A Survey on the Notion of Regulus in a Skew Space

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Introduction

These notes take origin from a series of lectures held by Prof. H. Havlicek at Università Cattolica del Sacro Cuore of Brescia (Italy) on June 2002.

The aim of these notes is to give an introduction to some basic topics in Geometry over noncommutative fields and, more in detail, to present the theory of reguli in projective spaces over such fields. As the title may suggest, the style of this work would recall the style of Prof. Havlicek who thought it was better to spend time on comments (the power of human speakers) rather than on technical details (the power of books), and so, writing these pages, we try to give the "ideas" and to underline relevant properties or constructions.

The whole survey will follow the guide of Segre (see [7, 6]) who first deals with these problems, adding to his foundamental work a bit of innovation and presenting the last evolutions of his ideas. In particular:

in **Section 1** we give the definition of vector space over a skew field and we present some examples with the aim of underlining the main differences between the skew and the commutative case, intending to give a short introduction to the reader that (as I was when I take part at the lectures) is new to these topics.

In **Section 2** we recall some important algebric results on skew fields, but trying not to loose the geometric point of view.

Section 3 is explicitely dedicated to present some geometric properties of vector spaces over skew fields, and another time the aim is to underline differences with the commutative case. In particular we deal with projective frames and cross-ratios.

With **Section 4** starts the most geometric part of these notes: we introduce the definition of a *regulus* in a 3-dimensional projective space and analyse the main properties of this object, in particular, the intersection of a regulus and a line. We present also briefly some ways to generalize the definition of regulus in higher dimensional spaces.

Section 5, finally, deals with planar sections of reguli. We distinguish between sections with planes that contains a generator and planes that do not, in the first case giving the definition of a *C-configuration*, in the second of a *conic*, and looking for their properties.

Before starting special thanks go to Hans Havlicek for his enthusiatic and attractive lectures of course, but also for his help with these notes, that passed through his clever pen. Many thanks also to Andrea Blunck, who read and corrected these pages during her stay in Brescia on November 2002 and helped with many useful observations and explanations, and to Silvia Pianta, who, in some sense, coordinated all this stuff from the backstage and, with great patience, taught me that a regulus is "made up of these lines" and not "made with this lines".

1 Vector spaces over skew fields

1.1 Definition. *Let F be a field (not necessarily a commutative one) and V a nonempty set. We say that V is a* right vector space *over the field F if:*

1. *On V there is a binary operation "$+$" such that $(V, +)$ is a group[1];*
2. *there exists a map*

$$\cdot : \begin{cases} V \times F & \longrightarrow & V \\ (v, \alpha) & \longmapsto & v \cdot \alpha \end{cases}$$

such that, for all $\alpha, \beta \in F$ and for all $u, v \in V$ the following conditions are fulfilled:

(a) $(u + v) \cdot \alpha = u \cdot \alpha + v \cdot \alpha$

[1] It is in fact not necessesary to demand that the group is abelian, we can obtain this by calculating $(u + v)(1 + 1)$ in two different ways.

(b) $v \cdot (\alpha + \beta) = v \cdot \alpha + v \cdot \beta$
(c) $(v \cdot \alpha) \cdot \beta = v \cdot (\alpha\beta)$
(d) $v \cdot 1 = v$

In a similar way we say V is a left vector space *over F if there are fulfilled properties* 1. *and*

2'. there exists a map

$$\cdot : \begin{cases} F \times V & \longrightarrow & V \\ (\alpha, v) & \longmapsto & \alpha \cdot v \end{cases}$$

such that, for all $\alpha, \beta \in F$ and for all $u, v \in V$ the following conditions are fulfilled:

(a) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$
(b) $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$
(c) $(\alpha\beta) \cdot v = \alpha \cdot (\beta \cdot v)$
(d) $1 \cdot v = v$

The notions of right and left vector space are a natural generalization of that of vector space to the skew field case, about which these notes deal. We assume a field F to be either a skew field or a commutative field and, if not specified, a vector space to be a right vector space over F. We omit to write the multiplication sign for external multiplication.

It is important to observe that, even if the definitions of right and left vector space are very similar, the structures obtained in the two cases are not, as to say, "interchangeable", in the sense that a right vector space cannot be naturally changed into a left one over the same skew field F, and, even more, when this is possible, the two structures obtained have deeply different behaviours, as will be shown in the following examples (see in particular example 1.3).

1.2 Example. Let F be a field and $n \in \mathbb{N} \setminus \{ 0 \}$; the set F^n can be naturally equipped with the structure of right or left vector space over itself through the applications:

$$\cdot_r : \begin{cases} F^n \times F & \longrightarrow & F^n \\ \big((\alpha_1, \ldots, \alpha_n), \beta\big) & \longmapsto & (\alpha_1\beta, \ldots, \alpha_n\beta) \end{cases}$$

and

$$\cdot_l : \begin{cases} F \times F^n & \longrightarrow & F^n \\ \big(\beta, (\alpha_1, \ldots, \alpha_n)\big) & \longmapsto & (\beta\alpha_1, \ldots, \beta\alpha_n) \end{cases}$$

1.3 Example. Let $\mathbb{H}$ be a four-dimensional vector space over the commutative field $\mathbb{R}$ of real numbers and let $\{ 1, i, j, k \}$ be a basis of $\mathbb{H}$. Let us consider the linear application $\cdot : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$ obtained as the linear extension of the map defined by the following action on the vectors of the basis of $\mathbb{H}$:

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

It is easy to check that $\mathbb{H}$ with this "multiplication" is a field, and in particular a non commutative one, considering for instance that

$$\begin{aligned} &i \cdot j = k \\ &j \cdot i = -k \end{aligned}$$

$(\mathbb{H}, +, \cdot)$ is one of the most classical examples of skew field and is called the *real quaternion field* (or the *real quaternion algebra*).
Let us now consider on $\mathbb{H}^2$ the structures of right and left vector space over $\mathbb{H}$ (as in 1.2). It is immediate to check that some properties deeply bound with the vector space structure, such as linear dependence and independece of vectors, are different in these two structures. Let us consider, for instance, the vectors $(1, i)$ and (j, k); these vectors are linearly dependent if considered as vectors of the *right* vector space $\mathbb{H}^2$, in fact:

$$(1, i)j = (j, ij) = (j, k)$$

while it is easy to see that they are independent in the *left* vector space $\mathbb{H}^2$:

$$x(1, i) = (x, xi) = (j, k) \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcl} x & = & j \\ xi & = & k \end{array} \right.$$

but $ji = -k \neq k$.

1.4 Definition. *Let V and W be two right vector spaces on the same field F. A map $f : V \longrightarrow W$ is a* homomorphism *if, for all u, v in V and for all α in F, the following properties are fulfilled:*

1. $f(u + v) = f(u) + f(v)$
2. $f(v\alpha) = f(v)\alpha.$

We will denote by $\mathrm{Hom}_F(V, W)$ *the set of all homomorphisms between the right vector spaces* V *and* W.

1.5 Example. Let V be a right vector space over F. We have already observed in 1.2 that F can be thought in a natural way both as right and left vector space over itself, and so we can consider

$$V^\wedge := \mathrm{Hom}_F(V, F).$$

For all $v^\wedge \in V^\wedge$, we will denote its action on a vector u of V by $\langle\, v^\wedge, u \,\rangle \in F$. $V^\wedge$ can be equipped with the structure of *left* vector space over F through the following maps: $\forall\, a^\wedge, b^\wedge \in V^\wedge$, $\forall\, v \in V$, $\forall\, \alpha \in F$

$$\begin{aligned}
&\langle\, a^\wedge + b^\wedge, v \,\rangle := \langle\, a^\wedge, v \,\rangle + \langle\, b^\wedge, v \,\rangle \\
&\langle\, \alpha a^\wedge, v \,\rangle := \alpha \langle\, a^\wedge, v \,\rangle.
\end{aligned}$$

The choice of considering $V^\wedge$ as *left* vector space is justifyed by necessity to guarantee the linearity of the multiplication by a scalar, in fact for all $a^\wedge \in V^\wedge$, $v \in V$, $\alpha, \beta \in F$, if we define:

$$\langle\, a^\wedge \alpha, v \,\rangle := \langle\, a^\wedge, v \,\rangle \alpha$$

we have

$$\begin{aligned}
&\langle\, a^\wedge \alpha, v\beta \,\rangle = \langle\, a^\wedge, v\beta \,\rangle \alpha = \langle\, a^\wedge, v \,\rangle \beta\alpha \\
&\langle\, a^\wedge \alpha, v\beta \,\rangle = \langle\, a^\wedge \alpha, v \,\rangle \beta = \langle\, a^\wedge, v \,\rangle \alpha\beta
\end{aligned}$$

and, in general, these two expressions are different. Neither the idea of defining

$$\langle\, a^\wedge \alpha, v \,\rangle := \alpha \langle\, a^\wedge, v \,\rangle$$

works fine, in fact in this case we have

$$\begin{aligned}
&\langle\, a^\wedge (\alpha\beta), v \,\rangle = \alpha\beta \langle\, a^\wedge, v \,\rangle \\
&\langle\, (a^\wedge \alpha)\beta, v \,\rangle = \beta \langle\, a^\wedge \alpha, v \,\rangle = \beta\alpha \langle\, a^\wedge, v \,\rangle
\end{aligned}$$

and also these two expressions are, in general, different.

The vector space $V^\wedge$ is called the *dual vector space* of V. As usual, if $\{\, e_1, \ldots, e_n \,\}$ is a basis of V the maps

$$e_i^\wedge : \begin{cases} V & \longrightarrow \quad F \\ \sum_{j=1}^{n} e_j \alpha_j & \longmapsto \quad \alpha_i \end{cases} \qquad i = 1, \ldots, n$$

belong to $V^\wedge$, are linearly independent and form a basis for $V^\wedge$.

Other differences in comparison with the commutative case, can be found also dealing with matrices. In particular we can no more define only one notion of rank of a matrix, but it is necessesary to distinguish between the maximum number of linear independent rows (*row rank*) and columns (*column rank*), and moreover between right linear independence (*right rank*) and left linear independence (*left rank*). These numbers in general can be different, as we can see easly considering in $\mathrm{Mat}_2(\mathbb{H})$ the matrix

$$\begin{pmatrix} 1 & i \\ j & k \end{pmatrix}$$

For the observations we have already done in 1.3 this matrix has right row rank 1, but left row rank 2. More in detail it is possible to prove that the following equalities hold:

$$\begin{aligned} \text{left row rank} &= \text{right column rank} \\ \text{right row rank} &= \text{left column rank} \end{aligned}$$

Other differences arise also when we search for eigenvalues and eigenvectors of an endomorphism f of a vector space V. So if $f \in \mathrm{End}_F(V) := \mathrm{Hom}_F(V, V)$ and $v \in V \setminus \{ 0 \}$ and $\lambda \in F$ are respectively an eigenvector of f and its eigenvalue, we have, for all $\alpha \in F \setminus \{ 0 \}$

$$f(v\alpha) = f(v)\alpha = (v\lambda)\alpha = (v\alpha)\alpha^{-1}\lambda\alpha$$

and so an eigenvalue is defined up to conjugation by elements of F. Moreover, if now g is an automorphism of V, λ is an eigenvalue also for the map $g^{-1}fg$ and the corresponding eigenvectors are $g^{-1}(v)$ for all eigenvectors v corresponding to λ in respect of f, in fact we have:

$$g^{-1}fg\left(g^{-1}(v)\right) = g^{-1}f(v) = g^{-1}(v\lambda) = g^{-1}(v)\lambda.$$

Let us conclude this short survey on analogies and differences between vector spaces on commutative fields and skew fields with another example of more geometrical nature, which involves projective geometries arising from vector spaces and the fundamental theorem on projectivities. In particular, while in the commutative case it is well known that in an n-dimensional projective space a projectivity that fixes $n+2$ points in general position is the identity, this property cannot be extended painless to the non commutative case. We will check this on a 1-dimensional example in 1.14, but before doing this we recall briefly what a projective space over a skew field is, and its main properties (a more detailed presentation can be found in [3]).

Let us consider a right vector space V of dimension $n+1$ over the skew-field F and define the following map:

$$\pi : \begin{cases} V & \longrightarrow & V/F^* \\ v & \longmapsto & vF^* \end{cases}$$

Note that the elements of V/F^* are defined up to a *right* proportional factor and that π is surjective.

1.6 Definition. *We call* $V^*/F^* = \pi(V^*)$ *the* right projective space *of dimension* n *over* F *associated to* V*, and we denote it by* $PG(V)$*. If* X *is a subspace of* V *of dimension* $m+1$ *we say that* $\pi(X^*)$ *is a* projective subspace *of* $PG(V)$ *of dimension* m*. As usual we call* point, line, plane *and* hyperplane *any projective subspace of* $PG(V)$ *of dimension 0, 1, 2 and* $n-1$ *respectively. If* $V = F^{n+1}$*, then we denote* $PG(V)$ *by* $PG(n, F)$*.*

Many of the "classical" definitions can be extended to projective geometries over skew fields, and many properties of projective spaces are still true; in particular:

1.7 Definition. *A* projective frame *of* $PG(V)$ *is an ordered set of* $n+2$ *points such that no* $n+1$ *are contained in a hyperplane.*

1.8 Theorem. *Let* V *be a right vector space of dimension* $n+1$ *over* F *and let* $\{v_1, \ldots, v_{n+1}\}$ *be a basis for* V*. Then*

$$\{\ \pi(v_1), \ldots, \pi(v_{n+1}), \pi(v_1 + \ldots + v_{n+1})\ \}$$

is a projective frame for $PG(V)$*. Viceversa if* $\{\ e_1, \ldots, e_{n+1}, u\ \}$ *is a projective frame of* $PG(V)$*, then there exists a basis* $\{\ v_1, \ldots, v_{n+1}\ \}$ *of* V *such that* $e_1 = \pi(v_1), \ldots,\ e_{n+1} = \pi(v_{n+1}),\ u = \pi(v_1 + \ldots + v_{n+1})$*.*

1.9 Definition. *An* isomorphism *between two projective spaces* $PG(V)$ *and* $PG(W)$ *over the same field* F *is a one-to-one mapping* f *of the subspaces of* $PG(V)$ *onto the subspaces of* $PG(W)$ *which preserves the containment relation. If* $PG(V) = PG(W)$ *then* f *is called a* collineation *or an* automorphism*; we denote the set of all collineations of* $PG(V)$ *by* $P\Gamma L(V)$*.*

Note that an isomorphism between $PG(V)$ and $PG(W)$ can exist only if V and W have the same dimension.

1.10 Theorem. *If* V *and* W *are two vector spaces over* F *of dimension at least 2, then any isomorphism of the projective space* $PG(V)$ *onto the* *projective space $PG(W)$ is induced by a semilinear transformation of V onto W.*

1.11 Corollary (Foundamental Theorem of Projective Geometry). *If V is a vector space over F of dimension at least 2, then the group of all collineation of the projective space $PG(V)$ is induced by the group of all non-singular semilinear transformations of V onto itself.*

1.12 Definition. *We call* projectivity *an isomorphism of $PG(V)$ onto $PG(W)$ which is induced by a linear transformation of V onto W. We denote the group of all projectivities of $PG(V)$ onto itself by $PGL(V)$.*

1.13 Theorem. *Let V be a right vector space of dimension $n+1$ over F and denote by $\Gamma L(V)$ the group of all semilinear transformations of V onto itself and by*

$$N := \{ \gamma \in \Gamma L(V) \mid \pi(\gamma(v)) = \pi(v) \text{ for all } v \in V \}.$$

Then $P\Gamma L(V) \cong \Gamma L(V)/N$ and $PGL(V) \cong GL(V)/N \cap GL(V)$.

This last theorem shows in particular that any projectivity of a projective space $PG(V)$ onto itself can be represented, as in commutative case, by a $n+1$ square matrix ($n \geqslant 2$), defined up to a right proportional factor chosen in the center of F.

We are now ready to show the announced example:

1.14 Example. Let f be a projectivity of the projective line $PG(1,F)$ onto itself and, for the sake of simplicity, let us suppose that f fixes the three points $(1,0)F^*$, $(0,1)F^*$ and $(1,1)F^*$ of a projective frame; as in the commutative case the matrix that represents the map f is, up to a proportional factor, of the form:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \qquad \text{where} \qquad \alpha \in F \setminus \{ 0 \}.$$

Let us now distinguish between two cases.
If α belongs to the center of F

$$Z(F) := \{ \alpha \in F \mid \forall\, \beta \in F : \ \alpha\beta = \beta\alpha \}$$

we have, for all points $(x,y)F^*$,

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} = \begin{pmatrix} x\alpha \\ y\alpha \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \alpha$$

and so $f = \mathrm{id}$.
If $\alpha \notin Z(F)$ there exists $\beta \in F$ such that $\alpha\beta \neq \beta\alpha$. Let us consider

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha\beta \end{pmatrix}$$

The vector $\begin{pmatrix} \alpha \\ \alpha\beta \end{pmatrix}$ cannot be proportional to $\begin{pmatrix} 1 \\ \beta \end{pmatrix}$ on the right side, because an element x of F that achieves this proportionality would have to fulfil

$$\begin{cases} \alpha = x \\ \alpha\beta = \beta x \end{cases}$$

while this is impossible, and so the application f cannot be the identity.

2 Some algebraic properties of skew fields

In this section F will denote a field and $Z(F)$ its *center*, i.e. the set

$$Z(F) := \{ \, a \in F \mid ab = ba \quad \forall\, b \in F \, \} \, .$$

It is easy to check that $Z(F)$ is a commutative subfield[2] of F, so F can always be equipped with the structure of vector space over its center.

2.1 Example. Let $\mathbb{H}$ be the algebra of real quaternions as in 1.3. Its center $Z(\mathbb{H})$ is the field $\mathbb{R}$ of real numbers and the dimension of $\mathbb{H}$ as vector space over $\mathbb{R}$ is $[\mathbb{H} : \mathbb{R}] = 4$.

Let now $t \in F$ be a fixed element of the field. We will denote by $Z_F(t)$ the *centralizer* in F of the element t, i.e. the set

$$Z_F(t) := \{ \, a \in F \mid at = ta \, \} \, .$$

This set is still a subfield of F, but, in general, it is not commutative, so it is still possible to consider F as a vector space over $Z_F(t)$, but now it is necessesary to distinguish between right vector space and left vector space; we will denote their dimensions respectively with $[F : Z_F(t)]_r$ and $[F : Z_F(t)]_l$, remembering that, in general, they can be different.

We will now state the principal results concerning the characters till now introduced.

[2] We will denote substructures with the symbol "$\leqslant$", so $Z(F) \leqslant F$.

2.2 Theorem (Centralizer Theorem). *Let F be a field, $Z(F)$ its center, C an intermediate subfield:*

$$Z(F) \leqslant C \leqslant F$$

and $Z_F(C)$ the set

$$Z_F(C) := \{ \, a \in F \mid ac = ca \quad \forall \, c \in C \, \} = \bigcap_{c \in C} Z_F(c)$$

If at least one of the dimensions $[C : Z(F)]$, $[F : Z_F(C)]_r$ or $[F : Z_F(C)]_l$ is finite, then all are finite and, moreover,

$$[C : Z(F)] = [F : Z_F(C)]_r = [F : Z_F(C)]_l.$$

This result (a proof can be found in [1, p. 49]) suggests that trying to find out examples in which the left and right dimensions of F over the centralizer of an element are distinct is quite difficult, in fact in all classical examples the hypothesis of finiteness of theorem 2.2 are always fulfilled.

2.3 Theorem. *Let F be an infinite field; then for all t in F we have (see [2])*

$$|Z_F(t)| = \infty.$$

2.4 Theorem (Skolem-Noether Theorem). *Let F be a non commutative field, E a proper subfield of F such that*

$$Z(F) < E < F \quad \textit{and}^3 \quad [E : Z(F)] < \infty$$

and let $\varphi : E \longrightarrow E$ be an automorphism of E such that $\varphi_{|_{Z(F)}} = \mathrm{id}$. Then there exists a $c \in F$ different from zero such that

$$\forall \, x \in E : \quad \varphi(x) = c^{-1}xc = x^c$$

and so φ is the restriction to E of an inner automorphism of F.

A proof of the previous result can be found in [5].

2.5 Example. Let us consider, as usual, the skew field of real quaternions $\mathbb{H}$, let us denote by $\mathbb{C}$ the subfield spanned by $\{ \, 1, i \, \}$ and consider the automorphism of $\mathbb{C}$ given by the conjugation of complex numbers: $\varphi(1) = 1$ and $\varphi(i) = -i$. We have $Z(\mathbb{H}) = \mathbb{R}$ and $[\mathbb{C} : \mathbb{R}] = 2$, so the hypothesis of theorem 2.4 are fulfilled and it turns out $c = j$, in fact

$$i^j = j^{-1}ij = j^{-1}k = -i.$$

[3] Let us observe that it is not important to specify if E is a right vector space or a left one because the subfield $Z(F)$ is commutative.

2.6 Theorem. *Let F be a field, $t \in F$ and let us denote by t^F the conjugacy class of t:*

$$t^F := \{ c^{-1}tc \mid c \in F \setminus \{ 0 \} \}.$$

Then we have that $\left|t^F\right| = 1$ if $t \in Z(F)$, $\left|t^F\right| = \infty$ otherwise.

Proof. If $t \in Z(F)$, for all $c \in F \setminus \{ 0 \}$ we have $c^{-1}tc = t$, and so $t^F = \{ t \}$ and $\left|t^F\right| = 1$.
Let now be $t \notin Z(F)$ and let c, d belong to $F \setminus \{ 0 \}$. It holds:

$$\begin{aligned} t^c = t^d \quad &\Longleftrightarrow \quad c^{-1}tc = d^{-1}td \quad \Longleftrightarrow \quad dc^{-1}tcd^{-1} = t \quad \Longleftrightarrow \\ &\Longleftrightarrow \quad cd^{-1} \in Z_F(t)^* \quad \Longleftrightarrow \quad c \in Z_F(t)^* d \end{aligned}$$

and so, considering F as a left vector space over the centralizer $Z_F(t)$ and building the corresponding (left) projective space, we can obtain that c and d are representatives of the same point of the projective space if and only if they determine the same conjugate element of t. Since $t \notin Z(F)$ there exists at least an element of F that does not commutate with t, and so that does not belong to $Z_F(t)$, so $[F : Z_F(t)] \geqslant 2$; this implies that in the projective space introduced above there is (at least) a line[4]. The number of distinct elements of F conjugate with t is therefore not smaller than the number of distinct points of a line of the projective space over the field $Z_F(t)$, and so, remebering theorem 2.3,

$$\left|t^F\right| \geqslant |Z_F(t)| + 1 = \infty. \qquad \square$$

2.7 Remark. In the proof of the last theorem we have underlined a meaningful property of conjugacy classes of elements $t \in F \setminus Z(F)$: the elements of t^F can be put in bijective correspondence with the points of a left projective space over the centralizer $Z_F(t)$.

We conclude this section with two well known theorems:

2.8 Theorem (Gordon-Motzkin Theorem). *Let F be a skew field and $p(x)$ be a polynomial of degree n of the form*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

The roots of $p(x)$ belong to at most n conjucacy classes of F. Moreover if two distinct roots of $p(x)$ belong to the same conjugacy class C, then there are infinitely many elements of C that are roots of $p(x)$.

[4] In the worst case it is a line itself.

2.9 Theorem (Wedderburn's Theorem). *Let F be a finite field; then F is commutative.*

A proof of the Gordon-Motzkin Theorem and of the Wedderburn's Theorem can be found in [4].

3 Some geometric properties of skew fields

Let us start dealing with the problem of coordinatization of a projective space on a skew field, in particular concerning with changing the canonical representants for a projective frame. In the commutative case this, of course, leaves all coordinates unchanged, but is this still true in the non commutative case?

Let V be a right vector space of finite dimension $k+1$ over the skew field F. Let us consider the projective space $PG(k, F)$ and let

$$\begin{aligned} e_1 &= (1,0,0,\ldots,0)F^* \\ e_2 &= (0,1,0,\ldots,0)F^* \\ &\ldots \\ e_{k+1} &= (0,0,\ldots,0,1)F^* \\ u &= (1,1,\ldots,1)F^* \end{aligned}$$

be a projective frame in $PG(k, F)$. In such a frame a point p has coordinates $(x_1, x_2, \ldots, x_{k+1})F^*$ where $x_i \in F,\ i = 1, \ldots, k+1$. All points of $PG(k, F)$ have coordinates defined up to a (right) proportional factor, so, as we said, we can change the canonical representants for our projective frame without changing the projective structure, i.e. obtaining for $e_1, \ldots, e_{k+1}, u$:

$$\begin{aligned} e_1 &= (c,0,0,\ldots,0)F^* \\ e_2 &= (0,c,0,\ldots,0)F^* \\ &\ldots \\ e_{k+1} &= (0,0,\ldots,0,c)F^* \\ u &= (c,c,\ldots,c)F^* \end{aligned}$$

where c lies in F^*. To determine the coordinates of the point p with respect to this new vectors of V we have to observe that:

$$\begin{aligned} (x_1, \ldots, x_{k+1}) = \left(cc^{-1}x_1, \ldots, cc^{-1}x_{k+1}\right) = \\ =& (c,0,\ldots,0)c^{-1}x_1 + \ldots + (0,\ldots,0,c)c^{-1}x_{k+1} \end{aligned}$$

so the coordinates of p are:

$$p = (x_1, \ldots, x_{k+1})F^* = \left(c^{-1}x_1c, \ldots, c^{-1}x_{k+1}c\right) F^*$$

and so they change through an inner automorphism of F.

Let us now consider a projective line over F, let us fix a projective frame e_1, e_2, u and a fourth point p that has in this frame coordinates $p = (x, y)F^*$. We all know the cross-ratio of the four points e_1, e_2, u, p is the number $t := xy^{-1}$ if $p \neq e_1$, $t := \infty$ if $p = e_1$. Remembering now the previous observations on the coordinatization, if we change the representants of the projective frame, we have that the cross-ratio of e_1, e_2, u, p is $t' = \left(c^{-1}xc\right)\left(c^{-1}y^{-1}c\right) = c^{-1}tc$, so we must conclude that also $c^{-1}tc$ for $c \in F^*$ can be the cross-ratio for p: if we want to have a consistent definition we must define the cross-ratio of the four points e_1, e_2, u, p as the whole conjugacy class of t, that we have denoted by t^F.

3.1 Definition. *Provided three distinct points e_1, e_2, u of the projective line $PG(1, F)$ and a fourth point p we define the* cross-ratio *of the four points e_1, e_2, u, p to be the conjugacy class of any element t of F that is the ratio between the first and the second projective coordinate of p in the projective frame that has e_1, e_2 as base points and u as unit point; we agree to say that the cross-ratio is infinite when the second coordinate of p is* 0.

Viceversa, if $t \notin Z(F)$, we can have many points (all conjugate with p) that have cross-ratio t^F: we can determine them by considering all projectivities that fix e_1, e_2, u and taking the orbit of p under the action of this group. So all points of $PG(1, F)$ are identified up to an inner automorphism by their cross-ratio with three distinct points e_1, e_2, u, and so the cross-ratio of four points is still a projective invariant.

The projective line can be divided into those point that have cross-ratio in the center of F (in fact those points that are fixed by all projectivities that fix the projective frame) and those points that have cross-ratio $t \notin Z(F)$, grouped into conjugacy classes that, for theorem 2.6, have infinitely many elements. If we now take three distinct points e_1', e_2', u' with cross-ratio (with respect to e_1, e_2, u) in the center, the cross-ratios t and t' of a point p with respect to the projective frames defined by e_1, e_2, u and, respectively, e_1', e_2', u' are bound by the relation:

$$t' = \frac{at + b}{ct + d}$$

where $a, b, c, d \in Z(F)$ and $ad - bc \neq 0$. This means that points that have cross-ratio t with respect to e_1, e_2, u are those and only those that have cross-ratio t' with respect to e'_1, e'_2, u', i.e. if we replace e_1, e_2, u with three distinct points with cross-ratio in $Z(F)$ we obtain the same structure of conjugacy classes on $PG(1, F)$.

4 Reguli over a skew field

Let F be a field and $PG(3, F)$ be the right projective space over F.

4.1 Definition. *Let D_1, D_2, D_3 be three pairwise skew lines; a* regulus *is the set of all lines meeting all the lines D_1, D_2, D_3. The lines of the regulus are called* generators *while the lines D_1, D_2, D_3, together with all lines of $PG(3, F)$ that meets all generators, are called* directrices.

4.2 Remark. A regulus can be obtained also by considering two skew lines D_1 and D_2 and a projectivity π between them: the lines of the regulus are those lines that join a point p of D_1 with its image $\pi(p)$ under the action of π. In fact, if we take a projectivity π of D_1 onto D_2, there always exists a third line D_3 such that the pencil of planes with axis D_3 cuts D_1 and D_2 in corresponding points, and so lines joining corresponding points form a regulus of directrices D_1, D_2 and D_3. To see this note that the lines D_1 and D_2 are 2-dimensional complementary subspaces of a 4-dimensional right vector space V over F, and the projectivity π, remembering theorem 1.11, is induced by a linear map f from D_1 to D_2. If we now fix a basis $\{ v_1, v_2 \}$ in D_1 and define $v_3 := f(v_1)$ and $v_4 := f(v_2)$ the set $\{ v_3, v_4 \}$ is a basis of D_2 and f is given by

$$f : (x_1, x_2, 0, 0) \longmapsto (0, 0, x_1, x_2).$$

Now let D_3 be the line

$$D_3 := \{ (x_1, x_2, x_1, x_2) \in V \mid x_1, x_2 \in F \}.$$

Each vector on this line splits naturally into

$$(x_1, x_2, x_1, x_2) = (x_1, x_2, 0, 0) + (0, 0, x_1, x_2)$$

and this shows that every line that joins corresponding points of the projectivity also meets D_3.

It is well known that through every point on one of the directrices there exists a line of the regulus and moreover, in the commutative case, the roles played by directrices and generators are interchangeable, in the sense that

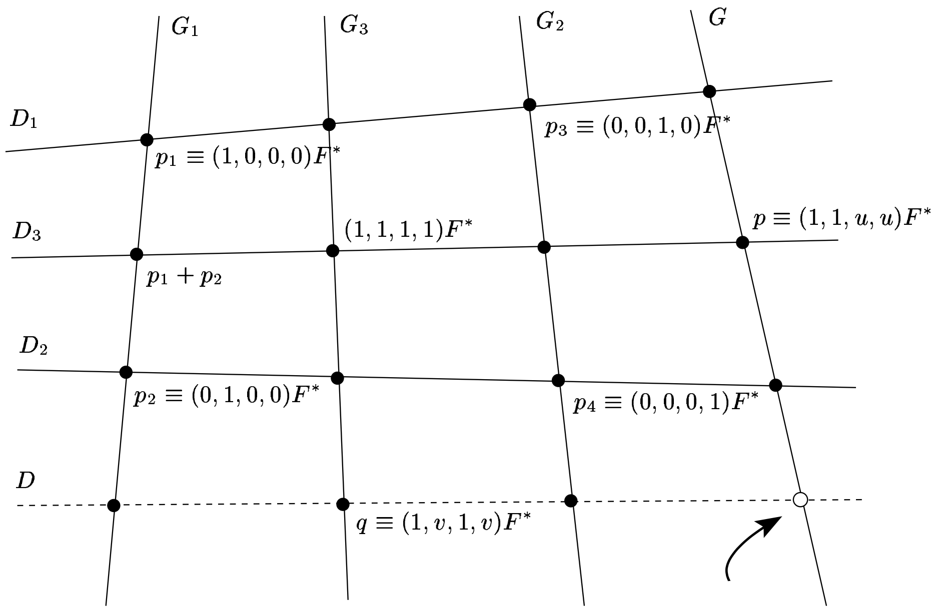

Figure 4.1: *Generators and directrices of a regulus.*

if we take three distinct generators G_1, G_2, G_3 and we consider the regulus defined by these lines, this regulus is independent on the lines we choose. This means that the three lines D_1, D_2 and D_3 belong to this regulus and for every point on one of the G_1, G_2, G_3 there exists a (unique) line of this new regulus (and so a directrix of the old one) through this point. This situation is no longer true in the skew case, where depending on the three lines G_1, G_2, G_3 we choose in general we obtain many reguli, all sharing the three starting lines D_1, D_2 and D_3. This in particular means that if we take any point on a generator in general a line through this point that meets all generators (and so a directrix) does not exist. We will achieve this result by proving the following statement:

4.3 Theorem. *The directrices of a regulus over F are the generators of a regulus of $PG(3, Z(F))$, considered as a subspace of $PG(3, F)$.*

Proof. Let D_1, D_2, D_3 be three distinct skew lines and G_1, G_2, G_3 be three distinct generators of the regulus having D_1, D_2 and D_3 as directrices. Without loss of generality we can fix the points of a projective frame in such a way that the four base points are, in the order, the points $D_1 \cap G_1$, $D_2 \cap G_1$, $D_1 \cap G_2$ and $D_2 \cap G_2$ (p_1, p_2, p_3 and p_4 respectively) and the unit point is $D_3 \cap G_3$ (see figure 4.1).

In such a frame the two lines D_1 and D_2 have equations

$$D_1: \quad x_2 = 0 = x_4$$
$$D_2: \quad x_1 = 0 = x_3$$

respectively, while the lines G_1 and G_2 have equations

$$G_1: \quad x_3 = 0 = x_4$$
$$G_2: \quad x_1 = 0 = x_2$$

It is easy to check that D_3, which is a line through $(1,1,1,1)F^*$ that meets both G_1 and G_2, has equations

$$D_3: \quad x_1 - x_2 = 0 = x_3 - x_4$$

and so a point p on this line[5] has coordinates $(1,1,u,u)F^*$, where $u \in F$, and the generator through p, being the intersection between the plane (D_1, p) and the plane (D_2, p), is the line

$$G: \quad ux_2 - x_4 = 0 = ux_1 - x_3.$$

Putting $u = 1$ in the equations of G we obtain for G_3 the equations

$$G_3: \quad x_1 - x_3 = 0 = x_2 - x_4$$

and so a point on G_3 has coordinates $(1, v, 1, v)F^*$, $v \in F$, and the line through this point that meets both G_1 and G_2 is the line of equation:

$$D: \quad vx_1 - x_2 = 0 = vx_3 - x_4.$$

It is now easy to check that the condition that this last line D meets the line G is equivalent to asking

$$uv = vu.$$

This implies that a line D of the regulus which has three of the generators as directrices meets all the generators (and so is a directrix of the first regulus) if and only if

$$uv = vu \quad \text{for all } u \in F,$$

that is if and only if $v \in Z(F)$. □

[5] In fact not all the points of D_3, the point $(0,0,1,1)F^*$ is missing.

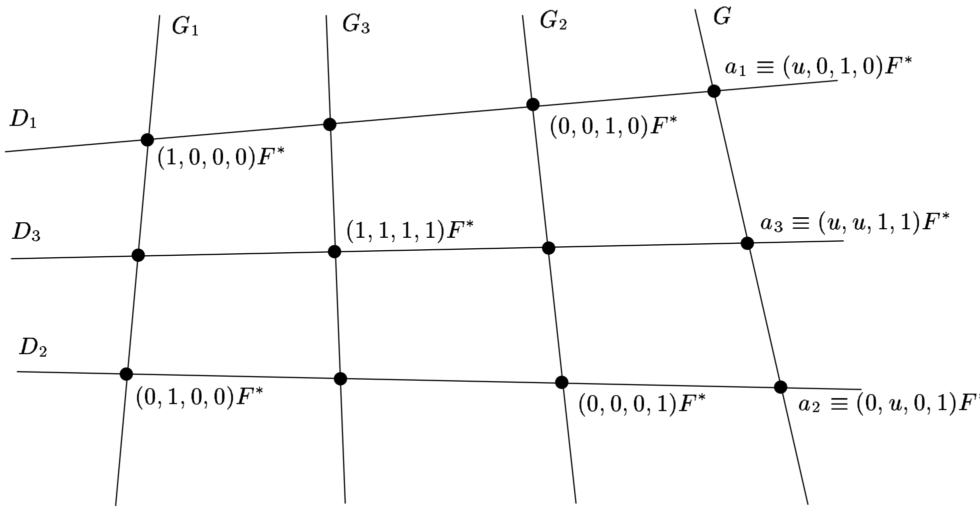

Figure 4.2:

We can observe that, as a consequence of this theorem, we can give to a generator G a sort of "structure", in the sense that its points can be divided into points through which there exists a line that meets all generators (in fact points that can be parametrized by a non homogeneous coordinate u belonging to the center of F), and points for which this condition is not fulfilled, grouped into conjugacy classes, as we did in § 3. This argument cannot be applied to points on a directrix which, in some sense, are "all the same" because from the point of view of the regulus they have the same properties (through all points on a directrix there is exactly a generator).

As we do in commutative case, we may also observe that if we take another line D that meets all generators and we take the point of intersection of this line with, for instance, G_1, we can consider the cross-ratio of this intersection point with the points p_1, p_2 and $p_1 + p_2$ and prove that this cross-ratio does not change if we take another generator instead of G_1. This allows us to speak of the cross-ratio of the four lines D_1, D_2, D_3, D and, in the skew case, we can restate the previous result saying that through a point p on a generator there is a directrix if and only if its cross-ratio belongs to the center of F.

Now we want to find out an expression for all points on a regulus (an hyperbolic quadric in the commutative case); let us consider the picture 4.2, where we have chosen a projective frame in the same way as we did before in the proof of theroem 4.3. We have already observed that the points of the line D_3 are those of the form $(u, u, 1, 1)F^*$ $(u \in F)$ plus an extra point, say "∞", with coordinates $(1, 1, 0, 0)F^*$. If we now take a point a_3 on D_3

different from ∞ and we consider the generator G passing through this point, it is easy to check that the points a_1 and a_2, intersection of this generator with D_1 and D_2 respectively, have coordinates:

$$a_1 \equiv (u, 0, 1, 0)F^*$$
$$a_2 \equiv (0, u, 0, 1)F^*.$$

So all points on G can be obtained as a linear combination of a_1 and a_2; if we take a combination made only by one non homogeneous parameter $t \in F$, we obtain for these points coordinates of the form:

$$(ut, u, t, 1)F^*.$$

Of course not all points of the regulus can be written in such a form: as we have already observed, for each directrix we "loose" its point at infinity and, moreover, choosing to make linear combination only with one parameter, implies that for each generator we loose another point (corresponding to the case $t = \infty$; in particular we cannot represent in this form, for instance, the point a_1); don't care about it for now, just remember that some points are missing.

Let us also underline the different roles played by u and t: both of them belong to F but changing u corresponds to changing the generator G we are considering, while changing t means changing the point on such G.

The form $(ut, u, t, 1)F^*$ we obtained for points on a regulus suggests us how we can obtain a quadratic homogeneous equation in the four indeterminates x_1, x_2, x_3 and x_4 that characterizes the points of the regulus:

$$(1) \quad x_1 x_3^{-1} = x_2 x_4^{-1}.$$

This works in the same way in the skew case because we take the inverse of x_3 and x_4: this allows us not to care about the right proportionality factor which is bound with coordinate representation of points, in fact if we take $\rho \in F^*$ and we replace the point x_i with $x_i\rho$ $(i = 1, 2, 3, 4)$ we obtain, for instance for $x_1 x_3^{-1}$:

$$(x_1\rho)(x_3\rho)^{-1} = x_1\rho\rho^{-1}x_3^{-1} = x_1 x_3^{-1}.$$

Again some points are missing in the representation (1); even if we forget we took these points away during our construction, we can see an evidence of this in the fact that the equation contains an inverse operation which, of course, is allowed only for non zero element of F.

As we have already observed in § 3, if we change the coordinates of the points of the projective frame (but, of course, without changing the points!) the parameters u and t change by an inner automorphism of F.

Let us now take in $PG(3,F)$ two triplets of pairwise skew lines D_1, D_2, D_3 and G_1, G_2, G_3 such that every line of the first triplet meets all lines of the second one, consider the two reguli $\mathscr{G}$ and $\mathscr{D}$ having D_1, D_2, D_3 and G_1, G_2, G_3 as directrices respectively and fix a projective frame as we have usually done before. If we take a line G of $\mathscr{G}$ and a line D of $\mathscr{D}$, we have already proved that, if we denote by v the cross-ratio of the four lines D_1, D_2, D_3 and D and by u the cross-ratio of G_1, G_2, G_3 and G, the lines G and D intersect if and only if u and v commute. Let us now take a new line of $\mathscr{G}$, say G', which determines, with respect to G_1, G_2 and G_3, a cross-ratio u' conjugate to u and a new line $D' \in \mathscr{D}$ which has a cross-ratio v' conjugate to v; the question is: if we know that u and v commute, what can we say about u' and v'? Well, if we take $c, d \in F^*$ we cannot always say that u^c and v^d commute (see example 4.5), but this is true if $c = d$, in fact

$$u^c v^c = (c^{-1}uc)(c^{-1}vc) = c^{-1}uvc = c^{-1}vuc = v^c u^c$$

and so we can conclude with the following statement:

4.4 Theorem. *If we take the set of all lines of $\mathscr{G}$ having cross-ratio u and the set of all lines of $\mathscr{D}$ having cross-ratio v, then these two sets are made up of lines either pairwise skew or such that every line from one set meets al least one line from the other set; in particular this last condition holds if $uv = vu$.*

4.5 Example. Let us take $F = \mathbb{H}$, $u = i = v$, $c = 1$ and $d = k - 1$. It is easy to check that

$$d^{-1} = (k-1)^{-1} = -\frac{1}{2} - \frac{1}{2}k$$

and so

$$d^{-1}id = \left(-\frac{1}{2} - \frac{1}{2}k\right) i(k-1) = j.$$

We have:

$$u^c v^d = id^{-1}id = ij = k$$
$$v^d u^c = d^{-1}idi = ji = -k.$$

4.1 Generalization to (odd) dimensions greater than 3

Here we want to give an outline on how the notion of regulus can be extended to dimensions greater than 3; we will only describe two possible constructions in $PG(5, F)$; the general case can be obtained from these examples.

Let us take in a 5-dimensional projective space three planes $\delta_1, \delta_2, \delta_3$ pairwise skew; we call a *regulus* with *director planes* $\delta_1, \delta_2, \delta_3$ the set of all lines of $PG(5, F)$ that meet all these planes. As we did in § 4 for the 3-dimensional case, we can observe that for each point on a director plane there exists only one line of the regulus through this point, while if we take a point p on a generator G_1 a director plane through p does not always exist. In particular, if we fix on G_1 a projective frame having $G_1 \cap \delta_1$, $G_1 \cap \delta_2$ and $G_1 \cap \delta_3$ as basic points and unit point respectively, a director plane through p exists if and only if the cross-ratio of p with respect to such a frame[6] lies in the center of F. This implies that, from regulus point of view, a director plane has no structure (all the points are "the same") while the points on a generator can be divided into these points belonging to $Z(F)$ and those that don't belong to $Z(F)$, grouped into conjugacy classes.

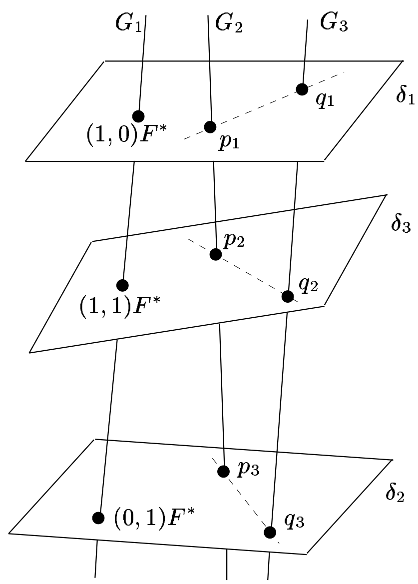

Figure 4.3: *A regulus in* $PG(5, F)$.

This construction is a natural generalization of the definition of reguli in 3-dimensional spaces, in fact if we consider the points p_1, p_2, p_3 and q_1, q_2, q_3 as in the figure 4.3, the lines $\overline{p_1, q_1}$, $\overline{p_2, q_2}$ and $\overline{p_3, q_3}$ span a 3-dimensional projective space and, in this space, they are the directrices of a regulus that, of course, contains G_2 and G_3.

We can also construct the regulus by giving a projectivity between two skew planes of $PG(5, F)$ and considering the lines joining corresponding points.

A similar construction can be done starting from four lines instead of three skew planes; we need a definition:

4.6 Definition. *Four lines of* $PG(5, F)$ *are in* general position *when, if we*

[6]As we did in 3-dimensional case, if we take 4 director planes, we can speak of the cross-ratio of this four planes; see page 19.

take arbitrarily three of them, they span the whole space.

So if we take four lines D_1, D_2, D_3, D_4 in general position, we can consider all planes of $PG(5, F)$ that meet these four lines in exactly one point; this planes, called *generator planes*, form a *regulus* having D_1, D_2, D_3 and D_4 as *directrices*. For each point p on a directrix there exists a generator plane through this point[7], and so directrices have "no structure", while for each generator plane G directrices exist only for those points that belong to the subplane over the center $Z(F)$ spanned by the four points $G \cap D_i$ $(i = 1, 2, 3, 4)$.

5 Planar sections of reguli

In this section we deal with the planar sections of a regulus in $PG(3, F)$. We will see that, in respect of the commutative case, the situation with skew fields is a bit more complicated, in fact, besides of the "classical" situation in which the planar sections are made up of a couple of distinct lines or of a conic, we have new and different possible configurations. In particular another time the center $Z(F)$ of the field plays a special role, suggesting if we will expect to obtain a classical configuration or a new one.

5.1 Planar sections with planes through a generator

Let D_1, D_2, D_3 be the directrices of a regulus $\mathscr{G}$, G_1 be a generator and π be a plane of $PG(3, F)$ through G_1. Each generator of the regulus $\mathscr{G}$ distinct from G_1 meets the plane π in a point.

5.1 Definition. *We call* C-configuration *the set of all points of the plane* π *that are the intersection of a generator distinct from* G_1 *with* π *and we denote it by* $\mathfrak{C}$.

As we have already observed in § 4, without loss of generality we can assume that the line G_1 has equations

$$G_1 : \; x_3 = 0 = x_4$$

and so the plane π has the equation

(1) $\quad \pi : \; x_3 = t x_4$

[7] We can obtain this plane for instance by intersecting all hyperplanes joining the point p with two of the other lines.

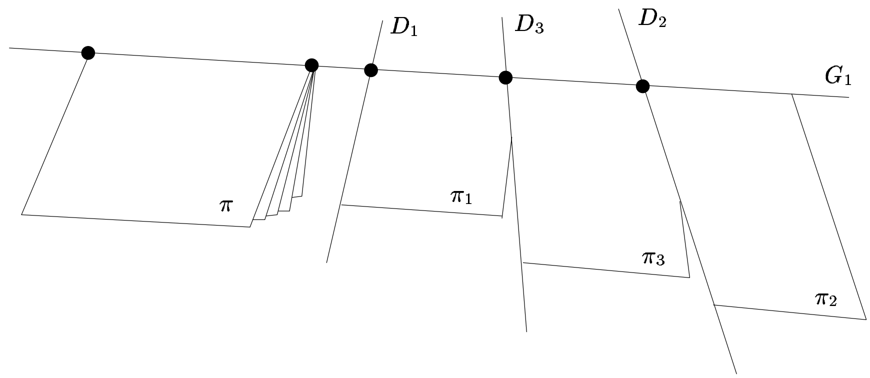

Figure 5.1: *Cross-ratio of four planes.*

where $t \in F$ is a non homogeneus parameter[8].

5.2 Definition. *The element $t \in F$ is called the* parameter of the C-configuration.

5.3 Remark. The parameter t in equation (1) is on the left even if we are still in a right vector space; this because in this way we can preserve the homogeneity of the equation: t won't "interfere" with the proportional factor of a point because they are on different side of x_4.

This situation is dual with that we have already seen in § 4 for the points on a generator, so we can do for π the same consideration we did there for such points. In particular (see figure 5.1) if we denote by π_1, π_2 and π_3 the planes through G_1 and, respectively, D_1, D_2 and D_3, t is the cross-ratio of the plane π with respect to π_1, π_2, π_3. Moreover we can consider all the planes that have cross-ratio t with respect to these three planes, and, in general, we won't obtain only π but a family of planes containing π, since, as we have already observed many times, the cross-ratio is not a single number, but a whole conjugacy class.

In § 4 we have determined that the points of the regulus are those of the form

(2) $\quad (ut, u, t, 1)F^* \qquad u, t \in F$

plus an extra line ($t = \infty$) and an extra point for every line ($u = \infty$). In the plane π the parameter t is constant, and so the same expression defines all

[8] Another time something is missing: the plane $x_4 = 0$, that is the plane corresponding to $t = \infty$.

the points of the C-configuration with free parameter $u \in F$, while t, fixed, is the parameter of the C-configuration. Let us now take two distinct points m and m' of the C-configuration, say

$$m \equiv (ut, u, t, 1)F^* \qquad m' \equiv (u't, u', t, 1)F^*, \quad c := (u - u')^{-1} \neq 0$$

and let us look for the point n intersection of the line trough m and m' and the line G_1. Remembering that G_1 is the line of equations $x_3 = 0 = x_4$, we obtain for n the following coordinates:

$$\begin{aligned} n \equiv \big(ut - u't, u - u', 0, 0\big)\, F^* &= \big((u - u')t, u - u', 0, 0\big)\, F^* = \\ &= \big((u - u')t(u - u')^{-1}, 1, 0, 0\big)\, F^* = \\ &= \big(c^{-1}tc, 1, 0, 0\big)\, F^*. \end{aligned} \tag{3}$$

If we now take two other points $\overline{m}$ and $\overline{m'}$ we will obtain a new point $\overline{n}$ of the form $\big(\overline{c}^{-1}t\overline{c}, 1, 0, 0\big)\, F^*$ that, if t is not in the center of F, is distinct from n. We have so proved the following statement:

5.4 Theorem. *If the parameter t of the C-configuration lies in the center $Z(F)$ of the field F the points of the C-configuration lie on a line that intersects the generator G_1 in exactly one point having cross-ratio t with respect to the points $p_1 = D_1 \cap G_1$, $p_2 = D_2 \cap G_1$ and $p_3 = D_3 \cap G_1$. If, on the contrary, $t \notin Z(F)$, the points of the configuration do not lie on a single line of π and the points obtained intersecting G_1 with the lines joining pairs of distinct points of the C-configuration are those points that have, with respect to p_1, p_2 and p_3 a cross-ratio which is conjugate to t.*

Let us now take the two points of the C-configuration corresponding to the parameter $u = 0$ and $u = 1$, i.e. the points $m \equiv (0, 0, t, 1)F^*$ and $m' \equiv (t, 1, t, 1)F^*$; by equation (3) we know that the point of G_1 that lies on the line through m and m' is $n \equiv (t, 1, 0, 0)F^*$. Of course all the points of the line M through m and m' can be obtained as a linear combination of m and n, so, taking a non homogeneous linear combination, they are of the form

$$(t, 1, 0, 0)u + (0, 0, t, 1) = (tu, u, t, 1). \tag{4}$$

By comparing this expression with the coordinates (2) we obtain that the points of M which belong to the C-configuration $\mathfrak{C}$ are those points for which $ut = tu$, and so such that $u \in Z_F(t)$. We have so proved that such points together with n are the points of a projective line over the centralizer $Z_F(t)$.

Let us now switch to the affine point of view. The points of $\mathfrak{C}$ are those of the form $(ut, u) \in F^2$, $u \in F$ and form a right vector space V over the centralizer $Z_F(t)$, so we can consider the following function:

$$f : \begin{cases} F & \longrightarrow & \mathfrak{C} = \{\, (ut, u) \mid u \in F \,\} \\ u & \longmapsto & (ut, u) \end{cases}$$

This map is bijective and, if we consider F as a right vector space over the centralizer $Z_F(t)$ of t, it is also linear, and so an isomorphism; this implies that the dimension of V is the same of the dimension $[F : Z_F(t)]$, even if all the points of V lie in the plane π.

5.5 Remark. Let us take a generator G, a plane π through G and a point p of G. A line $L \neq G$ through p that lies in π fulfils one of the following conditions:

- L meets the C-configuration in at least two distinct points m and m', if and only if the point p is one of the point n we described above;
- L meets the C-configuration in exactly one point;
- L meets the C-configuration in no points.

In the classical case (if the parameter t of the C-configuration lies in the center $Z(F)$) the C-configuration is a line R of π without a point (the point p), and so the first situation can take place only if $L = R$, while the third case cannot take place; in the skew case in general all the situations are possible.

5.6 Remark. Let us take two distinct generators G_1 and G, a plane π through G_1 and a point n on G_1 such that its cross-ratio is $t \notin Z(F)$. If we assume that the line G meets the plane π in a point m such that its cross-ratio is the same of n, then on the line joining m and n there are infinitely many points of the C-configuration and these points form a subline over the centralizer of t.

5.2 Planar sections with planes that do not contain a generator

Let π be a plane that does not contain any generator of the regulus $\mathscr{G}$. Let us start by observing that such a plane cannot contain a directrix either and that every generator has exactly one intersection point with π.

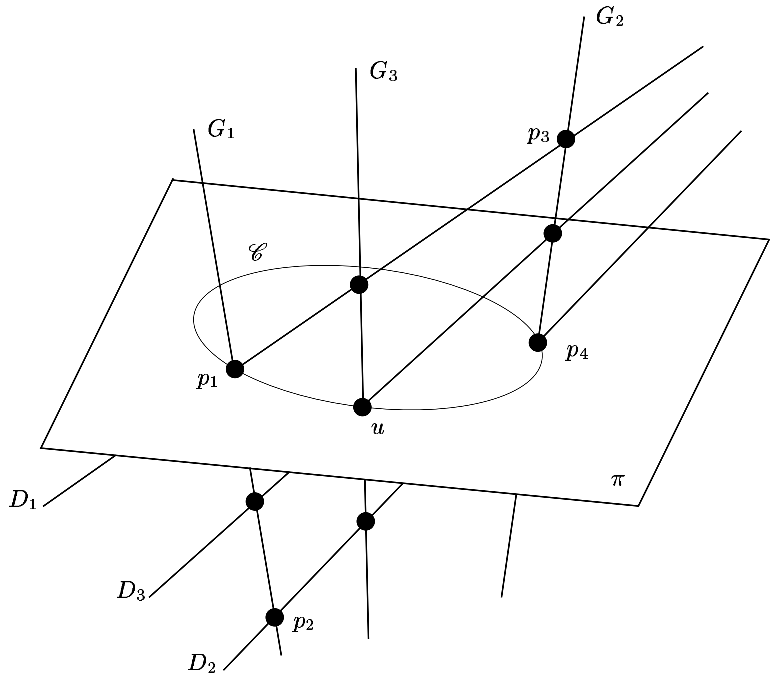

Figure 5.2: *A conic $\mathscr{C}$ on π.*

5.7 Definition. *We call the set of all common points of the plane π with the regulus $\mathscr{G}$ a* conic *and we denote it by $\mathscr{C}$.*

Let us determine an expression for the points of $\mathscr{C}$. The three directrices D_1, D_2 and D_3 of the regulus $\mathscr{G}$ meet the plane π in three distinct points (see figure 5.2), through each of this points there exists a generator and all these generators must be distinct because the plane π cannot contain any of them, so let us denote them by G_1, G_2 and G_3. If we fix a projective frame as we have usually done, i.e.

$$\begin{aligned}
p_1 &\equiv (1,0,0,0)F^* := G_1 \cap D_1 \\
p_2 &\equiv (0,1,0,0)F^* := G_1 \cap D_2 \\
p_3 &\equiv (0,0,1,0)F^* := G_2 \cap D_1 \\
p_4 &\equiv (0,0,0,1)F^* := G_2 \cap D_2 \\
u &\equiv (1,1,1,1)F^* := G_3 \cap D_3
\end{aligned}$$

we can easily obtain for the plane π the equation

$$\pi : \; x_2 = x_3.$$

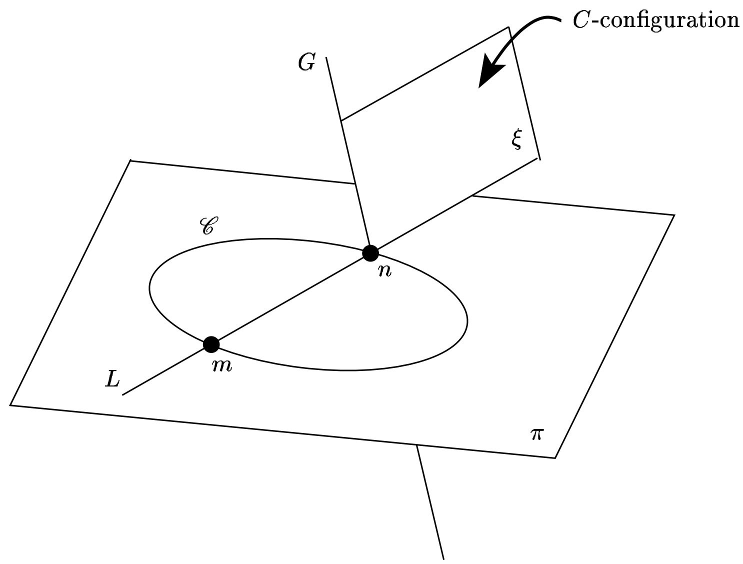

Figure 5.3: *Intersection between the conic $\mathscr{C}$ and a line L of π.*

On the other hand we know that the points of the regulus are those of coordinates $(ut, u, t, 1)F^*$, $u, t \in F$, so the points of the conic $\mathscr{C}$ are those points for which $u = t$ and so points that have coordinates

(5) $\quad (t^2, t, t, 1)F^*.$

As usual a point is missing, in particular the point we obtain when $t = \infty$, i.e. the point p_1. From the expression (5) we can obtain the equation of the conic on the plane π:

$$x_1 x_2^{-1} = x_2 x_4^{-1}.$$

Let us now take a point n on the conic $\mathscr{C}$ and denote by L a line of π through n. If G is the generator of $\mathscr{G}$ passing through n (G always exists and cannot lie in π), let us denote by ξ the plane determined by G and L. On such a plane the lines of the regulus determine a C-configuration $\mathfrak{C}$ and

$$\mathscr{C} \cap L = \mathscr{G} \cap L = \mathfrak{C} \cap L.$$

So if we now take another point m on $\mathscr{C}$ and assume L to be the line joining m and n we have two possibilities:

- m and n are the only points in common between $\mathscr{C}$ and L or
- there is at least a third common point, distinct from m and n.

Remembering 5.6, in the first case we can conclude that the two points m and n have different cross-ratios, while in the latter we can state that the cross-ratios of n and m belong to the same conjugacy class of F and, moreover, that there are infinitely many points in common between $\mathscr{C}$ and the line L.

5.8 Remark. This last construction suggests us a way to "identify" conjugacy classes of F: it is sufficient to find out in a plane π not containing any generator all secant lines to the conic $\mathscr{C} = \pi \cap \mathscr{G}$ (hence all lines that meet the regulus) in more than two points.

We can obtain this result also by an algebraic point of view. On the plane π the line L has equation

$$a_1x_1 + a_2x_2 + a_4x_4 = 0$$

for some $a_1, a_2, a_4 \in F$, and so the points of the conic $\mathscr{C}$ that belong to the line L are those that are roots of the quadratic equation

$$a_1t^2 + a_2t + a_4 = 0, \qquad a_1 \neq 0. \tag{6}$$

Remembering now the Gordon-Motzkin theorem (2.8), we can observe that if the equation (6) has two non conjugate solutions, then these solutions are all the roots, while if it has two conjugate solutions, then there are infinitely many conjugate solutions (in fact a subline over the centralizer of t); of course (6) can also have only one solution, or no one. So the points of the conic $\mathscr{C}$ can be divided into two groups:

points of 1$^{\text{st}}$ kind (or regular, central points): they are those points of $\mathscr{C}$ that have cross-ratio in the center of F; through each of these points there are both a generator and a directrix and any line through it has at most two intersections with the conic.

points of 2$^{\text{nd}}$ kind: they are those points that have cross-ratio $t \notin Z(F)$; through these points there are no directrices and through any of these points there is a line that meets $\mathscr{C}$ in infinitely many points belonging to the same conjugacy class of t.

5.9 Remark (what is a tangent?). In the skew case there are several different ways in which we can define what a *tangent line* is, depending on which property of the "classical" tangent we want to preserve and obtaining, of course, different objects in each case. We propose three different definitions.

1. (Artzy, [13]) The most intuitive idea is, of course, to define a tangent to be a line that meets the conic only in one point; this will produce the result that if we have a tangent in a point p, then there are as many tangents in p as there are elements in the centralizer of p.

2. (Segre, [7]) In the commutative case, if p is a point of a conic $\mathscr{C}$ of π and G is the generatrix through p, then it is well defined the (unique) tangent plane τ through p as the plane that contains G and the directrix D through p; the tangent in p to the conic $\mathscr{C}$ is the line $\tau \cap \pi$. Moreover it is easy to check that, if $p_1, p_2, p_3, p_4 \in G$, their cross-ratio is the same as the cross-ratio of their tangent planes (that belong to the pencil of planes through G).

   In the skew case, in general, a directrix through p does not exist, and so there is not such an obvious definition for the tangent plane as above. The idea is to consider the cross-ratio of p with respect to the points $G \cap D_1$, $G \cap D_2$ and $G \cap D_3$, and to take all the planes (in general there are infinitely many) which have this cross-ratio with respect to $\mathrm{pn}(G, D_1)$, $\mathrm{pn}(G, D_2)$ and $\mathrm{pn}(G, D_3)$; we can define a *tangent* to be any line cut on π by one of these planes.

   If we assume this definition, every time the cross-ratio t of p does not belong to the center $Z(F)$, we have to renounce to the idea that the tangent to the conic in p is unique: in this situation, in fact, the number of tangents is infinite.

3. (Berz) For all points of the first kind there is exactly one line that meets the conic in one point (as happens in the "classical" case) and so we decide this is the tangent. For all other points p the tangent is the (unique) line that has harmonic cross-ratio with respect to the lines that join this point p with two distinct first kind points p_1 and p_2 (for instance the points corresponding to the parameters 0 and ∞, which always lie in the center) and with the intersection point of the two tangents to the conic in these first kind points. This second situation generalizes a property of tangent lines to a conic in commutative case and has the advantage that through each point there is exactly one tangent, but in second kind points a tangent in the sense of this definiton could meet the conic in infinitely many points.

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If you want to deepen the topics covered by these notes, you would probably find useful the following references.

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