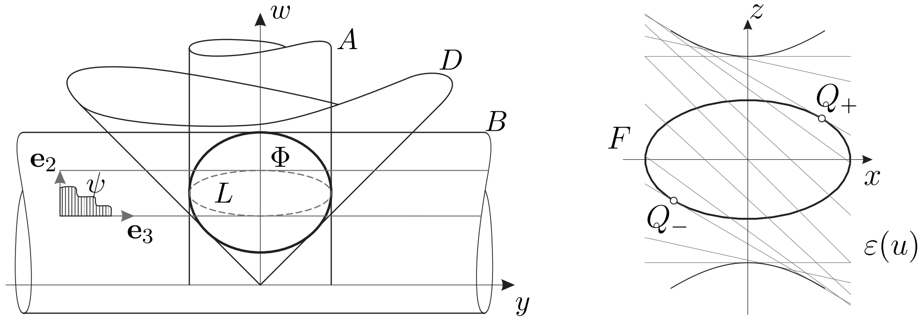
$$\begin{aligned} l_0(u)&=b^2\gamma^2\beta^2u^2(c^2(u^2-1)^2-a^2(u^2+1)^2),\\ l_1(u)&=(b^2(c^2(u^2-1)^2-a^2(u^2+1)^2)+4a^2c^2u^2)^2,\\ l_2(u)&=-b^2\gamma^2\beta^2u^2(b^2(c^2(u^2-1)^2-a^2(u^2+1)^2)+4a^2c^2u^2). \end{aligned} \tag{15}$$

The aim is to determine real rational functions $(v_1(u,t),v_2(u,t))$ satisfying equation (14) identically. It has already been shown in Schicho (1998) that these functions exist if some requirements are fulfilled. At first, the family of conics $L(u)$ has to contain real points for all $u\in\mathbb{R}$. If this is not the case, it is necessary to substitute $u=(u_0s^2+u_1)/(s^2+1)$ such that $L(s)$ satisfies this requirement for all $s\in\mathbb{R}$.

In the next step one computes the zeros of the polynomials $l_i(s)$. Real zeros appear with even multiplicity. In case that two of the polynomials $l_i(s)$ have common zeros, equation (14) can be simplified. Finally we end up with an equation $L(s)$ of the form (14) where no two polynomials have common zeros. In the present case these polynomials are of degrees ≤ 8. To construct real rational functions $(v_1(s,t),v_2(s,t))$ satisfying $L(s)$ identically, a linear system combined with a quadratic equation has to be solved. To our knowledge it is not possible to compute a symbolic solution for $v_1(s,t)$ and $v_2(s,t)$ in terms of the coefficients a,b,c of F, but only numeric solutions are available for particular choices of these coefficients. In addition, the degrees of the final parameterization of Φ are unnecessarily high.

Since this direct method does not result in a symbolic parameterization of Φ depending on the coefficients of the input quadric F, further geometric properties of the family of conics $L(u)$ have to be investigated. As a benefit for this extra work we can provide parameterizations of lower degrees. All computational steps proposed in the following, can be carried out symbolically with help of a computer-algebra-system.

- The rational family of conics $L(u)\subset\Phi\subset\mathbb{R}^4$ is transformed to a rational family of circles $\overline{C}(u)\subset S^2\subset\mathbb{R}^3$. A real rational parameterization of $\overline{C}(u)$ is constructed explicitly in Section 3.2.
- A real rational parameterization of Φ corresponds to a real rational polar representation of the quadric F; compare Section 3.3.



(a) Pencil of quadrics in $\mathbb{R}^4$.

(b) Envelope of the planes $\varepsilon(u)$.

Figure 3: Illustration of the situation in $\mathbb{R}^4$ and planes $\varepsilon(u)$.

3.1. Cones of revolution

Consider the top view projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ with $\pi(x, y, z, w) = (x, y, z)$. We intend to prove that the top-view projections $C(u) = \pi(L(u))$ are a family of conics which are contained in cones of revolution $\Gamma(u)$, with common vertex at the origin O. To achieve this we investigate at first the intersection of the cone $D \subset \mathbb{R}^4$ with a generic three-dimensional subspace E.

Lemma 6. *Consider the cone $D : x^2 + y^2 + z^2 - w^2 = 0$ and a hyperplane $E : a_1x + a_2y + a_3z - a_4w = 0$. If the intersection $K = D \cap E$ is a real cone $\subset \mathbb{R}^4$, its top view projection $\pi(K) = \Gamma$ is a cone of revolution with $\mathbf{a} = (a_1, a_2, a_3)$ as rotational axis and its half opening angle τ is determined by $\|\mathbf{a}\| \cos\tau = a_4$.*

Proof: The intersection $K = D \cap E$ is a quadratic cone with vertex $O \in \mathbb{R}^4$. Its projection $\Gamma = \pi(K)$ is a cone with vertex at O, given by

$$\Gamma : (a_1^2 - a_4^2)x^2 + (a_2^2 - a_4^2)y^2 + (a_3^2 - a_4^2)z^2 + 2(a_1a_2xy + a_2a_3yz + a_3a_1zx) = 0. \tag{16}$$

Since the origin in $\mathbb{R}^4$ coincides with the origin in $\mathbb{R}^3$, we use the same symbol O. Introducing the vector $\mathbf{x} = (x, y, z)$, we may write $D : \mathbf{x}^T \cdot \mathbf{x} - w^2 = 0$ and $E : \mathbf{a}^T \cdot \mathbf{x} - a_4w = 0$. Eliminating w from these two equations yields $\Gamma : \mathbf{x}^T \cdot M \cdot \mathbf{x} = 0$, with $M = \mathbf{a} \cdot \mathbf{a}^T - a_4^2 I$, and $I = \operatorname{diag}(1, 1, 1)$, which is just equation (16) in vector notation.

If $a_4 = 0$, it follows that rk $M = 1$, and Γ is the double plane $(\mathbf{x}^T \cdot \mathbf{a})^2 = 0$. If $a_4^2 = a_1^2 + a_2^2 + a_3^2$, E is tangent to D, and rk $M = 2$. The projection $\Gamma = \pi(D \cap E)$ consists of a real line carrying two conjugate complex planes.

Otherwise, rk $M = 3$ and its eigenvectors define the axes of symmetry of Γ. The eigenvalues and corresponding eigenvectors (eigenspaces) of M are

$$\begin{aligned} t_1 &= a_1^2 + a_2^2 + a_3^2 - a_4^2 &&\to \quad \mathbf{a} = (a_1, a_2, a_3), \\ t_2 &= t_3 = -a_4^2 &&\to \quad \lambda\mathbf{v}_1 + \mu\mathbf{v}_2, \ \text{with } \mathbf{v}_1, \mathbf{v}_2 \perp \mathbf{a}. \end{aligned}$$

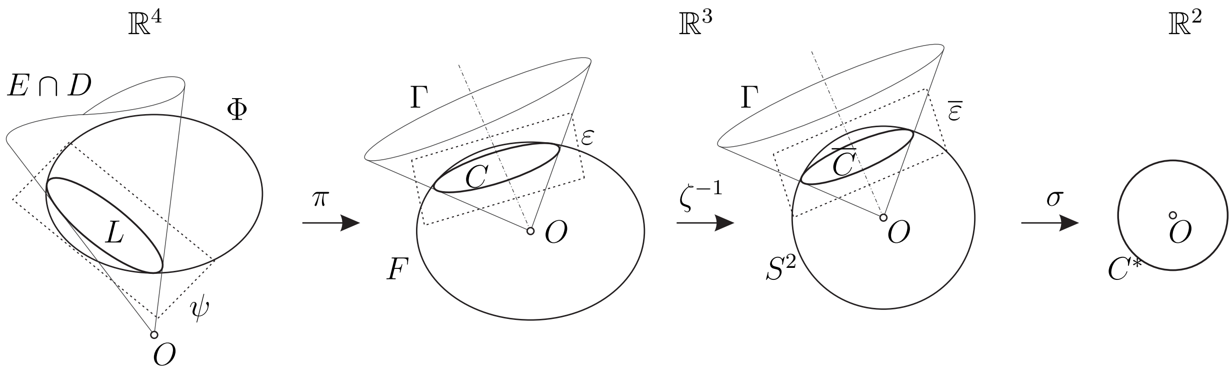


Figure 4: From left to right: Conics $L \subset \Phi \subset \mathbb{R}^4$, conics $C \subset F \subset \mathbb{R}^3$, circles $\overline{C} \subset S^2 \subset \mathbb{R}^3$, circles $C^* \subset \mathbb{R}^2$.

The eigenvalue t_1 corresponds to the axis $\mathbf{a}$ of Γ. The twofold eigenvalue $t_2 = t_3$ corresponds to a two-dimensional eigenspace spanned by two linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2$, both orthogonal to $\mathbf{a}$. Any plane passing through the axis with direction vector $\mathbf{a}$ is a plane of symmetry of Γ, and thus Γ is a cone of rotation. Intersecting $\Gamma : \mathbf{x}^T \cdot M \cdot \mathbf{x} = 0$ with the unit sphere $\mathbf{x}^T \cdot I \cdot \mathbf{x} = 1$ shows that the half opening angle τ of Γ satisfies $\|\mathbf{a}\| \cos \tau = a_4$. $\square$

Lemma 7. *Consider the cone $D : x^2 + y^2 + z^2 - w^2 = 0$. Let $\psi \subset \mathbb{R}^4$ be a plane with $O \notin \psi$ and assume that $L = D \cap \psi$ contains real points. Then the projection $\pi(L) = C$ is either a segment of a line or a conic contained in a rotational cone $\Gamma \subset \mathbb{R}^3$.*

Proof: The intersection $L = D \cap \psi$ is a conic in $\mathbb{R}^4$. Assume that its carrier plane ψ is not parallel to the w-axis, then the projection $C = \pi(L)$ is a conic as well. Consider the hyperplane E joining $O = (0, 0, 0, 0)$ and ψ. Lemma 6 says that the projection $\pi(D \cap E) = \Gamma$ is a cone of rotation. Since $\psi \subset E$, the projection $C = \pi(L)$ is a conic in the rotational cone Γ.

In case where the plane $\psi \subset \mathbb{R}^4$ is parallel to the w-axis, its projection $\pi(\psi) \subset \mathbb{R}^3$ is a line. Consequently the projection of the conic $L = D \cap \psi$ is a segment of that line. $\square$

In the remainder of the section we give the explicit representations of the conics $L(u) \subset \Phi$ and their projections $C(u) = \pi(L(u))$ being contained in cones of revolution $\Gamma(u)$ with common vertex O. The intersection of a cone of revolution with vertex at O and the unit sphere S^2 consists of two circles. It is possible to define a rational map $C(u) \mapsto \overline{C}(u)$ between the family of conics $C(u) \subset F$ and a family of circles $\overline{C}(u) \subset S^2$. An explicit representation of this map is finally given by the radius function $\rho(s,t)$ in equation (25). The motivation to proceed in that way is that the practical parameterization of a one-parameter family of circles on S^2 is easier than parameterizing a general one-parameter family of conics $C(u)$ in space. Moreover it turns out that the map $C(u) \mapsto \overline{C}(u)$ and its inverse do not raise the degree of the final parameterization.

The family of conics $L(u) = \psi(u) \cap D$ on the surface $\Phi \subset \mathbb{R}^4$ is represented by

$$\left.\begin{aligned} \psi(u) : \mathbf{b}(u, v_1, v_2) = \mathbf{e}_0(u) + v_1\mathbf{e}_1(u) + v_2\mathbf{e}_2 \\ D : x^2 + y^2 + z^2 - w^2 = 0 \end{aligned}\right\} L(u), \tag{17}$$

where $\psi(u)$ is one family of generating planes of the cylinder B from (12) and (13). Applying the top view projection π from (6) we obtain the family of conics $C(u) = \pi(L(u))$ on the ellipsoid $F = \pi(\Phi)$. These conics $C(u) \subset F$ are intersections of F with planes $\varepsilon(u) = \pi(\psi(u))$. Their representation reads

$$\left.\begin{aligned} \varepsilon(u) = \pi(\psi(u)) : -2u + \beta(u^2 - 1)x - \gamma(u^2 + 1)z = 0 \\ F = \pi(\Phi) : -1 + a^2x^2 + b^2y^2 + c^2z^2 = 0 \end{aligned}\right\} C(u). \tag{18}$$

Consider the hyperplanes $E(u)$ connecting planes $\psi(u)$ and the origin O in $\mathbb{R}^4$. According to Lemmas 6 and 7, the conics $C(u) = \Gamma(u) \cap \varepsilon(u)$ are intersections of rotational cones $\Gamma(u) = \pi(E(u) \cap D)$ and planes $\varepsilon(u)$. An illustration is given in Figure 4.

The cones $\Gamma(u)$ have common vertex O, and a direction vector of their rotational axes is $(\beta(u^2 + 1), 0, -\gamma(u^2 - 1))$. Since an expression for the half opening angle $\tau(u)$ is rather lengthy and not needed in the following, it is omitted here. The intersection $\Gamma(u) \cap S^2$ defines a family of circles $\overline{C}(u)$ represented by

$$\left.\begin{aligned} \overline{\varepsilon}(u) : -2ub + \beta(u^2 + 1)x - \gamma(u^2 - 1)z = 0 \\ S^2 : x^2 + y^2 + z^2 - 1 = 0 \end{aligned}\right\} \overline{C}(u). \tag{19}$$

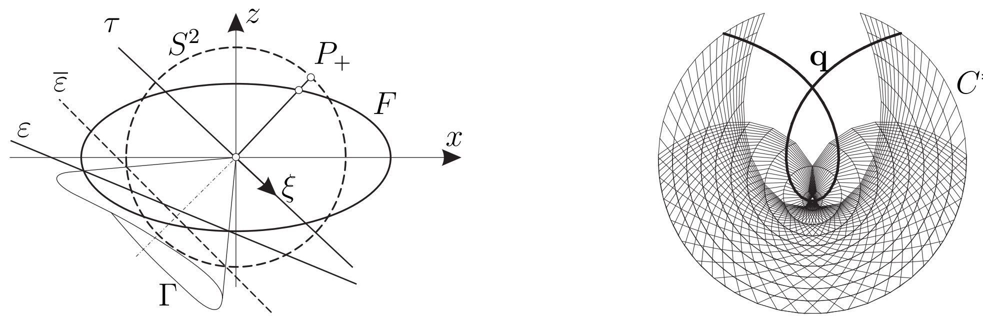
Remark. We note that the planes $\varepsilon(u)$ envelope the hyperbolic cylinder $-(b^2 - a^2)x^2 + (c^2 - b^2)z^2 - 1 = 0$, see Figure 3(b). It is obtained as intersection $B \cap \mathbb{R}^3 : w = 0$. Likewise, the carrier planes $\overline{\varepsilon}(u)$ of the circles $\overline{C}(u)$ envelope the hyperbolic cylinder $-(b^2 - a^2)x^2 + (c^2 - b^2)z^2 + b^2 = 0$.

3.2. Rational parameterization of a family of circles

In Section 3.1 the family of conics $L(u) \subset \Phi \subset \mathbb{R}^4$ has been transformed to a family of circles $\overline{C}(u) \subset S^2$. Not all planes $\overline{\varepsilon}(u)$ intersect S^2 in circles containing real points. In order to construct a real rational parameterization of the circles $\overline{C} = S^2 \cap \overline{\varepsilon}$, we have to restrict the parameter $u \in \mathbb{R}$ to a suitable proper interval $[u_0, u_1]$. This re-parameterization reads

$$u(s) = \frac{u_0 s^2 + u_1}{s^2 + 1}, \text{ with } u_0 = \frac{c - a}{\alpha},\ u_1 = -\frac{c - a}{\alpha}, \text{ and } \alpha = \sqrt{c^2 - a^2}. \tag{20}$$

The curves $\overline{C}(u_0), \overline{C}(u_1) \subset S^2$ degenerate to the points $P_\pm = \pm\frac{1}{b\alpha}(c\beta, 0, a\gamma)$, see Figure 5(a). Let τ be the symmetry plane of P_+ and P_-.



(a) Relation between F and S^2.

(b) Circles $C^\star \subset \tau$, with cubic trajectory $\mathbf{q}$.

Figure 5: Correspondence between the sphere and the quadric and the circles in the plane τ.

To gain a rational parameterization of the circles $\overline{C}(s)$, a stereographic projection $\sigma : S^2 \to \tau$ with projection center P_+ is performed. Since σ is a conformal map, it transfers circles $\overline{C} \subset S^2$ to circles $C^\star \subset \tau$. An implicit representation for $C^\star(s)$ is obtained by choosing a Cartesian coordinate system $\{O, \xi, \eta\}$ in τ, with $\eta = y$ and $\xi = \eta \times \overrightarrow{OP_+}$. This gives

$$C^*(s) : (\xi - m(s))^2 + \eta^2 - r(s)^2 = 0, \text{ with} \tag{21}$$

$$m(s) = \frac{s^2\gamma\beta}{b^2 + s^2ac}, \text{ and } r(s)^2 = \frac{b^2s^2(s^2a + c)(a + cs^2)}{(b^2 + s^2ac)^2}. \tag{22}$$

The denominator of $r(s)^2$ is a square of a polynomial. Its numerator is a non-negative polynomial and therefore it is the sum of the two squares $h_1(s)^2 = (s^2b(a + c))^2$ and $h_2(s)^2 = (sb\sqrt{ac}(s^2 - 1))^2$. The terms $h_1(s)$ and $h_2(s)$ together with $m(s)$ define a rational cubic trajectory $\mathbf{q}(s)$ of the family of circles $C^*(s)$ with the property that $\mathbf{q}(s) \in C^\star(s)$ for all $s \in \mathbb{R}$. A parameterization reads

$$\mathbf{q}(s) = \frac{1}{b + s^2ac}\begin{pmatrix} s^2\gamma\beta + h_1(s) \\ h_2(s) \end{pmatrix} = \frac{1}{b + s^2ac}\begin{pmatrix} s^2(b(c + a) + \gamma\beta) \\ sb(s^2 - 1)\sqrt{ac} \end{pmatrix}$$

We construct a parameterization of that part of the plane τ being covered by the circles $C^\star(s)$ of bi-degree $(3, 2)$, with the property that $\mathbf{c}^\star(s_0, t)$ represents the fixed circle $C^\star(s_0)$, with $s_0 = \text{const}$. Using the abbreviation $\mu(s, t) = 2\sqrt{ac}t(s^2 - 1) - (c + a)s(t^2 - 1)$, this parameterization reads

$$\mathbf{c}^*(s, t) = \frac{1}{(b^2 + s^2ac)(1 + t^2)}\begin{pmatrix} s(-b\mu(s, t) + s\gamma\beta(t^2 + 1)) \\ -sb(\sqrt{ac}(s^2 - 1)(t^2 - 1) + 2ts(c + a)) \end{pmatrix}.$$

The inverse stereographic projection $\sigma^{-1} : \tau \to S^2$ maps $\mathbf{c}^\star(s, t)$ to a rational parameterization $\overline{\mathbf{c}}(s, t)$ of S^2 of bi-degree $(6, 2)$,

$$\overline{\mathbf{c}}(s, t) = (\overline{c}_1(s, t), \overline{c}_2(s, t), \overline{c}_3(s, t)) = \frac{1}{n(s, t)}\overline{\mathbf{g}}(s, t), \tag{23}$$

whose numerator $\overline{\mathbf{g}} = (\overline{g}_1, \overline{g}_2, \overline{g}_3)$ and denominator n are the polynomials

$$
\begin{aligned}
\overline{g}_1(s,t) &= -2b\gamma s(a+cs^2)\mu(s,t) - \beta(1+t^2)(c(s^4+1)(b^2-s^2ac)+2s^2(b^2a-c^3s^2)) \\
\overline{g}_2(s,t) &= -2\alpha s(b^2+s^2ac)(\sqrt{ac}(1-t^2)(1-s^2)+2st(a+c)) \\
\overline{g}_3(s,t) &= 2b\beta s(as^2+c)\mu(s,t) - \gamma(1+t^2)(a(s^4+1)(b^2-s^2ac)+2s^2(b^2c-a^3s^2)) \\
n(s,t) &= \alpha(-2\beta\gamma s^3\mu(s,t)+b(1+t^2)((1-s^4)(b^2-acs^2)+2s^2(s^2(a^2+c^2)+2ac))).
\end{aligned} \tag{24}
$$

3.3. Rational polar representation of F and its conchoid surfaces

According to Definition 1, a polar representation $\mathbf{f}(s,t) = \rho(s,t)\mathbf{k}(s,t)$ of a surface F consists of a radius function $\rho(s,t)$ and a parameterization $\mathbf{k}(s,t)$ of S^2. The parameterization $\overline{\mathbf{c}}(s,t)$ from equation (23) is already the spherical part of the rational polar representation of the ellipsoid F. To determine the radius function $\rho(s,t)$, the family of conics $C(s) \subset \varepsilon(s)$ has to be parameterized, compare equation (18). Using the substitution (20), the coefficients e_i and $\overline{e}_i$ of the planes $\varepsilon : e_0 + e_1x + e_2y + e_3z = 0$ and $\overline{\varepsilon} : \overline{e}_0 + \overline{e}_1x + \overline{e}_2y + \overline{e}_3z = 0$ read

$$
\begin{aligned}
&e_0 = \alpha(s^4-1), \quad && e_1 = \beta(a(s^4+1)+2cs^2), \quad && e_2 = 0, \quad e_3 = \gamma(c(s^4+1)+2as^2), \text{ and} \\
&\overline{e}_0 = be_0, && \overline{e}_1 = -\tfrac{\beta}{\gamma}e_3, && \overline{e}_2 = 0, \quad \overline{e}_3 = -\tfrac{\gamma}{\beta}e_1.
\end{aligned}
$$

The conics $C(s) \subset F$ are computed as intersection curves $C = \Gamma \cap \varepsilon$. Thus we have

$$
\mathbf{f}(s,t) = \mathbf{c}(s,t) = \rho(s,t)\overline{\mathbf{c}}(s,t), \text{ with}
$$
$$
\rho(s,t) = \frac{-e_0}{e_1\overline{c}_1 + e_2\overline{c}_2 + e_3\overline{c}_3} = \frac{-e_0 n}{e_1\overline{g}_1 + e_2\overline{g}_2 + e_3\overline{g}_3}. \tag{25}
$$

We note that $\overline{\mathbf{c}} \subset \overline{\varepsilon}$ for all $s \in \mathbb{R}$. In case that $\varepsilon(s_0) = \overline{\varepsilon}(s_0)$, it follows that $\overline{\mathbf{c}} \subset \varepsilon(s_0)$, and the denominator and numerator of (25) have the common factor $(s - s_0)$. The condition $\varepsilon(s_0) = \overline{\varepsilon}(s_0)$ holds for the zeros of s^2-1, corresponding to $u = 0$, and for the zeros of s^2+1, corresponding to $u = \infty$. This implies that the polynomial $\varepsilon : e_0 n + e_1\overline{g}_1 + e_2\overline{g}_2 + e_3\overline{g}_3 = 0$ is divisible by $s^4 - 1$. Since $e_0 = \alpha(s^4 - 1)$, also the denominator $e_1\overline{g}_1 + e_2\overline{g}_2 + e_3\overline{g}_3$ is divisible by $s^4 - 1$. Thus the radius function (25) is represented by

$$
\begin{aligned}
\rho(s,t) &= \frac{n(s,t)}{p(s,t)} \text{ with } n(s,t) \text{ from (24) and} \\
p(s,t) &= \alpha(-2\beta\gamma bs\mu(s,t) + (1+t^2)(ac(1-s^4)(b^2-s^2ac)+2s^2b^2(a^2+c^2+2s^2ac))).
\end{aligned} \tag{26}
$$

Combining equations (23), (25) and (26) leads to real rational polar representations of the ellipsoid F and its conchoid surfaces F_d at distance $d \in \mathbb{R}$,

$$
F: \quad \mathbf{f}(s,t) = \frac{1}{p(s,t)}\overline{\mathbf{g}}(s,t) = \rho(s,t)\overline{\mathbf{c}}(s,t), \tag{27}
$$

$$
F_d: \quad \mathbf{f}_d(s,t) = \frac{n(s,t) + d\,p(s,t)}{p(s,t)}\overline{\mathbf{c}}(s,t) = \frac{n(s,t) + d\,p(s,t)}{n(s,t)\,p(s,t)}\overline{\mathbf{g}}(s,t). \tag{28}
$$

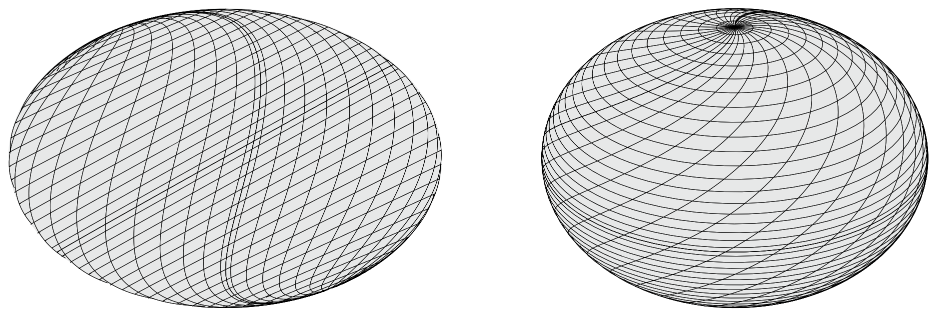


Figure 6: Rational parameterization with rational distance of an ellipsoid.

The parameterization $\mathbf{f}(s,t)$ of F is of bi-degree $(6,2)$ whereas the parameterization $\mathbf{f}_d(s,t)$ of F_d is typically of bi-degree $(12,4)$. Numerical examples show that the degree of improperness of the parameterization (27) is four. A rational parameterization of the del Pezzo surface $\Phi = D \cap A$ is of bi-degree $(6,2)$ and reads

$$\varphi(s,t) = \frac{1}{p(s,t)}(\overline{g}_1(s,t), \overline{g}_2(s,t), \overline{g}_3(s,t), \pm n(s,t)). \tag{29}$$

Theorem 8. *A quadric $F \subset \mathbb{R}^3$ admits a rational polar representation $\mathbf{f}(s,t)$ of bi-degree at most $(6,2)$ with respect to an arbitrarily chosen reference point O. Its conchoid surfaces F_d at distance d with respect to O admit rational polar representations $\mathbf{f}_d(s,t)$ of bi-degree at most $(12,4)$.*

4. Special cases

In Sections 2 and 3 we have given a detailed investigation of real rational polar representations of regular quadrics F with respect to a reference point O in general position. What remains is a discussion of the cases having been excluded so far. These are

- F is a singular quadric,
- the reference point O lies on F,
- the reference point O lies on a focal conic of F or coincides with a focal point of a rotational quadric F.

The rational polar representations of these cases are typically of lower degree.

4.1. Singular quadrics

The quadric $F : X^T \cdot M \cdot X = 0$, with $X = (1, x, y, z) \in \mathbb{R}^4$, $M = M^T \in \mathbb{R}^{4\times4}$, is called *singular* if $\det(M) = 0$. Singular quadrics are cones and cylinders in case that rk $(M) = 3$ or pairs of planes if rk $(M) = 2$, or a double plane if rk $(M) = 1$. We again assume that F contains more than one real point. For the construction of rational polar representations of planes, cylinders and cones we refer to Peternell et al. (2011).

4.2. Reference point lies on the quadric

Consider the quadric $F : X^T \cdot M \cdot X = 0$, with $X = (1, x, y, z)$. The reference point $O = (0, 0, 0)$ is contained in F if and only if the constant term of $F(x, y, z) = 0$ vanishes. We have a look at two methods to find a real rational polar representation of F.

On the one hand, consider a parameterization $\mathbf{f}(u, v) = \rho(u, v)\mathbf{k}(u, v)$ with an arbitrary rational parameterization $\mathbf{k}(u, v)$ of S^2. To determine the unknown radius function $\rho(u, v)$, one inserts $\mathbf{f}(u, v)$ into $F(x, y, z) = 0$. This gives the trivial solution $\rho(u, v) = 0$, and besides this a rational function $\rho(u, v)$, expressed by the coordinates of $\mathbf{k}(u, v)$.

On the other hand, a quadric F is mapped by a perspective collineation κ of the form (9) to a quadric F'. By choosing the denominator of κ as tangent plane of F we can assume that F' becomes a paraboloid. By an admissible rotation we can achieve $F' : z = ax^2 + by^2$, with $a, b \in \mathbb{R} \setminus 0$. Thus F' is either an elliptic or a hyperbolic paraboloid, depending whether $ab > 0$ or $ab < 0$.

We consider a one-parameter family of cones of rotation $\Gamma(v)$ with vertex O and axis z. An implicit equation of these cones is $\Gamma(v) : \sinh v^2(x^2 + y^2) - z^2 = 0$, and a possible parameterization reads

$$\mathbf{g}(u, t, v) = u \left(\frac{2t}{\sinh v}, \frac{1-t^2}{\sinh v}, 1 + t^2 \right).$$

Intersecting $\Gamma(v)$ with F' determines the function $u(v, t) = \sinh^2 v(1+t^2)/(4at^2+b(1-t^2)^2)$. The t-lines of the final parameterization $\mathbf{f}(t, v) = \mathbf{g}(u(t, v), t, v)$ are rational quartic curves, the intersection curves $\Gamma(v) \cap F'$. This polar representation of F' reads, see Figure 7(a),

$$\mathbf{f}(v, t) = \frac{(1 + t^2)\sinh v}{4at^2 + b(1 - t^2)^2} \begin{pmatrix} 2t \\ (1 - t^2) \\ (1 + t^2)\sinh v \end{pmatrix}, \text{ with } \|\mathbf{f}(v, t)\| = \frac{(1 + t)^2 \sinh v \cosh v}{4at^2 + b(1 - t^2)^2}.$$

Performing the rational substitutions $\cosh v = (1+s^2)/(1-s^2)$ and $\sinh v = 2t/(1-s^2)$ yields a rational polar representation of F' of bi-degree $(4, 2)$. In case that $F' : z = a(x^2 + y^2)$ is a paraboloid of rotation, the parameterization simplifies and is of bi-degree $(2, 2)$, and its norm is independent on t,

$$\mathbf{f}(v, t) = \frac{\sinh v}{a(1 + t^2)} \left(2t, 1 - t^2, (1 + t^2)\sinh v\right), \text{ with } \|\mathbf{f}(v, t)\| = \frac{\sinh v \cosh v}{a}.$$

4.3. Reference point lies on a focal conic of the quadric

Given a quadric $F : X^T \cdot M \cdot X = 0$, the perspective collineation κ from equation (9) maps F to the quadric $F' : \pm 1 + \mathbf{x}'^T \cdot M' \cdot \mathbf{x}' = 0$, centered at $O = (0, 0, 0)$. The tangential cone Δ of F' with vertex at O is fixed with respect to κ. Thus *Delta* is the tangential cone of F and F', and reads

$$\Delta : \mathbf{x}'^T \cdot M' \cdot \mathbf{x}' = 0.$$

The eigenvectors and eigenvalues of M' define the coordinate transformation to achieve the normal form (10). The case of pairwise distinct eigenvalues is discussed already. It remains to investigate the cases of coinciding eigenvalues, namely $b = c$ for all affine types of Table 1 and additionally $a = b$ and $a = b = c$ in case that F' is an ellipsoid. We discuss these particular cases exemplarily for an ellipsoid F'.

F' is a rotational quadric. Consider the rotational ellipsoid $F' : a^2x^2 + b^2(y^2 + z^2) = 1$ with axis x. We substitute $b = c$ in the parameterization (24) and in (26). The parameterization of the spherical part $\overline{\mathbf{c}}(s, t)$ and the rational polar representation $\mathbf{f}(s, t)$ of F' are

$$\overline{\mathbf{c}}(s,t) = \frac{1}{n(s,t)}\overline{\mathbf{g}}(s,t), \text{ and } \mathbf{f}(s,t) = \frac{1}{p(s,t)}\overline{\mathbf{g}}(s,t) = \rho(s,t)\overline{\mathbf{c}}(s,t). \tag{30}$$

with coordinate functions $\overline{g}_i(s, t)$ of $\overline{\mathbf{g}}(s, t)$ and polynomials $n(s, t)$ and $p(s, t)$,

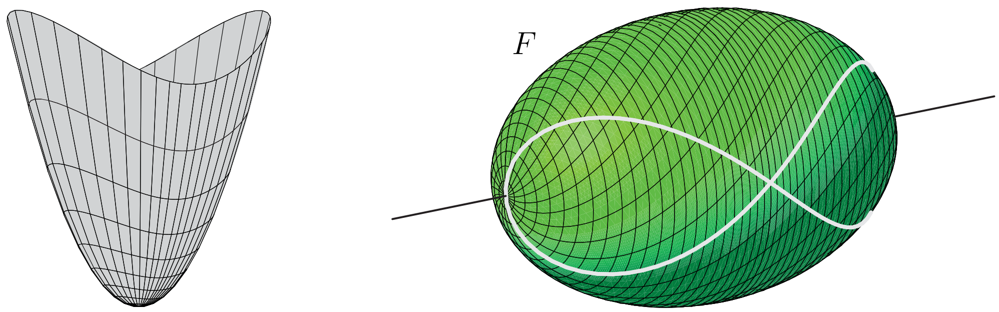
$$\begin{aligned}
\overline{g_1}(s,t) &= b(1+t^2)(s^4-1)\\
\overline{g_2}(s,t) &= -2s(\sqrt{ab}(1-t^2)(1-s^2) + 2st(a+b))\\
\overline{g_3}(s,t) &= 2s(2t\sqrt{ab}(s^2-1) - (a+b)s(t^2-1)) = 2s\mu(s,t)\\
p(s,t) &= b(1+t^2)(2bs^2 + a(1+s^4))\\
n(s,t) &= (1+t^2)(2as^2 + b(1+s^4)).
\end{aligned} \tag{31}$$

The t-lines of $\mathbf{f}(s, t)$ are parallel circles, the s-lines are rational curves of degree four. Figure 7(b) shows an illustration of an ellipsoid with one highlighted s-line. The parameterization $\mathbf{f}(s, t)$ is of bi-degree $(4, 2)$ and has rational length $\rho(s)$ independent on t,

$$\rho(s) = \frac{n(s,t)}{p(s,t)} = \frac{2as^2 + b(1+s^4)}{b(2bs^2 + a(1+s^4))}. \tag{32}$$

Alternatively, the parameterization $\mathbf{f}(s, t)$ could be derived using the cone model, see Section 3. The corresponding pencil of quadrics $\mathcal{B}$ contains a cylinder over the conic $-(b^2 - a^2)x^2 + b^2w^2 = 1$, and its generating planes are parallel to the $yz-$plane.

Two or more coinciding eigenvalues of M' imply, that F' is a rotational quadric and that the tangential cone Δ to F and F', with vertex O, is a *rotational cone*. This can happen only for special positions of O with respect to F. Section 4.4 will show that Δ is a rotational tangential cone of F with vertex O, exactly if O is contained in a so called *focal conic* of F, see Figure 8(a).



(a) Paraboloid.

(b) Rotational ellipsoid.

Figure 7: Special cases.

F' is a sphere. Consider a sphere $F' : a^2(x^2 + y^2 + z^2) = 1$ of radius $1/a$, centered at O. Any rational parameterization $\mathbf{f}(u, v) = 1/a\mathbf{k}(u, v)$, with $\|\mathbf{k}\| = 1$, of F' has rational length trivially. The pre-image F with respect to κ is a rotational quadric with O as focal point. Since F' admits a proper rational polar representation of bi-degree $(2, 2)$, the same holds for the pre-image F. The conchoid surfaces of F with respect to O are reducible and each component admits proper rational polar representations.

4.4. Focal conics of quadrics

Let $F \subset \mathbb{R}^3$ be a quadric and let $P \in \mathbb{R}^3$ be a fixed point with $P \notin F$. We want to determine possible positions for P, such that the quadratic tangential cone Δ with vertex P is a rotational cone. Let δ be the polar plane of P with respect to F. Then Δ is defined as cone connecting P and the conic $F \cap \delta$. If P is outside of F, Δ consists of all real tangent lines of F passing through P. If P is inside F, the tangent lines of F through P are conjugate complex and Δ does not contain real points except P. In the excluded case $P \in F$, the polar plane δ is tangent to F, and $F \cap \delta$ is not a conic.

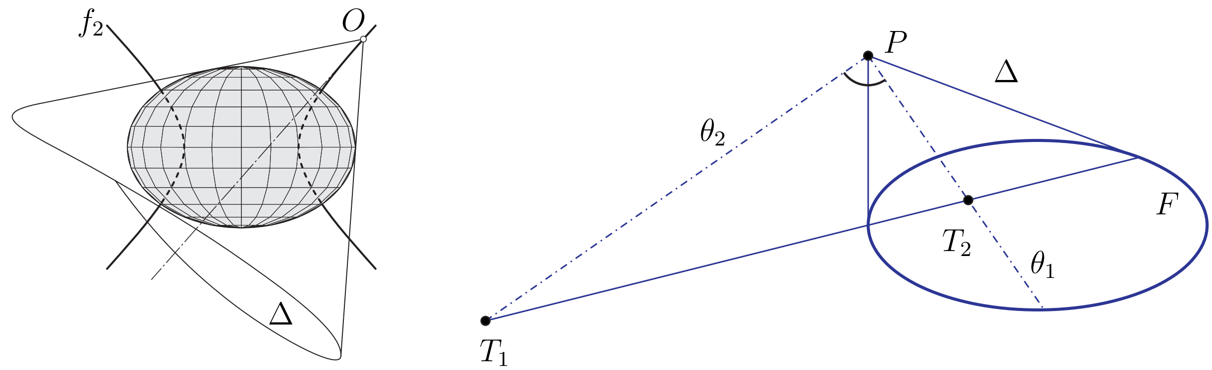
To define the points with a rotational tangential cone Δ we outline the method presented in McCrea (1960). Let us consider the quadric

$$F : \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} - 1 = X^T \cdot M \cdot X = 0, \text{ with } M = \mathrm{diag}(-1, \frac{1}{a}, \frac{1}{b}, \frac{1}{c}), \tag{33}$$

and $X = (1, x, y, z)$ and $a \geq b \geq c$. This normal form of F differs from those used in Table 1, but it has the advantage that one or two coefficients may be negative.

Consider an arbitrary point $P = (p_1, p_2, p_3) = \mathbf{p} \in \mathbb{R}^3$ and planes $\theta : (\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0$ containing P, where $\mathbf{n} \in \mathbb{R}^3$ denotes θ's normal vector. The polarity with respect to F maps θ to its pole T, whose Cartesian coordinates are

$$\mathbf{t} = \frac{1}{\mathbf{p} \cdot \mathbf{n}} (an_1, bn_2, cn_3) = \frac{1}{\mathbf{p} \cdot \mathbf{n}} \mathrm{diag}(a, b, c) \cdot \mathbf{n}. \tag{34}$$



(a) Ellipsoid with focal conic f_2 and tangential cone Δ.

(b) Symmetry planes θ_i of a tangential cone Δ.

Figure 8: Tangential cone Δ of an ellipsoid.

The quadratic cone Δ with vertex P typically has three symmetry planes $\theta_1, \theta_2, \theta_3$ and its normal vectors are the axes of Δ. A plane of symmetry θ has the property that PT is parallel to $\mathbf{n}$, thus PT is perpendicular to θ, see Figure 8(b). This gives the equation $(\mathbf{p} \cdot \mathbf{p}^T - \operatorname{diag}(a, b, c)) \cdot \mathbf{n} = M \cdot \mathbf{n} = \lambda \mathbf{n}$, with a symmetric matrix

$$M = \begin{pmatrix} p_1^2 - a & p_1 p_2 & p_1 p_3 \\ p_1 p_2 & p_2^2 - b & p_2 p_3 \\ p_1 p_3 & p_2 p_3 & p_3^2 - c \end{pmatrix}. \tag{35}$$

This is an eigenvalue problem of the symmetric matrix M. If the characteristic polynomial of M has three different zeros, then there are three different orthogonal eigenvectors determining the axes and the symmetry planes $\theta_1, \theta_2, \theta_3$ of Δ. By construction, the planes θ_i are conjugate with respect to F. This expresses the fact that the pole T_i of θ_i is contained in the other planes θ_j, θ_k, for $i \neq j, k$ and $i, j, k = \{1, 2, 3\}$. The points P, T_1, T_2, T_3 form a special polar tetrahedron of F, with orthogonal planes θ_i.

The cone Δ has rotation symmetry if and only if the matrix M has multiple eigenvalues. The characteristic polynomial of M reads

$$\begin{aligned} \det(M - \lambda I) &= -(\lambda + a)(\lambda + b)(\lambda + c) + p_1^2(\lambda + b)(\lambda + c) \\ &\quad + p_2^2(\lambda + a)(\lambda + c) + p_3^2(\lambda + a)(\lambda + b). \end{aligned} \tag{36}$$

Consider the case $a > b > c$. We first assume $p_1, p_2, p_3 \neq 0$ and insert $\lambda = \{\infty, -c, -b, -a\}$ in (36). This gives the signs $-, +, -, +$, hence the determinant (36) has three different zeros. To achieve a double eigenvalue we consider $p_1 = 0$, and (36) factorizes to

$$(\lambda + a)(-(\lambda + b)(\lambda + c) + p_2^2(\lambda + c) + p_3^2(\lambda + b)). \tag{37}$$

This polynomial has a double zero if $\lambda = -a$ is a zero of the second factor. By letting $x = p_1, y = p_2$ and $z = p_3$, this gives the conjugate complex conic

$$f_1 : x = 0 \ \cap \ -\frac{y^2}{a-b} - \frac{z^2}{a-c} = 1.$$

Analogously, for $p_2 = 0$ and $p_3 = 0$, we obtain the conics

$$f_2 : y = 0 \ \cap \ \frac{x^2}{a-b} - \frac{z^2}{b-c} = 1, \quad \text{and} \quad f_3 : z = 0 \ \cap \ \frac{x^2}{a-c} + \frac{y^2}{b-c} = 1. \tag{38}$$

These three conics f_1, f_2 and f_3 are called *focal conics* of the quadric F. They are the locus of points having a rotational tangential cones Δ with respect to F. Since we are only interested in cones with real vertices, there remain the hyperbola f_2 and the ellipse f_3. For points at f_2, f_3 which are inside of F, the cone Δ is conjugate complex but has a real vertex and is defined by a quadratic equation with real coefficients.

Consider the case $b = c$. Then F is a rotational quadric with axis x. The characteristic polynomial (36) of the matrix M simplifies to

$$\det(M - \lambda I) = (\lambda + b)[(\lambda + b)(p_x^2 - (\lambda + a)) + (\lambda + a)(p_y^2 + p_z^2)] = 0. \tag{39}$$

This equation has multiple zeros if either $\lambda = -b$ and thus $f_1 : y = 0 \ \cap \ z = 0$ or if $\lambda = -a$ and thus $f_2 : x = 0 \ \cap \ y^2 + z^2 = -(a - b)$. The first case defines the rotational axis as focal conic f_1, the second case defines the conjugate complex circle f_2. Since f_2 does not contain real points, it remains the rotational axis as real focal curve.

5. Example

To conclude the paper we discuss an example which is used for the Figures 3(b), 5(b) and 6. Let an ellipsoid be given by

$$F : -2 + 2x^2 + 4x - 2xy + 2y^2 + z^2 = 0.$$

The polar plane $\delta : x - 1 = 0$ of the origin intersects F in a complex conic. The transformations to achieve a normal form are the following:

- Perspective collineation:

$$\kappa : \mathbf{x}' = \frac{1}{x-1}\mathbf{x}$$

- Rotation about O with $\omega = \frac{3}{8}\pi$ combined with a re-ordering of the coordinate axes:

$$\widetilde{\mathbf{x}} = R \cdot \kappa(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 1 \\ \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \end{pmatrix} \cdot \kappa(\mathbf{x}). \tag{40}$$

Applying these two transformations to F leads to the normal form, again

$$F' : x^2 + (3 - \sin 2\omega + \cos 2\omega)y^2 + (3 + \sin 2\omega - \cos 2\omega)z^2 - 2. \tag{41}$$

Since the coefficients of F' are trigonometric functions of the rotation angle ω, and because of the fact that the final parameterization $\mathbf{f}(s,t)$ contains square roots of these coefficients, e.g. $\gamma = \sqrt{c^2 - b^2}$, we use floating point numbers as approximations. Inserting the coefficients $a = 1/2, b = (3 - \sin 2\omega + \cos 2\omega)/2$ and $c = (3 + \sin 2\omega - \cos 2\omega)/2$ into the solution (29) and inverting the transformations to get the following rational parameterization of the quadric F,

$$\begin{aligned}
\mathbf{f}(s,t) = \; & \tfrac{1}{p(s,t)}(x(s,t), y(s,t), z(s,t)), \text{ with} \\
x(s,t) = \; & (-1.07s^6 + 0.80)(t^2+1) - 1.41(t^2 + 1.20t - 1)s^5 + (1.59t^2 - 2.01 - 6.02t)s^4 \\
& +0.34(t^2 - 5.47t - 1)s^3 + (6.11t^2 - 4.54t - 1.47)s^2 + 1.06(t^2 + 3.34t - 1)s \\
y(s,t) = \; & (0.44s^6 - 0.33)(t^2+1) - 3.40(t^2 - 0.20t - 1)s^5 + (-0.66t^2 - 14.53t + 0.83)s^4 \\
& +0.83(t^2 + .93t - 1)s^3 + (-2.53t^2 - 10.97t + 0.61)s^2 + 2.56(t^2 - 0.57t - 1)s \\
z(s,t) = \; & (1.10s^6 - 0.83)(t^2+1) - 8.43ts^5 + (12.82t^2 - 5.21)s^4 + 4.42ts^3 \\
& +(4.60t^2 - 3.98)s^2 + 4.01ts, \\
p(s,t) = \; & (0.82s^6 + 2.23)(t^2+1) - 1.41(t^2 + 1.20t - 1)s^5 + (5.86t^2 - 6.02t + 2.25)s^4 \\
& +0.34(t^2 - 17.26t - 1)s^3 + (15.84t^2 - 4.54t - 0.32)s^2 + 1.06(t^2 + 7.12t - 1)s.
\end{aligned}$$

The norm of $\mathbf{f}(s,t)$ reads

$$\|\mathbf{f}(s,t)\| = \tfrac{1}{p(s,t)}((1.60s^6 + 4.79s^2 + 1.21)(t^2+1) - 4.50s^3t(s^2-1) + (11.84t^2 + 2.21)s^4).$$

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References

Aigner, M., Jüttler, B., Gonzalez-Vega, L., Schicho, J., 2009. Parameterizing surfaces with certain special support functions, including offsets of quadrics and rationally supported surfaces. Journal of Symbolic Computation 44, 180–191.

Albano, A., Roggero, M., 2010. Conchoidal transform of two plane curves. Applicable Algebra in Engineering, Communication and Computingn 21, 309–328.

Gibson, C., 1998. Elementary Geometry of Algebraic Curves. Cambridge University Press.

Griffiths, P., Harris, J., 1978. Principles of algebraic geometry. John Wiley & Sons, New York.

Kunz, E., 2000. Introduction to plane algebraic curves. Birkhäuser.

Manin, Y.I., 1974. Cubic Forms. North-Holland, Amsterdam.

McCrea, W.H., 1960. Analytical Geometry of Three Dimensions. Oliver and Boyd LTD., Edinburgh.

Peternell, M., 1997. Rational Parametrizations for Envelopes of Quadric Families. Ph.D. thesis. University of Technology. Vienna.

Peternell, M., Gruber, D., Sendra, J., 2011. Conchoid surfaces of rational ruled surfaces. Comp. Aided Geom. Design 28, 427–435.

Peternell, M., Gruber, D., Sendra, J., 2012. Conchoid surfaces of spheres. In press, doi: 10.1016/j.cagd.2012.06.005.

Schicho, J., 1998. Rational parametrization of surfaces. J. Symbolic Computation 26, 1–29.

Sendra, J., Sendra, J.R., 2010. Rational parametrization of conchoids to algebraic curves. Applicable Algebra in Engineering, Communication and Computing 21, 413–428.

Sendra, J.R., Sendra, J., 2008. An algebraic analysis of conchoids to algebraic curves. Applicable Algebra in Engineering, Communication and Computing 19, 285–308.

Wieleitner, H., 1908. Spezielle Ebene Kurven. G.J. Göschen'sche Verlagshandlung, Leipzig.