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Architectural freeform surfaces designed for cost-effective paneling through mold re-use

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Abstract

The realization of architectural freeform skins is a big challenge, in particular if one desires a smooth appearance and uses curved panels. These have to be brought into shape by special manufacturing technologies, most of which require the costly production of molds. Previous approaches to mold re-use relied on optimization algorithms which play with the available tolerances and allowed deviations from the reference geometry. One aims at a good trade-off between fabrication cost and a visual appearance which comes close to the original design intent.

For general freeform surfaces, there may be no other ways to computationally solve the paneling problem. However, we will show in this paper that there is a rich class of surfaces which very much look like freeform shapes, but have significant advantages over totally unrestricted freeform geometry. These surfaces are known as Weingarten surfaces. They are characterized by a relation between their principal curvatures, leading to a just one-parametric family of curvature elements and thus local surface shapes. This allows one to fabricate N panels with a number of molds which is roughly just $\sqrt{N}$. Moreover, if the panels are fabricated from material which is not rigid after panel production, one can exploit the allowed deformations through bending and further increase the accessible shape variety or reduce the number of molds even more. We also provide an overview of computational techniques for the computation of Weingarten surfaces and their deformation through bending and illustrate the approach through a number of architectural case studies.

Keywords: Paneling, architectural freeform skin, mold re-use, Weingarten surface, optimization, discrete differential geometry

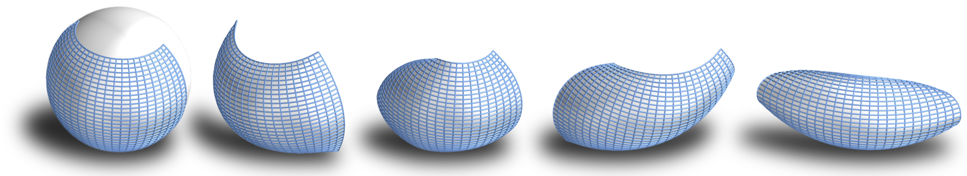

Figure 1: Isometric deformations of a spherical patch. All these surfaces can be cladded by bending panels formed on the same spherical mold. A sample building designed with these surfaces is shown in Fig. 2 (right).

1 Introduction

Paneling is a highly important topic in freeform architecture, especially if the panels are not flat and need to be brought into shape by special manufacturing technologies. For double curved panels, this is mostly done with help of molds. Their fabrication is typically more expensive than the production of the panel with that mold.

A general freeform shape, no matter which layout of panel seams is chosen, will lead to panels all of which are at least slightly different from each other. This lack of repetition in panel shapes is a severe problem and a major factor in cost explosion. One obvious way to deal with this problem is to give up on the smoothness of the overall skin and work with simpler panel shapes, in particular flat ones. A large portion of architectural geometry deals with this problem and provides solutions that have already found their way into architectural practice; for a survey see Pottmann et al. (2015).

Paneling is an optimization problem which has discrete and continuous variables. The discrete variables include the ones which select the type of panel (planar, cylindrical, various types of double curved panels, etc.) and assigns molds to those where molds are needed. This facilitates mold re-use when possible. The continuous variables are those which determine the exact parameters of a panel, which depend on their type, and their exact position in space. A solution for this problem has been presented by Eigensatz et al. (2010): Given a design surface and a layout of panel seams, it minimizes the cost under provided tolerances on the allowed deviation from the design surface and on gaps between panels (which will be hidden in the seams) and kink angles (angles between normals) at adjacent panels. In this way one can find a balance between the quality of the architectural skin and its fabrication cost. By the nature of the optimization problem one has to apply heuristics and thus it is not guaranteed that a minimum is found.

In the present paper we take a slightly different perspective. We aim at special shapes which facilitate mold re-use, but look very much like freeform shapes and should be sufficient in terms of possible shape varieties for the architectural application. This amounts to a search for surfaces where one has a precise or nearly precise agreement of local surface shapes at the size of a panel. This depends on

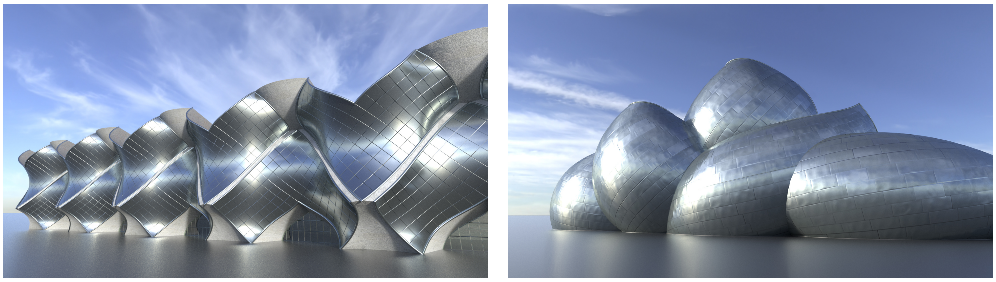

Figure 2: Architectural cladding with intrinsic repetition. On the left, a building composed by a repeated shape that is isometric to a Weingarten surface. For this kind of shapes, if dealing with flexible cladding materials, N panels can be formed with approximately $\sqrt{N}$ molds and applied over the surface by isometric bending. On the right, a building made of surfaces isometric to the same sphere. In this case, all panels can be formed on a single mold and bent on the surface.

the type of material one is using:

If the panel is rigid after production, one needs *local extrinsic repetition*, meaning that there exist many instances of local neighborhoods on the surface which are congruent to each other. An example is provided by surfaces of revolution or helical surfaces. They can be moved in themselves and thus one has this local extrinsic repetition along the trajectories of the generating motion, i.e., along parallel circles or helical paths. Giving up a bit on that, we will argue that surfaces which possess curves along which the curvature behavior is the same (the two principal curvatures are constant along these curves), offer similar advantages for paneling.

If the panel is not rigid and still can be bent within some limits (but not stretched), one can look for *local intrinsic repetition*. This means that certain local neighborhoods of the size of a patch can be matched by an isometric (length-preserving) deformation. Obviously, all surfaces which arise from those with extrinsic repetition by an isometric deformation, are in this class. Here, one gets quite easily shapes that are generated from simple ones, but have a freeform appearance (see Fig. 1). This is due to the human eyes not recognizing intrinsic repetition as well as extrinsic repetition.

1.1 Overview and contributions

- We show that so-called Weingarten surfaces, whose principal curvatures κ_1, κ_2 are related by a function, $F(\kappa_1, \kappa_2) = 0$, offer advantages in cost effective paneling. They possess an extrinsic repetition property, namely for their curvature elements.

- We present an overview of existing, partially very recent contributions to the computation of important classes of Weingarten surfaces.

- We show how to effectively perform an isometric deformation of a given shape using a very recent approach to discrete surface mappings based on quad meshes.

- We accompany our work with examples and design studies and outline promising directions for future research.

2 Basic geometry

2.1 Surfaces with extrinsic repetition

The simplest way of obtaining repetition in panels on a surface is the presence of symmetries. There may be a part of the surface which after applying the present symmetries (e.g. reflections at planes) yields the entire surface. One could call this part the fundamental domain F, which is common terminology in the study of tilings. If the fundamental domain F is covered with m panels and there are k copies of F which make up the overall surface S, then the total number of panels is $N = km$. Since k is usually a small integer, one does not gain too much in this way, as the number m of molds will still be high for a sufficiently complex design. In particular, the more extrinsic symmetries are present, the less the surface S will have a real freeform appearance.

As already mentioned above, there is a case where m and k can be of the same order of magnitude. It happens if the surface allows for a motion in itself. This is the case if S is either a rotational surface, a helical surface or a general cylinder. The latter case is a special single curved surface and does not deserve much attention here, as our focus is on double curved panels. For a rotational or helical surface, $k+1$ positions of a profile curve p (needs not be planar) and the $m+1$ trajectories of $m+1$ points on p determine a curve network with $N = km$ faces. Since panels along trajectories are congruent, this requires only m molds. However, now k and m can be both large. For example, we may have $k = m$ and thus a reduction from N panels to $\sqrt{N}$ molds. Note that so far we would achieve a perfectly smooth skin, but have the disadvantage of a surface which is clearly not freeform.

To come closer to freeform surfaces while keeping some extrinsic repetition of panels, we have to give up a bit on the quality of the resulting surface by allowing small gaps and kinks between adjacent panels. However, there is still a chance to get pretty close to the appearance of a smooth surface. Usually, an architectural skin does not exhibit strong and sudden curvature variations. This means that the curvature element at a point $\mathbf{p}$, which may be represented by the osculating paraboloid (see Pottmann et al. (2007)), will fit well in some neighborhood of $\mathbf{p}$. We make then the assumption that, on architectural surfaces, this neighborhood has approximately the size of a single panel. As we want again a reduction from N to $\sqrt{N}$ molds, we have to make sure that curvature elements agree along curves on the surface. We want then such curves to be a one-parameter family of curves that cover the entire surface. This means that we have just a one-parameter family of different curvature elements, or equivalently, pairs (κ_1, κ_2) of principal curvatures. These pairs may be seen as points in the (κ_1, κ_2)-plane, where they form a curve. A curve has an implicit representation

$$F(\kappa_1, \kappa_2) = 0. \tag{1}$$

Hence, we have a functional relation between the principal curvatures on the surface S. Such surfaces are called *Weingarten surfaces*, named after Julius Weingarten (1861) who studied them first. In fact, his study has been about surfaces with intrinsic repetition, namely those which are isometric to surfaces of revolution. He showed that the focal surfaces (formed by the principal curvature centers) of these surfaces exhibit a relation between their principal curvatures. Hence, extrinsic and intrinsic repetition are closely connected topics.

Let us point out that the agreement of curvature pairs (κ_1, κ_2) happens along the *isolines of curvature*. These are curves along which (κ_1, κ_2) are constant. Due to relation (1), it suffices to require that κ_1 or κ_2 or another function of them (different from F) is constant. Since curvature elements agree along isolines of curvature, panels which can be formed by the same mold are roughly aligned along them (see Fig. 3).

Let us briefly address some familiar classes of Weingarten surfaces. Of course, helical and rotational surfaces are Weingarten surfaces. Although one is usually not concerned so much about relation (1), it could be even prescribed for rotational and helical surfaces mathematically (leading to an ordinary differential equation), but the relation between equation and shape is not intuitive. Another class of Weingarten surfaces are tubes with constant radius r around space curves. There, one principal curvature, say κ_1, equals $1/r$ and thus $F = \kappa_1 - 1/r$. The simplest and most important functions of the principal curvatures are *mean curvature* $H = (\kappa_1 + \kappa_2)/2$ and *Gaussian curvature* $K = \kappa_1\kappa_2$. Surfaces with constant values of H or K have been extensively studied in differential geometry. In particular, we point to minimal surfaces $H = 0$ and developable surfaces $K = 0$.

There is a considerable amount of mathematical research going on studying so called linear Weingarten surfaces. These are surfaces with an (affine) linear relation between the Gaussian and mean curvature (see e.g., Pámpano (2020)).

Particularly interesting for applications are surfaces with a constant ratio c of principal curvatures, i.e., $F(\kappa_1, \kappa_2) = \kappa_1 - c\kappa_2 = 0$. Here, all molds for manufacturing panels are geometrically similar to each other. Additionally, for $c < 0$, such surfaces allow for mounting a curved support structure consisting of bent rectangular strips orthogonally on the surface. This structure follows the network of asymptotic curves with constant intersection angle (see Jimenez et al. (2020), Schling et al. (2018)).

2.2 Surfaces with intrinsic repetition

Let us assume that the panels are not rigid and still allow for some bending without stretching. Then, we can apply isometric deformations to panelizations which have extrinsic repetition and obtain ones with intrinsic repetition. The molds can be the same as for the extrinsically repetitive surface. Since isometric mappings allow for the generation of a large shape variety, one could actually realize very different architectural skins with the same set of molds.

Let us first discuss *isometric mappings* between surfaces. They have the attractive property of preserving all lengths of curves, hence also angles between tangents and areas of domains. In fact, they even preserve the Gaussian curvature K. Hence intrinsic repetition happens along iso-lines of Gaussian curvature.

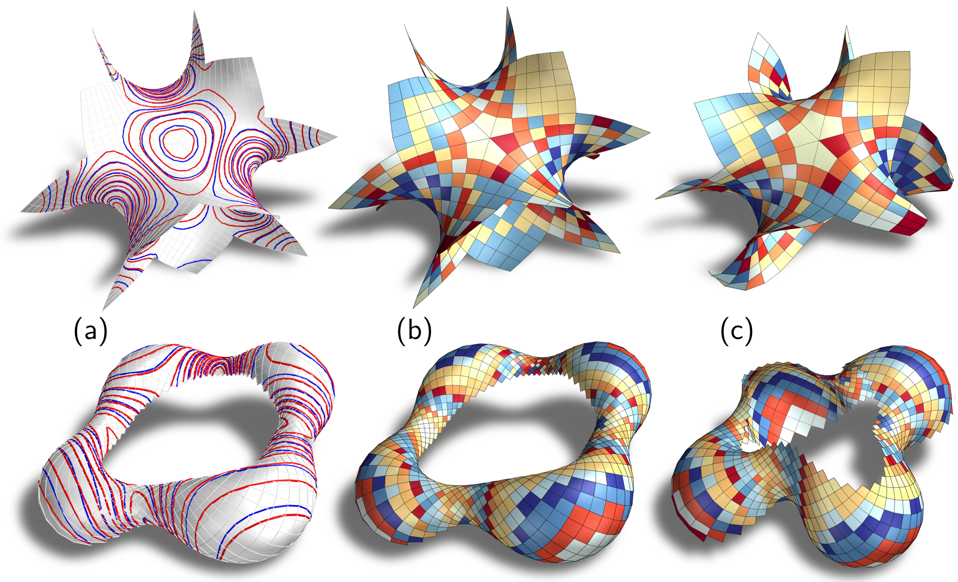

Figure 3: Panels design for mold re-use. (a) Weingarten surfaces designed with (Pellis et al., 2020). Isolines of curvatures κ_1 and κ_2 are shown in red and blue respectively. Coincident isolines layouts indicate that if one of the principal curvatures is constant, so is the other one. (b) Extrinsic repetition. Panels are clustered according to curvature isolines. Panels belonging to the same cluster (shown with the same color) can be formed on the same mold. (c) Intrinsic repetition. The surface is deformed isometrically with (Jiang et al., 2020). If realized with a flexible material, panels clustered on (b) can take their shape on the surface (c) by isometric deformation. Architectural applications are shown in Fig. 8.

If we are fine with an intrinsic counterpart to the Weingarten surfaces discussed above, we simply have to apply isometric mappings to them. This can change their appearance significantly as demonstrated in Fig. 4. It is well known that a rotational surface can be mapped isometrically to a two-parameter family of different rotational surfaces and a one-parameter family of helical surfaces (Bour's theorem). A beautiful constructive proof with help of strip models formed by rotational cones or parts of developable helical surfaces can be found in the first book on discrete differential geometry (difference geometry) by R. Sauer (1970).

3 Algorithms and computational tools

3.1 Computation of Weingarten surfaces

This subsection is an overview of possible approaches to the computation of Weingarten surfaces. It would lead too far to discuss these methods in detail. Note that our focus is on the demonstration of the potential which Weingarten surfaces provide for paneling architectural skins.

Generating Weingarten surfaces by analytical descriptions is a challenging mathematical research topic. However, for applications it is important to get hands on computational tools that enable a designer to work directly with the shape

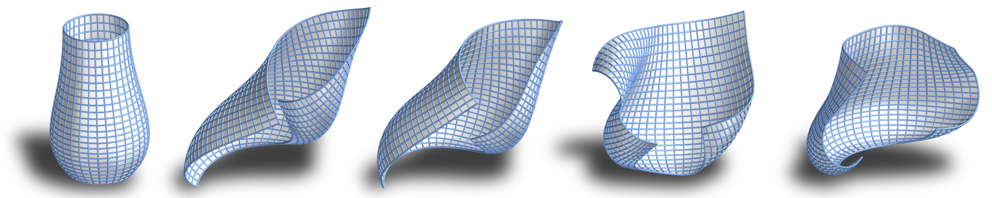

Figure 4: Isometric deformations of a rotational surface with (Wang et al., 2019). We can observe the high design freedom allowed by isometric deformations of a given shape.

incorporating also handle-based editing strategies. To that end, it is advisable to model Weingarten surfaces as discrete nets/meshes which are also well suited for computation by optimization.

Smooth Weingarten surfaces such as minimal surfaces, CMC (constant mean curvature) surfaces, and surfaces with constant Gaussian curvature, on which there is a vast amount of theory, have been discretized in various ways. Discretizations of such surfaces which focus on preserving integrability lie at the heart of structure preserving discrete differential geometry (Bobenko and Suris, 2008).

Robust computation methods of discrete CMC surfaces with fixed given or free boundaries performs computations on a type of Voronoi tesselation (Pan et al., 2012). This method works very well to generate the shape of CMC surfaces, but naturally neglects the mesh layout as part of the design. This however can be very important particularly for architectural applications such as paneling.

Studying methods for modeling developable surfaces, which are also Weingarten surfaces, is an active research topic (Rabinovich et al., 2018a,b; Jiang et al., 2020). Architectural applications reach from famous designs by F. Gehry to cost effective paneling to curved support structures (Schling et al., 2018).

Weingarten surfaces, defined by an affine linear relation $aH + bK - c = 0$ between mean curvature H and Gaussian curvature K have been recently studied by Tellier (2020), both from a computational perspective and with a view towards applications in architecture. We add here their advantage in connection with paneling.

Weingarten surfaces with a linear relation between the principal curvatures have the property that all molds for manufacturing panels are geometrically similar to each other. On the theoretical side these surfaces can be generated as PQ-nets by a Christoffel-type dualization process out of special spherical PQ-nets (Jimenez et al., 2020). More important for applications however are such surfaces in the context of A-nets when these surfaces are negatively curved. These A-nets assume a constant intersection angle of parameter lines along which one can attach a curved support structure (Jimenez et al., 2020). Here the supporting strips sit orthogonally on the surface and can be unfolded into the plane becoming elongated rectangles which makes fabrication by bendable material quite efficient.

Mold re-use with bendable material is also achieved when paneling surfaces that are isometric to a surface of revolution. Discrete models perfectly suitable to model surfaces that are isometric to rotational surfaces are described by discrete orthogonal geodesic coordinates. They utilize the fact that the meridian curves and parallel circles of a surface of revolution constitute special orthogonal geodesic coordinates on the surface (Wang et al., 2019). Handle-based editing allows for modeling surfaces that are isometric to rotational surfaces without knowing the latter.

Another recent approach to the design of Weingarten surfaces, also addressing mold re-use, is to model surfaces by special discrete S-nets (Pellis et al., 2020). S-nets are, apart from singular vertices, regular quadrilateral meshes where each vertex and its four connected neighbors lie on a sphere (see also Schling et al. (2018)). This carries a lot of curvature information of the net. By solving an optimization problem, the discrete principal curvatures can be constrained to fulfill affine linear relations.

3.2 Computing isometric deformations

Isometric or near isometric deformations have received a lot of interest in Geometry Processing and Computer Graphics; see e.g. Chern et al. (2018), Pietroni et al. (2010), Sorkine and Alexa (2007). We use here the probably simplest approach to isometric deformations due to Jiang et al. (2020). It represents the surface to be deformed as a quad mesh S and encodes the isometry condition into the quadrilateral faces. Let $\mathbf{v}_1, \dots, \mathbf{v}_4$ be a quad before deformation and $\mathbf{v}_1', \dots, \mathbf{v}_4'$ its image after deformation (Fig. 5). Isometry requires that (i) the lengths of diagonals in corresponding quads are the same,

$$(\mathbf{v}_1 - \mathbf{v}_3)^2 = (\mathbf{v}_1' - \mathbf{v}_3')^2, \quad (\mathbf{v}_2 - \mathbf{v}_4)^2 = (\mathbf{v}_2' - \mathbf{v}_4')^2, \tag{2}$$

and that (ii) the angle between the diagonals remains unchanged during deformation. In view of Equation (2), this can be expressed as

$$(\mathbf{v}_1 - \mathbf{v}_3) \cdot (\mathbf{v}_2 - \mathbf{v}_4) = (\mathbf{v}_1' - \mathbf{v}_3') \cdot (\mathbf{v}_2' - \mathbf{v}_4'). \tag{3}$$

Hence, one has very simple quadratic constraints which can nicely be satisfied using a Levenberg-Marquardt optimization algorithm (see Jiang et al. (2020)).

3.3 Paneling

The state of the art method of Eigensatz et al. (2010) for computing cost optimal paneling solutions on freeform surfaces relies on a time-consuming discrete optimization algorithm to identify panel repetition, i.e., to find extrinsically similar regions of a reference surface where the same panel can be used. On a Weingarten surface such regions occur along isolines of curvature. This allows us to replace the expensive discrete optimization by a simple clustering step and directly proceed with non-linear optimization to minimize gaps and kink angles between neighboring panels as proposed by Eigensatz et al. (2010).

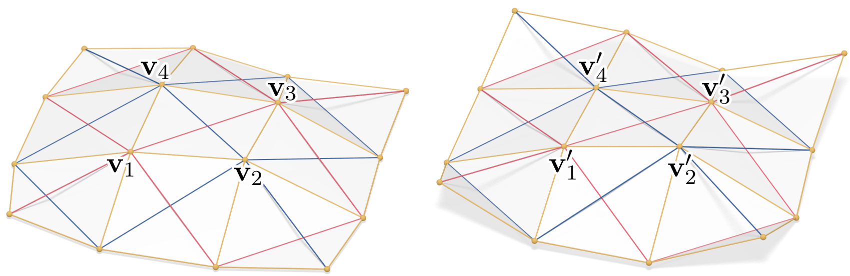

Figure 5: Isometric deformation of a surface represented as a quad mesh (yellow). In each pair of corresponding quads (in general not planar), corresponding diagonals (red, blue) have the same length and they form the same angle.

Given a curve network of panel seams with N faces (each such face has to be covered by a panel) on a reference surface, we cluster the faces according to curvature to form roughly $\sqrt{N}$ clusters, see Section 4.1. Each cluster contains all panels that are manufactured using the same mold. Computationally, panels that stem from the same mold share their shape parameters, for example the coefficients of the defining polynomial when dealing with paraboloids and cubics. We compare this approach with Eigensatz et al. (2010) by tuning the parameters in their algorithm to obtain approximately $\sqrt{N}$ unique molds. In the examples shown in Figures 6 and 7 we restrict the admissible panel types to cubics.

4 Applications

We outline now a possible workflow for the design of freeform shapes with intrinsic and extrinsic panel repetition.

4.1 Design with extrinsic panel repetition

The first step is to design a Weingarten surface, following one of the approaches presented in Section 3.1. In our examples, we modeled such surfaces through a quad mesh with (Pellis et al., 2020). Once we have a suitable shape, a desired panel layout can be designed over the surface. There are no particular restrictions on the layout. Hence, individual panels shall be clustered in groups that can be formed on the same mold. Clustering can be done according to the average of the principal curvatures within each panel. Since on Weingarten surfaces principal curvatures are in functional relation, panel clusters will occur approximately along the curvature isolines (see Fig. 3). Once we have the panel clusters, the shape of each mold can be computed trough optimization as described in Section 3.3.

4.2 Design with intrinsic panel repetition

If the cladding is realized with a flexible material, one can aim at intrinsic repetition. In this case, for shape design, a Weingarten surface can be further modified through isometric deformation. As shown in Figures 1 and 4, isometric deformation allows us significantly greater design freedom. To this end, the method of Jiang et al. (2020) can be used for interactive modeling. A panel layout can then be designed on

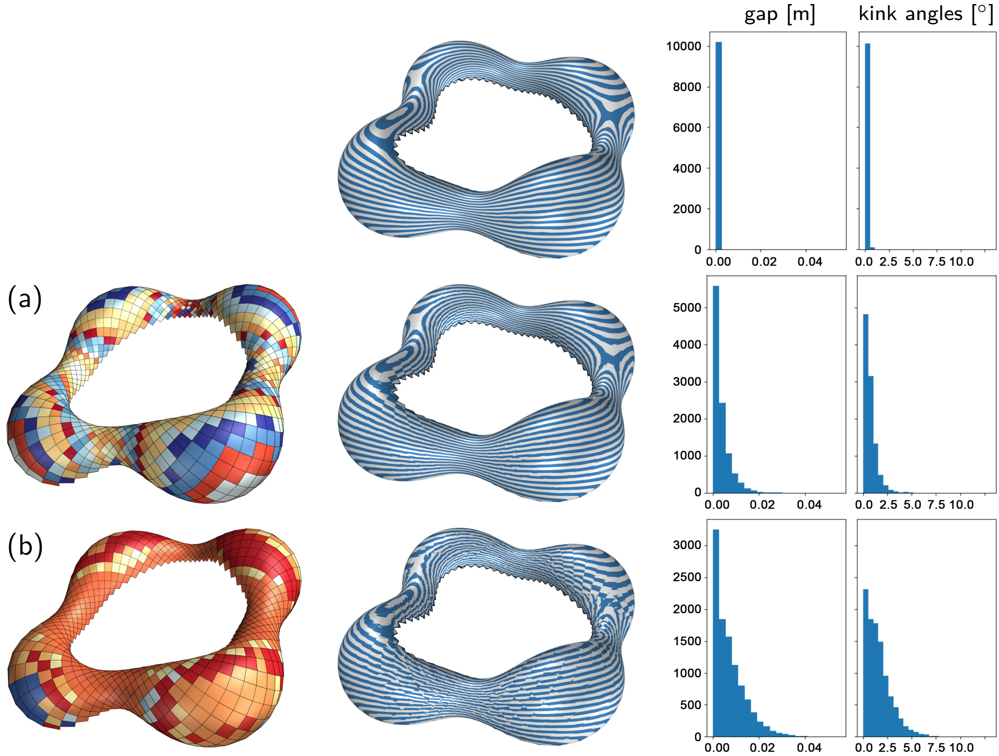

Figure 6: Comparison of paneling solutions on Weingarten surfaces obtained with (a) our method and (b) Eigensatz et al. (2010). The top row shows a solution with 960 unique cubic panels. From the left, the panels clusters and the resulting zebra striping of the panelized surface are shown. The histograms display the corresponding gaps and kink angles between adjacent panels, measured along the network of seam curves at 10216 regularly spaced locations. See Table 1 for further statistical data.

the final shape. Since extrinsic repetition of local shape occurs on the undeformed surface, for clustering and mold design the panel layout shall be mapped back to the starting Weingarten surface. We can then proceed as in Section 4.1. Since made with bendable material, the resulting panels will take their final shape on the design surface by (approximately) isometric deformation.

While the majority of examples in our paper follow a quadrilateral panel layout, this is not necessary, as illustrated by a hexagonal paneling in Figure 9.

It is important to note the following: The panelizations in Figures 9 and 2 are smooth even across panel boundaries, since the original rotational surface (sphere) has a precise extrinsic repetitive structure. This is not true for panelizations of other Weingarten surfaces, whether extrinsic or intrinsic. Depending how well the panelization algorithm outlined in Section 3.3 has performed, there will be kink angles and small gaps of a size so that they can be hidden in the seams.

5 Conclusion and future research

We have proposed Weingarten surfaces as preferable shapes for the design of architectural skins due to their advantage in paneling them. While these surfaces

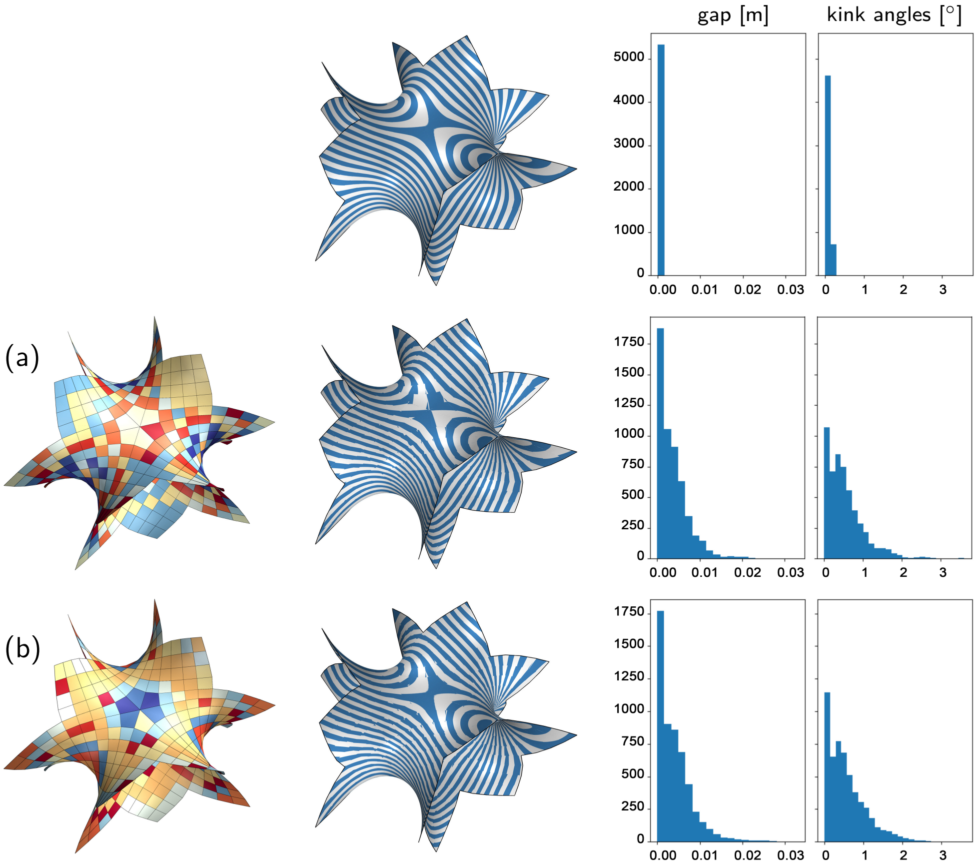

Figure 7: Comparison of paneling solutions on Weingarten surfaces obtained with (a) our method and (b) Eigensatz et al. (2010). The top row shows a solution with 480 unique cubic panels. From the left, the panels clusters and the resulting zebra striping of the panelized surface are shown. The histograms display the corresponding gaps and kink angles between adjacent panels, measured along the network of seam curves at 5336 regularly spaced locations. See Table 1 for further statistical data.

		#molds	#panels	med (max) gap	med (max) kink
Fig. 6	(a)	30	960	0.0023 (0.0544) m	0.549° (12.954°)
	(b)	31	960	0.0055 (0.0445) m	1.305° (8.004°)
Fig. 7	(a)	20	480	0.0029 (0.0280) m	0.438° (3.536°)
	(b)	22	480	0.0033 (0.0322) m	0.454° (2.810°)

Table 1: Divergence (gap) and kink angle analysis for the examples shown in Figures 6 and 7. The respective median values as well as the maximum are listed.

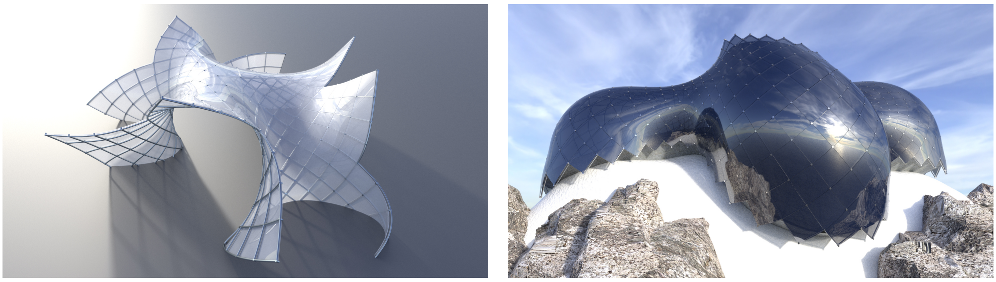

Figure 8: Intrinsic repetition. Architectural skins with panel layouts shown in Fig. 3 (c).

look like freeform shapes, they are repetitive in curvature elements (small surface patches). This yields a significant reduction in the number of molds, namely roughly $\sqrt{N}$ molds for the production of N panels. If one uses panels which after production can still be bent, one can enrich the class of preferred design surfaces by those which are isometric to Weingarten surfaces. They still have the same advantages in terms of mold re-use.

In mathematics, there is ongoing research on Weingarten surfaces, also on discrete models which may be directly useful for architectural applications. On the computational side, it may be very interesting to come up with an algorithm which approximates an arbitrary freeform surface by a Weingarten surface. The functional relation $F(\kappa_1, \kappa_2) = 0$ would not be prescribed, but emerge as a result of an optimization algorithm. That algorithm needed to take as input a surface S which is not Weingarten, which means that the set of principal curvature pairs (κ_1, κ_2) forms a certain domain D in the (κ_1, κ_2)-plane. During optimization, S needed to be modified minimally to a new surface S' whose associated curvature domain D' is a curve or at least very close to a curve. For the paneling application, it may be even better to directly combine this with local surface approximations (of the size of panels) rather than working with curvatures.

The presented approach to paneling with flexible material is more special than required from a purely geometric perspective. One could nicely cover arbitrary freeform surfaces S with panels from flexible material. There, mold repetition should occur roughly along the curves of constant Gaussian curvature of S. However, we are currently lacking a panelization algorithm in the style of Eigensatz et al. (2010), which exploits isometric deformations of panels. The efficient computation of isometries according to Jiang et al. (2020) should make this possible. Since isometric deformations have more degrees of freedom than rigid body motions, the results on arbitrary surfaces with isometrically bent panels could be even better than those for Weingarten surfaces with rigid panels.

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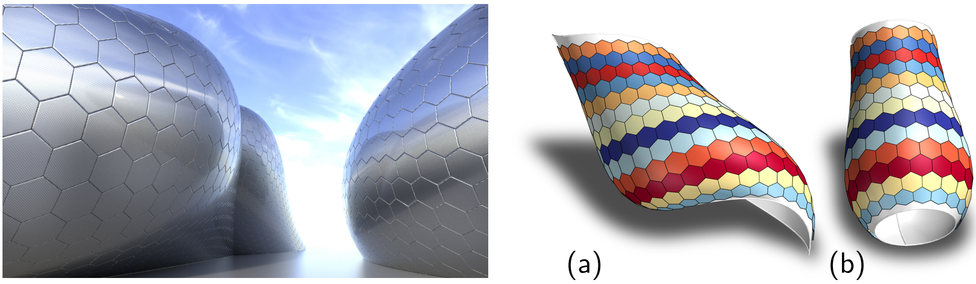

Figure 9: Hexagonal paneling with mold re-use. (a) A hexagonal panel layout designed on a shape isometric to a rotational surface. (b) For panel clustering and mold design, the panel layout shall be mapped back to the corresponding undeformed rotational surface. On the left, we illustrate shapes from Figures 4 and 1, cladded with hexagonal flexible panels.

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References

Bobenko, A. I. and Y. B. Suris (2008). *Discrete differential geometry. Integrable structure*, Volume 98 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.

Chern, A., F. Knöppel, U. Pinkall, and P. Schröder (2018). Shape from metric. *ACM Trans. Graph. 37*(4), 63:1–63:17.

Eigensatz, M., M. Kilian, A. Schiftner, N. J. Mitra, H. Pottmann, and M. Pauly (2010). Paneling architectural freeform surfaces. *ACM Trans. Graph. 29*(4), 45:1–45:10.

Jiang, C., C. Wang, F. Rist, J. Wallner, and H. Pottmann (2020). Quad-mesh based isometric mappings and developable surfaces. *ACM Trans. Graphics 39*(4), 128:1–128:13.

Jimenez, M. R., C. Müller, and H. Pottmann (2020). Discretizations of Surfaces with Constant Ratio of Principal Curvatures. *Discrete Comput. Geom. 63*(3), 670–704.

Pámpano, A. (2020). A variational characterization of profile curves of invariant linear Weingarten surfaces. *Differential Geom. Appl. 68*, 101564, 27.

Pan, H., Y.-K. Choi, Y. Liu, W. Hu, Q. Du, K. Polthier, C. Zhang, and W. Wang (2012). Robust modeling of constant mean curvature surfaces. *ACM Trans. Graph. 31*(4), 85:1–85:11.

Pellis, D., H. Wang, F. Rist, M. Kilian, H. Pottmann, and C. Müller (2020). Principal symmetric meshes. *ACM Trans. Graphics 39*(4), 127:1–127:17.

Pietroni, N., M. Tarini, and P. Cignoni (2010). Almost isometric mesh parameterization through abstract domains. *IEEE Trans. Vis. Comp. Graphics 16*(4), 621–635.

Pottmann, H., A. Asperl, M. Hofer, and A. Kilian (2007). *Architectural geometry*. Bentley Institute Press.

Pottmann, H., M. Eigensatz, A. Vaxman, and J. Wallner (2015). Architectural geometry. *Computers and Graphics 47*, 145–164.

Rabinovich, M., T. Hoffmann, and O. Sorkine-Hornung (2018a). Discrete geodesic nets for modeling developable surfaces. *ACM Trans. Graph. 37*(2), 16:1–16:17.

Rabinovich, M., T. Hoffmann, and O. Sorkine-Hornung (2018b). The shape space of discrete orthogonal geodesic nets. *ACM Trans. Graph. 37*(6), 228:1–228:17.

Sauer, R. (1970). *Differenzengeometrie*. Springer.

Schling, E., M. Kilian, H. Wang, D. Schikore, and H. Pottmann (2018). Design and construction of curved support structures with repetitive parameters. In L. H. et al. (Ed.), *Adv. in Architectural Geometry*, pp. 140–165. Klein Publ. Ltd.

Sorkine, O. and M. Alexa (2007). As-rigid-as-possible surface modeling. In *Proc. Symposium Geometry Processing*, pp. 109–116.

Tellier, X. (2020). *Morphogenesis of curved structural envelopes under fabrication constraints*. Ph. D. thesis, Univ. Paris-Est.

Wang, H., D. Pellis, F. Rist, H. Pottmann, and C. Müller (2019). Discrete geodesic parallel coordinates. *ACM Trans. Graph. 38*(6), 173:1–173:13. Proc. SIGGRAPH Asia.

Weingarten, J. (1861). Über eine Klasse aufeinander abwickelbarer Flächen. *J. reine u. angewandte Mathematik 59*, 382–393.