Affine arc length polylines and curvature continuous uniform B-splines

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Abstract

We study the recently introduced notion of polylines that form a discrete version of planar curves in affine arc length parametrization, showing that they match the control polylines of curvature continuous uniform quadratic B-splines (with analogous results in $\mathbb{R}^n$). It is demonstrated how inflection-free planar curves may be approximated by such affine arc length polylines in a way that the polyline is close to an affinely equidistant discretization of the curve and allows good approximations of the smooth affine curvature.

Keywords: Discrete affine differential geometry, quadratic B-spline, geometric continuity, curve approximation

1. Introduction and Overview

Affine geometry is the study of geometric objects and properties that are invariant under (equi-)affine transformations. Many of the concepts from Euclidean planar curve geometry can be reproduced in an analogous way in planar affine geometry; thus we have affine arc length, affine curve normal, affine curvature, affine evolutes, affine osculating conics (analogous to the Euclidean osculating circle). Inflection-free planar curves can be parametrized by the affine arc length. This parametrization has simple expressions for the affine normal, curvature etc., as opposed to the rather unwieldy formulae in a general parametrization.

There is a geometrically simple discrete analogon to planar curves in affine arc length parametrization: Polylines where the area spanned by three consecutive vertices is constant, that is $\det(\mathbf{x}_i - \mathbf{x}_{i-1}, \mathbf{x}_{i+1} - \mathbf{x}_i) = A$, or equivalently, where $\mathbf{x}_{i+1} - \mathbf{x}_i \parallel \mathbf{x}_{i+2} - \mathbf{x}_{i-1}$. We term those *affine arc length polylines*. They were studied recently by [1]. Using some of the theory contained in that work, we will extend it to include notions not studied before. These include the discrete osculating conic and the connection to curvature continuous quadratic B-splines.

For most applications cubic B-splines are preferred over quadratic B-splines because they generically have C^2 continuity as opposed to only C^1, which is often not enough. For a given control polyline or data to be interpolated the smoothness of quadratic B-splines can be increased by either using non-uniform parameters or allowing non-uniform control vertex weights, which results in rational B-splines (NURBS). This involves a tradeoff, as the main advantage of using *uniform* B-splines lies in the algorithmic simplicity and speed of evaluation.

There is, however, a different approach to constructing splines with the desired smoothness, which lies in the concept of imposing conditions on the control polyline (the data) itself. This work will show how the condition for curvature continuity (equivalent to *geometrical* G^2 continuity in the plane) is such that the control polylines coincide with the aforementioned affine arc length polylines.

A frequently occuring task is the approximation of a given curve by some type of curve governed by discrete data, in our case a curvature continous uniform quadratic B-spline. At a first glance the geometric constraints on the control polygon might be thought as being too restrictive to allow the satisfactory approximation of arbitrary curves. A simple approach will

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result in affine arc length polylines with strong oscillations of the discrete affine curvatures. This makes the polyline visually uneven, because the affine arc lengths of the segments of the smooth curve between two consecutive discrete polyline vertices oscillate as well. However, in Section 4 we present an algorithm which, by means of an uncomplicated iterative process, optimizes the smoothness of the discrete affine curvatures, thus yielding polylines with relatively few vertices and desirable geometric properties.

Generally affine arc length polylines have higher vertex density where the (Euclidean) curvature is higher, thus offering a good compromise between simplicity (few vertices) and quality of approximation (many vertices). The advantage of smoothing the affine curvatures is that it makes the polyline locally as affinely regular as possible. This ensures that the affine arc length polyline is nearly an affinely equidistant discretization of the cuve. It also provides good approximations of the smooth affine curvature of the original curve.

Previous work

Affine differential geometry is a classical field of study going back to the early 1900s due to the research of G. Pick, W. Blaschke, G. Thomson and others. Well-known textbooks on the subject are e.g. [2, 3]. At that time, discretization of those smooth notions was not considered.

Discrete affine differential geometry has mostly focused on surfaces in $\mathbb{R}^3$, in particular discrete affine minimal surfaces, which in the case of negative curvature can described by A-nets with an added geometric constraint [4, 5]. Using those nets as control nets of bilinear B-spline surfaces yields geometrically smooth surfaces. Thus the principle of imposing geometric conditions on the discrete data in order to make the resulting B-spline object (curve resp. surface) geometrically smoother that it would generically be is the same as in this paper.

The recent work [1] studied the same model of discrete curves in affine arc length parametrization (called *equal area polygons*) as we have undertaken, focusing on the notion of the discrete affine evolute. They also proved a discrete version of the well-known classical *six-sextactic-point theorem*, which is the affine version of the four-vertex theorem. Affine osculating conics were not considered, and neither was the connection to curvature continuous quadratic B-splines.

Splines with higher smoothness than C^1, in particular curvature continuity were studied extensively in classical B-spline literature (geometric splines, ν-splines, uniform cubic B-splines), see e.g. [6, 7]. For example, [8] stated the general conditions for the curvature continuous joining of Bézier curve segments. The majority of these works did not look at quadratic splines in particular, and most assumed a degree ≥ 3.

However, there is also a number of works were G^2 quadratic spline interpolation was studied explicitly, like [9]. The splines in that paper allow a representation as non-uniform B-splines, where the non-uniformity is necessary in order to do without geometric constraints on the control polygon (apart from the obvious one of convexity).

Although the equal-area condition for G^2 continuity of uniform quadratic B-splines clearly was realized in the literature, its interpretation in terms of discrete affine differential geometry provides new insights into its geometric meaning.

Contributions and overview

1. Section 2 introduces a way of describing polylines in an affine-invariant way and uses that notation to describe *affine arc length polylines*, which can be seen as discrete affine arc length parametrized curves. Building upon the work done by [1] on the discrete affine evolute, we study discretizations of conics and introduce the notion of the discrete osculating conic.
   Finally we show the connection between affine arc length polylines and G^2 continuous uniform quadratic B-splines.
2. Section 3 generalizes some of the results of the previous section to curves in $\mathbb{R}^n$ and uniform B-splines of order n.
3. In Section 4 it is shown how inflection-free planar curves can be approximated by discrete affine arc length polylines such that the affine arc length polyline is close to an affinely equidistant discretization of the original curve. Those approximations have desirable geometric qualities, as the vertex density of the polyline is higher where the Euclidean curvature is higher and the polyline is locally as near affinely regular as possible. They also allow good approximations of the affine smooth affine curvature.

2. Discrete planar affine curve geometry

There are several geometries that are commonly termed affine geometry. The one we will be using is (planar) *equiaffine geometry*, that is the study of geometric notions that are invariant under the group of *equiaffine transformations* (also called the special affine group) of the plane which comprises the transformations

$$\mathbf{x} \mapsto M\mathbf{x} + \mathbf{v}, \qquad M \in \mathbb{R}^{2\times 2} \text{ with } \det M = 1, \mathbf{v} \in \mathbb{R}^2.$$

Generally the invariant geometric properties of equiaffine geometry are parallelity, ratios of parallel lengths and areas.

On the other hand, *general affine geometry* allows all transformations $\mathbf{x} \mapsto M\mathbf{x}+\mathbf{v}$ with $\det M \neq 0$. In this paper *affine* will always refer to equiaffine geometry, because it is the main object of study. General affine geometry will always be referred to as such.

The affine polyline coordinates introduced below belong to equiaffine geometry, but the notion of the affine arc length polyline as well as its affine curvature, osculating conics and the algorithm for approximating curves by affine arc length polylines in Section 4 are fully invariant under the general affine group, thus being objects of general affine geometry.

2.1. Affine polyline coordinates

Let $\mathbf{x}_i \in \mathbb{R}^2$, $i = 0, \ldots, N$ be a polyline. We will use the *forward difference operator* frequently, defined as $\delta\alpha_i = \alpha_{i+1} - \alpha_i$ for any sequence α. We also write $\mathbf{v}_i := \delta\mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$ as an abbreviation. Then we define *affine polyline coordinates* A_i, B_i by

$$A_i = \det(\mathbf{v}_{i-1}, \mathbf{v}_i) \qquad \text{and} \qquad B_i = \det(\mathbf{v}_{i-1}, \mathbf{v}_{i+1}). \tag{1}$$

They are illustrated in Figure 1. A_i and B_i are invariant under equiaffine transformations, and they associate $2(N+1) - 5$ numbers with the polyline consisting of $N+1$ vertices, which describe the polyline uniquely up to equiaffine transformations: $\mathbf{x}_0$ and $\mathbf{x}_1$ can be freely chosen, and $\mathbf{x}_2$ has one degree of freedom left under the condition $\det(\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_1) = A_1$. After that, the rest of the polyline can be constructed uniquely by way of the equation

$$B_i\mathbf{v}_i = A_{i+1}\mathbf{v}_{i-1} + A_i\mathbf{v}_{i+1}, \tag{2}$$

as long as $B_i \neq 0$. This formula can be easily verified by$B_i \det(\mathbf{v}_i, \mathbf{v}_{i+1}) = A_{i+1} \det(\mathbf{v}_{i-1}, \mathbf{v}_{i+1}) + A_i \cdot 0$ and $B_i \det(\mathbf{v}_i, \mathbf{v}_{i-1}) = A_{i+1} \cdot 0 + A_i \det(\mathbf{v}_{i+1}, \mathbf{v}_{i-1})$. The 5 degrees of freedom in choosing $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$ correspond to the 5 degrees of freedom of the equiaffine group.

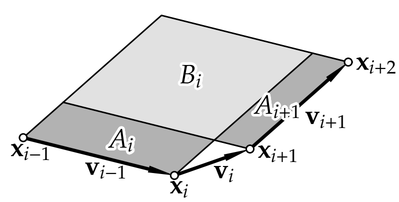

Figure 1: Affine polyline coordinates: A polyline $\mathbf{x}_i$, $i = 0, \ldots, N$ is determined up to equiaffine transformations by the areas A_i, $i = 1, \ldots, N-1$ and B_i, $i = 1, \ldots, N-2$.

2.2. Affine arc length polylines

The affine arc length of a curve $f\colon \mathbb{R} \to \mathbb{R}^2$ is defined (e.g. in [2, §4]) as

$$s(t) = \int_0^t \det\left(\dot{f}(\tilde{t})\ddot{f}(\tilde{t})\right)^{\frac{1}{3}} d\tilde{t}.$$

Therefore noninflecting curves possess a parametrization with respect to the affine arc length.

Using affine polyline coordinates this definition can be discretized to define the affine length of a polyline in an obvious way:

$$L_2(\mathbf{x}_0, \ldots, \mathbf{x}_N) := \sum_{i=1}^{N-1} A_i^{\frac{1}{3}}. \tag{3}$$

Lengths are therefore carried by triples of consecutive polyline vertices $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$. A polyline can be said to be in discrete affine arc length parametrization if all lengths are the same, that is $A := A_i = \text{const.} \neq 0$ We call such a polyline an *affine arc length polyline*. See Figure 2 for an example. As A_i is the area of the parallelogram spanned by $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$, [1] (who only considered closed curves) called them *equal-area polygons*. Throughout the paper we will assume $A > 0$, meaning we are restricted to polylines that are noninflecting in a discrete sense. However, in Section 4.4 we demonstrate how several affine arc length polylines may be concatenated to discretize curves with inflection points.

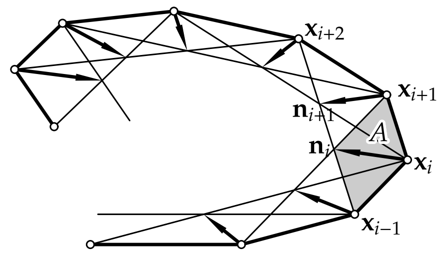

Figure 2: Affine arc length polyline: The polyline edges $\mathbf{x}_i\mathbf{x}_{i+1}$ are parallel to the diagonals $\mathbf{x}_{i-1}\mathbf{x}_{i+2}$, which is equivalent to the parallelogram areas spanned by 3 vertices all being equal (shown in gray for $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$). The arrows depict the discrete affine normals $\mathbf{n}_i$.

2.3. *Affine curvatures of affine arc length polylines*

The affine curvature is defined by the equation $f''' + \kappa f' = 0$, or equivalently by $\kappa := \det(f'', f''')$, where f'' is the affine curve normal [2]. Analogously to the Euclidean case the envelope of the normal lines $f(t) + \lambda f(t)''$ is called the affine evolute.

In the discrete setting of affine arc length polylines the affine normal lines are

$$\mathbf{x}_i + \lambda \mathbf{n}_i \quad \text{with} \quad \mathbf{n}_i := \delta^2 \mathbf{x}_{i-1} = \mathbf{x}_{i-1} - 2\mathbf{x}_i + \mathbf{x}_{i+1}$$

and the vertices of the (discrete) affine evolute result as intersection points of consecutive affine normal lines, as shown in Fig. 3. One can now define the *affine curvature* k_i of an affine arc length polyline $\mathbf{x}_{i-1}, \dots, \mathbf{x}_{i+2}$ by straightforward discretization through $\delta^3 \mathbf{x}_{i-1} + k_i \, \delta \mathbf{x}_i = 0$. Useful equivalent characterisations are

$$\mathbf{x}_{i+2} - \mathbf{x}_{i-1} = (3 - k_i)(\mathbf{x}_{i+1} - \mathbf{x}_i), \quad \text{or} \quad k_i = \frac{\det(\mathbf{n}_i, \mathbf{n}_{i+1})}{A}, \quad \text{or} \quad k_i = 2 - \frac{B_i}{A}. \tag{4}$$

From this it is easy to see that the affine evolute ξ_i of an affine arc length polyline can be expressed by

$$\xi_i = \mathbf{x}_i + \tfrac{1}{k_i} \mathbf{n}_i = \mathbf{x}_{i+1} + \tfrac{1}{k_i} \mathbf{n}_{i+1}. \tag{5}$$

The discrete affine evolute was studied extensively by [1]. We will focus on the notion of the osculating conic (Section 2.4).

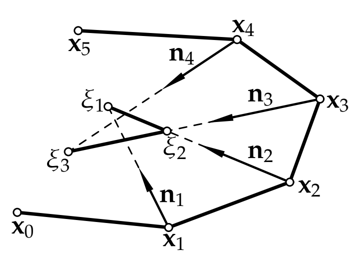

Figure 3: Affine evolute ξ_i of an affine arc length polyline $\mathbf{x}_i$. The evolute points ξ_i are the intersections of consecutive affine normals $\mathbf{n}_i = \mathbf{x}_{i+1} - 2\mathbf{x}_i + \mathbf{x}_{i-1}$ attached to $\mathbf{x}_i$, and thus form the discrete envelope of the normal map. ξ_i is associated with the curvature k_i and the quadruple of vertices $\mathbf{x}_{i-1}, \dots, \mathbf{x}_{i+2}$ (or the edge $\mathbf{x}_i\mathbf{x}_{i+1}$).

Affinely regular polylines were introduced by [10] as the orbits of equiaffine transformations. In particular they arise as equidistant discretizations of conics in affine arc length parametrization. [1] show how an affine arc length polyline is affinely regular if and only if its affine evolute is a single point. Due to eq. (5) that is equivalent to k_i being constant.

This is the discrete analogon to the well-known result from classical affine differential geometry, that a planar curve with constant affine curvature is a conic, see e.g. [2].

That result can be somewhat generalized:

Theorem 1. *An affine arc length polyline $\mathbf{x}_i$ discretizes a conic if and only if the sequence of its affine curvatures alternates between two constant values, that is, $k_{i-1} = k_{i+1}\ \forall\, i$. (For an example see Fig. 5.)*

Proof. We need to show that 6 consecutive vertices $\mathbf{x}_{i-2}, \dots, \mathbf{x}_{i+3}$ of an affine arc length polyline lie on a conic if and only if $k_{i-1} = k_{i+1}$.

We lay the unique conic through $\mathbf{x}_{i-2}, \dots, \mathbf{x}_{i+2}$ and apply an equiaffine transformation such that the x-axis is a symmetry axis of the conic and the symmetry maps $\mathbf{x}_i \leftrightarrow \mathbf{x}_{i+1}$. See Figure 4.

The affine arc length polyline property $A_i = A_{i+1}$ implies that also $\mathbf{x}_{i-1} \leftrightarrow \mathbf{x}_{i+2}$. Then the line $\mathbf{x}_{i-2}\mathbf{x}_{i+1}$ (which is parallel to $\mathbf{x}_{i-1}\mathbf{x}_i$) is symmetric to the line through $\mathbf{x}_i$ parallel to $\mathbf{x}_{i+1}\mathbf{x}_{i+2}$, on which $\mathbf{x}_{i+3}$ must lie. Then due to eq. (4), $k_{i-1} = k_{i+1} \Leftrightarrow \|\mathbf{x}_{i+3} - \mathbf{x}_i\| = (3 - k_{i+1})\|\mathbf{x}_{i+2} - \mathbf{x}_{i+1}\| = (3 - k_{i-1})\|\mathbf{x}_{i-1} - \mathbf{x}_i\| = \|\mathbf{x}_{i-2} - \mathbf{x}_{i+1}\| \Leftrightarrow$ the symmetry maps $\mathbf{x}_{i-2} \leftrightarrow \mathbf{x}_{i+3} \Leftrightarrow \mathbf{x}_{i+3}$ lies on the conic through $\mathbf{x}_{i-2}, \dots, \mathbf{x}_{i+2} \Leftrightarrow \mathbf{x}_{i-2}, \dots, \mathbf{x}_{i+3}$ lie on a conic.

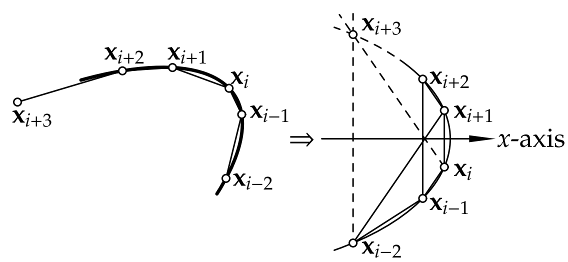

Figure 4: Proof of Theorem 1 by means of symmetry. $\mathbf{x}_{i+3}$ lies on the conic through $\mathbf{x}_{i-2}, \dots, \mathbf{x}_{i+2}$ iff $k_{i-1} = k_{i+1}$.

□

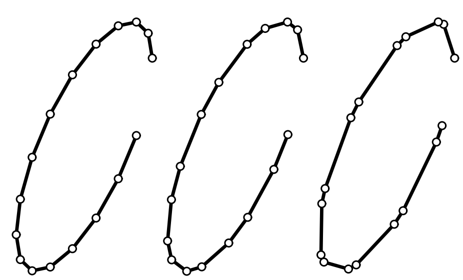

Figure 5: The same conic discretized by different affine arc length polylines with the same value A and the same first vertex $\mathbf{x}_0$. The left one has constant k_i and is therefore affinely regular; the others have k_i alternating between two values.

2.4. *Osculating conics of affine arc length polylines*

In planar affine differential geometry the affine evolute point is also the center of a unique conic which has 5-point (4th order) contact with the curve, called the *osculating conic*. For $\kappa = 0$ the affine evolute point is ideal and the osculating conic a parabola.

Because of $\mathbf{x}_{i+1} - \mathbf{x}_i \parallel \mathbf{x}_{i+2} - \mathbf{x}_{i-1}$ the set of the centers of all conics that contain $\mathbf{x}_{i-1}, \dots, \mathbf{x}_{i+2}$ is the line through $(\mathbf{x}_i + \mathbf{x}_{i+1})/2$ and $(\mathbf{x}_{i-1} + \mathbf{x}_{i+2})/2$ (including the ideal point), which passes through the affine evolute point. Thus we can define a unique conic through $\mathbf{x}_{i-1}, \dots, \mathbf{x}_{i+2}$ with the affine evolute point as its center, which we call the *outer osculating conic*.

Another way of defining a discrete osculating conic (again with ξ_i as center point) is to make the three edges involved ($\mathbf{x}_{i-1}\mathbf{x}_i$, $\mathbf{x}_i\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+1}\mathbf{x}_{i+2}$) tangents of the conic. This can be termed the *inner osculating conic*. Both conics are displayed in Figure 6.

Just like in the classical smooth setting, we can infer the type of the conic from the sign of the affine curvature:

$$\text{The inner and outer osculating conics are} \begin{cases} \text{hyperbolas} \\ \text{parabolas} \\ \text{ellipses} \end{cases} \Leftrightarrow \begin{cases} k_i < 0 \\ k_i = 0 \\ k_i > 0 \end{cases} . \tag{6}$$

Both choices of discrete osculating conics, however, are not ideal when one considers the (constant) affine curvatures of those conics as planar curves. The rationale of an osculating conic is that its affine curvature matches that of the given curve at the point of osculation. This is not the case with either of the conics defined above, but a better solution suggests itself:

Theorem 2. *For an affine arc length polyline we define three types of osculating conics of $\mathbf{x}_{i-1}, \ldots, \mathbf{x}_{i+2}$:*

osculating conic	*definition*	*affine curvature*
outer	*center ξ_i, through $\mathbf{x}_{i-1}, \ldots, \mathbf{x}_{i+2}$*	κ_i^o
inner	*center ξ_i, tangential to $\mathbf{x}_{i-1}\mathbf{x}_i$, $\mathbf{x}_i\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+1}\mathbf{x}_{i+2}$*	κ_i^i
middle	*center ξ_i, geometric mean of inner and outer conic*	κ_i^m

Then the inner and outer osculating conic are related by uniform scaling around the center (or a translation in the case $k_i = 0$), which allows us to define the middle osculating conic as their geometric mean. $\kappa_i^o, \kappa_i^i, \kappa_i^m$ are the conics' classical smooth affine curvatures (see Section 2.3), which are constants for each conic. Then they have the following values:

$$\begin{aligned} \kappa_i^o &= A^{-\frac{2}{3}}\, k_i \left(1 - \tfrac{k_i}{4}\right)^{\frac{1}{3}} \\ \kappa_i^i &= A^{-\frac{2}{3}}\, k_i \left(1 - \tfrac{k_i}{4}\right)^{-\frac{1}{3}} \\ \kappa_i^m &= A^{-\frac{2}{3}}\, k_i. \end{aligned} \tag{7}$$

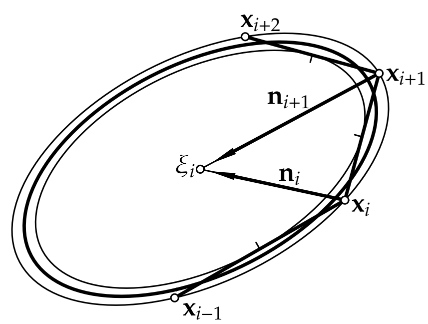

Figure 6: Affine osculating conics of an affine arc length polyline $\mathbf{x}_{i-1}, \ldots, \mathbf{x}_{i+2}$. Each one has the affine evolute point ξ_i as center. The *outer osculating conic* interpolates the four vertices, the *inner osculating conic* has the three edges as tangents and the *middle osculating conic* is the geometric mean of both. Among those three conics the middle osculating conic has the advantage of being the best approximation of the polygon (the maximum distance of the polygon from the conic is lowest), as well as having its smooth affine curvature match the discrete affine curvature by having the value $\kappa_i^m = k_i A^{-\frac{2}{3}}$.

Proof. Depending on the sign of k_i, every polyline $\mathbf{x}_{i-1}, \ldots, \mathbf{x}_{i+2}$ can be brought into one of the forms

hyperbola	parabola	ellipse
$\mathbf{x}_i = r\begin{pmatrix} \cosh \lambda i \\ -\sinh \lambda i \end{pmatrix}$	$\mathbf{x}_i = \begin{pmatrix} i \\ \lambda i^2 \end{pmatrix}$	$\mathbf{x}_i = r\begin{pmatrix} \cos \lambda i \\ \sin \lambda i \end{pmatrix}$
$A = -2r^2 \sinh \lambda (1 - \cosh \lambda)$	$A = 2\lambda$	$A = 2r^2 \sin \lambda (1 - \cos \lambda)$
$k_i = 2(1 - \cosh \lambda) < 0$	$k_i = 0$	$k_i = 2(1 - \cos \lambda) > 0$

with $r, \lambda > 0$ by an equiaffine transformation. The points of contact of the polyline edges with the inner osculating conic are the polyline edge midpoints $\frac{1}{2}(\mathbf{x}_{i-1} + \mathbf{x}_i)$ etc. for reasons of symmetry.

If we consider the elliptic case, in the special form from the table above the osculating conics are circles with center at 0. The outer conic has radius r. The inner conic's radius is $\|(\mathbf{x}_i + \mathbf{x}_{i+1})/2\| = r\cos\frac{\lambda}{2} = r\sqrt{(1+\cos\lambda)/2} = \sqrt{1 - k_i/4}$ (with the value of k_i from the table). This gives us a scaling factor $\sqrt{1-k_i/4}$ between the outer and inner osculating conic. In the hyperbolic case we analogously find the same factor. The scaling factor "inner : middle : outer osculating conic" is therefore $\left(1 - \frac{k_i}{4}\right)^{1/4}$ ($k_i \neq 0$). In the case $k_i = 0$ symmetry shows that the inner osculating conic is also an x-symmetric parabola, and inserting the edge midpoints proves that it has the form $(i, \lambda i^2 + \lambda/4)^T$. It is therefore related to the outer osculating conic by translation with vector $(0, A/8)^T$. Thus they have the same smooth affine curvature.

Keeping in mind that the (constant) smooth affine curvatures for hyperbola, parabola and ellipse are $-r^{-\frac{4}{3}}$, 0 and $r^{-\frac{4}{3}}$, respectively, the desired results follow from straight-forward calculation. $\square$

2.5. Planar discrete affine arc length polylines and B-splines

Trying to connect planar affine geometry with the theory of B-splines the obvious choice is to look at quadratic B-splines, because they consist of parabolae which have vanishing curvature from the point of view of affine geometry.

So we will take a polyline $\mathbf{x}_0, \ldots, \mathbf{x}_N$ to be the control polyline of a uniform quadratic B-spline. The algorithm of *Chaikin* [11] is the version of the *Lane-Riesenfeld* subdivision algorithm [12] that goes with quadratic B-splines. It replaces the polyline with $\mathbf{x}_{\frac{1}{4}}, \mathbf{x}_{\frac{3}{4}}, \ldots, \mathbf{x}_{N-\frac{1}{4}}$, where $\mathbf{x}_{i\pm\frac{1}{4}} := \frac{3}{4}\mathbf{x}_i + \frac{1}{4}\mathbf{x}_{i\pm1}$. Now it is easy to see (Fig. 7) that the Chaikin algorithm does not change the affine arc length of the polyline:

$$L_2(\mathbf{x}_0, \ldots, \mathbf{x}_N) = L_2(\mathbf{x}_{\frac{1}{4}}, \ldots, \mathbf{x}_{N-\frac{1}{4}})$$

In addition an affine arc length polyline will maintain that property under subdivision, where the new constant A' has the value $A' = \frac{1}{8}A$.

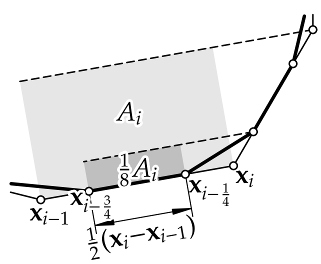

Figure 7: Chaikin subdivision of a polyline results in $A_{i-\frac{1}{4}} = A_{i+\frac{1}{4}} = \frac{1}{8}A_i$.

In fact we find that affine arc length polylines have a distinguished role in the context of uniform quadratic B-splines. The two consecutive parabola segments of the B-spline with control polyline $\mathbf{x}_0, \ldots, \mathbf{x}_3$ are parametrized by $p_1(t) = \mathbf{x}_1 - \frac{(1-t)^2}{2}\mathbf{v}_0 + \frac{t^2}{2}\mathbf{v}_1$ and $p_2(t) = \mathbf{x}_2 - \frac{(1-t)^2}{2}\mathbf{v}_1 + \frac{t^2}{2}\mathbf{v}_2$ (with $0 \leq t \leq 1$ each). Calculating their (Euclidean) curvatures at the intersection point we find $\kappa_1(1) = \kappa_2(0) \Leftrightarrow A_1\|\mathbf{v}_1\|^{-3} = A_2\|\mathbf{v}_1\|^{-3} \Leftrightarrow A_1 = A_2$. Thus we see that *a uniform quadratic B-spline is curvature continuous if and only if its control polyline is an affine arc length polyline.*

3. Affine arc length polylines in $\mathbb{R}^n$

Affine differential geometry differs from its Euclidean counterpart by the fact that the notion of the affine arc length of a curve is explicitly dependent on the dimension of the containing space.

As in the planar case, we use as affine equivalence transformations those linear maps which leave n-dimensional volumes constant, that is, those maps of the form $\mathbf{x} \mapsto M\mathbf{x} + \mathbf{v}$ with $\det(M) = 1$, $\mathbf{v} \in \mathbb{R}^n$. Thus the obvious way of defining an affinely-invariant arc length element is to use a determinant of derivatives of the curve f, which is $\det(\dot{f}, \ddot{f}, \ldots, f^{(n)})$ in $\mathbb{R}^n$. To make this term invariant under changes of variables, however, it is necessary to add a suitable exponent (1/3 in the planar case). Thus one gets for the affine arc length in n dimensions

$$s(t) = \int \det\left(\dot f(t), \dots, f^{(n)}(t)\right)^{\frac{2}{n(n+1)}} dt.$$

Therefore affine arc length parametrization has $s = t \Rightarrow \det(\dot f, \ddot f, \dots, f^{(n)}) \equiv 1$. Allowing (as in the planar case) any constant value $\neq 0$ instead of 1, we now define

Definition 3. *A polyline* $\mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^n$ *consists of* $N - n + 1$ discrete affine length elements, *which are the subpolylines* $\mathbf{x}_i, \dots, \mathbf{x}_{i+n}$, *which have the affine lengths*

$$L_n(\mathbf{x}_i, \dots, \mathbf{x}_{i+n}) = \det(\delta\mathbf{x}_i, \dots, \delta\mathbf{x}_{i+n-1})^{\frac{2}{n(n+1)}}, \tag{8}$$

in particular $L_2(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}) = A_i^{1/3}$. *The polyline is called an* affine arc length polyline *iff all values of* L_n *are the same* $\neq 0$, *that is*

$$\exists A \neq 0\colon\ \det(\delta\mathbf{x}_i, \dots, \delta\mathbf{x}_{i+n-1}) = A \ \forall\, i. \tag{9}$$

In the plane it was advantageous to use affine arc length polylines as control polylines of uniform quadratic B-splines; in $\mathbb{R}^n$ that role is played by uniform B-splines of degree n:

Theorem 4. *Affine arc length polylines are exactly those polylines* $\mathbf{x}_i \in \mathbb{R}^n$ *which when taken as control polylines of uniform B-spline curves of degree n result in all Frenet curvatures* $k_1, \dots, k_{n-1}$ *being continuous at the transitions of the B-spline segments.*

Proof. The B-spline basis functions $\mathcal{B}_j^k(t)$ for equidistant knots $t_j = j$ are recursively defined by

$$\mathcal{B}_j^k(t) = \frac{t-j}{k}\mathcal{B}_j^{k-1}(t) + \frac{j+k+1-t}{k}\mathcal{B}_{j+1}^{k-1}(t) \quad \text{and} \quad \mathcal{B}_j^0(t) = \begin{cases} 1, & t \in [j, j+1) \\ 0, & \text{else} \end{cases}$$

and form the B-spline curve $f(t) = \sum_{i=0}^N \mathcal{B}_i^n(t)\,\mathbf{x}_i$ with control points $\mathbf{x}_i$ (see e.g. [14]). As B-spline curves are translation invariant, in the useable region $t \in [n, N-n]$ the basis functions form a partition of unity.

The curvatures $k_1, \dots, k_{n-2}$ of f are continuous because they only depend on the first $n-1$ derivatives of f and the basis functions $\mathcal{B}_i^n$ are C^{n-1}. The curvature k_{n-1} (called the *torsion* τ in $\mathbb{R}^3$) is calculated in [13] to be

$$k_{n-1} = \frac{\det\left(\dot f, \dots, f^{(n)}\right) \cdot \left\|\left[\dot f, \dots, f^{(n-2)}\right]\right\|}{\|\dot f\| \cdot \left\|\left[\dot f, \dots, f^{(n-1)}\right]\right\|^2},$$

where $[a_0, \dots, a_k]$ denotes the *multivector* spanned by $a_0, \dots, a_k$, whose length is equal to the k-dimensional volume of the parallelotope spanned by $a_0, \dots, a_k$. The only term in the formula which has to be tested for continuity is the determinant, as it is the only one which contains $f^{(n)}$. Using [14, Thm. 6.18] with equidistant knots to calculate the derivatives of the first B-spline segment we get

$$f^{(k)}(t) = \sum_{i=0}^{n-k} \mathcal{B}_{i+k}^{n-k}(t)\,\delta^k\mathbf{x}_i \quad \text{for } n \le t \le n+1.$$

Using the calculus of multivectors we can now employ induction for k starting from $k = n+1$ down to 1.

$$\left[f^{(k)}(t), f^{(k+1)}(t), \dots, f^{(n)}(t)\right] = \left[f^{(k)}(t), \delta^{k+1}\mathbf{x}_0, \dots, \delta^{k+1}\mathbf{x}_{n-k-1}\right] = \sum_{i=0}^{n-k} \mathcal{B}_{i+k}^{n-k}(t)\left[\delta^k\mathbf{x}_i, \delta^{k+1}\mathbf{x}_0, \dots, \delta^{k+1}\mathbf{x}_{n-k-1}\right] =$$

$$= \sum_{i=0}^{n-k} \mathcal{B}_{i+k}^{n-k}(t)(-1)^i\left[\delta^k\mathbf{x}_1 - \delta^k\mathbf{x}_0, \dots, \delta^k\mathbf{x}_i - \delta^k\mathbf{x}_{i-1}, \delta^k\mathbf{x}_i, \delta^k\mathbf{x}_{i+1} - \delta^k\mathbf{x}_i, \dots, \delta^k\mathbf{x}_{n-k} - \delta^k\mathbf{x}_{n-k-1}\right] =$$

$$= \sum_{i=0}^{n-k} \mathcal{B}_{i+k}^{n-k}(t)(-1)^i\left[-\delta^k\mathbf{x}_0, \dots, -\delta^k\mathbf{x}_{i-1}, \delta^k\mathbf{x}_i, \delta^k\mathbf{x}_{i+1}, \dots, \delta^k\mathbf{x}_{n-k}\right] =$$

$$= \sum_{i=0}^{n-k} \mathcal{B}_{i+k}^{n-k}(t)\left[\delta^k\mathbf{x}_0, \dots, \delta^k\mathbf{x}_{n-k}\right] = \left[\delta^k\mathbf{x}_0, \dots, \delta^k\mathbf{x}_{n-k}\right].$$

With e.g. [15, p.16] and $k = 1$ we then see $\det(\dot{f}(t), \ldots, f^{(n)}(t)) \cdot [\mathbf{e}_1, \ldots, \mathbf{e}_n] = \left[\dot{f}(t), \ldots, f^{(n)}(t)\right] = [\delta\mathbf{x}_0, \ldots, \delta\mathbf{x}_{n-1}] = \det(\delta\mathbf{x}_0, \ldots, \delta\mathbf{x}_{n-1}) \cdot [\mathbf{e}_1, \ldots, \mathbf{e}_n]$ for all $n \le t \le n+1$. Therefore the determinant is continuous over segment transitions (even constant) if and only if the control polyline $\mathbf{x}_i$ fulfils equation (9). □

In addition, we saw in Section 2.5 that if we subdivide a general planar polyline $\mathbf{x}_0, \ldots, \mathbf{x}_N$ by the Chaikin algorithm to a polyline $\mathbf{x}'_0, \ldots, \mathbf{x}'_{2N-1}$, the overall affine length will stay the same: $\sum_{i=1}^{N-1} A_i^{1/3} = \sum_{i=1}^{2N-2} A_i'^{1/3}$. We now find that the same holds true in $\mathbb{R}^n$ if we use the Lane-Riesenfeld algorithm of degree n, as is appropriate for B-Splines of degree n.

Theorem 5. *Let $\mathbf{x}_0, \ldots, \mathbf{x}_N$ be the vertices of the polyline $\mathbf{x}$ in $\mathbb{R}^n$. Let $\mu\mathbf{x} := \left(\frac{\mathbf{x}_0+\mathbf{x}_1}{2}, \ldots, \frac{\mathbf{x}_{N-1}+\mathbf{x}_N}{2}\right)$ denote the averaging operator for finite sequences and $\delta\mathbf{x} := (\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_N - \mathbf{x}_{N-1})$ the forward difference operator. The Lane-Riesenfeld refinement of degree n [12] of $\mathbf{x}$ produces a polyline $\mathbf{x}'$, which is described by*

$$\mathbf{x}' = (\mathbf{x}'_0, \ldots, \mathbf{x}'_{N+1}) := \mu^n(\mathbf{x}_0, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{x}_N).$$

Then the discrete affine arc lengths (as defined by eq. (8)) of $\mathbf{x}'$ are

$$L_n(\mathbf{x}'_0, \ldots, \mathbf{x}'_n) = L_n(\mathbf{x}'_1, \ldots, \mathbf{x}'_{n+1}) = \tfrac{1}{2} L_n(\mathbf{x}_0, \ldots, \mathbf{x}_n). \tag{10}$$

Proof. Let $\mathbf{v} = (\mathbf{v}_0, \ldots, \mathbf{v}_{2n}) := (0, \mathbf{x}_1 - \mathbf{x}_0, 0, \mathbf{x}_2 - \mathbf{x}_1, \ldots, 0, \mathbf{x}_n - \mathbf{x}_{n-1}, 0)$ and thus $\delta\mathbf{x}' = \mu^n\mathbf{v}$. We also note that $(\delta\mu\mathbf{v})_{2i} = \frac{1}{2}(\mathbf{v}_{2i+2} - \mathbf{v}_{2i}) = 0$, from which $(\delta^m\mu^m\mathbf{v})_{2i} = 0$, $m \ge 0$ follows by iteration, as the operators δ and μ commute.

We will use induction for k to show

$$\left[(\mu^k\mathbf{v})_j, (\mu^k\mathbf{v})_{j+1}, \ldots, (\mu^k\mathbf{v})_{j+(k-1))}\right] = 2^{-\frac{k(k+1)}{2}}[\mathbf{v}_{j+1}, \mathbf{v}_{j+3}, \ldots, \mathbf{v}_{j+2k-1}] \tag{11}$$

for $k \le n$ and $j = 0, 2, \ldots, 2(n-k)$. For $k = 1$ the statement is true, because $\mathbf{v}_j = 0$ for even j. Assume therefore that it holds for some $k < n$, and pass to $k+1$.

As an abbreviation we introduce the sequence $\mathbf{w} = (\mathbf{w}_0, \ldots, \mathbf{w}_n) := \mu^k\mathbf{v}$.

$$\begin{aligned}
&\left[(\mu^{k+1}\mathbf{v})_j, (\mu^{k+1}\mathbf{v})_{j+1}, \ldots, (\mu^{k+1}\mathbf{v})_{j+k}\right] = 2^{-(k+1)}[\mathbf{w}_j + \mathbf{w}_{j+1}, \ldots, \mathbf{w}_{j+k} + \mathbf{w}_{j+k+1}] = \\
&\quad = 2^{-(k+1)}\Big[\sum_{i=0}^{k}\Big((\mathbf{w}_{j+i} + \mathbf{w}_{j+i+1})(-1)^i \sum_{l=0}^{i}\tbinom{k}{l}\Big), \mathbf{w}_{j+1} + \mathbf{w}_{j+2}, \ldots, \mathbf{w}_{j+k} + \mathbf{w}_{j+k+1}\Big] = \\
&\quad = 2^{-(k+1)}\Big[\underbrace{\sum_{i=0}^{k}(-1)^i\tbinom{k}{i}\,\mathbf{w}_{j+i}}_{=(-1)^k\delta^k\mathbf{w}_j = 0} + (-2)^k\mathbf{w}_{j+k+1}, \mathbf{w}_{j+1} + \mathbf{w}_{j+2}, \ldots, \mathbf{w}_{j+k} + \mathbf{w}_{j+k+1}\Big] = \\
&= 2^{-1}[\mathbf{w}_{j+1}, \mathbf{w}_{j+2}, \ldots, \mathbf{w}_{j+k+1}] \underset{\substack{\uparrow \\ \text{eq. (11) with } j+2 \text{ in place of } j}}{=} 2^{-1-\frac{k(k+1)}{2}}[\mathbf{w}_{j+1}, \mathbf{v}_{j+3}, \mathbf{v}_{j+5}, \ldots, \mathbf{v}_{j+2k+1}] = 2^{-\frac{(k+1)(k+2)}{2}}[\mathbf{v}_{j+1}, \mathbf{v}_{j+3}, \ldots, \mathbf{v}_{j+2k+1}].
\end{aligned}$$

To complete the proof we let $k = n$ in (11) to see $[(\delta\mathbf{x}')_0, \ldots, (\delta\mathbf{x}')_n] = 2^{-n(n+1)/2}[(\delta\mathbf{x})_0, \ldots, (\delta\mathbf{x})_n]$, which is the proposition according to Definition 3. The length of $(\mathbf{x}'_1, \ldots, \mathbf{x}'_{n+1})$ is the same, as can be seen by reversing the ordering of the vertices. □

4. Curve approximation by affine arc length polylines

We now want to study the problem of approximating an inflection-free planar curve f by an affine arc length polyline $\mathbf{x}$ such that $\mathbf{x}_i = f(t_i)$. Let $S_i := \int_{t_i}^{t_{i+1}} \det(\dot{f}(t), \ddot{f}(t))^{1/3}\,dt$ be the affine arc length from $\mathbf{x}_i$ to $\mathbf{x}_{i-1}$. Curves with inflection points can be treated by allowing a jump of the affine arc length constant A at each inflection point (including a sign change). See Section 4.4 for a demonstration of that approach.

A simple approach to the problem of discretizing a given inflection-free planar curve calls for the choice of two initial points $\mathbf{x}_0$ and $\mathbf{x}_1$ on the curve. With a likewise fixed value A the method proceeds by imposing $\det(\mathbf{x}_{i+1}-\mathbf{x}_i, \mathbf{x}_{i+2}-\mathbf{x}_{i+1}) = A$, thus finding $\mathbf{x}_{i+2}$ at the intersection of the curve with a line parallel to $\mathbf{x}_{i+1} - \mathbf{x}_i$. This is the path taken in [1].

Figure 5 demonstrates this procedure for different values of t_1 but identical A in discretizing a conic. All of these discretizations have the sequence of discrete affine curvatures k_i alternate between two constant values (see Theorem 1), except for the one with constant k_i, which is affinely regular. Clearly this discretization is the most desirable one, because it has $S_i = \text{const.}$, making it an affinely equidistant discretization of the conic.

For a generic curve f discretizations by affine arc length polylines cannot be affinely equidistant, but we will try to make the differences $S_{i+1} - S_i$ as small as possible. The path we will be taking to achieve this is to minimize $\sum_i (\delta k_i)^2$. It is motivated by the following theorem:

Theorem 6. *Consider the affine arc length polyline $\mathbf{x}_i = f(t_i)$, with S_i being the affine arc length from $\mathbf{x}_i$ to $\mathbf{x}_{i+1}$. Then if S_i is sufficiently small $(S_{i+1} - S_i)/S_i$ is a monotonous function of δk_i. In particular asymptotically for $S_i \to 0$ the function $(S_{i+1} - S_i)/S_i$ attains 0 at $\delta k_i = 0$.*

Proof. We consider a quintuple of an affine arc length polyline with vertices lying on f such that

$$(t_{-2}, t_{-1}, t_0, t_1, t_2) := \Big(-t(2 + z(t)),\ -t,\ 0,\ t(1 + w(t)),\ t(2 + y(t)) \Big)$$

Assume f is parametrized by the affine arc length and let $\mathbf{x}_i := f(t_i)$, $i = -2, \ldots, 2$. Then $S_i = t_{i+1} - t_i$. We calculate Taylor polynomials of the discrete polyline coordinates (Section 2.1) by using the following table

$\det(.,.)$	f_0'	f_0''	f_0'''	$f_0^{(4)}$	$f_0^{(5)}$
f_0'	0	1	0	$-\kappa_0$	$-2\kappa_0'$
f_0''	-1	0	κ_0	κ_0'	
f_0'''	0	$-\kappa_0$	0		
$f_0^{(4)}$	κ_0	$-\kappa_0'$			
$f_0^{(5)}$	$2\kappa_0'$				

where $f_0^{(j)} := f^{(j)}(0)$, $\kappa_0 := \kappa(0)$ etc.

We know that in the Taylor series of $\mathbf{x}_i$ derivatives $f_0^{(j)}$ only appear in terms where the exponent of t is $\geq j$, and $\delta\mathbf{x}_i = O(t)$. Therefore the remainder of the Taylor series of $A_i(t) = \det(\delta\mathbf{x}_{i-1}, \delta\mathbf{x}_i)$ which is not calculable from the table, coming from sums of determinants $\det(f_0^{(j)}, f_0^{(k)})$, $j + k \geq 7$, is $O(t^7)$. This means that we can calculate the coefficients of $A_i(t) = A_{i,3}t^3 + A_{i,4}t^4 + A_{i,5}t^5 + A_{i,6}t^6 + O(t^7)$. For instance

$$\begin{aligned}A_{-1}(t) = t^3\tfrac{2+3z_0+z_0^2}{2} + t^4\tfrac{3+2z_0}{2}z_0' + t^5\Big(-\kappa_0\tfrac{6+15z_0+14z_0^2+6z_0^3+z_0^4}{24} + \tfrac{1}{2}(z_0')^2 + \tfrac{3+2z_0}{4}z_0''\Big) + \\ + t^6\Big(\kappa_0'\tfrac{60+176z_0+205z_0^2+120z_0^3+35z_0^4+4z_0^5}{240} - \kappa_0 z_0'\tfrac{15+28z_0+18z_0^2+4z_0^3}{24} + \tfrac{1}{2}z_0'z_0'' + \tfrac{3+2z_0}{12}z_0'''\Big) + O(t^7).\end{aligned}$$

Now we demand that $\mathbf{x}_{-2}, \ldots, \mathbf{x}_2$ is an affine arc length polyline $\Leftrightarrow A_{-1}(t) = A_0(t) = A_1(t)$. Equating coefficients of those conditions yields equations in the Taylor coefficients of $w(t), y(t), z(t)$. E.g. $A_{-1,3} = A_{0,3} \Rightarrow z_0 = w_0$ or $z_0 = -3 - w_0$. The second solution is excluded because it implies $\mathbf{x}_{-2} = \mathbf{x}_1$. Under the assumption $z_0 = w_0$ the equation $A_{-1,4} = A_{0,4}$ results in $z_0' = w_0'$. Proceeding like this to consecutively solve the eight equations $A_{-1,j} = A_{0,j} = A_{1,j}$, $j = 3, 4, 5, 6$ we find

$$\begin{aligned}A_{-1}(t) = A_0(t) &\Rightarrow w(t) - z(t) = \tfrac{2+w_0}{60}(1 + w_0)(5 + 6w_0 + 2w_0^2)\kappa_0'\, t^3 + O(t^4)\\ A_0(t) = A_1(t) &\Rightarrow y(t) - w(t) = \tfrac{2+w_0}{60}(5 + 4w_0 + w_0^2)\kappa_0'\, t^3 + O(t^4)\end{aligned}$$

Now we can do the analogous calculation for the equation of constant discrete affine curvature, $k_{-1} = k_0 \Leftrightarrow B_{-1}(t) = B_0(t)$, inserting the results from above for $z(t)$ and $y(t)$ to get

$$\det(\delta\mathbf{x}_{-2}, \delta\mathbf{x}_0) = \det(\delta\mathbf{x}_{-1}, \delta\mathbf{x}_1) \Rightarrow w(t) = -\tfrac{\kappa_0'}{6}t^3 + O(t^4)$$

and thus $S_0 - S_{-1} = -\frac{\kappa_0'}{6}t^4 + O(t^5)$. If the discrete affine curvature is not constant, we use the equation $B_0 - B_{-1} + A_0\,\delta k_{-1} = 0$ (see (4)) to obtain

$$S_0 = t\Big(\delta k_{-1} + \sqrt{16 + (\delta k_{-1})^2}\Big)/4 + t^3\kappa_0\,\alpha(\delta k_{-1}) - t^4\tfrac{\kappa_0'}{6}\beta(\delta k_{-1}) + O(t^5).$$

Here α and β are abbreviations for differentiable terms in δk_{-1} with $\alpha(0) = 0$ and $\beta(0) = 1$, whose exact forms are not necessary to know. For slightly simpler notation we may define ϕ by $\delta k_{-1} = 4\sinh(\phi)$, then we equivalently get

$$S_0 = t\,e^{\phi} + t^3\kappa_0\,\tilde{\alpha}(\phi) - t^4\tfrac{\kappa_0'}{6}\tilde{\beta}(\phi) + O(t^5), \tag{12}$$

where $\tilde{\alpha}(0) = 0$ and $\tilde{\beta}(0) = 1$ like α, β above. Thus the relative change of the affine arc lengths is

$$\frac{S_0 - S_{-1}}{S_{-1}} \approx e^{\phi} - 1 \qquad \text{for } S_i \to 0. \tag{13}$$

□

This shows that minimizing the $(\delta k_j)^2$ is effective in order to make an affine arc length polyline a near affine equidistant discretization of f. In fact it turns out that the $(\delta k_j)^2$ are a very sensitive measure for the non-constantness of the S_i: If $\mathbf{x}_0 = f(0)$ and $\mathbf{x}_1 = f(t_1)$, even a small deviation of t_1 from the optimum value ($(S_1 - S_0)/S_0 \ll 1\%$) will cause strong oscillations in k_i, and thus large values of $(\delta k_j)^2$, as evidenced by Figure 11. We find the optimal value of t_1 by minimizing $\sum(\delta k_j)^2$. This optimization process, as well as the initial value for t_1 is discussed in detail in Section 4.2.

Another way of characterising the optimal affine arc length polyline in our problem is to define them it be (locally) "as affinely regular as possible". By this we mean that locally its deviation from an affinely regular polyline is as small as possible.

To justify this we will denote the sector areas with respect to the discrete affine evolute point ξ_i of $\mathbf{x}_{i-1}, \ldots, \mathbf{x}_{i+2}$ by $\mathcal{A}_j$, as shown in Figure 8. The polyline $\mathbf{x}_{i-1}, \ldots, \mathbf{x}_{i+2}$ is affinely regular with respect to ξ_i, and so we have $\mathcal{A}_{-1} = \mathcal{A}_0 = \mathcal{A}_1$. Calculating the areas of the next triangles outside

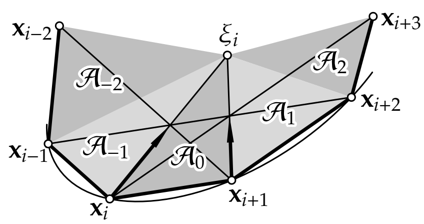

Figure 8: If ξ_i is the affine evolute point of $\mathbf{x}_{i-1}, \ldots, \mathbf{x}_{i+2}$, the three areas $\mathcal{A}_{-1}$, $\mathcal{A}_0$, $\mathcal{A}_1$ are the same. The next areas outside of them have the relative changes $(\mathcal{A}_{-2} - \mathcal{A}_0)/\mathcal{A}_0 = k_i - k_{i-1}$ and $(\mathcal{A}_2 - \mathcal{A}_0)/\mathcal{A}_0 = k_i - k_{i+1}$. Therefore minimizing $\sum_i (k_{i+1} - k_i)^2$ makes the polygon (locally) as affinely regular as possible.

those three we get the relative changes

$$\frac{\mathcal{A}_{-2} - \mathcal{A}_0}{\mathcal{A}_0} = k_i - k_{i-1} \qquad \text{and} \qquad \frac{\mathcal{A}_2 - \mathcal{A}_0}{\mathcal{A}_0} = k_i - k_{i+1}.$$

Thus for $\mathbf{x}_{i-2}, \ldots, \mathbf{x}_{i+3}$ to be near affinely regular we want $k_i - k_{i-1}$ and $k_{i+1} - k_i$ as near zero as possible. So finally we can state our problem clearly:

We wish to discretize f by an affine arc length polyline $\mathbf{x}_i$ such that equivalently

$$\begin{array}{ccccc} \mathbf{x}_i \textit{ discretizes } f \textit{ in a nearly} & \Leftrightarrow & \mathbf{x}_i \textit{ is locally near} & \Leftrightarrow & \displaystyle\sum_i (\delta k_i)^2 = \mathit{min} \\ \textit{affinely equidistant way} & & \textit{affinely regular} & & \end{array} \tag{14}$$

The method we will be using to minimize the $(\delta k_i)^2$ is a simple iterative algorithm called *golden section search*. It is explained in detail in Section 4.2.

It turns out that if we approximate the curve f by a affine arc length polyline with minimal $\sum_i (k_{i+1} - k_i)^2$, the affine curvatures of the middle osculating conics $\kappa_i^m = k_i\,A^{-\frac{2}{3}}$ form good approximations of the smooth affine curvature of f. See Figs. 11 and 13 as well as table 1 for examples. For non-optimal choices of t_1 this approximation is destroyed by the strong oscillations of k_i.

4.1. *Preprocessing*

As mentioned in the previous section, the construction of an affine arc length polyline $\mathbf{x}_0, \ldots, \mathbf{x}_n$ with $\mathbf{x}_i = f(t_i)$ calls for frequent calculation of intersection points of the given curve f and straight lines; it needs to be performed for every vertex $\mathbf{x}_i$ in every optimization step. This can be nontrivial and thus time-consuming. Therefore it is proposed here to preprocess f, *replacing* it by a curve $\tilde{f}$ which is a very close approximation of f, and whose structure makes those intersection operations easy.

This can be achieved by obtaining a fine sampling $\mathbf{p}_j$, $j = 0, \ldots, N$ of f and interpolating this polyline by parabolic arcs. It is appropriate to construct those arcs in an affine-invariant way: Make the tangent direction of the arc from $\mathbf{p}_j$ to $\mathbf{p}_{j+1}$ in $\mathbf{p}_j$ to be $\mathbf{p}_{j+1} - \mathbf{p}_{j-1}$ and in $\mathbf{p}_{j+1}$ to be $\mathbf{p}_{j+2} - \mathbf{p}_j$. It is illustrated in Fig. (9). We will always assume that $\tilde{f}\colon [0, N] \to \mathbb{R}^2$ is parametrized in such a way that $\tilde{f}(j) = \mathbf{p}_j$ (which lies on f). Thus for $\tilde{f}(t)$ the integral part of t specifies the parabolic arc this point is on, and the fractional part the position on that arc.

It is important to note that $\mathbf{p}$ is *not* an affine arc length polyline, because it does not have curvature continuity. It is only a tool to facilitate the construction of the affine arc length polyline $\mathbf{x}$.

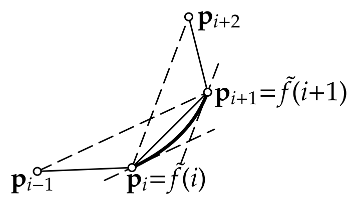

Figure 9: Before constructing the affine arc length polyline we replace f by a curve $\tilde{f}$, which is constructed from a fine sampling $\mathbf{p}$ of f, interpolated by parabolic segments as shown in the figure. This is done because the construction of $\mathbf{x}_i$ involves frequent intersecting of f (resp. $\tilde{f}$) with straight lines, which is simplified considerably by using a piecewise parabolic replacement for f.

It turns out that the density (number of vertices) of the polyline $\mathbf{p}$ strongly affects the quality of the end result $\mathbf{x}$: it must be sufficiently high. We suggest $|\mathbf{p}| \approx 20 \cdot |\mathbf{x}|$. In the example below we aimed to approximate f by an affine arc length polyline $\mathbf{x}$ with 60 vertices, so we replaced f by a curve made from 1000 parabolic segments. See Table 1 for details.

number of vertices of $\mathbf{p}$	200	400	600	800	1000
max	90%	14%	2.9%	1.3%	0.9%
mean	19%	2.4%	0.7%	0.37%	0.25%

Table 1: Effect of the vertex density in the preprocessing step on the quality of affine curvature approximation shown for the curve from Example 1 (Section 4.3): The deviations of the affine curvatures of the middle osculating conics $A^{-2/3}k_i$ of the affine arc length polyline $\mathbf{x}$ compared to the exact affine curvature of the underlying curve f increases steeply for lower numbers of vertices of $\mathbf{p}$.

While it is also possible to replace f with a piecewise linear curve instead of a piecewise quadratic one, due to the lower approximation quality the vertex count of $\mathbf{p}$ must be much higher to enable the same smoothness of the final affine arc length polyline $\mathbf{x}$.

4.2. *Initialisation and golden section search*

After choosing a first vertex on $\tilde{f}$, e.g. $\mathbf{x}_0 := \tilde{f}(0)$, we need an initial value t_1, $\mathbf{x}_1 := \tilde{f}(t_1)$ to start the optimization. We do this by calculating the exact affine arc length of the replacement curve $\tilde{f}$, which is easy (and another advantage of a piecewise parabolic replacement of f). If we take the affine polyline coordinates of $\mathbf{p}_j$ to be A_j^* and B_j^*, then the segment from $\mathbf{p}_j$ to $\mathbf{p}_{j+1}$ has the affine arc length

$$S_j := \left(\frac{4A_j^* A_{j+1}^*}{A_j^* + B_j^* + A_{j+1}^*} \right)^{1/3}. \tag{15}$$

The total length of $\tilde{f}$ is $S := \sum_{j=0}^{N-1} S_j$. If we want to approximate the curve by the affine arc length polyline $\mathbf{x}_0, \ldots, \mathbf{x}_n$ it is a good choice to use $A := (S/n)^3$. As initial value for t_1 we want to place $\mathbf{x}_1 = \tilde{f}(t_1)$ such that the affine arc length from $\mathbf{x}_0$ to $\mathbf{x}_1$ is roughly $1/n$ of the total affine arc length. For simplicity we initialize t_1 with an integral value:

$$t_1 \text{ such that } \sum_{j=0}^{t_1-1} S_j \approx S/n.$$

From this the rest of the affine arc length polyline $\mathbf{x}_0, \ldots, \mathbf{x}_n$ can be constructed easily as described in Section 4.

Now we want to perform optimization on t_1 in order to minimize the error function $\psi(t_1) := \sum_i (k_{i+1} - k_i)^2$, keeping $\mathbf{x}_0$ and A fixed. The error function is unimodal with respect to t_1. Therefore we can use the iterative method called *golden section search* (see [16]). This method is a variant of ternary search which finds a minimum of a unimodal function of one variable in an interval. As initial interval we can use e.g. $[0, 2t_1] =: [a, d]$. It is split into three parts with length ratios $\phi : 1 : \phi$, where ϕ is the golden section. This gives us four points $a < b < c < d$. Now the function ψ to be minimized is evaluated at the middle points b and c. If $\psi(b) < \psi(c)$ the minimum cannot lie in the right interval $(c, d]$, which is discarded. The remaining interval $[a, c]$ is once again subdivided into 3 intervals with the length ratios $\phi : 1 : \phi$. Here the properties of the golden section guarantee that ϕ needs only be evaluated one more time to achieve this: $(a, b, c, d) \mapsto (a, a{+}c{-}b, b, c)$. If $\psi(b) \geq \psi(c)$, the left interval is the one discarded. Every iteration decreases the total interval size by a factor of $\phi - 1 \approx 0.618$ until the interval is judged small enough. In the end we simply set $t_1 := \frac{1}{2}(b + c)$. Examples below were created in that way.

4.3. *Summary and example*

Below is the summary of the proposed method of approximating a noninflecting planar curve f with an affine arc length polyline $\mathbf{x}_0, \ldots, \mathbf{x}_n$.

1. *Preprocessing:* Replace f by a piecewise parabolic curve $\tilde{f}$, to facilitate the constructions in the following steps. $\tilde{f} \colon [0, N] \to \mathbb{R}^2$, $\tilde{f}(j) = \mathbf{p}_j$, where $\mathbf{p}_j$ lies on f; the tangent directions of $\tilde{f}(j)$ are chosen as shown in Figure 9. It is important that the number of parabolic segments is sufficiently high, to make the replacement curve $\tilde{f}$ a good approximation of f.
2. If S_j is the affine length of the parabolic segment of $\tilde{f}$ from $\mathbf{p}_j$ to $\mathbf{p}_{j+1}$ (as given by eq. (15)), the total affine length of $\tilde{f}$ is $S := \sum_{j=0}^{N-1} S_j$. Let $\mathbf{x}_0 := \tilde{f}(0)$. We set t_1 to an initial integral value such that $\sum_{j=0}^{t_1-1} S_j$ is as close to S/n as possible. Thus $\mathbf{x}_1 := \tilde{f}(t_1) = \mathbf{p}_{t_1}$.
3. The affine arc length polyline $\mathbf{x}_0, \ldots, \mathbf{x}_n$ can now be constructed uniquely from $\mathbf{x}_0$, $\mathbf{x}_1$ and the constant A, which we set to the value $A := (S/n)^3$. The construction consecutively finds $\mathbf{x}_i$ at the intersection of the curve $\tilde{f}$ and the straight line of all points $\mathbf{z}$ with $\det(\mathbf{x}_{i-1} - \mathbf{x}_{i-2}, \mathbf{z} - \mathbf{x}_{i-1}) = A$. The affine arc length polyline determines the affine curvatures $k_1, \ldots, k_{n-2}$.
4. Golden section search is performed to find the optimal value t_1 which minimizes $\psi(t_1) := \sum_i (k_{i+1} - k_i)^2$. The affine arc length polyline is constructed anew from $\mathbf{x}_0$, A and the updated t_1 to get the new value of $\psi(t_1)$ in every optimization step. The optimization terminates after a fixed number of iterations.

As an example we take f to be a Bézier curve of degree 8, to be approximated by an affine arc length polyline with 60 vertices. Fig 11 and Table 1 compare the affine curvatures of the middle osculating conics $k_i A^{-2/3}$ with the exact affine curvatures of the original Bézier curve at that point, $\kappa((t_{i-1} + t_i + t_{i+1} + t_{i+2})/4)$.

4.4. *Approximating curves with inflection points*

Curves with inflection points do not have an affine arc length parametrization in the classical sense and can therefore not be represented by affine arc length polylines. It is however possible to split them at the inflection points and treat the segments separately, with visually pleasing results. Below is a description of the procedure.

Assume that we want to approximate a curve $f(t)$, $0 \leq t \leq 1$ which has k inflection points by a polyline $\mathbf{x}_i$ containing $n + 1$ vertices.

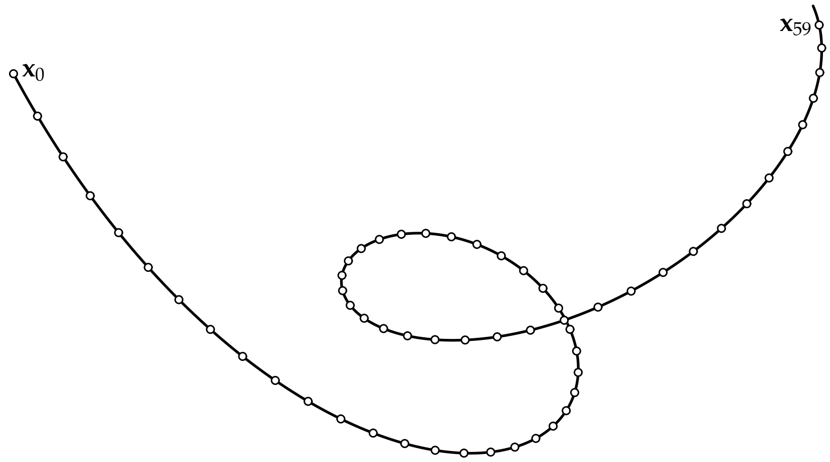

Figure 10: Bézier curve of degree 8 and its approximation by a 60 vertex affine arc length polyline, optimized for $\sum_i (k_{i+1} - k_i)^2 = \min$.

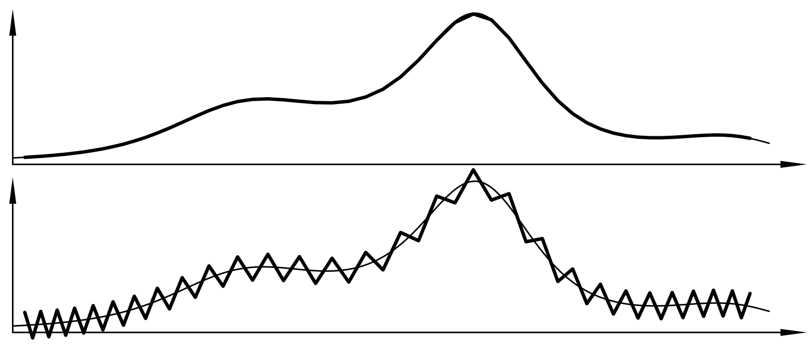

Figure 11: Affine curvatures $\kappa_i^m = k_i A^{-2/3}$ of the middle osculating conics of the example depicted in Figure 10. The exact affine curvature of the smooth curve is the thin curve. The top diagram is for the value of t_1 (where $\mathbf{x}_1 = \tilde{f}(t_1)$) which minimizes $\sum_i (k_{i+1} - k_i)^2$, the bottom one has t_1 increased by 0.2% to show the sensitivity of the smoothness of k_i (and thus of the local affine regularity of $\mathbf{x}$) to t_1.

1. Identify the inflection points $\tilde{f}(s_i)$, $0 < s_1 < \cdots < s_k < N$ of the piecewise parabolic replacement curve $\tilde{f}$. This is easy, as $s_i \in \mathbb{Z}$.
2. Let $\mathbf{x}_0 := \tilde{f}(0)$ and $n_0 := 0$. We want to place vertex $\mathbf{x}_{n_i}$ close to the i-th inflection point for each $i \geq 1$. To find a good choice for the indices n_i we calculate $\tilde{S}_i := \int_0^{s_i} |\det(\dot{\tilde{f}}, \ddot{\tilde{f}})|^{1/3} dt$, $\tilde{S} := \int_0^N |\det(\dot{\tilde{f}}, \ddot{\tilde{f}})|^{1/3} dt$ and then choose the n_i as to fulfil
$$\tilde{S}_1 : \tilde{S}_2 : \cdots : \tilde{S}_k : \tilde{S} \approx n_1 : n_2 : \cdots : n_k : n$$
as closely as possible.
3. Now we consecutively construct the sub-polylines $\mathbf{x}_{n_{i-1}}, \ldots, \mathbf{x}_{n_1}$, constructing $\mathbf{x}_{n_{i-1}+1}, \ldots, \mathbf{x}_{n_i}$ for the already fixed $\mathbf{x}_{n_{i-1}}$ and $A := (\tilde{S}_i - \tilde{S}_{i-1})/(n_i - n_{i-1})$. For the smoothing optimization in this step it is advisable to disregard a few vertices that are closest to the inflection points. The sub-polyline $\mathbf{x}_{n_{i-1}}, \ldots, \mathbf{x}_{n_1}$ determines the affine curvatures $k_{n_{i-1}+1}, \ldots, k_{n_i-2}$. Therefore, if we skip l vertices at each end of the sub-polyline this means that we will minimize $\sum_{j=n_{i-1}+1+l}^{n_i-3-l} (k_{j+1} - k_j)^2$, for e.g. $l = 2$.
This may be redone with adjusted value for A if the resulting $\mathbf{x}_{n_i}$ is too far from the inflection point, but an exact coincidence is not necessary.

An example of this situation is shown in Figures 12 and 13.

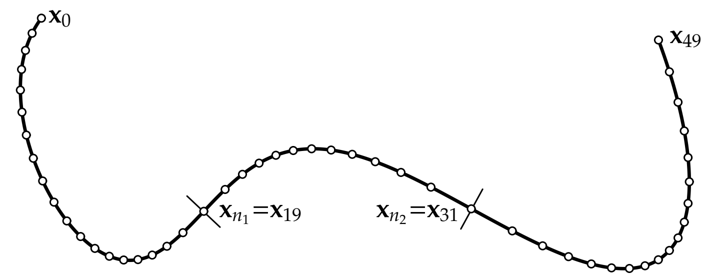

Figure 12: Curve with two inflection points and its approximation by a 50 vertex polyline, which consists of three concatenated affine arc length polylines, each optimized for $\sum_i (k_{i+1} - k_i)^2 = \min$. The vertices close to the inflection points ($\mathbf{x}_{31}$ and $\mathbf{x}_{19}$) are shared between two affine arc length polylines each.

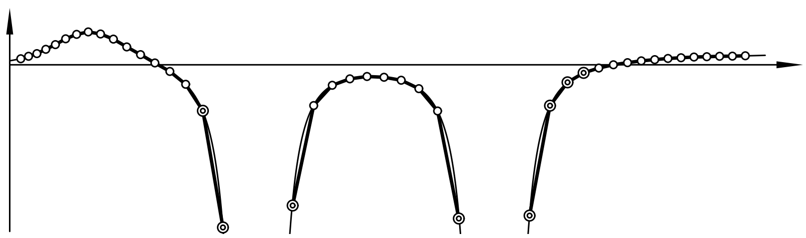

Figure 13: Affine curvatures $k_i\, A^{-2/3}$ of the middle osculating conics of a curve with two inflection points. The exact affine curvature of the smooth curve is the thin graph. Values that depend on vertices on *both* sides of an inflection point are meaningless and therefore omitted. The highlighted curvature values were disregarded in the optimizations $\sum_i (k_{i+1} - k_i)^2 = \min$ for being too close to the inflection points.

Conclusion and future work

We have shown that the theory of affine arc length polylines, the discretization of curves in affine arc length parametrization, can be extended beyond the previous work by [1]. In particular we defined discrete osculating conics. In addition we showed how affine arc length polylines equivalently arise as control polygons of curvature continuous uniform quadratic B-splines. Some of our results were shown to also hold in $\mathbb{R}^n$ for uniform B-splines of degree n. Finally, we presented an algorithm for the approximation of inflection-free planar curves by affine arc length polylines in a way that the smoothness of the discrete affine curvature was optimized, making it a good approximation of the smooth affine curvature of the original curve. This demonstrates how the added geometric constraints on the discrete data pose no obstacle to the practical problem of approximating an arbitrary given curve.

There are several directions for future research that look promising. One possibility is to try to extend discrete affine curve geometry in $\mathbb{R}^3$ and $\mathbb{R}^n$. Another idea is to further study the concept of imposing geometric constraints on discrete data in order to make piecewise smooth objects deriving from it (geometrically) smoother than in the generic case. In particular, we need to see whether the notion affine arc length polylines can be transferred into the theory of surfaces in $\mathbb{R}^3$.

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This research was supported by grant No. 23735-N13 of the Austrian Science Fund (FWF) and by the DFG-Collaborative Research Center, TRR 109, *Discretization in Geometry and Dynamics* through grant I 706-N26 of the Austrian Science Fund (FWF). I thank Helmut Pottmann for supervision and important advice during the creation of this paper.

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