CONFORMAL HEXAGONAL MESHES

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ABSTRACT. We explore discrete conformal and discrete minimal surfaces all of whose faces are planar hexagons. Discrete conformal meshes are built of conformal hexagons for which we establish a dual construction. We apply this dual construction to conformal hexagonal meshes covering the sphere and get discrete hexagonal minimal surfaces via a discrete analogue of the Christoffel dual construction.

We compare the smooth and the discrete settings by means of limit considerations and also by a discussion of Möbius invariants.

1. INTRODUCTION

The present paper considers hexagonal meshes from the viewpoint of *discrete differential geometry*. Discrete differential geometry is a wide field which considers objects like polygons, meshes, and polytopes with the aim of finding discrete analogues of classical (i.e., smooth) differential geometry. In this sense not only objects but also properties and notions of the smooth setting are carried over to the discrete theory. Also the other way round is of great interest, which means that one explores attributes assigned to discrete objects which survive a refinement process to a continuous limit. A first treatise of discrete differential geometry can be found in the monograph *Differenzengeometrie* by R. Sauer [14] whereas a modern approach is contained in *Discrete Differential Geometry: Integrable Structure* by A. I. Bobenko and Yu. B. Suris [5]. Discrete differential geometry is not only interesting within pure mathematics (see e.g. [1] or [5]) but also in computer graphics and geometry processing (see e.g. [15]) and architectural design (see e.g. [12]). The general idea here is to get notions like curvature, offset surface, and conformal equivalence for discrete objects which are of great importance in applications.

A well studied class of surfaces in differential geometry are minimal surfaces. Different but equivalent definitions of minimal surfaces in the smooth setting can be discretized to different definitions of discrete minimal surfaces. For example Plateau's problem has been considered in the discrete setting e.g. by U. Pinkall and K. Polthier [11] using a discrete Dirichlet energy of triangle meshes. Another definition which however does not work for triangle meshes is the discrete curvature theory of A. I. Bobenko, H. Pottmann, and J. Wallner [4] which is based on the concept of edgewise parallelity between mesh and Gauss image. Here, a discrete minimal surface is characterized by vanishing mean curvature, analogous to the smooth case. An incidence geometric characterization of such discrete minimal surfaces can be found in [10]. A further way to find minimal surfaces is via the so-called Christoffel dual construction [7]. The dual of an isothermic parametrisation of a sphere is an isothermic parametrisation of a minimal surface and vice versa. For quadrilateral meshes, A. I. Bobenko et al. discuss this in [1, 3].

The present paper establishes a discrete Christoffel dual construction for special hexagonal meshes, namely conformal ones. A mesh with vertices in a sphere where each face is a conformal hexagon will be called *discrete isothermic* like in the smooth setting. As mentioned before, the Christoffel dual of a discrete isothermic surface covering a sphere can be seen as a discrete minimal surface consisting of planar hexagons.

2. Multi-ratio and vertex offset meshes

Following [12], two meshes $\mathcal{M}$ and $\mathcal{M}'$ with the same combinatorics are called *parallel* if all corresponding edges are parallel. Trivial pairs of parallel meshes can be found by translation or dilation of a fixed mesh. With a vertex-wise addition and scalar multiplication the space of all meshes which are parallel to a given one is a vector space.

To define offset meshes we need an appropriate notion of distance. Possibilities are the following: A mesh $\mathcal{M}^d$ is a *vertex (edge, face, resp.) offset* of $\mathcal{M}$ at constant distace d if $\mathcal{M}$ and $\mathcal{M}^d$ are parallel and all corresponding vertices (edges, faces, resp.) are at constant distance d.

For quad meshes H. Pottmann et al. [12, 13] showed a connection between the existence of vertex and face offsets on the one hand, and circular and conical meshes on the other hand. In the present paper the circular meshes are the more important ones. A *circular polygon* is a polygon with a circumcircle and a *quasi-circular polygon* is edge-wise parallel to a polygon with a circumcircle (see e.g. [9]).

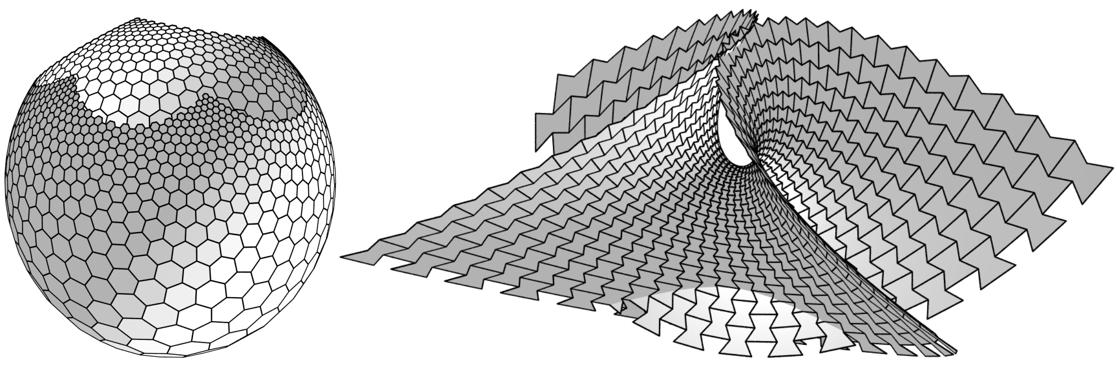

Figure 1. A discrete Enneper's surface (right) and its discrete Gauss image (see Example 1).

A *circular mesh* is a mesh where each face is a circular polygon and a *quasi-circular mesh* is a mesh where each face is a quasi-circular polygon.

For four complex numbers $z_0, \ldots, z_3$, we have the *cross-ratio*

$$\operatorname{cr}(z_0, z_1, z_2, z_3) = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_0)},$$

which is Möbius invariant and characterizes Möbius equivalence classes of quadrilaterals. A quadrilateral is circular if and only if its cross-ratio is real. A generalization of the cross-ratio to polygons with an even number of vertices $z_0, \ldots, z_{n-1}$ is the so-called *multi-ratio* (see e.g. [2])

$$\operatorname{q}(z_0, \ldots, z_{n-1}) := \frac{(z_0 - z_1)(z_2 - z_3) \cdot \ldots \cdot (z_{n-2} - z_{n-1})}{(z_1 - z_2)(z_3 - z_4) \cdot \ldots \cdot (z_{n-1} - z_0)}.$$

Obviously, the multi-ratio $\operatorname{q}(z_0, \ldots, z_{n-1})$ is Möbius invariant since it is invariant under translations, dilations, rotations and transformations of the form $z \mapsto 1/z$.

Lemma 1. *Let $(z_i) = (z_0, \ldots, z_{n-1})$ with n even be a polygon in the complex plane. Further let α_i denote the angles between $z_{i-2} - z_{i-3}$ and $z_{i+1} - z_i$ and let θ_i be the internal angles between $z_{i-1} - z_i$ and $z_{i+1} - z_i$, where indices are taken modulo n. Then the following statements are equivalent.*

(i) There exists a polygon parallel to (z_i) with all vertices on a circle.

(ii) The angles α_i fulfill $\sum_{i \text{ even}} \alpha_i \in \pi\mathbb{Z}$, or, which is equivalent, $\sum_{i \text{ odd}} \alpha_i \in \pi\mathbb{Z}$.

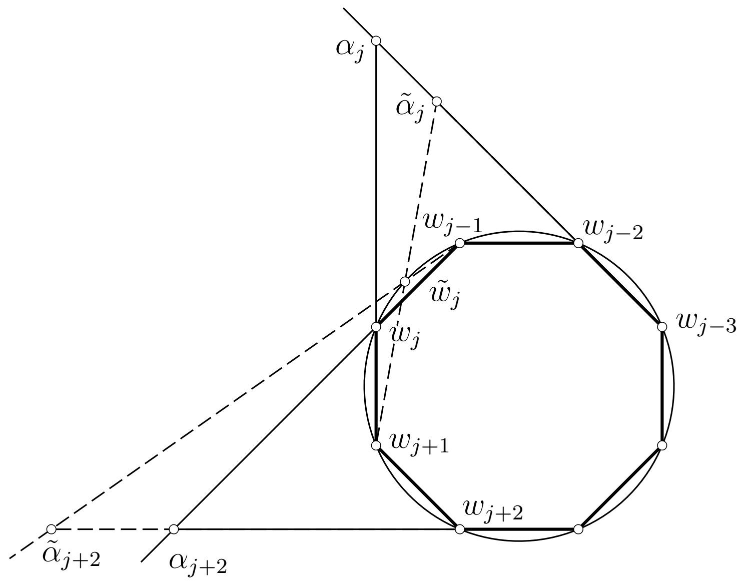

FIGURE 2. For a regular 8-gon we have an angle of $3\pi/4$ for all $\alpha_i = \angle(w_{i-2} - w_{i-3}, w_{i+1} - w_i)$. When changing a vertex w_j to $\tilde{w}_j$ on the circumcircle the corresponding angles α_{j-1}, α_j, α_{j+2} and α_{j+3} are replaced by α into $\tilde{\alpha}_{j-1} = \alpha_{j-1} - \alpha$, $\tilde{\alpha}_j = \alpha_j - \alpha$, $\tilde{\alpha}_{j+2} = \alpha_{j+2} + \alpha$ and $\tilde{\alpha}_{j+3} = \alpha_{j+3} + \alpha$, respectively.

(iii) The multi-ratio $\mathrm{q}(z_0, \dots, z_{n-1})$ *is real.*

(iv) The interior angles θ_i *have the property* $\sum_{i \text{ even}} \theta_i = \sum_{i \text{ odd}} \theta_i$.

Proof. (i) $\Longrightarrow$ (ii): Without loss of generality let (z_i) lie on a circle. For regular n-gons (w_i) (see Figure 2) we have $\alpha_i = 3\frac{2\pi}{n}$ for all i, which yields $\sum_{i \text{ even}} \alpha_i = 3\pi$. When changing one single vertex of (w_i) on the circle, e.g. $w_j \longmapsto \tilde{w}_j$, the directions $w_{j+1} - w_j$ and $w_j - w_{j-1}$ will change about the same angle α, which follows from the inscribed angle theorem. We get a new n-gon $(\tilde{w}_i)$ with $\tilde{w}_i = w_i$ for all i except for $i = j$. For the corresponding angles we have $\tilde{\alpha}_{j-1} = \alpha_{j-1} - \alpha$, $\tilde{\alpha}_j = \alpha_j - \alpha$, $\tilde{\alpha}_{j+2} = \alpha_{j+2} + \alpha$ and $\tilde{\alpha}_{j+3} = \alpha_{j+3} + \alpha$, where all others remain unchanged. Therefore $\sum_{i \text{ even}} \tilde{\alpha}_i = \sum_{i \text{ even}} \alpha_i = 3\pi \in \pi\mathbb{Z}$. If we change each vertex of the regular polygon (w_i) until they come to the positions of (z_i) the sum of the considered angles remains in $\pi\mathbb{Z}$.

(ii) $\Longrightarrow$ (i): Now we start with a polygon (z_i) with corresponding angles α_i such that $\sum_{i \text{ even}} \alpha_i \in \pi\mathbb{Z}$. Starting with an arbitrary vertex w_0 on a circle we construct vertices $w_1, \dots, w_n$ on this circle with edges $w_i - w_{i-1}$ parallel to $z_i - z_{i-1}$ for all $i \in \{0, \dots, n\}$, where indices are taken modulo n only for z_i but not yet for the points w_i (see Figure 3 with $w_i = z_i$). Until now we do not know whether $w_n = w_0$ or not. From "(i) $\Longrightarrow$ (ii)" we know that the polygon $(w_i)_{i=0}^{n-1}$ fulfills condition (ii). Using the assumptions we can conclude that also $w_n - w_{n-1}$ is parallel to $w_0 - w_{n-1}$ which yields $w_n = w_0$.

(i) $\Longleftrightarrow$ (iii): With $z_k - z_{k-1} = a_k \exp(i\varphi_k)$ and $\alpha_k = \varphi_{k-2} - \varphi_{k-1}$ we get

$$\begin{aligned}\mathrm{q}(z_0, \dots, z_{n-1}) &= \prod_{k \text{ even}} \frac{(z_{k-2} - z_{k-3})}{(z_{k+1} - z_k)} = \prod_{k \text{ even}} \frac{a_{k-2} e^{i\varphi_{k-2}}}{a_{k+1} e^{i\varphi_{k+1}}} = \\ &= \prod_{k \text{ even}} \frac{a_{k-2}}{a_{k+1}} e^{i\alpha_k} = \Big(\prod_{k \text{ even}} \frac{a_{k-2}}{a_{k+1}} \Big) \exp\big(i \sum_{j \text{ even}} \alpha_j\big).\end{aligned}$$

This yields

$$\mathrm{q}(z_0, \dots, z_{n-1}) \in \mathbb{R} \Longleftrightarrow \sum_{j \text{ even}} \alpha_j \in \pi\mathbb{Z}.$$

(i) $\Longleftrightarrow$ (iv): see [9]. □ □

Corollary 2. *Let M be a Möbius transformation and let (z_i) be an n-gon with n even. Then (z_i) is quasi-circular if and only if $(M(z_i))$ is.*

Proof. $\mathrm{q}(z_0, \dots, z_{n-1}) = \mathrm{q}(M(z_0), \dots, M(z_{n-1})) \in \mathbb{R}$ follows from the Möbius invariance of the multi-ratio. □ □

Remark 3. *The construction of a parallel polygon with vertices on a circle either is closing for all starting points z_0, or for none of them (see Figure 3). Let $(w_i)_{i=0}^{n-1}$ be an arbitrary polygon and let $(z_i)_{i=0}^{n}$ be contained in the circle $\mathcal{S}^1$ where $z_i - z_{i+1}$ is parallel to $w_i - w_{i+1}$ (indices taken modulo n only for w_i). Define $\mu : \mathcal{S}^1 \longrightarrow \mathbb{R}^2$ by $\mu(z) = [z_1 - z] \cap [z_{n-1} - z_n]$. Then the set $\mu(\mathcal{S}^1)$ is a conic section, which* *is a consequence of properties of projective mappings between pencils of lines (see Figure 3, right).*

Theorem 4 applies the above properties of polygons to meshes. The multi-ratios which are mentioned are computed w.r.t. an arbitrary choice of Cartesian coordinate system in each face.

Theorem 4. *Each of the following statements concerning a mesh $\mathcal{M}$ with planar faces implies the other five.*
(i) $\mathcal{M}$ has a vertex offset.
(ii) Every face of $\mathcal{M}$ is a quasi-circular polygon.
(iii) For each face, the angles α_i fulfill both, $\sum_{i \ even} \alpha_i \in \pi\mathbb{Z}$ and $\sum_{i \ odd} \alpha_i \in \pi\mathbb{Z}$, respectively.
(iv) For each face, the angles α_i fulfill $\sum_{i \ odd} \alpha_i \in \pi\mathbb{Z}$.
(v) The multi-ratio of each face is real.
(vi) For each face, the interior angles θ_i have the property $\sum_{i \ even} \theta_i = \sum_{i \ odd} \theta_i$.

The equivalence of statements (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) can be found in [9], and (i) $\Leftrightarrow$ (ii) for quad meshes can be found in [12]. The rest follows immediately from Lemma 1.

3. Conformal hexagons

Both conformal and curvature line parametrisations play a fundamental role in the theory of minimal surfaces. On the one hand, the Weierstrass representation converts a pair of holomorphic functions (i.e., conformal parametrizations of $\mathcal{S}^2$) to a certain parametrization of a minimal surface. On the other hand, Christoffel duality converts a conformal parametrization of the sphere to an isothermic parametrization of a minimal surface and vice versa. We now aim at a discrete analogue of a continuous conformal surface, using hexagonal meshes.

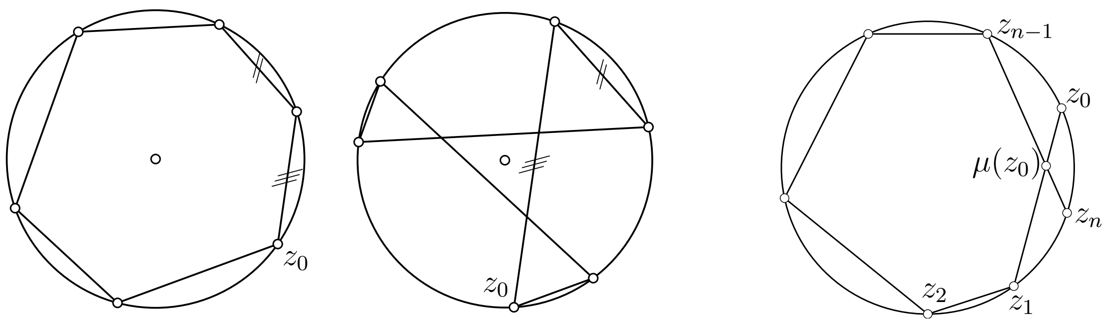

FIGURE 3. *Left:* The construction of a parallel polygon with vertices on a circle is closing for all starting points z_0 or for none. *Right:* The set of all points $\mu(z_0)$ which arise from the construction of Remark 3 are located on a conic section.

Definition 5. *A hexagon* $(z_0, \ldots, z_5)$ *is called* conformal *if both* $\mathrm{cr}(z_0, z_1, z_2, z_3) = -1/2$ *and* $\mathrm{cr}(z_0, z_5, z_4, z_3) = -1/2$.

The prototype of a conformal hexagon is a regular hexagon. Since Möbius transformations leave the cross-ratio of four points invariant, we immediately see that conformality of hexagons is Möbius invariant and each hexagon which is Möbius equivalent to a regular hexagon is conformal.

For a conformal hexagon both quadrilaterals z_0, z_1, z_2, z_3 and z_0, z_5, z_4, z_3 are circular since their cross-ratios are real (see Figure 4). The multi-ratio of conformal hexagons is

$$\mathrm{q}(z_0, \ldots, z_5) = \frac{(z_0 - z_1)(z_2 - z_3)(z_4 - z_5)}{(z_1 - z_2)(z_3 - z_4)(z_5 - z_0)} = -\frac{1}{2}\frac{(z_3 - z_0)(z_4 - z_5)}{(z_3 - z_4)(z_5 - z_0)} = -1,$$

which implies, with Lemma 1, that (z_i) is quasi-circular.

Definition 6. *Let* $f : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *be a smooth regular mapping (i.e., the partial derivatives* f_x*,* f_y *are linearly independent in* U*). Then the hexagon*

$$\begin{aligned} f_0 &= f + \varepsilon f_x & f_3 &= f - \varepsilon f_x \\ f_1 &= f + \frac{\varepsilon}{2} f_x + \frac{\sqrt{3}}{2}\varepsilon f_y & f_4 &= f - \frac{\varepsilon}{2} f_x - \frac{\sqrt{3}}{2}\varepsilon f_y \\ f_2 &= f - \frac{\varepsilon}{2} f_x + \frac{\sqrt{3}}{2}\varepsilon f_y & f_5 &= f + \frac{\varepsilon}{2} f_x - \frac{\sqrt{3}}{2}\varepsilon f_y \end{aligned}$$

is called infinitesimal hexagon *at* (x, y) *where* $f = f(x, y) \in \mathbb{R}^3$*,* $f_x = \partial f/\partial x$ *and* $f_y = \partial f/\partial y$*.*

Note that in general infinitesimal hexagons are not planar. According to the Taylor expansion the vertices f_i of the infinitesimal hexagon differ from $f(z_i)$ in terms of order $o(\varepsilon)$, where z_i are the vertices of a regular hexagon with radius ε and centered at (x, y).

For the following we extend the cross-ratio to points of $\mathbb{R}^3$ (see e.g. [3]). Four points in $\mathbb{R}^3$, define a plane or sphere, which is identified with $\mathbb{C} \cup \infty$ via stereographic projection.

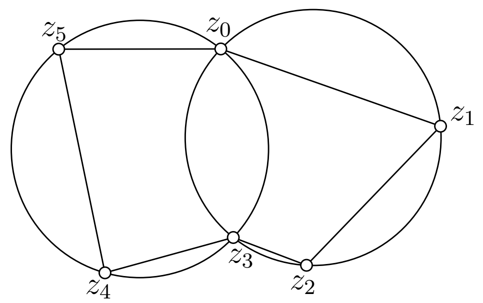

FIGURE 4. An arbitrary conformal hexagon (z_i) with its two circles.

Theorem 7. *Consider a regular mapping $f : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and the associated infinitesimal hexagon $f_0, \ldots, f_5$ for a point $(x, y) \in U$. Then $\operatorname{cr}(f_0, f_1, f_2, f_3) = -1/2 + o(\varepsilon)$ and $\operatorname{cr}(f_0, f_5, f_4, f_3) = -1/2 + o(\varepsilon)$ for all $(x, y) \in U$ if and only if f is a conformal mapping.*

Proof. Translation $x \mapsto x - f - \varepsilon f_x$ and scaling $x \mapsto 2x/\varepsilon$ transform the vertices to

$$\begin{aligned} \hat{X}_0 &= 0 & \hat{X}_3 &= -4f_x \\ \hat{X}_1 &= -f_x + \sqrt{3}f_y & \hat{X}_4 &= -3f_x - \sqrt{3}f_y \\ \hat{X}_2 &= -3f_x + \sqrt{3}f_y & \hat{X}_5 &= -f_x - \sqrt{3}f_y. \end{aligned}$$

The inversion $\tilde{X}_i = \hat{X}_i / \|\hat{X}_i\|^2$ sends $\hat{X}_0$ to ∞. As all three transformations do not change the cross-ratio we get

$$\operatorname{cr}(f_0, f_1, f_2, f_3) = \operatorname{cr}(\hat{X}_0, \hat{X}_1, \hat{X}_2, \hat{X}_3) = \operatorname{cr}(\infty, \tilde{X}_1, \tilde{X}_2, \tilde{X}_3) = \frac{\tilde{X}_3 - \tilde{X}_2}{\tilde{X}_1 - \tilde{X}_2}.$$

We start with the first cross-ratio condition

$$\operatorname{cr}(f_0, f_1, f_2, f_3) = -\frac{1}{2} + o(\varepsilon) \Longleftrightarrow \tilde{X}_3 - \tilde{X}_2 = -\frac{1}{2}(\tilde{X}_1 - \tilde{X}_2) + o(\varepsilon), \tag{1}$$

which is equivalent to

$$-\frac{f_x}{2C} - 3\frac{-3f_x + \sqrt{3}f_y}{A} + \frac{-f_x + \sqrt{3}f_y}{B} = o(\varepsilon),$$

where $A = \|-3f_x + \sqrt{3}f_y\|^2$, $B = \|-f_x + \sqrt{3}f_y\|^2$ and $C = \|f_x\|^2$. Collecting the coefficients of f_x and f_y we get

$$f_x \cdot \Big(\frac{-1/2}{C} + \frac{9}{A} - \frac{1}{B}\Big) + f_y \cdot \Big(\frac{-3\sqrt{3}}{A} + \frac{\sqrt{3}}{B}\Big) = o(\varepsilon),$$

which is equivalent to

$$-\frac{1}{2}AB + 9BC - AC = 0 \quad \text{and} \quad -3B + A = 0$$

because of the linear independence of $\{f_x, f_y\}$. It is easy to see that $-3B + A = 0 \Leftrightarrow \|f_x\| = \|f_y\|$ and $-\frac{1}{2}AB + 9CB - CA = 0 \Leftrightarrow 3\langle f_x, f_x\rangle^2 - 6\langle f_y, f_y\rangle\langle f_x, f_x\rangle + 12\langle f_x, f_y\rangle^2 + 3\langle f_y, f_y\rangle^2 - 8\sqrt{3}\langle f_x, f_y\rangle\langle f_y, f_y\rangle = 0$. Further, it is easy to see that Equation (1) is equivalent to

$$\|f_x\| = \|f_y\| \quad \text{and} \quad \Big[\langle f_x, f_y\rangle = 0 \text{ or } \langle f_x, f_y\rangle = \frac{2\sqrt{3}\langle f_y, f_y\rangle}{3}\Big]. \tag{2}$$

Since $\|f_y\|^2 \geq \langle f_x, f_y\rangle = 2\sqrt{3}/3\|f_y\|^2$ would impliy $1 \geq 2\sqrt{3}/3$, which is a contradiction, f_x and f_y must be orthogonal. We get

$$\operatorname{cr}(f_0, f_1, f_2, f_3) = -1/2 + o(\varepsilon) \quad \text{and} \quad \operatorname{cr}(f_0, f_5, f_4, f_3) = -1/2 + o(\varepsilon)$$

is equivalent to $\|f_x\| = \|f_y\|$ and $\langle f_x, f_y\rangle = 0$, which is further equivalent to the conformality of f. □ □

4. A dual construction for conformal hexagons

With a view towards the smooth Christoffel dual construction of Section 5, we introduce a dual construction for conformal hexagons.

Definition 8. *For a conformal hexagon (z_i), let $a_i := z_{i+1} - z_i$ be the edge vectors, where indices are taken modulo n. A hexagon (z_i^*) is called* dual *to (z_i) if*

$$\begin{aligned} z_1^* - z_0^* &= -1/(\overline{z_1 - z_0}) = -1/\overline{a_0} & z_4^* - z_3^* &= -1/(\overline{z_4 - z_3}) = -1/\overline{a_3} \\ z_2^* - z_1^* &= 2/(\overline{z_2 - z_1}) = 2/\overline{a_1} & z_5^* - z_4^* &= 2/(\overline{z_5 - z_4}) = 2/\overline{a_4} \\ z_3^* - z_2^* &= -1/(\overline{z_3 - z_2}) = -1/\overline{a_2} & z_0^* - z_5^* &= -1/(\overline{z_0 - z_5}) = -1/\overline{a_5} \end{aligned}$$

(see Figure 5).

Proposition 9. *Let (z_i) be a conformal hexagon, $a_i := z_{i+1} - z_i$ and $b := z_0 - z_3$. Then*

(i) $\sum_{i=0}^5 a_i = 0, \quad a_0 + a_1 + a_2 + b = 0, \quad \dfrac{a_0 a_2}{a_1 b} = -\dfrac{1}{2}, \quad \dfrac{a_3 a_5}{a_4 b} = \dfrac{1}{2}.$
(ii) $z_0^* - z_3^* = 2/\overline{b}$ and in particular $z_0 - z_3$ is parallel to $z_0^* - z_3^*$.
(iii) The hexagon (z_i) posesses a dual hexagon.
(iv) The dual (z_i^*) is a conformal hexagon, and is unique up to translation.
(v) Non-corresponding diagonals of both quadrilaterals z_0, z_1, z_2, z_3 and z_0, z_5, z_4, z_3

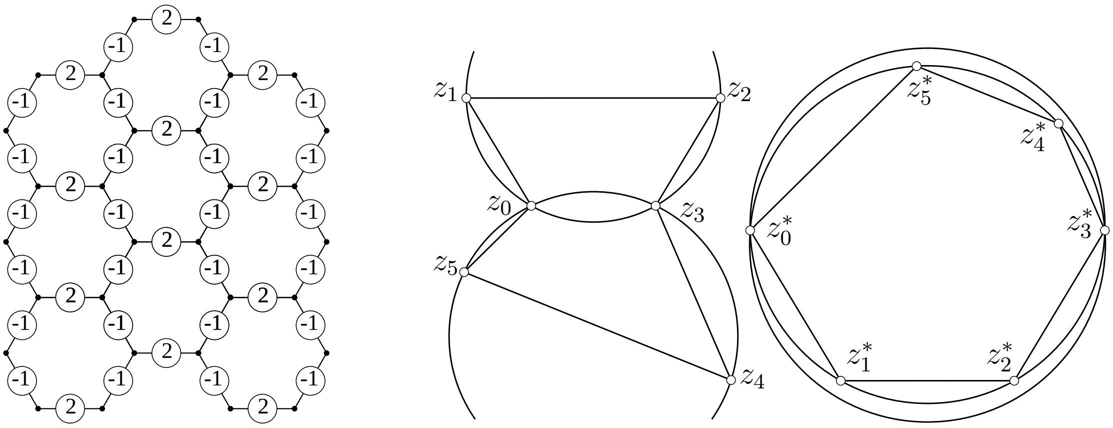

FIGURE 5. *Left:* Edge coefficients in the discrete dual construction (Definition 8). *Right:* A conformal hexagon and its dual. For each conformal hexagon $z_0, \dots, z_5$ both quadrilaterals z_0, z_1, z_2, z_3 and z_0, z_3, z_4, z_5 are circular.

are transformed according to

$$z_1^* - z_3^* = 3\frac{z_0 - z_2}{|z_0 - z_2|^2}, \qquad z_2^* - z_0^* = 3\frac{z_3 - z_1}{|z_3 - z_1|^2},$$
$$z_5^* - z_3^* = 3\frac{z_0 - z_4}{|z_0 - z_4|^2}, \qquad z_4^* - z_0^* = 3\frac{z_3 - z_5}{|z_3 - z_5|^2}.$$

In particular they are parallel:

$$z_2 - z_0 \parallel z_1^* - z_3^*, \qquad z_1 - z_3 \parallel z_0^* - z_2^*,$$
$$z_4 - z_0 \parallel z_5^* - z_3^*, \qquad z_5 - z_3 \parallel z_4^* - z_0^*.$$

(vi) Applying the duality twice yields the original hexagon, up to translation: $(z_i^{**}) = (z_i)$.

Proof. The statements follow immediately from the definition or are straightforward except for (v), which is [6, Corollary 31]. □ □

5. Christoffel dual construction

The following theorem by E. B. Christoffel [7] characterizes isothermic surfaces via a dual construction. An *isothermic* parametrization is a conformal cuvature line parametrization. It is known that all minimal surfaces can be expressed in isothermic parameters. For the unit sphere $\mathcal{S}^2$, every conformal parametrization is isothermic.

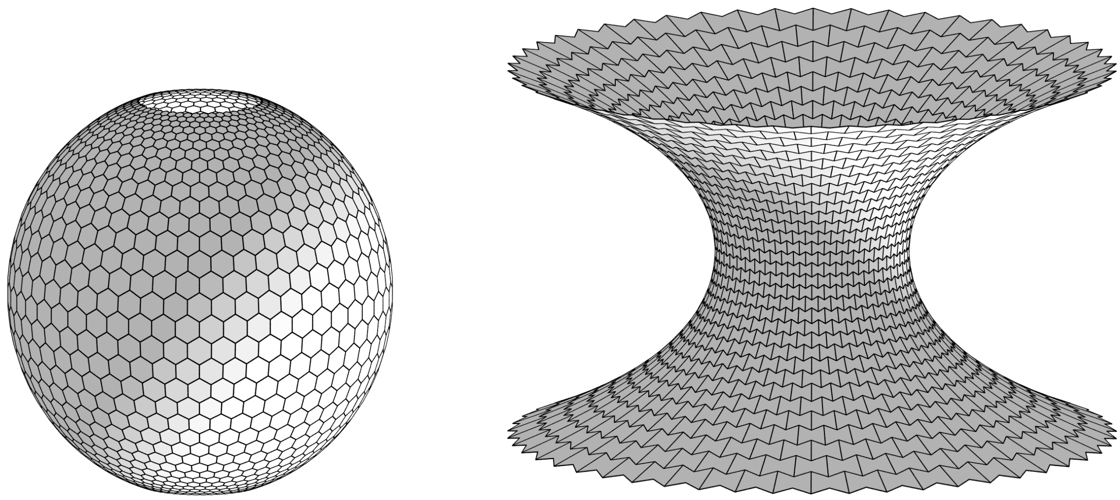

Figure 6. A discrete catenoid and its discrete Gauss image (see Example 2).

Theorem 10 (Christoffel). *Let f be an isothermic parametrisation. Then the* Christoffel dual *f^*, defined by the formulas*

$$f_x^* = \frac{f_x}{\|f_x\|^2} \quad \textit{and} \quad f_y^* = -\frac{f_y}{\|f_y\|^2}$$

exists and is isothermic again. The dual f^* is a minimal surface if and only if f is a sphere.

The next two propositions state properties of the smooth Christoffel dual construction.

Proposition 11. *Let $f : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be an isothermic parametrisation, let f^* be its dual, and consider the ball $B_r(x, y)$ with radius r centered at (x, y). Then*

$$\lim_{r \to 0} \frac{\mathcal{A}(f(B_r(x,y)))}{\mathcal{A}(f^*(B_r(x,y)))} = \|f_x(x,y)\|^2 \|f_y(x,y)\|^2,$$

where $\mathcal{A}$ is the surface area.

Proposition 12. *Let $f : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be an isothermic parametrisation, $\varepsilon > 0$ and $r = \varepsilon/\sqrt{\pi}$. Further let $\varphi : U' \longrightarrow U$ with $\varphi(x, y) = (\varepsilon x, \varepsilon y)$ be a parameter transformation, $f_\varepsilon := f \circ \varphi$ the transformed function and f_ε^* its dual. Then*

$$\lim_{\varepsilon \to 0} \mathcal{A}(f_\varepsilon(B_r(x,y)))\mathcal{A}(f_\varepsilon^*(B_r(x,y))) = 1. \tag{3}$$

We consider the dual construction of discrete isothermic surfaces as quad meshes [3, 1]. The dual $z_0^*, \ldots, z_3^*$ of a conformal square with vertices $z_0, \ldots, z_3$ is defined via $z_{i+1}^* - z_i^* = (-1)^i/\overline{(z_{i+1} - z_i)}$. For a square P with edge length l and its dual P^*, which then has edge length $1/l$, we obtain $\mathcal{A}(P) = l^2$, $\mathcal{A}(P^*) = 1/l^2$ and therefore

$$\frac{\mathcal{A}(P)}{\mathcal{A}(P^*)} = l^4 \quad \text{and} \quad \mathcal{A}(P)\mathcal{A}(P^*) = 1,$$

which are discrete analogues of Propositions 11 and 12.

Computing the area of hexagons is more involved than the rectangle case. For a conformal hexagon (z_i) and its dual (z_i^*) we have $z_{i+1}^* - z_i^* = -1/\overline{(z_{i+1} - z_i)}$ for $i \in \{0, 2, 3, 5\}$ and $z_{i+1}^* - z_i^* = 2/\overline{(z_{i+1} - z_i)}$ for $i \in \{1, 4\}$. Multiplying each vertex z_i with $r \in \mathbb{R} \setminus 0$ we get the hexagon $(w_i) := (rz_i)$ and its dual (w_i^*) with $w_{i+1} - w_i = r(z_{i+1} - z_i)$ and $w_{i+1}^* - w_i^* = -r^{-1}/\overline{(z_{i+1} - z_i)}$ for $i \in \{0, 2, 3, 5\}$ and $w_{i+1}^* - w_i^* = 2r^{-1}/\overline{(z_{i+1} - z_i)}$ for $i \in \{1, 4\}$. Consequently $w_{i+1}^* - w_i^* = r^{-1}(z_{i+1}^* - z_i^*)$. The areas of (z_i) and (z_i^*), are denoted by $\mathcal{A}(z_i) = p$ and $\mathcal{A}(z_i^*) = q$ respectively. Then

$$\mathcal{A}(w_i) = r^2 \mathcal{A}(z_i) = r^2 p \quad \text{and} \quad \mathcal{A}(w_i^*) = \mathcal{A}\big(\frac{1}{r} z_i^*\big) = \frac{1}{r^2} q.$$

This yields disretizations of Propositions 11 and 12, namely

$$\frac{\mathcal{A}(v_i)}{\mathcal{A}(v_i^*)} = \frac{r^2 p}{\frac{1}{r^2} q} = r^2 r^2 \frac{p}{q} \quad \text{and} \quad \mathcal{A}(v_i)\mathcal{A}(v_i^*) = pq. \tag{4}$$

The fact that r does not occur in (4) means that $\mathcal{A}(v_i)\mathcal{A}(v_i^*)$ does not depend on the discrete parametrisation.

6. Discrete conformal and discrete minimal surfaces

Definition 13. *A* discrete (hexagonal) conformal surface *is a mesh with regular hexagonal combinatorics where each hexagon is conformal in the sense of Definition 5.*

This definition is motivated by the following statements:

(i) The definition of planar discrete conformal surfaces (Definition 13) is Möbius invariant (see Section 3).
(ii) According to Theorem 7, the limit of cross-ratios of the two quadrilaterals of the infinitesimal hexagon both equal $-1/2$ if and only if the considered mapping is conformal.

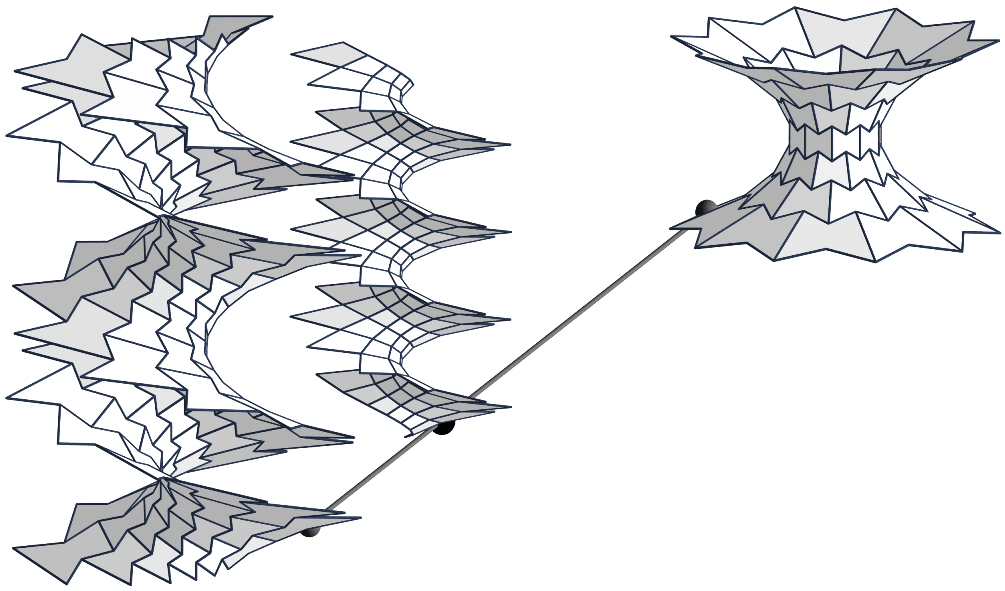

Figure 7. Linear combinations of parallel meshes, where one is a discrete catenoid and the second is a discrete helical surface are members of the corresponding associated family of minimal surfaces. Special combinations can lead to quad meshes. The case illustrated here is in fact a discrete helicoid, which means that it discretizes a surface generated by the helical motion of a straight line which orthogonally intersects the helical axis (see Example 4).

(iii) The discrete dual construction fulfils the property $(z_i^{**}) = (z_i)$, which is analogous to the smooth dual construction (see Proposition 9 (vi)).
(iv) The discrete dual construction always closes for conformal hexagons and transforms a discrete conformal surface into another one (see Proposition 9).
(v) The discrete dual construction of Definition 8 fulfils discrete analogues of properties of the smooth dual construction (see Propositions 11 and 12 and Equations (4)).

Consequently, a discrete conformal surface can be seen as a discrete analogue of a smooth conformal parametrized surface.

Remark 14.

(i) *Since each face of a conformal mesh has multi-ratio -1, it posesses the vertex offset property (see Theorem 4).*
(ii) *All vertices of a circular hexagonal mesh are always contained in a sphere or in a plane. We can say that a circular hexagonal surface is a discrete analogue of a surface with umbilic points only.*

The next definition is motivated by Theorem 10.

Definition 15. *A* discrete (hexagonal) minimal surface *is the dual of a conformal mesh covering the unit sphere.*

Here the word "covering" can be understood as "inscribed", "edge-wise tangent", i.e., circumscribed or "face-wise tangent", i.e., circumscribed. Later we show that a certain notion of discrete mean curvature vanishes for all such minimal surfaces.

Remark 16. *A more general definition of conformal hexagonal meshes can be derived from the dual construction for quad meshes in* [6]*. Instead of taking cross-ratios $-1/2$ in Definition 5, we take fractions $-\alpha_n/\beta_m$ where m and n identify the row and the column of the position of the quadrilateral. This can be interpreted as a discrete reparametrization of the standard conformal mesh. The dual construction then must be modified in the following way:*

$$\begin{aligned}
z_1^* - z_0^* &= -\beta_m/(\overline{z_1 - z_0}) & \qquad z_4^* - z_3^* &= -\beta_m/(\overline{z_4 - z_3})\\
z_2^* - z_1^* &= \alpha_n/(\overline{z_2 - z_1}) & z_5^* - z_4^* &= \alpha_n/(\overline{z_5 - z_4})\\
z_3^* - z_2^* &= -\beta_m/(\overline{z_3 - z_2}) & z_0^* - z_5^* &= -\beta_m/(\overline{z_0 - z_5}).
\end{aligned}$$

After this change Proposition 9 is still valid.

7. A construction of planar conformal meshes

This section describes an explicit construction of a conformal hexagonal mesh, which we are going to use later.

Proposition 17. *Let (z_i) be a conformal hexagon and let α and β be two similarities, which map (z_4, z_5) to (z_2, z_1) and (z_3, z_4) to (z_1, z_0), respectively. Then α and β commute, i.e., $\alpha \circ \beta = \beta \circ \alpha$ and $\beta^k \circ \alpha^l(z_i) = \alpha^l \circ \beta^k(z_i)$ is a conformal mesh, with no gaps, where $(k,l) \in \mathbb{Z}^2$ (Figure 8).*

PROOF. There exist $\varphi, \psi \in \mathbb{R}$ and $v, w \in \mathbb{C}$ such that $\alpha(z) = re^{i\varphi}z + v$ and $\beta(z) = se^{i\psi}z + w$. We obtain

$$\begin{aligned} \alpha(z_5) &= re^{i\varphi}z_5 + v = z_1 & \qquad \beta(z_4) &= se^{i\psi}z_4 + w = z_0 \\ \alpha(z_4) &= re^{i\varphi}z_4 + v = z_2 & \qquad \beta(z_3) &= se^{i\psi}z_3 + w = z_1, \end{aligned}$$

which implies $re^{i\varphi}(z_5 - z_4) = z_1 - z_2$ and $se^{i\psi}(z_4 - z_3) = z_0 - z_1$. It follows that

$$\frac{r}{s}e^{i(\varphi-\psi)}\frac{z_4 - z_5}{z_3 - z_4} = \frac{z_1 - z_2}{z_0 - z_1} \iff \frac{r}{s}e^{i(\varphi-\psi)}\frac{z_5 - z_0}{z_0 - z_3} = -\frac{z_2 - z_3}{z_3 - z_0},$$

because of the cross-ratio condition of conformal hexagons. Further,

$$se^{i\psi}z_2 + w - \underbrace{(se^{i\psi}z_3 + w)}_{=z_1} = re^{i\varphi}z_0 + v - \underbrace{(re^{i\varphi}z_5 + v)}_{=z_1},$$

which implies $\beta(z_2) = \alpha(z_0)$. Since $\beta^{-1}(z) = s^{-1}e^{-i\psi}z - s^{-1}e^{-i\psi}w$, $\alpha^{-1}(z) = r^{-1}e^{-i\varphi}z - r^{-1}e^{-i\varphi}v$ and $\alpha^{-1}(z_2) = \beta^{-1}(z_0)$ we have

$$r^{-1}e^{-i\varphi}z_2 - r^{-1}e^{-i\varphi}v = s^{-1}e^{-i\psi}z_0 - s^{-1}e^{-i\psi}w, \quad re^{i\varphi}z_0 + v = se^{i\psi}z_2 + w.$$

We multiply the last two equations and get

$$z_0z_2 - z_0v + v\alpha^{-1}(z_2) = z_0z_2 - wz_2 + w\beta^{-1}(z_0) \iff v = w\frac{(z_4 - z_2)}{(z_4 - z_0)}.$$

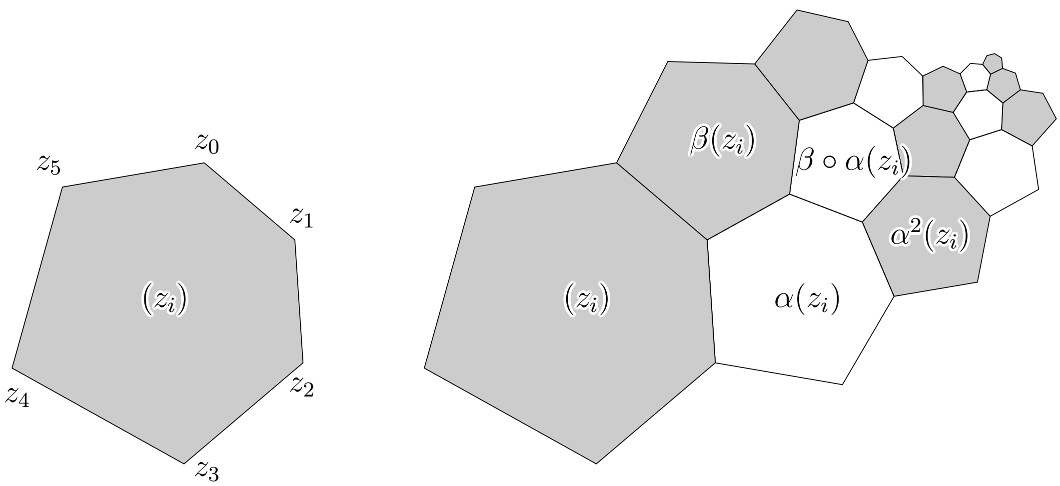

FIGURE 8. *Left:* An arbitrary conformal hexagon (z_i). *Right:* A planar conformal mesh $\beta^k \circ \alpha^l(z_i) = \alpha^l \circ \beta^k(z_i)$ (see Proposition 17).

We want to show that $\beta\circ\alpha=\alpha\circ\beta$. Therefore we compute $\alpha\circ\beta(z)=rse^{i(\varphi+\psi)}z+re^{i\psi}w+v$ and $\beta\circ\alpha(z)=rse^{i(\varphi+\psi)}z+se^{i\varphi}v+w$. Consequently,

$$\alpha\circ\beta=\beta\circ\alpha \iff (re^{i\varphi}-1)w=(se^{i\psi}-1)v. \tag{5}$$

Replacing v by $w(z_4-z_2)/(z_4-z_0)$, the last equation is further equivalent to

$$(re^{i\varphi}-1)=(se^{i\psi}-1)\frac{(z_4-z_2)}{(z_4-z_0)} \iff \beta(z_2)=\alpha(z_0),$$

which we have already shown to be true. □ □

To obtain circular conformal meshes we have to start with a circular conformal hexagon and then apply Proposition 17.

A similarity which is no translation is decomposable into a dilation and a rotation. The center of rotation of α and β is $v/(1-re^{i\varphi})$ and $w/(1-se^{i\psi})$, respectively. From equation (5) we obtain that both centers must be the same if and only if $\alpha\circ\beta=\beta\circ\alpha$. In $\mathbb{R}^2$ we take the 3×3 matrices

$$A=\left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \begin{matrix}\operatorname{Re} v\\ \operatorname{Im} v\end{matrix} & rD_\varphi \end{array}\right) \quad\text{and}\quad B=\left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \begin{matrix}\operatorname{Re} w\\ \operatorname{Im} w\end{matrix} & sD_\psi \end{array}\right)$$

for α and β, where D_ω is the 2×2 rotation matrix by an angle of ω. Since A and B commute, i.e., $AB=BA$, $\alpha^l\circ\beta^k$ can be written in the form $\exp(l\log A+k\log B)$.

7.1. **Discrete holomorphic functions.** Now, we consider a mesh $\mathcal{M}$ with arbitrary combinatorics stored in the graph $\mathcal{G}$. The *double* $\mathcal{D}$ of $\mathcal{G}$ is a quad graph defined such that the new vertices $V(\mathcal{D})$ are the old ones $V(\mathcal{G})$ combined with the vertices of the dual graph $V(\mathcal{G}^*)$ (see e.g. [5]). A quadrilateral of $\mathcal{D}$ consists of the two vertices incident with an edge of $\mathcal{G}$ and the two vertices incident with the corresponding edge of $\mathcal{G}^*$. A function f is *discrete holomorphic with (possibly complex) weights* ν if for each quadrilateral (z_0,w_0,z_1,w_1) of the double graph $\mathcal{D}$ the equation

$$\frac{f(w_1)-f(w_0)}{f(z_1)-f(z_0)}=i\nu(z_0,z_1)=-\frac{1}{i\nu(w_0,w_1)} \tag{6}$$

holds. A *discrete Laplacian operator with (in general different) weights* ν of a complex function f is

$$(\Delta f)(z)=\sum_{w\in\operatorname{star}(z)}\nu(w,z)(f(w)-f(z)),$$

where $\operatorname{star}(z)$ consists of all vertices which are connected with z by an edge of $\mathcal{G}$. Further, f is called *discrete harmonic* if $(\Delta f)(z)=0$ for all verticies z of the graph.

We consider complex functions defined an a graph $\mathcal{G}$ with values coming from the embedding in $\mathbb{C}$.

Proposition 18. *The mesh $\mathcal{M}$ with double graph combinatorics derived from a conformal hexagonal mesh generated with Proposition 17 are function values of a discrete holomorphic function defined on a regular hexagonal graph.*

Proof. We consider points w_i generated with the same similarities as the mesh $\mathcal{M}$, starting with one arbitrary point. These new points are the function values of the vertices of the dual mesh $\mathcal{M}^*$. Since the new mesh with vertices $V(\mathcal{M}) \cup V(\mathcal{M}^*)$ and double graph combinatorics is generated via similarities, the ratio $(z_1 - z_0)/(w_1 - w_0)$ is constant for each quadrilateral. Therfore exists a “nice” weight function ν that fulfills (6) and is constant at (z_0, z_1) and (w_0, w_1) on the mesh. □ □

Corollary 19. *The conformal hexagonal mesh $\mathcal{M}$ generated with Proposition 17 and its dual mesh $\mathcal{M}^*$ are function values of a discrete harmonic function defined on a regular hexagonal graph.*

Proof. This follows immediately from Proposition 18 and [5, Theorem 7.3]. □

Corollary 20. *Let (z_i) be a hexagon which is Möbius equivalent to a regular one. Then, the conformal hexagonal mesh $\mathcal{M}$ generated with Proposition 17 is discrete harmonic with constant weights.*

Proof. Let (z_i) be a hexagon of the mesh $\mathcal{M}$. Then, without loss of generality we have to show that the sum of the vectors of the edges emanating from z_2 is the zero vector. If (z_i) is a regular hexagon, then the proposition is obvious. For the non-regular case let us assume further without loss of generality that $\alpha(z) = re^{i\varphi}z$ which means that the center of rotation is 0. Therefore we have to show that $(z_1 - z_2) + (z_3 - z_2) + (re^{i\varphi}z_3 - z_2) = 0$.

$$\begin{aligned} re^{i\varphi}z_3 - z_2 = re^{i\varphi}(z_3 - z_4) = -\frac{1}{2}re^{i\varphi}\frac{(z_2 - z_3)(z_4 - z_1)}{(z_1 - z_2)} &= \\ = -\frac{1}{2}\frac{(z_2 - z_3)(z_4 - z_1)}{(z_5 - z_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_5 - z_4)}&. \end{aligned}$$

To finish the proof we must show that $p := (z_1 - z_2)(z_5 - z_4) + (z_3 - z_2)(z_5 - z_4) + (z_1 - z_2)(z_3 - z_4)$ is zero. The cross-ratio conditions of four successive vertices of (z_i) are equivalent to $\mathrm{crp}(k) = 0$ for all $k \in \{0, \ldots, 5\}$, where

$$\mathrm{crp}(k) := (z_k - z_{k+1})(z_{k+2} - z_{k+3}) + \frac{1}{2}(z_{k+1} - z_{k+2})(z_{k+3} - z_k).$$

A careful computation shows that

$$\begin{aligned} p^2 = &\frac{4}{3}\mathrm{crp}(0)\mathrm{crp}(2) - \frac{8}{3}\mathrm{crp}(1)\mathrm{crp}(2) + \frac{4}{3}\mathrm{crp}(1)\mathrm{crp}(3) \\ &- \frac{2}{3}\mathrm{crp}(0)p + \frac{4}{3}\mathrm{crp}(1)p + 2\mathrm{crp}(2)p - \frac{4}{3}\mathrm{crp}(3)p + \frac{2}{3}\mathrm{crp}(4)p = 0, \end{aligned}$$

which we wanted to show. □ □

8. Polygons with vanishing mixed area and discrete minimal surfaces

The *mixed area* of two parallel polygons $P = (p_i)$ and $Q = (q_i)$ $(i = 0, \dots, n-1)$ is defined by

$$\operatorname{area}(P, Q) = \frac{1}{4} \sum\nolimits_{0 \le i < n} (\det(p_i, q_{i+1}) + \det(q_i, p_{i+1})),$$

where indices are taken modulo n. A discrete curvature theory based on mixed areas [4] takes this notion to define a *discrete mean curvature* of a face P by

$$H_P = -\frac{\operatorname{area}(P, \sigma(P))}{\operatorname{area}(P)},$$

where $\sigma(P)$ denotes the corresponding face of a discrete Gauss image.

In smooth differential geometry a minimal surface can be defined as a surface with vanishing mean curvature in each point. A discrete minimal surface in this setting is a mesh where the discrete mean curvature H_P is zero for all faces P of the mesh. Incidence geometric properties of the polygons with vanishing mixed

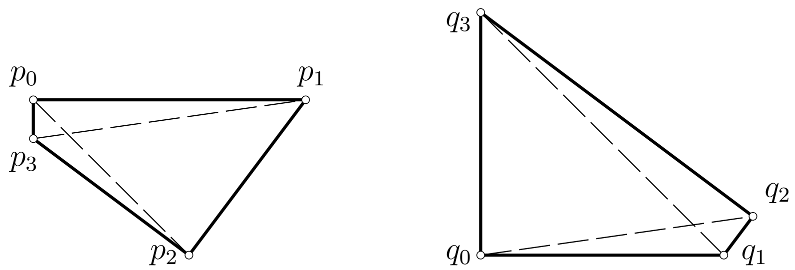

Figure 9. A pair of parallel quadrilaterals $p_0, \dots, p_3$ and $q_0, \dots, q_3$ has vanishing mixed area if and only if their non-corresponding diagonals are parallel: $q_0q_2 \parallel q_1q_3$ and $q_1q_3 \parallel q_0q_2$.

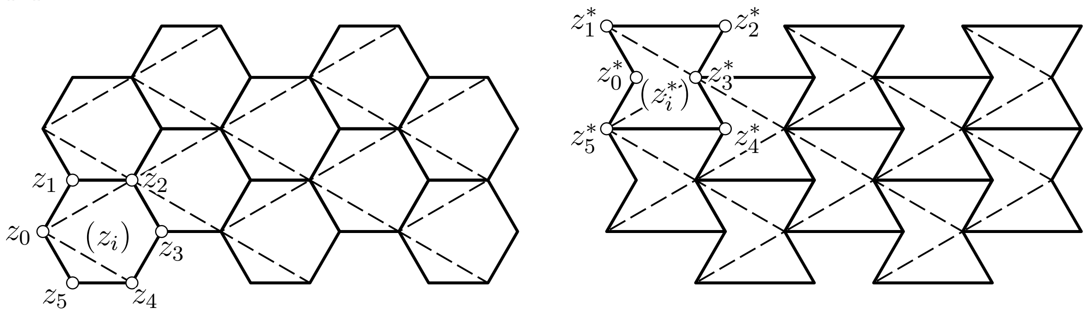

Figure 10. A pair of dual hexagonal meshes $\mathcal{M}$ and $\mathcal{M}^*$. The mesh $\mathcal{M}$ on the left hand side consists of hexagons (z_i) where the union of the vertices z_0, z_2, z_4 form a quad mesh $\overline{\mathcal{M}}$ (dashed). The dual hexagonal mesh $\mathcal{M}^*$ consists of hexagons (z_i^*), where the union of the vertices z_1^*, z_3^*, z_5^* form a quad mesh $\overline{\mathcal{M}}^*$. According to Remark 22 and Proposition 9, (v), $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}^*$ are reciprocal parallel.

area were studied in [10]. A result of [4] is that two parallel quadrilaterals have vanishing mean curvature if and only if their non-corresponding diagonals are parallel (see Figure 9).

According to Proposition 9, (v) the non-corresponding diagonals of the quadrilaterals of a conformal hexagon and its dual are parallel. This yields

Proposition 21. *A discrete minimal surface in the sense of Definition 15 has vanishing discrete mean curvature and therefore is a discrete minimal surface in the sense of* [4]*.*

Remark 22. *A pair of meshes $\mathcal{M}$ and $\mathcal{M}'$ is called* reciprocal parallel*, if their combinatorics are dual (correspondences are vertex-face, face-vertex and edge-edge) and corresponding edges are parallel. The connection between the existence of a reciprocal parallel mesh and infinitesimal flexibility was studied in* [16]*.*

Proposition 9, (v) says that non-corresponding diagonals of conformal hexagons of the quadrilaterals with cross-ratio equal to $-1/2$ are parallel.

From a discrete conformal surface and its dual we can derive two pairs of reciprocal parallel quad *meshes by choosing the edges $z_2 - z_0 \parallel z_3^* - z_1^*$ and $z_4 - z_0 \parallel z_3^* - z_5^*$ for the first pair and $z_1 - z_3 \parallel z_0^* - z_2^*$ and $z_5 - z_3 \parallel z_0^* - z_4^*$ for the second pair (see Figure 10).*

9. Examples of discrete minimal surfaces

As a preparation to the construction of examples, we have to discuss the relation between Christoffel duality and Weierstrass representation. We start with an arbitrary *conformal map* which is a holomorphic function $g : U \subset \mathbb{C} \longrightarrow \mathbb{C}$ where $g'(z_0) \neq 0$ for all $z_0 \in U$. This is a conformal parametrisation of a part of the plane. With the stereographic projection

$$\Phi(z) := \frac{1}{(|z^2| + 1)}(2z, |z|^2 - 1)$$

we get $n := \Phi \circ g$ as a conformal parametrisation of the sphere. By applying the Christoffel duality (Theorem 10) to n we get an isothermic parametrisation f^* of a minimal surface with

$$f_x^* = \frac{n_x}{\|n_x\|^2} \quad \text{and} \quad f_y^* = -\frac{n_y}{\|n_y\|^2}. \tag{7}$$

On the other hand we get minimal surfaces f with the same Gauss image as f^* via the Weierstrass representation (see e.g. [8]):

Theorem 23 (Weierstrass representation)**.** *For a holomorphic function h and a meromorphic function g (with some restrictions) the map*

$$f = \operatorname{Re} \int h \cdot \left(\frac{1}{2}(\frac{1}{g} - g), -\frac{1}{2i}(\frac{1}{g} + g), 1 \right) \tag{8}$$

is a parametrisation of a minimal surface. $\Phi \circ g$ is the Gauss image of f.

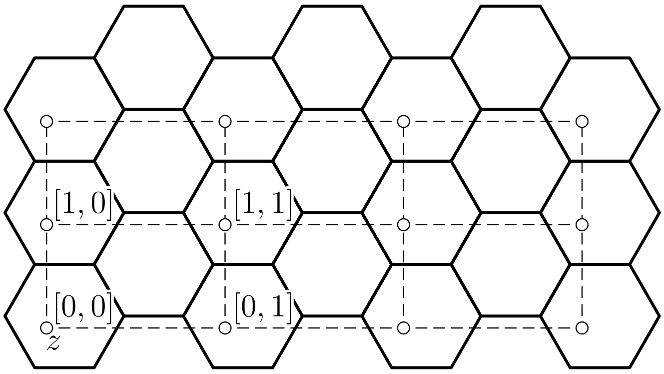

FIGURE 11. The derived quad mesh (dashed) $[m, n] := \alpha^{m-n} \circ \beta^{2n}(z)$ with $(m, n) \in \mathbb{Z}^2$ represents a discrete parametrization of the conformal hexagonal mesh generated with two similarities α and β following Proposition 17. We chose z arbitrarily.

As any holomorphic function $f(x + iy)$ satisfies $\mathrm{Re}(f') = \frac{\partial}{\partial x} \mathrm{Re}\, f$, Equations (7) and (8) produce the same result if and only if $n = \Phi \circ g$ satisfies the condition that $n_x/\|n_x\|^2$ equals the integrand in (8).

9.1. **Examples.** For our examples, we start with a circular hexagonal conformal mesh in $\mathbb{C}$ and apply the discrete Christoffel duality to the stereographic projection of the mesh.

Let α and β be two similarities which generate a conformal hexagonal mesh as explained in Section 7 and choose $z \in \mathbb{C}$ such that $\alpha(z) \neq z \neq \beta(z)$. We call a mesh $\alpha^{m-n} \circ \beta^{2n}(z)$ with $(m, n) \in \mathbb{Z}^2$ a *derived quad mesh* (Figure 11). The derived quad mesh represents a discrete parametrization assigned to the conformal hexagonal surface. We basically distinguish three cases:

(i) Both, α and β are translations.
(ii) α is a rotation and β is a dilation with the same fixed point.
(iii) Both, α and β are similarities with the same fixed point but different from a pure translation, rotation, or dilation.

For $(m, n) \in \mathbb{Z}^2$ and after an appropriate change of parameters, the derived quad mesh is of the form $m + in$ in (i), $e^{a(m+in)}$ in (ii), and $e^{(a+ib)(m+in)}$ in (iii), where $a, b \in \mathbb{R}$, $a, b \neq 0$. Therefore the meshes discretize the meppings $z \mapsto z$, $z \mapsto e^{az}$, and $z \mapsto e^{(a+ib)z}$, respectively.

Example 1 (Discrete Enneper's surface)**.** Letting $g(z) = z$ and $h(z) = z$ yields

$$n(x + iy) = \frac{1}{x^2 + y^2 + 1}(2x, 2y, x^2 + y^2 - 1),$$

and it is easy to verify that $n_x/\|n_x\|^2$ is equal to the real part of the integrand of (8). This is exactly the case of Enneper's surface (see Figures 1 and 12). We see that Christoffel duality of the regular hexagonal mesh generates a discrete Enneper's suface.

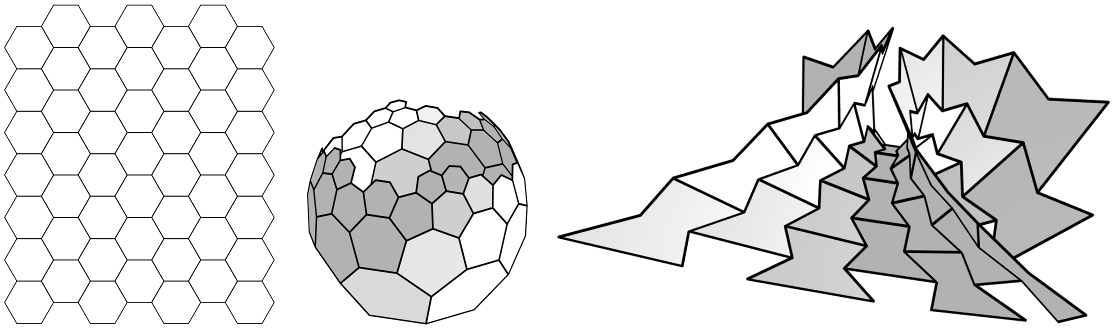

FIGURE 12. *Left:* Circular conformal mesh which discretizes $z \longmapsto z$. *Center:* Discrete Gauss image, which is the stereographic projection of the circular conformal mesh. *Right:* Discrete minimal surface generated as the discrete Christoffel dual of the Gauss image. According to Example 1 the hexagonal mesh is a discrete Enneper's surface.

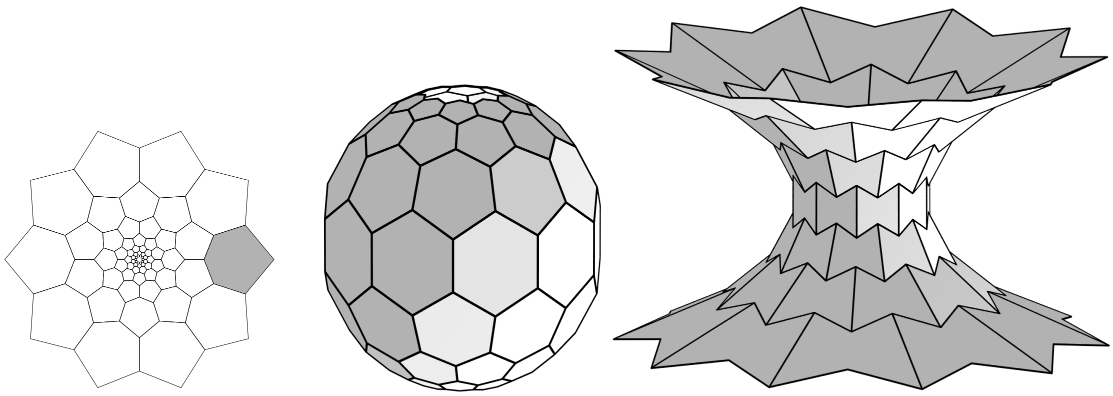

FIGURE 13. *Left:* Circular conformal mesh which discretizes $z \longmapsto e^{\lambda z}$ ($\lambda > 0$) (a symmetric hexagon, which is Möbius equivalent to a regular one is marked). *Center:* Discrete Gauss image, which is the stereographic projection of the circular conformal mesh. *Right:* Discrete minimal surface generated as the discrete Christoffel dual of the Gauss image. Referring to Example 2 the hexagonal mesh is a discrete catenoid.

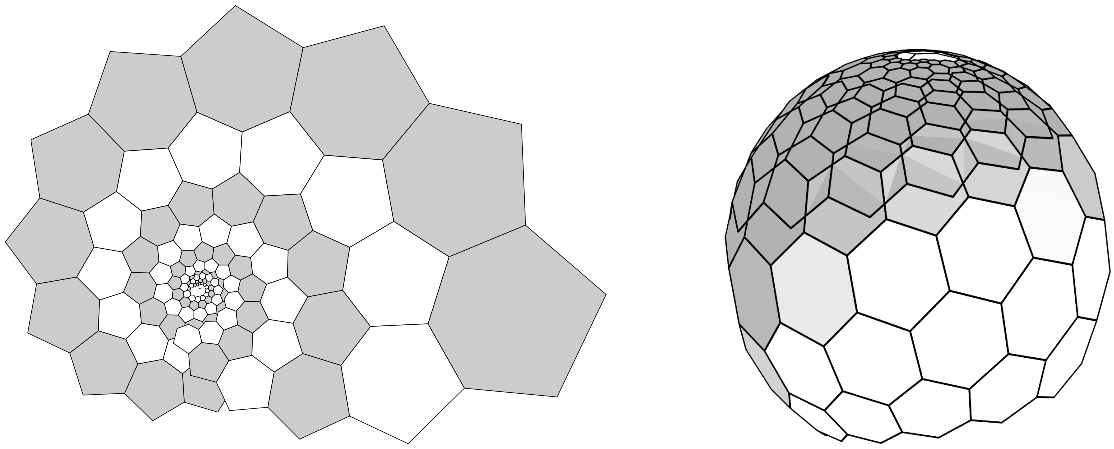

FIGURE 14. *Left:* Circular conformal mesh $\mathcal{M}$ which discretizes $z \longmapsto e^{az}$ ($a \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$). According to Proposition 17 we start with an arbitrary hexagon which is Möbius equivalent to a regular hexagon and apply similarities. Here the mesh $\mathcal{M}$ overlaps itself (multi-valued function). *Right:* Stereographic projection of the mesh $\mathcal{M}$.

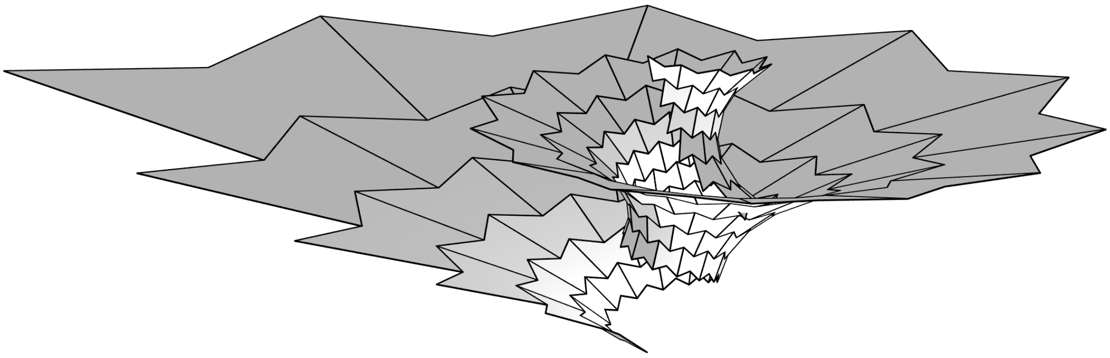

FIGURE 15. A discrete minimal surface which discretizes the smooth helical surface given by (9). The corresponding Gauss image is shown by Figure 14, right.

Example 2 (Discrete catenoid)**.** We start with a symmetric hexagon which is Möbius equivalent to a regular hexagon but not itself regular (see Figure 13, left) and apply Proposition 17 to get a circular conformal mesh with rotational symmetry. This mesh discretizes the holomorphic function $g(z) = e^{\lambda z}$ with an appropriate choice of $\lambda > 0$. We compute $n(z) = (\Phi \circ g)(z) = \Phi(e^{\lambda z})$ and see that $n_x/\|n_x\|^2$ equals the real part of the integrand of (8) for $h(z) = 1/\lambda = \text{const}$. The resulting minimal surface is the catenoid. We see that Christoffel duality of a hexagonal mesh with rotational symmetries as described generates a discrete catenoid (see Figures 6 and 13).

Example 3 (Helical surface)**.** We start with an arbitrary hexagon, which is not regular, but Möbius equivalent to a regular hexagon and apply Proposition 17 to get a mesh which discretizes the function $g(z) = e^{az}$, where $a \in \mathbb{C} \setminus 0$ (see Figure 14, left). With $h(z) = 1/a$ it is easy to verify that $n_x/\|n_x\|^2$ equals the real part of the integrand of (8). For $a \in \mathbb{R}$ we obtain the catenoid (see Example 2) and for $a \in i\mathbb{R}$ we obtain the helicoid and especially for $a = i$ the helicoid with the parametrization

$$f(x, y) = (\sin(x)\sinh(y), \cos(x)\sinh(y), x).$$

For $a = a_1 + ia_2$ $(a_1, a_2 \neq 0)$ we get the surface

$$f(u, v) = D_{\omega u} \cdot D_{\alpha v} \begin{pmatrix} (-a_1^2 + a_2^2)/(a\overline{a})^2 \cosh v \\ + \quad 2a_1a_2/(a\overline{a})^2 \sinh v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u/(a\overline{a}) \end{pmatrix}, \tag{9}$$

where D_t is the rotation matrix for rotation around the z-axis by an angle of t, $\omega = a\overline{a}/(2a_1a_2)$ and $\alpha = (a_2^2 - a_1^2)/(2a_1a_2)$. We see that this is a helical surface too.

The Christoffel dual of the considered hexagonal mesh generates a discrete minimal surface illustrated in Figure 15.

Example 4 (Associated family, helicoid)**.** The spherical hexagonal mesh of Example 2 can be dualized in yet another way. We interchange the coefficients 2 and -1 in the discrete dual construction and get a helical surface. The edges of all faces of this mesh are parallel to the corresponding edges of the catenoid given in Example 2. Linear combinations of these two discrete surfaces give all members of the associated family of this minimal surface. A special combination yields a quad mesh which discretizes the helicoid (see Figures 5 and 16).

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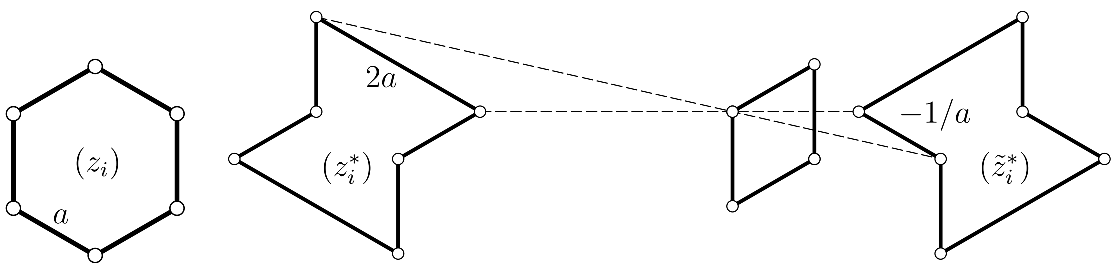

Figure 16. Each hexagon (z_i), which is Möbius equivalent to a regular one (left) can be dualized in three different ways, by interchanging the coefficients 2 and -1 in the discrete dual construction (Definition 8). Two of them, (z_i^*) and $(\tilde{z}_i^*)$ are illustrated here. The linear combination $1/3z_i^* + 2/3\tilde{z}_i^*$ yields a quadrilateral.

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