Affine Metric Geometry and Weak Orthogonal Groups

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Abstract

By following the ideas underpinning the well-established "homogeneous model" of an n-dimensional Euclidean space, we investigate whether the motion group or the weak motion group of an n-dimensional affine metric space on a vector space $\boldsymbol{V}$ over an arbitrary field admits a specific faithful linear representation as weak orthogonal group of an $(n+1)$-dimensional metric vector space. Apart from a few exceptions, such a representation exists precisely when the metric structure on $\boldsymbol{V}$ is given by a quadratic form with a non-degenerate polar form.
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1 Introduction

There is a widespread literature on the problem of describing the motion group of the Euclidean space $\mathbb{R}^n$ (equipped with the standard inner product) by means of a Clifford algebra. One of the known approaches makes use of the so-called "homogeneous model". It is based upon the introduction of homogeneous coordinates or, said differently, the embedding of $\mathbb{R}^n$ in the projective space $\mathbb{P}(\mathbb{R}^{n+1})$, and it fits into the following more general construction: First, $\mathbb{R}^n$ is equipped with an inner product of signature $(p, n-p, 0)$. Then the dual vector space of $\mathbb{R}^{n+1}$, in symbols $(\mathbb{R}^{n+1})^*$, is equipped with an inner product of signature $(p, n-p, 1)$, and the corresponding Clifford algebra is being used. So, the inner product on $(\mathbb{R}^{n+1})^*$ is degenerate with a one-dimensional radical. See, for example, C. G. Gunn [12], D. Klawitter [16], D. Klawitter and M. Hagemann [17]. We also refer to J. M. Selig [22], [23], where in the Euclidean case $(p = n)$ the signature $(0, n, 1)$

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is used instead of $(n, 0, 1)$. The cited sources contain a wealth of references to previous work.

We are interested in the generalisation of the above results to arbitrary affine metric spaces of finite dimension. In Sections 2 and 3, we collect some basic facts from linear algebra and we establish auxiliary results about transvections and dilatations, which are employed in Section 5. Our starting point in Section 4 is the affine space $\mathbb{A}(\boldsymbol{V})$ on a finite-dimensional vector space $\boldsymbol{V}$ over an arbitrary field F. By analogy to the real case, we consider the F-vector spaces $\boldsymbol{V}^*$ (*i.e.* the dual vector space of $\boldsymbol{V}$), $F \times \boldsymbol{V}$ and $F \times \boldsymbol{V}^*$; we identify the latter with the dual vector space of $F \times \boldsymbol{V}$. Our main tool is a faithful linear representation $\beta\colon \mathrm{AGL}(\boldsymbol{V}) \to \mathrm{GL}(F \times \boldsymbol{V}^*)$, where $\mathrm{AGL}(\boldsymbol{V})$ denotes the group of all affinities of $\boldsymbol{V}$ onto itself. Then we recall the notion of an affine metric space $\mathbb{A}(\boldsymbol{V}, Q)$, which arises by equipping $\mathbb{A}(\boldsymbol{V})$ with a quadratic form $Q\colon \boldsymbol{V} \to F$.

In Section 5, we address the main problem: Find all dyads of metric vector spaces $(\boldsymbol{V}, Q)$ and $(F \times \boldsymbol{V}^*, \tilde{Q})$ such that $\mathbb{A}(\boldsymbol{V}, Q)$ has a motion group or a weak motion group whose β-image coincides with the weak orthogonal group of $(F \times \boldsymbol{V}^*, \tilde{Q})$. The transformations of the latter group allow for a neat description in terms of the corresponding Clifford algebra. However, this topic is beyond the scope of the present note; see [13] for further details and an extensive bibliography. Proposition 5.2 provides solutions to the above problem under the extra assumptions that, firstly, the polar form of Q is non-degenerate and, secondly, $\tilde{Q}$ is a non-zero scalar multiple of a quadratic form $Q^{\uparrow}$ arising from Q by an explicit construction; see Proposition 5.1. The polar form of $Q^{\uparrow}$ has a particular one-dimensional radical and, moreover, $Q^{\uparrow}$ maps all vectors of the radical to 0. In Remarks 5.3 and 5.4, we refer to closely related outcomes by F. Bachmann [1], E. W. Ellers [7], E. W. Ellers and H. Hähl [9], J. Helmstetter [15], E. M. Schröder [20], H. Struve and R. Struve [24], H. Wolff [26], [27]. Remark 5.7 contains the transition from Q to $Q^{\uparrow}$ in terms of coordinates. Our main result is Theorem 5.9. Apart from quite a few exceptional cases, which occur when both $\dim \boldsymbol{V}$ and $|F|$ are "very small", there are only the solutions as in Proposition 5.2. All exceptional cases are itemised in Remark 5.10, Tables 1–4; in some of these cases the polar form of Q fails to be non-degenerate. In conclusion, we switch to the "projective point of view" by going over to the projective space $\mathbb{P}(F \times \boldsymbol{V}^*)$. It will turn out that this merely leads us to yet another (trivial) exceptional case, but otherwise does not give rise to new results.

2 Preliminaries

Throughout this article, we consider only *finite-dimensional* vector spaces over a (commutative) field F. In what follows, we fix our notation and we collect some basic facts; see [2, Ch. II], [11], [14] and the sources listed below.

Let $\boldsymbol{V}$ be a vector space. We write $\boldsymbol{V}^*$ for its *dual vector space*, $\langle\cdot,\cdot\rangle\colon \boldsymbol{V}^*\times\boldsymbol{V}\to F$ for the *canonical pairing*, $\operatorname{Char} F$ for the characteristic of F and we put $F^\times := F\setminus\{0\}$. The zero vector of $\boldsymbol{V}$ (resp. $\boldsymbol{V}^*$) is denoted by $\boldsymbol{o}$ (resp. $\boldsymbol{o}^*$). Each subset $\boldsymbol{S}\subseteq\boldsymbol{V}$ determines its *annihilator* $\boldsymbol{S}^\circ := \{\boldsymbol{a}^*\in\boldsymbol{V}^* \mid \langle\boldsymbol{a}^*,\boldsymbol{s}\rangle = 0 \text{ for all } \boldsymbol{s}\in\boldsymbol{S}\}$, which is a subspace of $\boldsymbol{V}^*$. In particular, given any subspace $\boldsymbol{T}$ of $\boldsymbol{V}$, in symbols $\boldsymbol{T}\leq\boldsymbol{V}$, we have $\dim\boldsymbol{T}^\circ = \dim\boldsymbol{V}-\dim\boldsymbol{T}$. We consider $\boldsymbol{V}$ as the dual vector space of $\boldsymbol{V}^*$ by identifying $\boldsymbol{x}\in\boldsymbol{V}$ with the linear form $\langle\cdot,\boldsymbol{x}\rangle\colon\boldsymbol{V}^*\to F$. In this way our results apply, *mutatis mutandis*, to $\boldsymbol{V}^*$.

Let $\tilde{\boldsymbol{V}}$ be a vector space, too, and let $\eta\colon\tilde{\boldsymbol{V}}\to\boldsymbol{V}$ be a linear mapping. The *transpose* of η is given as $\eta^{\mathrm{T}}\colon\boldsymbol{V}^*\to\tilde{\boldsymbol{V}}^*\colon\boldsymbol{a}^*\mapsto\boldsymbol{a}^*\circ\eta$. Thus, for all $\boldsymbol{a}^*\in\boldsymbol{V}^*$ and all $\tilde{\boldsymbol{x}}\in\tilde{\boldsymbol{V}}$, we have $\langle\eta^{\mathrm{T}}(\boldsymbol{a}^*),\tilde{\boldsymbol{x}}\rangle = \langle\boldsymbol{a}^*,\eta(\tilde{\boldsymbol{x}})\rangle$. The mapping η^{T} is linear and satisfies $(\eta^{\mathrm{T}})^{\mathrm{T}}=\eta$. The image of η^{T} and the kernel of η are related by $\eta^{\mathrm{T}}(\boldsymbol{V}^*) = (\ker\eta)^\circ$.

All linear bijections of $\boldsymbol{V}$ onto itself form the *general linear group* $\mathrm{GL}(\boldsymbol{V})$. Any pair $(\boldsymbol{c}^*,\boldsymbol{f})\in\boldsymbol{V}^*\times\boldsymbol{V}$ such that $\langle\boldsymbol{c}^*,\boldsymbol{f}\rangle\neq -1$ gives rise to the linear bijection

$$\delta_{\boldsymbol{c}^*,\boldsymbol{f}}\colon\boldsymbol{V}\to\boldsymbol{V}\colon\boldsymbol{x}\mapsto\boldsymbol{x}+\langle\boldsymbol{c}^*,\boldsymbol{x}\rangle\boldsymbol{f}. \tag{2.1}$$

If $\boldsymbol{c}^*=\boldsymbol{o}^*$ or $\boldsymbol{f}=\boldsymbol{o}$, then $\delta_{\boldsymbol{c}^*,\boldsymbol{f}}$ equals the identity $\mathrm{id}_{\boldsymbol{V}}$. Otherwise, $\delta_{\boldsymbol{c}^*,\boldsymbol{f}}\neq\mathrm{id}_{\boldsymbol{V}}$ fixes precisely the vectors of the hyperplane $\ker\boldsymbol{c}^*\leq\boldsymbol{V}$ and $\delta_{\boldsymbol{c}^*,\boldsymbol{f}}$ is called a *transvection* (resp. *dilatation*) provided that $\langle\boldsymbol{c}^*,\boldsymbol{f}\rangle=0$ (resp. $\langle\boldsymbol{c}^*,\boldsymbol{f}\rangle\neq 0$); see [8], [25, p. 20].[1]

Upon choosing a vector $\boldsymbol{f}\in\boldsymbol{V}\setminus\{\boldsymbol{o}\}$, we put

$$\Delta(\boldsymbol{V},\boldsymbol{f}) := \{\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \mid \boldsymbol{a}^*\in\boldsymbol{V}^* \text{ and } \langle\boldsymbol{a}^*,\boldsymbol{f}\rangle\neq -1\}, \tag{2.2}$$

which is a subgroup of $\mathrm{GL}(\boldsymbol{V})$. It is easily checked that there is a bijective mapping

$$\{\boldsymbol{a}^*\in\boldsymbol{V}^* \mid \langle\boldsymbol{a}^*,\boldsymbol{f}\rangle\neq -1\}\to\Delta(\boldsymbol{V},\boldsymbol{f})\colon\boldsymbol{a}^*\mapsto\delta_{\boldsymbol{a}^*,\boldsymbol{f}}. \tag{2.3}$$

Next, let $Q\colon\boldsymbol{V}\to F$ be a quadratic form. So $(\boldsymbol{V},Q)$ is a *metric vector space* as in [21, 1.1]; see also [3, Ch. IX], [13, Sect. 1], [20, § 7], [25]. Then $B\colon\boldsymbol{V}\times\boldsymbol{V}\to F\colon(\boldsymbol{x},\boldsymbol{y})\mapsto Q(\boldsymbol{x}+\boldsymbol{y})-Q(\boldsymbol{x})-Q(\boldsymbol{y})$ denotes the *polar form* of Q. We have $B(\boldsymbol{x},\boldsymbol{x})=2Q(\boldsymbol{x})$ for all $\boldsymbol{x}\in\boldsymbol{V}$. From B being bilinear, we get

$$D\colon\boldsymbol{V}\to\boldsymbol{V}^*\colon\boldsymbol{x}\mapsto D(\boldsymbol{x}) := B(\boldsymbol{x},\cdot) \tag{2.4}$$

as the *induced linear mapping* of B. The transpose of D takes the form $D^{\mathrm{T}}\colon\boldsymbol{V}\to\boldsymbol{V}^*$. Using (2.4) and the fact that B is a symmetric bilinear form, it follows $\langle D^{\mathrm{T}}(\boldsymbol{y}),\boldsymbol{x}\rangle = \langle D(\boldsymbol{x}),\boldsymbol{y}\rangle = B(\boldsymbol{x},\boldsymbol{y}) = B(\boldsymbol{y},\boldsymbol{x}) = \langle D(\boldsymbol{y}),\boldsymbol{x}\rangle$ for all $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{V}$. Hence we have

$$D = D^{\mathrm{T}}. \tag{2.5}$$

[1]The term "dilatation" appears with a different meaning, among others, in [19, p. 26].

Vectors $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$ are *orthogonal*, in symbols $\boldsymbol{x} \perp \boldsymbol{y}$, precisely when $B(\boldsymbol{x}, \boldsymbol{y}) = 0$. Given $\boldsymbol{S} \subseteq \boldsymbol{V}$ the set $\boldsymbol{S}^{\perp} := \{\boldsymbol{x} \in \boldsymbol{V} \mid \boldsymbol{x} \perp \boldsymbol{s} \text{ for all } \boldsymbol{s} \in \boldsymbol{S}\}$ is a subspace of $\boldsymbol{V}$. In particular, $\boldsymbol{V}^{\perp}$ is called the *radical* of B. Then

$$\boldsymbol{V}^{\perp} = \{\boldsymbol{x} \in \boldsymbol{V} \mid \langle D(\boldsymbol{x}), \boldsymbol{y}\rangle = 0 \text{ for all } \boldsymbol{y} \in \boldsymbol{V}\} = \ker D. \tag{2.6}$$

Also, (2.6) and (2.5) imply

$$(\boldsymbol{V}^{\perp})^{\circ} = (\ker D)^{\circ} = D^{\mathrm{T}}(\boldsymbol{V}) = D(\boldsymbol{V}). \tag{2.7}$$

The *rank* of B is defined as $\dim D(\boldsymbol{V})$. If $\operatorname{Char} F = 2$, then B is an alternating bilinear form and its rank turns out to be even. The linear mapping D is bijective if, and only if, $\boldsymbol{V}^{\perp} = \{\boldsymbol{o}\}$. Under these circumstances B is said to be *non-degenerate*.

The above notation (Q, B, D, $\perp$) will be maintained throughout. In the presence of several quadratic forms, a common subscript or superscript will be added to these symbols.

Again, let a linear mapping $\eta\colon \tilde{\boldsymbol{V}} \to \boldsymbol{V}$ be given. Then the *pullback* of Q along η, that is $Q \circ \eta$, is a quadratic form, say $\tilde{Q}$, and $\tilde{B}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = B(\eta(\tilde{\boldsymbol{x}}), \eta(\tilde{\boldsymbol{y}}))$ for all $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}} \in \tilde{\boldsymbol{V}}$. The left hand side of the last equation can be rewritten as $\langle \tilde{D}(\tilde{\boldsymbol{x}}), \tilde{\boldsymbol{y}}\rangle$; the right hand side equals $\langle (D \circ \eta)(\tilde{\boldsymbol{x}}), \eta(\tilde{\boldsymbol{y}})\rangle = \langle (\eta^{\mathrm{T}} \circ D \circ \eta)(\tilde{\boldsymbol{x}}), \tilde{\boldsymbol{y}}\rangle$. Hence

$$\tilde{D} = \eta^{\mathrm{T}} \circ D \circ \eta. \tag{2.8}$$

A mapping $\varphi \in \mathrm{GL}(\boldsymbol{V})$ is called an *isometry* of $(\boldsymbol{V}, Q)$ if $Q = Q \circ \varphi$. All isometries of $(\boldsymbol{V}, Q)$ make up the *orthogonal group* $\mathrm{O}(\boldsymbol{V}, Q)$. The formula

$$(\varphi^{\mathrm{T}})^{-1} \circ D = D \circ \varphi \ \text{ for all } \ \varphi \in \mathrm{O}(\boldsymbol{V}, Q) \tag{2.9}$$

follows by replacing η with φ in (2.8) and by taking into account $\tilde{D} = D$. The *weak orthogonal group* $\mathrm{O}'(\boldsymbol{V}, Q)$ consists of all isometries of $(\boldsymbol{V}, Q)$ that fix the radical $\boldsymbol{V}^{\perp}$ elementwise. The group $\mathrm{O}'(\boldsymbol{V}, Q)$ appears in the literature under various names; our terminology and notation follows [6]. If $\boldsymbol{r} \in \boldsymbol{V}$ satisfies $Q(\boldsymbol{r}) \neq 0$, then the *Q-reflection* in the direction of $\boldsymbol{r}$, that is the mapping

$$\xi_{\boldsymbol{r}}\colon \boldsymbol{V} \to \boldsymbol{V}\colon \boldsymbol{x} \mapsto \boldsymbol{x} - Q(\boldsymbol{r})^{-1} B(\boldsymbol{r}, \boldsymbol{x})\boldsymbol{r}, \tag{2.10}$$

belongs to $\mathrm{O}'(\boldsymbol{V}, Q)$. If, moreover, $\boldsymbol{r} \in \boldsymbol{V}^{\perp}$, which implies $\operatorname{Char} F = 2$, then $D(\boldsymbol{r}) = \boldsymbol{o}^{*}$ and so $\xi_{\boldsymbol{r}} = \mathrm{id}_{\boldsymbol{V}}$. Otherwise, $\xi_{\boldsymbol{r}}$ is of order two. Each $\varphi \in \mathrm{O}'(\boldsymbol{V}, Q)$ is a product of Q-reflections, unless F and $(\boldsymbol{V}, Q)$ satisfy one of the conditions (2.11) or (2.12) for some basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ of $\boldsymbol{V}$ and all $\boldsymbol{x} = \sum_{h=1}^{n} x_h \boldsymbol{e}_h$ with $x_h \in F$:

$$|F| = 2,\ \dim \boldsymbol{V} > 2 \ \text{ and } \ Q(\boldsymbol{x}) = x_1 x_2; \tag{2.11}$$

$$|F| = 2,\ \dim \boldsymbol{V} \geq 4 \ \text{ and } \ Q(\boldsymbol{x}) = x_1 x_2 + x_3 x_4; \tag{2.12}$$

see [13, Sect. 2] for numerous references.[2]

[2]The conditions on "$\dim \boldsymbol{V}$" as in (2.11) and (2.12) have been written down incorrectly in [13].

3 Lemmata on transvections and dilatations

Let $(\boldsymbol{V}, Q)$ be a metric vector space. According to (2.2), any $\boldsymbol{f} \in \boldsymbol{V} \setminus \{\boldsymbol{o}\}$ gives rise to the group $\Delta(\boldsymbol{V}, \boldsymbol{f})$, which in turn determines a subgroup of the orthogonal group $\mathrm{O}(\boldsymbol{V}, Q)$ and a subgroup of the weak orthogonal group $\mathrm{O}'(\boldsymbol{V}, Q)$, namely

$$\Delta\mathrm{O}(\boldsymbol{V}, Q, \boldsymbol{f}) := \Delta(\boldsymbol{V}, \boldsymbol{f}) \cap \mathrm{O}(\boldsymbol{V}, Q) \text{ and } \Delta\mathrm{O}'(\boldsymbol{V}, Q, \boldsymbol{f}) := \Delta(\boldsymbol{V}, \boldsymbol{f}) \cap \mathrm{O}'(\boldsymbol{V}, Q). \tag{3.1}$$

Furthermore, if $Q(\boldsymbol{f}) \neq 0$, then $\boldsymbol{f}$ yields the Q-reflection $\xi_{\boldsymbol{f}} \in \mathrm{O}'(\boldsymbol{V}, Q)$; see (2.10). We proceed with an explicit description of the groups appearing in (3.1).

Lemma 3.1. *Let $\boldsymbol{f}$ be a non-zero vector of a metric vector space $(\boldsymbol{V}, Q)$.*

(a) *If $\boldsymbol{f} \notin \boldsymbol{V}^\perp$ and $Q(\boldsymbol{f}) \neq 0$, then $\Delta\mathrm{O}(\boldsymbol{V}, Q, \boldsymbol{f}) = \Delta\mathrm{O}'(\boldsymbol{V}, Q, \boldsymbol{f}) = \{\mathrm{id}_{\boldsymbol{V}}, \xi_{\boldsymbol{f}}\}$ and $\xi_{\boldsymbol{f}} \neq \mathrm{id}_{\boldsymbol{V}}$.*

(b) *If $\boldsymbol{f} \notin \boldsymbol{V}^\perp$ and $Q(\boldsymbol{f}) = 0$, then $\Delta\mathrm{O}(\boldsymbol{V}, Q, \boldsymbol{f}) = \Delta\mathrm{O}'(\boldsymbol{V}, Q, \boldsymbol{f}) = \{\mathrm{id}_{\boldsymbol{V}}\}$.*

(c) *If $\boldsymbol{f} \in \boldsymbol{V}^\perp$ and $Q(\boldsymbol{f}) \neq 0$, then $\Delta\mathrm{O}(\boldsymbol{V}, Q, \boldsymbol{f}) = \Delta\mathrm{O}'(\boldsymbol{V}, Q, \boldsymbol{f}) = \{\mathrm{id}_{\boldsymbol{V}}\}$.*

(d) *If $\boldsymbol{f} \in \boldsymbol{V}^\perp$ and $Q(\boldsymbol{f}) = 0$, then*

$$\Delta\mathrm{O}(\boldsymbol{V}, Q, \boldsymbol{f}) = \Delta(\boldsymbol{V}, \boldsymbol{f}) \text{ and } \Delta\mathrm{O}'(\boldsymbol{V}, Q, \boldsymbol{f}) = \{\delta_{\boldsymbol{a}^*, \boldsymbol{f}} \mid \boldsymbol{a}^* \in (\boldsymbol{V}^\perp)^\circ\}. \tag{3.2}$$

Furthermore, by putting $n := \dim \boldsymbol{V}$ and $k := \dim \boldsymbol{V}^\perp$, it follows

$$\left|\Delta\mathrm{O}(\boldsymbol{V}, Q, \boldsymbol{f})\right| = |F^\times| \cdot |F|^{n-1} \text{ and } \left|\Delta\mathrm{O}'(\boldsymbol{V}, Q, \boldsymbol{f})\right| = |F|^{n-k}. \tag{3.3}$$

Proof. We pick any $\boldsymbol{a}^* \in \boldsymbol{V}^*$ subject to $\langle \boldsymbol{a}^*, \boldsymbol{f} \rangle \neq -1$. Then $\delta_{\boldsymbol{a}^*, \boldsymbol{f}} \in \mathrm{O}(\boldsymbol{V}, Q)$ is equivalent to

$$Q(\delta_{\boldsymbol{a}^*, \boldsymbol{f}}(\boldsymbol{x})) - Q(\boldsymbol{x}) = \langle \boldsymbol{a}^*, \boldsymbol{x} \rangle B(\boldsymbol{x}, \boldsymbol{f}) + \langle \boldsymbol{a}^*, \boldsymbol{x} \rangle^2 Q(\boldsymbol{f}) = 0 \text{ for all } \boldsymbol{x} \in \boldsymbol{V}. \tag{3.4}$$

Also, $\delta_{\boldsymbol{a}^*, \boldsymbol{f}} \in \mathrm{O}'(\boldsymbol{V}, Q)$ is satisfied if, and only if, (3.4) holds alongside with

$$\boldsymbol{a}^* \in (\boldsymbol{V}^\perp)^\circ. \tag{3.5}$$

If $Q(\boldsymbol{f}) \neq 0$, then (2.1), (2.4) and (2.10) show

$$\xi_{\boldsymbol{f}} = \delta_{\boldsymbol{c}^*, \boldsymbol{f}}, \quad \text{where } \boldsymbol{c}^* := -Q(\boldsymbol{f})^{-1} D(\boldsymbol{f}). \tag{3.6}$$

The claims in (a)–(c) and (3.2) now follow easily from (3.4), (3.5), (3.6), $\boldsymbol{f} \in \boldsymbol{V}^\perp$ being equivalent to $D(\boldsymbol{f}) = \boldsymbol{o}^*$ and $\mathrm{id}_{\boldsymbol{V}} = \delta_{\boldsymbol{o}^*, \boldsymbol{f}}$. By virtue of the bijection (2.3), the equations in (3.3) are immediate from (2.2), $\dim(\boldsymbol{V}^\perp)^\circ = n - k$ and (3.2). □

Next, we present a crucial result about a particular subgroup of the group $\Delta(\boldsymbol{V}, \boldsymbol{f})$; compare Corollary 5.8.

Lemma 3.2. *Let $\boldsymbol{f}$ be a non-zero vector of a metric vector space $(\boldsymbol{V}, Q)$. Then the following are equivalent.*

(a) *The group $\{\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \mid \boldsymbol{a}^* \in \{\boldsymbol{f}\}^\circ\}$ is contained in* $\mathrm{O}'(\boldsymbol{V}, Q)$.

(b) *One of the subsequent conditions holds:*

$$Q(\boldsymbol{f}) = 0 \ \textit{and}\ \boldsymbol{V}^\perp = F\boldsymbol{f}; \tag{3.7}$$

$$\dim \boldsymbol{V} = 1; \tag{3.8}$$

$$\dim \boldsymbol{V} = 2,\ Q(\boldsymbol{f}) \neq 0,\ \dim \boldsymbol{V}^\perp = 0 \ \textit{and}\ |F| = 2. \tag{3.9}$$

Proof. (a) $\Rightarrow$ (b) If $\dim \boldsymbol{V} \leq 1$, then (3.8) holds due to $\boldsymbol{f} \neq \boldsymbol{o}$. Otherwise, $\{\boldsymbol{f}\}^\circ$ is at least one-dimensional and so $|F| \leq \left|\{\boldsymbol{f}\}^\circ\right|$. The bijection (2.3) shows that the group $\{\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \mid \boldsymbol{a}^* \in \{\boldsymbol{f}\}^\circ\}$ is of order $\left|\{\boldsymbol{f}\}^\circ\right|$. By our assumption, $\{\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \mid \boldsymbol{a}^* \in \{\boldsymbol{f}\}^\circ\} \subseteq \Delta\mathrm{O}'(\boldsymbol{V}, Q, \boldsymbol{f})$; therefore

$$2 \leq |F| \leq \left|\{\boldsymbol{f}\}^\circ\right| = \left|\{\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \mid \boldsymbol{a}^* \in \{\boldsymbol{f}\}^\circ\}\right| \leq \left|\Delta\mathrm{O}'(\boldsymbol{V}, Q, \boldsymbol{f})\right|. \tag{3.10}$$

Case 1: $Q(\boldsymbol{f}) = 0$. Then (3.10) implies that $\boldsymbol{f}$ meets the hypotheses of Lemma 3.1 (d). Hence $\boldsymbol{f} \in \boldsymbol{V}^\perp$ and so $F\boldsymbol{f} \leq \boldsymbol{V}^\perp$. On the other hand, the second equation in (3.2) yields $\{\boldsymbol{f}\}^\circ \leq (\boldsymbol{V}^\perp)^\circ$. This gives, by going over to annihilators on either side, $F\boldsymbol{f} \geq \boldsymbol{V}^\perp$. All in all, (3.7) holds.

Case 2: $Q(\boldsymbol{f}) \neq 0$. Now (3.10) implies that $\boldsymbol{f}$ meets the hypotheses of Lemma 3.1 (a). Hence $\boldsymbol{f} \notin \boldsymbol{V}^\perp$. Furthermore, $\left|\Delta\mathrm{O}'(\boldsymbol{V}, Q, \boldsymbol{f})\right| = 2$ and so, together with (3.10), we get $|F| = \left|\{\boldsymbol{f}\}^\circ\right| = 2$. Consequently, $\dim \{\boldsymbol{f}\}^\circ = 1$, whence $(F\boldsymbol{f})^\circ = \{\boldsymbol{f}\}^\circ$ results in $\dim \boldsymbol{V} = \dim(F\boldsymbol{f}) + \dim(F\boldsymbol{f})^\circ = 2$. From $\operatorname{Char} F = 2$ and $\dim \boldsymbol{V} = 2$ being even, the radical $\boldsymbol{V}^\perp$ has even dimension ≤ 2. Due to $\boldsymbol{f} \notin \boldsymbol{V}^\perp$, we cannot have $\dim \boldsymbol{V}^\perp = 2$. Thus $\dim \boldsymbol{V}^\perp = 0$. To sum up, (3.9) is satisfied.

(b) $\Rightarrow$ (a) First, suppose that (3.7) holds. Then Lemma 3.1 (d) applies together with $(\boldsymbol{V}^\perp)^\circ = \{\boldsymbol{f}\}^\circ$. By the second equation in (3.2), we have $\{\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \mid \boldsymbol{a}^* \in \{\boldsymbol{f}\}^\circ\} \subseteq \mathrm{O}'(\boldsymbol{V}, Q)$.

Next, suppose that (3.8) holds. Here the group $\{\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \mid \boldsymbol{a}^* \in \{\boldsymbol{f}\}^\circ\}$ coincides with $\{\mathrm{id}_{\boldsymbol{V}}\}$ and so it is contained in $\mathrm{O}'(\boldsymbol{V}, Q)$.

Finally, suppose that (3.9) holds. Due to $|F| = 2$, $Q(\boldsymbol{f}) \neq 0$ actually means $Q(\boldsymbol{f}) = -1 = 1$. From $\boldsymbol{V}^\perp = \{\boldsymbol{o}\}$ and $\operatorname{Char} F = 2$, the polar form B of Q is non-degenerate and alternating. Since $\dim \boldsymbol{V} = 2$ and $|F| = 2$, the annihilator $\{\boldsymbol{f}\}^\circ$ comprises only $D(\boldsymbol{f}) = -Q(\boldsymbol{f})^{-1}B(\boldsymbol{f}, \cdot)$ and $\boldsymbol{o}^*$. Thus, by $\delta_{\boldsymbol{o}^*,\boldsymbol{f}} = \mathrm{id}_{\boldsymbol{V}}$ and (2.10), $\{\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \mid \boldsymbol{a}^* \in \{\boldsymbol{f}\}^\circ\} = \{\mathrm{id}_{\boldsymbol{V}}, \xi_{\boldsymbol{f}}\} \subseteq \mathrm{O}'(\boldsymbol{V}, Q)$. □

The following lemma will take us to Corollary 5.11, which will be used in the proof of Theorem 5.12.

Lemma 3.3. *Let $\boldsymbol{f}$ be a non-zero vector of a metric vector space $(\boldsymbol{V}, Q)$. Then, for all $s \in F \setminus \{0,1\}$ and all non-zero $\boldsymbol{a}^* \in \{\boldsymbol{f}\}^\circ$, the mapping $s\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \in \mathrm{GL}(\boldsymbol{V})$ does not belong to $\mathrm{O}'(\boldsymbol{V}, Q)$.*

Proof. Suppose, by way of contradiction, that $s\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \in \mathrm{O}'(\boldsymbol{V}, Q)$ with s and $\boldsymbol{a}^*$ as above. So, $\dim \boldsymbol{V} \geq 2$ and $|F| > 2$. The only eigenvalue of the transvection $\delta_{\boldsymbol{a}^*,\boldsymbol{f}}$ is $1 \in F$ and the corresponding eigenspace equals the hyperplane $\ker \boldsymbol{a}^*$ of $\boldsymbol{V}$. Consequently, $s \neq 0, 1$ is the only eigenvalue of $s\delta_{\boldsymbol{a}^*,\boldsymbol{f}}$ and so the radical $\boldsymbol{V}^\perp$, which is fixed elementwise under $s\delta_{\boldsymbol{a}^*,\boldsymbol{f}}$, turns out to be $\{\boldsymbol{o}\}$. From $s\delta_{\boldsymbol{a}^*,\boldsymbol{f}}(\boldsymbol{f}) = s\boldsymbol{f}$ and $s\delta_{\boldsymbol{a}^*,\boldsymbol{f}} \in \mathrm{O}'(\boldsymbol{V}, Q)$, the hyperplane $\{\boldsymbol{f}\}^\perp$ coincides with $s\delta_{\boldsymbol{a}^*,\boldsymbol{f}}(\{\boldsymbol{f}\}^\perp)$. A hyperplane of $\boldsymbol{V}$ is fixed (as a set) under $s\delta_{\boldsymbol{a}^*,\boldsymbol{f}}$ if, and only if, it contains $\boldsymbol{f}$. Therefore $\boldsymbol{f} \in \{\boldsymbol{f}\}^\perp$ or, said differently, $B(\boldsymbol{f}, \boldsymbol{f}) = 0$.

Case 1: $Q(\boldsymbol{f}) \neq 0$. Then $0 = B(\boldsymbol{f}, \boldsymbol{f}) = 2Q(\boldsymbol{f})$ forces $\mathrm{Char}\, F = 2$. Now $Q(s\delta_{\boldsymbol{a}^*,\boldsymbol{f}}(\boldsymbol{f})) = s^2 Q(\boldsymbol{f}) = Q(\boldsymbol{f})$ shows $s^2 = 1$. This implies $s = 1$, an absurdity.

Case 2: $Q(\boldsymbol{f}) = 0$. As $\boldsymbol{f} \notin \boldsymbol{V}^\perp = \{\boldsymbol{o}\}$ and $\boldsymbol{a}^* \neq \boldsymbol{o}^*$, there exists a vector $\boldsymbol{u} \in \boldsymbol{V}$ with $B(\boldsymbol{f}, \boldsymbol{u}) = 1$ and $\langle \boldsymbol{a}^*, \boldsymbol{u} \rangle \neq 0$. We put $\boldsymbol{v} := \boldsymbol{u} - Q(\boldsymbol{u})\boldsymbol{f}$, so that $\delta_{\boldsymbol{a}^*,\boldsymbol{f}}(\boldsymbol{v}) = \boldsymbol{u} + (\langle \boldsymbol{a}^*, \boldsymbol{u} \rangle - Q(\boldsymbol{u}))\boldsymbol{f}$. Then, by straightforward calculations, $Q(\boldsymbol{v}) = 0$ and $Q(s\delta_{\boldsymbol{a}^*,\boldsymbol{f}}(\boldsymbol{v})) = s^2 \langle \boldsymbol{a}^*, \boldsymbol{u} \rangle \neq 0$. This contradicts $s\delta_{\boldsymbol{a}^*,\boldsymbol{f}}$ being an isometry. □

Our final lemma relies on a result by E. M. Schröder [18, (1.25) Satz]. It will be an essential tool for proving Theorem 5.9.

Lemma 3.4. *Let $(\boldsymbol{V}, Q_1)$ be a metric vector space such that $\boldsymbol{V}^{\perp_1} = \{\boldsymbol{o}\}$. Furthermore, suppose that none of the subsequent conditions applies:*

$$\dim \boldsymbol{V} = 1 \ \text{ and } \ |F| = 3; \tag{3.11}$$

$$\dim \boldsymbol{V} = 2 \ \text{ and } \ |F| = 2. \tag{3.12}$$

If a quadratic form $Q_2 \colon \boldsymbol{V} \to F$ satisfies

$$\mathrm{O}(\boldsymbol{V}, Q_1) = \mathrm{O}(\boldsymbol{V}, Q_2) \ \text{ or } \ \mathrm{O}(\boldsymbol{V}, Q_1) = \mathrm{O}'(\boldsymbol{V}, Q_2), \tag{3.13}$$

then $Q_1 = cQ_2$ for some $c \in F^\times$.

Proof. Our first goal is to establish that, whenever $\dim \boldsymbol{V} \geq 1$, any Q_1-reflection is also a Q_2-reflection and vice versa. To this end, let us pick any vector $\boldsymbol{f} \in \boldsymbol{V} \setminus \{\boldsymbol{o}\}$, whence $\boldsymbol{f} \notin \boldsymbol{V}^{\perp_1}$. Also, for $i \in \{1, 2\}$, we put (within this proof only)

$$d_i(\boldsymbol{f}) := \left|\Delta\mathrm{O}(\boldsymbol{V}, Q_i, \boldsymbol{f})\right| \ \text{ and } \ d_i'(\boldsymbol{f}) := \left|\Delta\mathrm{O}'(\boldsymbol{V}, Q_i, \boldsymbol{f})\right|;$$

compare (3.1).

Case 1: $Q_1(\boldsymbol{f}) \neq 0$. Then $\boldsymbol{f}$ and Q_1 satisfy the hypotheses of Lemma 3.1 (a), which gives $d_1(\boldsymbol{f}) = 2$. Hence (3.13) implies

$$d_2(\boldsymbol{f}) = 2 \ \text{ or } \ d_2'(\boldsymbol{f}) = 2. \tag{3.14}$$

We use this intermediate result in order to find out which of the hypotheses appearing in Lemma 3.1 (a)–(d) are met by $\boldsymbol{f}$ and Q_2.

Obviously, the hypotheses of (b) do not hold, since under these circumstances we would get $d_2(\boldsymbol{f}) = d_2'(\boldsymbol{f}) = 1$, a contradiction to (3.14). Likewise, the hypotheses of (c) cannot be fulfilled. We claim that the hypotheses of (d) are not satisfied either. For a verification, we assume that the contrary holds. So (3.3), with $n := \dim \boldsymbol{V}$ and $k := \dim \boldsymbol{V}^{\perp_2}$, gives $d_2(\boldsymbol{f}) = |F^\times| \cdot |F|^{n-1}$ and $d_2'(\boldsymbol{f}) = |F|^{n-k}$. If $d_2(\boldsymbol{f}) = 2$, then either (3.11) or (3.12) holds; both cases are contradictory, since they have been excluded. Thus (3.14) means $d_2'(\boldsymbol{f}) = 2$, whence $|F| = 2$ and $n - k = 1$. Then, due to $\operatorname{Char} F = 2$, the polar form of Q_2 is alternating and so its rank $n - k = 1$ turns out to be even, which is also contradictory.

By the above, $\boldsymbol{f}$ and Q_2 are compelled to satisfy the hypotheses of Lemma 3.1 (a), that is $\boldsymbol{f} \notin \boldsymbol{V}^{\perp_2}$ and $Q_2(\boldsymbol{f}) \neq 0$. Consequently, $d_2(\boldsymbol{f}) = d_2'(\boldsymbol{f}) = 2$ and so, by (3.13) and Lemma 3.1 (a), the Q_1-reflection in the direction of $\boldsymbol{f}$ coincides with the Q_2-reflection in the direction of $\boldsymbol{f}$.

Case 2: $Q_1(\boldsymbol{f}) = 0$. So there is no Q_1-reflection in the direction of $\boldsymbol{f}$. Clearly, $\boldsymbol{f}$ and Q_1 satisfy the hypotheses of Lemma 3.1 (b), whence $d_1(\boldsymbol{f}) = 1$. Now (3.13) implies

$$d_2(\boldsymbol{f}) = 1 \ \text{ or } \ d_2'(\boldsymbol{f}) = 1. \tag{3.15}$$

Since $\boldsymbol{f} \notin \boldsymbol{V}^{\perp_1} = \{\boldsymbol{o}\}$, there is an auxiliary vector $\boldsymbol{u} \in \boldsymbol{V}$ with $B_1(\boldsymbol{f}, \boldsymbol{u}) \neq 0$. Also, $Q_1(\boldsymbol{f}) = 0$ implies $B_1(\boldsymbol{f}, \boldsymbol{f}) = 0$. Therefore $B_1(\boldsymbol{f}, \boldsymbol{f} + \boldsymbol{u}) = B_1(\boldsymbol{f}, \boldsymbol{u}) \neq 0$ and $Q_1(\boldsymbol{f} + \boldsymbol{u}) = B_1(\boldsymbol{f}, \boldsymbol{u}) + Q_1(\boldsymbol{u}) \neq Q_1(\boldsymbol{u})$. We put $\boldsymbol{v} := \boldsymbol{u}$ if $Q_1(\boldsymbol{u}) \neq 0$ and $\boldsymbol{v} := \boldsymbol{f} + \boldsymbol{u}$ otherwise. Thus $Q_1(\boldsymbol{v}) \neq 0$ and $B_1(\boldsymbol{f}, \boldsymbol{v}) \neq 0$. Consequently, the Q_1-reflection in the direction of $\boldsymbol{v}$ does not fix $\boldsymbol{f}$. From the previous case, this Q_1-reflection is also a Q_2-reflection, which in turn entails $\boldsymbol{f} \notin \boldsymbol{V}^{\perp_2}$. Therefore and by (3.15), $\boldsymbol{f}$ and Q_2 satisfy the hypotheses of Lemma 3.1 (b). Hence $Q_2(\boldsymbol{f}) = 0$ and a Q_2-reflection in the direction of $\boldsymbol{f}$ does not exist either.

Finally, let us verify our assertion concerning Q_1 and Q_2: If $\dim \boldsymbol{V} = 0$, then $Q_1 = Q_2$ is the zero form and so $Q_1 = cQ_2$ holds for $c := 1$. Otherwise, by the above, the set of all Q_1-reflections coincides with the set of all Q_2-reflections and, clearly, $\{\boldsymbol{o}\} = \boldsymbol{V}^{\perp_1} \neq \boldsymbol{V}$. Under these premises, [18, (1.25) Satz] (see also [20, (7.81) Satz], [21, 1.7.4]) establishes $Q_1 = cQ_2$ for some $c \in F^\times$. □

4 Affine metric spaces

Throughout this section $\boldsymbol{V}$ denotes a vector space. First, we collect some well-known results about affine spaces and affine mappings. Our terminology is close to the one in [4, p. 33] and [11, Ch. 2, Ch. 3]. For proofs we refer also to [2, Ch. II], [14, Ch. 6], [19, § 5], [25, Ch. 2], even though the terminology from there may be different from ours.

If $\boldsymbol{u} \in \boldsymbol{V}$ and $\boldsymbol{T} \leq \boldsymbol{V}$, then the coset $\boldsymbol{u} + \boldsymbol{T}$ will be addressed as an *affine subspace*[3] of $\boldsymbol{V}$. The *affine space* $\mathbb{A}(\boldsymbol{V})$ is understood to be the set comprising all affine subspaces of $\boldsymbol{V}$. The *dimension* $\dim \mathbb{A}(\boldsymbol{V})$ is defined as $\dim \boldsymbol{V}$. The cosets of subspaces $\boldsymbol{T} \leq \boldsymbol{V}$ with dimension 0, 1, 2 and $\dim \boldsymbol{V} - 1$ are the *affine points*, *affine lines*, *affine planes* and *affine hyperplanes* of $\boldsymbol{V}$. Given any $\boldsymbol{x} \in \boldsymbol{V}$ we shall usually write $\boldsymbol{x}$ for the affine point $\boldsymbol{x} + \{\boldsymbol{o}\}$. Also, we shall drop the adjective "affine" when speaking about points if no confusion is to be expected. By analogy to the above, each affine subspace $\boldsymbol{u} + \boldsymbol{T}$ of $\boldsymbol{V}$ gives rise to the *affine space* $\mathbb{A}(\boldsymbol{u} + \boldsymbol{T})$. It comprises all affine subspaces of $\boldsymbol{V}$ that are contained in $\boldsymbol{u} + \boldsymbol{T}$, and we put $\dim \mathbb{A}(\boldsymbol{u} + \boldsymbol{T}) := \dim \boldsymbol{T}$.

Let $\tilde{\boldsymbol{V}}$ also be a vector space. We consider affine spaces $\mathbb{A}(\boldsymbol{u}+\boldsymbol{T})$ and $\mathbb{A}(\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{T}})$ with $\boldsymbol{u} \in \boldsymbol{V}$, $\boldsymbol{T} \leq \boldsymbol{V}$, $\tilde{\boldsymbol{u}} \in \tilde{\boldsymbol{V}}$ and $\tilde{\boldsymbol{T}} \leq \tilde{\boldsymbol{V}}$. A mapping $\gamma\colon \boldsymbol{u} + \boldsymbol{T} \to \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{T}}$ is said to be *affine* provided that it can be written in the form

$$\gamma\colon \boldsymbol{u} + \boldsymbol{T} \to \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{T}}\colon \boldsymbol{x} \mapsto \gamma(\boldsymbol{w}) + \gamma_+(\boldsymbol{x} - \boldsymbol{w}) \tag{4.1}$$

for some point $\boldsymbol{w} \in \boldsymbol{u} + \boldsymbol{T}$ and some linear mapping $\gamma_+\colon \boldsymbol{T} \to \tilde{\boldsymbol{T}}$. An *affinity* is understood to be a bijective affine mapping.

Let us briefly recall a few properties of the affine mapping γ appearing in (4.1): Under γ, the affine space $\mathbb{A}(\boldsymbol{u} + \boldsymbol{T})$ is mapped into the affine space $\mathbb{A}(\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{T}})$. We have $\gamma_+(\boldsymbol{x} - \boldsymbol{y}) = \gamma(\boldsymbol{x}) - \gamma(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{u} + \boldsymbol{T}$, so that γ_+ is uniquely determined by γ, whereas any point of $\mathbb{A}(\boldsymbol{u} + \boldsymbol{T})$ may take over the role of $\boldsymbol{w}$ in (4.1). Also, the affine mapping γ is bijective if, and only if, γ_+ is a linear bijection.

The group of all affinities of $\boldsymbol{V}$ onto itself is denoted by $\mathrm{AGL}(\boldsymbol{V})$ and acts faithfully on $\mathbb{A}(\boldsymbol{V})$. Any $\gamma \in \mathrm{AGL}(\boldsymbol{V})$ can be written in a *unique* way as

$$\gamma\colon \boldsymbol{V} \to \boldsymbol{V}\colon \boldsymbol{x} \mapsto \boldsymbol{t} + \gamma_+(\boldsymbol{x}) \text{ with } \gamma_+ \in \mathrm{GL}(\boldsymbol{V}) \text{ and } \boldsymbol{t} \in \boldsymbol{V}. \tag{4.2}$$

Indeed, it suffices to rewrite (4.1) with $\boldsymbol{w} := \boldsymbol{o} \in \boldsymbol{V}$ and $\boldsymbol{t} := \gamma(\boldsymbol{o})$. In particular, (4.2) defines a *translation* if, and only if, $\gamma_+ = \mathrm{id}_{\boldsymbol{V}}$.

In order to obtain a linear representation of the group $\mathrm{AGL}(\boldsymbol{V})$, we change over from $\boldsymbol{V}$ to the affine hyperplane $\{1\} \times \boldsymbol{V} = (1, \boldsymbol{o}) + \{0\} \times \boldsymbol{V}$ of the vector space

[3] Some authors consider also the empty set as being an affine subspace of $\boldsymbol{V}$. We refrain from following this convention.

$F \times \boldsymbol{V}$. Thereby we make use of the affinity

$$\varepsilon \colon \boldsymbol{V} \to \{1\} \times \boldsymbol{V} \colon \boldsymbol{x} \mapsto (1, \boldsymbol{o}) + (0, \boldsymbol{x}) = (1, \boldsymbol{x}). \tag{4.3}$$

If γ is given as in (4.2), then $\varepsilon \circ \gamma \circ \varepsilon^{-1}$ is an affinity of $\{1\} \times \boldsymbol{V}$. Furthermore,

$$\gamma^{\zeta} \colon F \times \boldsymbol{V} \to F \times \boldsymbol{V} \colon (x_0, \boldsymbol{x}) \mapsto \big(x_0, x_0 \boldsymbol{t} + \gamma_+(\boldsymbol{x})\big) \tag{4.4}$$

is the *only* linear mapping of $F \times \boldsymbol{V}$ to itself that extends $\varepsilon \circ \gamma \circ \varepsilon^{-1}$. This γ^{ζ} is bijective. We therefore obtain that

$$\zeta \colon \operatorname{AGL}(\boldsymbol{V}) \to \operatorname{GL}(F \times \boldsymbol{V}) \colon \gamma \mapsto \gamma^{\zeta} \tag{4.5}$$

is a faithful linear representation of $\operatorname{AGL}(\boldsymbol{V})$. Its image will be written as $\operatorname{AGL}(\boldsymbol{V})^{\zeta}$. The pairing

$$\langle \cdot, \cdot \rangle \colon (F \times \boldsymbol{V}^*) \times (F \times \boldsymbol{V}) \to F \colon \big((a_0, \boldsymbol{a}^*), (x_0, \boldsymbol{x})\big) \mapsto a_0 x_0 + \langle \boldsymbol{a}^*, \boldsymbol{x} \rangle$$

allows us to consider $F \times \boldsymbol{V}^*$ as being the dual vector space of $F \times \boldsymbol{V}$. There is another faithful linear representation of $\operatorname{AGL}(\boldsymbol{V})$, which is known as the *dual* of (4.5); see [10, p. 4]. It reads

$$\beta \colon \operatorname{AGL}(\boldsymbol{V}) \to \operatorname{GL}(F \times \boldsymbol{V}^*) \colon \gamma \mapsto \gamma^{\beta} := \big((\gamma^{\zeta})^{\mathrm{T}}\big)^{-1} \tag{4.6}$$

and we denote its image by $\operatorname{AGL}(\boldsymbol{V})^{\beta}$. If γ is given as in (4.2), then

$$\gamma^{\beta}(a_0, \boldsymbol{a}^*) = \Big(a_0 - \langle (\gamma_+^{\mathrm{T}})^{-1}(\boldsymbol{a}^*), \boldsymbol{t} \rangle, (\gamma_+^{\mathrm{T}})^{-1}(\boldsymbol{a}^*)\Big) \text{ for all } (a_0, \boldsymbol{a}^*) \in F \times \boldsymbol{V}^*. \tag{4.7}$$

Lemma 4.1. $\operatorname{AGL}(\boldsymbol{V})^{\beta}$ *is the elementwise stabiliser of* $F(1, \boldsymbol{o}^*)$ *in* $\operatorname{GL}(F \times \boldsymbol{V}^*)$.

Proof. From (4.7), any $\gamma^{\beta} \in \operatorname{AGL}(\boldsymbol{V})^{\beta}$ fixes all linear forms in $F(1, \boldsymbol{o}^*)$. Conversely, any mapping belonging to $\operatorname{GL}(F \times \boldsymbol{V}^*)$ can be written as $(\varkappa^{\mathrm{T}})^{-1}$ with $\varkappa \in \operatorname{GL}(F \times \boldsymbol{V})$. If $(\varkappa^{\mathrm{T}})^{-1}$ fixes $F(1, \boldsymbol{o}^*)$ elementwise, then

$$\big\langle (1, \boldsymbol{o}^*), \varkappa(1, \boldsymbol{o}) \big\rangle = \big\langle \varkappa^{\mathrm{T}}(1, \boldsymbol{o}^*), (1, \boldsymbol{o}) \big\rangle = \big\langle (1, \boldsymbol{o}^*), (1, \boldsymbol{o}) \big\rangle = 1$$

implies $\varkappa(1, \boldsymbol{o}) = (1, \boldsymbol{t})$ for some $\boldsymbol{t} \in \boldsymbol{V}$. From $\{0\} \times \boldsymbol{V} = \ker(1, \boldsymbol{o}^*)$ being invariant under $\varkappa$, there exists a $\gamma_+ \in \operatorname{GL}(\boldsymbol{V})$ such that $\varkappa(0, \boldsymbol{x}) = \big(0, \gamma_+(\boldsymbol{x})\big)$ for all $\boldsymbol{x} \in \boldsymbol{V}$. The affinity γ arising from γ_+ and $\boldsymbol{t}$ according to (4.2) satisfies $\gamma^{\beta} = (\varkappa^{\mathrm{T}})^{-1}$. □

Remark 4.2. The vector space $F \times \boldsymbol{V}^*$ can be identified with the vector space consisting of all affine functions $\boldsymbol{V} \to F$ as follows: Any $(a_0, \boldsymbol{a}^*) \in F \times \boldsymbol{V}^*$ is taken for the affine function $\boldsymbol{V} \to F \colon \boldsymbol{x} \mapsto a_0 + \langle \boldsymbol{a}^*, \boldsymbol{x} \rangle$. The linear forms belonging to $F(1, \boldsymbol{o}^*)$ turn into the constant functions $\boldsymbol{V} \to F$. Furthermore, the faithful linear representation $\beta \colon \operatorname{AGL}(\boldsymbol{V}) \to \operatorname{GL}(F \times \boldsymbol{V}^*)$ in (4.6) may be described in the following way: For any $\gamma \in \operatorname{AGL}(\boldsymbol{V})$, the image of the affine function $(a_0, \boldsymbol{a}^*)$ under γ^{β} is given by the product function $(a_0, \boldsymbol{a}^*) \circ \gamma^{-1}$; see (4.7).

The *projective space* $\mathbb{P}(F \times \boldsymbol{V})$ is understood to be the set of all subspaces of $F \times \boldsymbol{V}$. The (projective) *dimension* of $\mathbb{P}(F \times \boldsymbol{V})$ is one less than the dimension of $F \times \boldsymbol{V}$. We adopt the usual geometric terms: *points*, *lines*, *planes* and *hyperplanes* of $\mathbb{P}(F \times \boldsymbol{V})$ are the subspaces of $F \times \boldsymbol{V}$ with (vector) dimension 1, 2, 3 and $\dim(F \times \boldsymbol{V}) - 1$, respectively; see [4, p. 30], [11, Ch. 2, Ch. 3]. Furthermore, we refer to [2, Ch. II § 9], [14, Ch. 6], [19, § 6], [25, Ch. 3]. The general linear group $\mathrm{GL}(F \times \boldsymbol{V})$ acts in a canonical way on $\mathbb{P}(F \times \boldsymbol{V})$: Any $\varkappa \in \mathrm{GL}(F \times \boldsymbol{V})$ determines a *projective collineation* on $\mathbb{P}(F \times \boldsymbol{V})$, which is given as $\boldsymbol{X} \mapsto \varkappa(\boldsymbol{X})$ for all $\boldsymbol{X} \in \mathbb{P}(F \times \boldsymbol{V})$. This action of $\mathrm{GL}(F \times \boldsymbol{V})$ has the kernel $F^{\times} \mathrm{id}_{F \times \boldsymbol{V}}$.

Using the affinity ε as in (4.3), the *embedding* of the affine space $\mathbb{A}(\boldsymbol{V})$ in the projective space $\mathbb{P}(F \times \boldsymbol{V})$ takes the form

$$\iota \colon \mathbb{A}(\boldsymbol{V}) \to \mathbb{P}(F \times \boldsymbol{V}) \colon \boldsymbol{x} + \boldsymbol{T} \mapsto \mathrm{span}(\varepsilon(\boldsymbol{x} + \boldsymbol{T})) = \mathrm{span}(\{1\} \times (\boldsymbol{x} + \boldsymbol{T})).$$

An element of $\mathbb{P}(F \times \boldsymbol{V})$ is said to be *at infinity* if it is not in the image of ι. In particular, $\{0\} \times \boldsymbol{V}$ is the only *hyperplane at infinity* of $\mathbb{P}(F \times \boldsymbol{V})$. The group $\mathrm{AGL}(\boldsymbol{V})^{\zeta}$ (see (4.5)) acts on $\mathbb{P}(F \times \boldsymbol{V})$ as a group of projective collineations, which allows us to deal with affinities in projective terms. Furthermore, by sending any subspace of $F \times \boldsymbol{V}$ to its annihilator, a bijection of $\mathbb{P}(F \times \boldsymbol{V})$ onto $\mathbb{P}(F \times \boldsymbol{V}^*)$ is obtained, which reverses inclusions. For example, the hyperplane at infinity corresponds to the point $F(1, \boldsymbol{o}^*)$. The action of the group $\mathrm{AGL}(\boldsymbol{V})^{\beta}$ (see (4.6)) on the point set of $\mathbb{P}(F \times \boldsymbol{V}^*)$ mirrors the action of $\mathrm{AGL}(\boldsymbol{V})$ on the set of affine hyperplanes of $\boldsymbol{V}$.

A quadratic form $Q \colon \boldsymbol{V} \to F$ makes $\mathbb{A}(\boldsymbol{V})$ into an *affine metric space* $\mathbb{A}(\boldsymbol{V}, Q)$, which can be equipped with a wealth of additional structure [20, § 9], [21, Sect. 3]. If an affinity $\mu \in \mathrm{AGL}(\boldsymbol{V})$ is given by analogy to (4.2), but with $\mu_+ \in \mathrm{O}(\boldsymbol{V}, Q)$ (resp. $\mu_+ \in \mathrm{O}'(\boldsymbol{V}, Q)$) and arbitrary $\boldsymbol{t} \in \boldsymbol{V}$, then μ is called a *motion* (resp. a *weak motion*) of $(\boldsymbol{V}, Q)$. All such motions (resp. weak motions) comprise the *motion group* $\mathrm{AO}(\boldsymbol{V}, Q)$ (resp. the *weak motion group* $\mathrm{AO}'(\boldsymbol{V}, Q)$).[4]

Given a point $\boldsymbol{p}$ in $\mathbb{A}(\boldsymbol{V})$ and a vector $\boldsymbol{r} \in \boldsymbol{V}$ such that $Q(\boldsymbol{r}) \neq 0$, the mapping

$$\xi_{\boldsymbol{p},\boldsymbol{r}} \colon \boldsymbol{V} \to \boldsymbol{V} \colon \boldsymbol{x} \mapsto \boldsymbol{x} - Q(\boldsymbol{r})^{-1} B(\boldsymbol{r}, \boldsymbol{x} - \boldsymbol{p})\boldsymbol{r} \tag{4.8}$$

is the *affine Q-reflection* with axis $\boldsymbol{p} + \{\boldsymbol{r}\}^{\perp}$ in the direction of $\boldsymbol{r}$; see [20, p. 99] or [21, p. 976], where the term *affine Q-symmetry* is used instead. Then, with the notation as in (4.2), $(\xi_{\boldsymbol{p},\boldsymbol{r}})_+ = \xi_{\boldsymbol{r}}$ and $\boldsymbol{t} = Q(\boldsymbol{r})^{-1} B(\boldsymbol{r}, \boldsymbol{p})\boldsymbol{r}$. This establishes $\xi_{\boldsymbol{p},\boldsymbol{r}} \in \mathrm{AO}'(\boldsymbol{V}, Q)$.

From (4.5) and (4.6), we obtain—by restriction—faithful linear representations of $\mathrm{AO}(\boldsymbol{V}, Q)$ and $\mathrm{AO}'(\boldsymbol{V}, Q)$. Their images are written as $\mathrm{AO}(\boldsymbol{V}, Q)^{\zeta}$, $\mathrm{AO}(\boldsymbol{V}, Q)^{\beta}$, $\mathrm{AO}'(\boldsymbol{V}, Q)^{\zeta}$ and $\mathrm{AO}'(\boldsymbol{V}, Q)^{\beta}$.

[4]There is no widely accepted terminology for these groups.

5 Main results

Let $\boldsymbol{V}$ be a vector space. Thus there is the faithful linear representation $\beta\colon \mathrm{AGL}(\boldsymbol{V}) \to \mathrm{GL}(F \times \boldsymbol{V}^*)$ as in (4.6). Also, we recall our standard notation $(Q, B, D, \perp)$ for dealing with quadratic forms; see Section 2. Our aim is to find *all* dyads of metric vector spaces $(\boldsymbol{V}, Q)$ and $(F \times \boldsymbol{V}^*, \tilde{Q})$ such that the β-image of the motion group $\mathrm{AO}(\boldsymbol{V}, Q)$ or the β-image of the weak motion group $\mathrm{AO}'(\boldsymbol{V}, Q)$ coincides with the weak orthogonal group $\mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$. That is, we require that one of the following holds:

$$\mathrm{AO}(\boldsymbol{V}, Q)^\beta = \mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q}); \tag{5.1}$$

$$\mathrm{AO}'(\boldsymbol{V}, Q)^\beta = \mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q}). \tag{5.2}$$

The next two propositions provide *most, but not all* of the solutions of the above problem. Therein, we make use of two auxiliary linear mappings together with their transposes:

$$\nu\colon \boldsymbol{V} \to F \times \boldsymbol{V}\colon \boldsymbol{x} \mapsto (0, \boldsymbol{x}), \quad \nu^{\mathrm{T}}\colon F \times \boldsymbol{V}^* \to \boldsymbol{V}^*\colon (a_0, \boldsymbol{a}^*) \mapsto \boldsymbol{a}^*,$$
$$\pi\colon F \times \boldsymbol{V} \to \boldsymbol{V}\colon (x_0, \boldsymbol{x}) \mapsto \boldsymbol{x}, \quad \pi^{\mathrm{T}}\colon \boldsymbol{V}^* \to F \times \boldsymbol{V}^*\colon \boldsymbol{a}^* \mapsto (0, \boldsymbol{a}^*).$$

Proposition 5.1. *Let* $\boldsymbol{V}$ *be a vector space.*

(a) *If* $Q\colon \boldsymbol{V} \to F$ *is a quadratic form with non-degenerate polar form* B*, then the quadratic form*

$$Q^\uparrow := Q \circ D^{-1} \circ \nu^{\mathrm{T}}\colon F \times \boldsymbol{V}^* \to F\colon (a_0, \boldsymbol{a}^*) \mapsto (Q \circ D^{-1})(\boldsymbol{a}^*) \tag{5.3}$$

satisfies $Q^\uparrow(1, \boldsymbol{o}^*) = 0$*, and the polar form* $B^\uparrow$ *of* $Q^\uparrow$ *has* $F(1, \boldsymbol{o}^*)$ *as its radical.*

(b) *If* $\tilde{Q}\colon F \times \boldsymbol{V}^* \to F$ *is a quadratic form with* $\tilde{Q}(1, \boldsymbol{o}^*) = 0$ *and with* $F(1, \boldsymbol{o}^*)$ *being the radical of its polar form* $\tilde{B}$*, then the linear mapping* $\pi \circ \tilde{D} \circ \pi^{\mathrm{T}}\colon \boldsymbol{V}^* \to \boldsymbol{V}$ *is invertible, and the quadratic form*

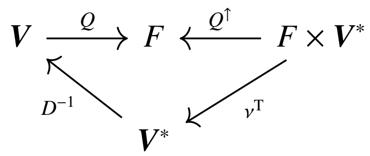
$$\tilde{Q}^\downarrow := \tilde{Q} \circ \pi^{\mathrm{T}} \circ (\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})^{-1}\colon \boldsymbol{V} \to F\colon \boldsymbol{x} \mapsto \tilde{Q}(0, (\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})^{-1}(\boldsymbol{x})) \tag{5.4}$$

has a non-degenerate polar form $\tilde{B}^\downarrow$.

(c) *If* Q *is given as in* (a)*, then* $(Q^\uparrow)^\downarrow = Q$ *and* $cQ^\uparrow = (cQ)^\uparrow$ *for all* $c \in F^\times$.

(d) *If* $\tilde{Q}$ *is given as in* (b)*, then* $(\tilde{Q}^\downarrow)^\uparrow = \tilde{Q}$ *and* $c\tilde{Q}^\downarrow = (c\tilde{Q})^\downarrow$ *for all* $c \in F^\times$.

Proof. (a) From (2.6), $\ker D = \boldsymbol{V}^\perp = \{\boldsymbol{o}\}$. So, D is bijective and $Q^\uparrow$ is well defined:



It is clear that $Q^{\uparrow}(1, \boldsymbol{o}^*) = Q(\boldsymbol{o}) = 0$. Substituting $\eta := D^{-1} \circ \nu^{\mathrm{T}}$ and $\tilde{D} := D^{\uparrow}$ in (2.8) and then using $(D^{-1})^{\mathrm{T}} = (D^{\mathrm{T}})^{-1} = D^{-1}$, which follows from (2.5), gives

$$D^{\uparrow} = \nu \circ D^{-1} \circ \nu^{\mathrm{T}} \colon F \times \boldsymbol{V}^* \to F \times \boldsymbol{V} \colon (a_0, \boldsymbol{a}^*) \mapsto (0, D^{-1}(\boldsymbol{a}^*)) \qquad (5.5)$$

as the induced linear mapping of $B^{\uparrow}$. From (5.5), $\ker D^{\uparrow} = F(1, \boldsymbol{o}^*)$. According to (2.6), the latter kernel equals the radical of $B^{\uparrow}$.

(b) By our assumption on the radical of $\tilde{B}$ and from (2.6), $\ker \tilde{D} = F(1, \boldsymbol{o}^*)$. Thus $\tilde{D}(F \times \boldsymbol{V}^*) = \tilde{D}\big(F(1, \boldsymbol{o}^*) \oplus (\{0\} \times \boldsymbol{V}^*)\big) = \tilde{D}(\{0\} \times \boldsymbol{V}^*)$. On the other hand, from (2.7), $\tilde{D}(F \times \boldsymbol{V}^*) = \big(F(1, \boldsymbol{o}^*)\big)^{\circ} = \{0\} \times \boldsymbol{V}$, whence $\tilde{D}(\{0\} \times \boldsymbol{V}^*) = \{0\} \times \boldsymbol{V}$. Therefore $(\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})(\boldsymbol{V}^*) = \boldsymbol{V}$. Hence the inverse mapping $(\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})^{-1}$ exists and $\tilde{Q}^{\downarrow}$ is well defined:

$$\begin{array}{ccccc} \boldsymbol{V} & \xrightarrow{\tilde{Q}^{\downarrow}} & F & \xleftarrow{\tilde{Q}} & F \times \boldsymbol{V}^* \\ & {\scriptstyle (\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})^{-1}} \searrow & & \nearrow {\scriptstyle \pi^{\mathrm{T}}} & \\ & & \boldsymbol{V}^* & & \end{array}$$

As $\tilde{D}$ coincides with its transpose, so does $(\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})^{-1}$. By analogy with (2.8),

$$\tilde{D}^{\downarrow} = ((\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})^{-1} \circ \pi) \circ \tilde{D} \circ (\pi^{\mathrm{T}} \circ (\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})^{-1}) = (\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})^{-1}. \qquad (5.6)$$

The radical of $\tilde{B}^{\downarrow}$ equals $\ker \tilde{D}^{\downarrow} = \{\boldsymbol{o}\}$ and so $\tilde{B}^{\downarrow}$ is non-degenerate.

(c) The first assertion follows from (5.3), (5.4), (5.5), $\nu^{\mathrm{T}} \circ \pi^{\mathrm{T}} = \mathrm{id}_{\boldsymbol{V}^*}$ and $\pi \circ \nu = \mathrm{id}_{\boldsymbol{V}}$:

$$(Q^{\uparrow})^{\downarrow} = \underbrace{Q \circ D^{-1} \circ \nu^{\mathrm{T}}}_{= Q^{\uparrow}} \circ \pi^{\mathrm{T}} \circ (\pi \circ \underbrace{\nu \circ D^{-1} \circ \nu^{\mathrm{T}}}_{= D^{\uparrow}} \circ \pi^{\mathrm{T}})^{-1} = Q.$$

The second assertion holds trivially.

(d) By our assumptions on $\tilde{Q}$ and $\tilde{B}$, we have $\tilde{Q} = \tilde{Q} \circ \pi^{\mathrm{T}} \circ \nu^{\mathrm{T}}$. So, from (5.4), (5.3) and (5.6), we obtain

$$(\tilde{Q}^{\downarrow})^{\uparrow} = \underbrace{\tilde{Q} \circ \pi^{\mathrm{T}} \circ (\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})^{-1}}_{= \tilde{Q}^{\downarrow}} \circ \underbrace{(\pi \circ \tilde{D} \circ \pi^{\mathrm{T}})}_{= (\tilde{D}^{\downarrow})^{-1}} \circ \nu^{\mathrm{T}} = \tilde{Q} \circ \pi^{\mathrm{T}} \circ \nu^{\mathrm{T}} = \tilde{Q}.$$

Again, the second assertion holds trivially. □

Proposition 5.2. *Let $(\boldsymbol{V}, Q)$ be a metric vector space such that the polar form B of Q is non-degenerate, let $Q^{\uparrow}$ be given as in* (5.3) *and let $c \in F^{\times}$. Then, with β is as in* (4.6), $\mathrm{AO}(\boldsymbol{V}, Q)^{\beta} = \mathrm{O}'(F \times \boldsymbol{V}^*, cQ^{\uparrow})$.

Proof. From $\mathrm{O}'(F \times \boldsymbol{V}^*, Q^\uparrow) = \mathrm{O}'(F \times \boldsymbol{V}^*, cQ^\uparrow)$, it suffices to verify the claim for $c = 1$. Let any motion $\mu \in \mathrm{AO}(\boldsymbol{V}, Q)$ be given by analogy to (4.2), that is with $\mu_+ \in \mathrm{O}(\boldsymbol{V}, Q)$ and $\boldsymbol{t} \in \boldsymbol{V}$. Then $\mu^\beta \in \mathrm{O}(F \times \boldsymbol{V}^*, Q^\uparrow)$ follows from

$$\underbrace{Q \circ D^{-1} \circ \nu^{\mathrm{T}}}_{=Q^\uparrow} \circ \mu^\beta = Q \circ D^{-1} \circ (\mu_+^{\mathrm{T}})^{-1} \circ \nu^{\mathrm{T}} = Q \circ \mu_+ \circ D^{-1} \circ \nu^{\mathrm{T}} = \underbrace{Q \circ D^{-1} \circ \nu^{\mathrm{T}}}_{=Q^\uparrow};$$

thereby we argue as follows. First, we use (5.3). Next, we read off from (4.7), with $\gamma := \mu$, that $\nu^{\mathrm{T}} \circ \mu^\beta = (\mu_+^{\mathrm{T}})^{-1} \circ \nu^{\mathrm{T}}$. Then we take into account $D^{-1} \circ (\mu_+^{\mathrm{T}})^{-1} = \mu_+ \circ D^{-1}$, which follows from rewriting (2.9) with $\varphi := \mu_+$. By our assumption, $Q \circ \mu_+ = Q$ and, finally, we apply (5.3) again. From Lemma 4.1, the radical of $B^\uparrow$ is elementwise invariant under μ^β. Hence $\mu^\beta \in \mathrm{O}'(F \times \boldsymbol{V}^*, Q^\uparrow)$.

Conversely, let any isometry belonging to $\mathrm{O}'(F \times \boldsymbol{V}^*, Q^\uparrow)$ be given. This isometry fixes $(1, \boldsymbol{o}^*)$ and, by Lemma 4.1, it can be written in the form γ^β with $\gamma \in \mathrm{AGL}(\boldsymbol{V})$ as in (4.2), that is with $\gamma_+ \in \mathrm{GL}(\boldsymbol{V})$ and $\boldsymbol{t} \in \boldsymbol{V}$. It remains to establish that $Q \circ \gamma_+ = Q$. The calculation

$$\begin{aligned} Q \circ \gamma_+ &= Q \circ \gamma_+ \circ \pi \circ \nu = Q \circ \pi \circ \gamma^\zeta \circ \underbrace{\nu \circ D^{-1} \circ \nu^{\mathrm{T}}}_{=D^\uparrow} \circ \pi^{\mathrm{T}} \circ D \\ &= Q \circ \pi \circ \nu \circ D^{-1} \circ \nu^{\mathrm{T}} \circ \gamma^\beta \circ \pi^{\mathrm{T}} \circ D \\ &= \underbrace{Q \circ D^{-1} \circ \nu^{\mathrm{T}}}_{=Q^\uparrow} \circ \gamma^\beta \circ \pi^{\mathrm{T}} \circ D = Q \circ D^{-1} \circ \nu^{\mathrm{T}} \circ \pi^{\mathrm{T}} \circ D = Q \end{aligned}$$

relies on the following reasoning: First, we multiply by $\mathrm{id}_{\boldsymbol{V}} = \pi \circ \nu$ and we use $\gamma_+ \circ \pi \circ \nu = \pi \circ \gamma^\zeta \circ \nu$, which follows from (4.4). Then we multiply by $\mathrm{id}_{\boldsymbol{V}} = D^{-1} \circ \nu^{\mathrm{T}} \circ \pi^{\mathrm{T}} \circ D$. Applying formula (2.9) to $\gamma^\beta \in \mathrm{O}'(F \times \boldsymbol{V}^*, Q^\uparrow)$ and $D^\uparrow$ yields $\gamma^\zeta \circ D^\uparrow = D^\uparrow \circ \gamma^\beta$; compare (4.6). Next, we remove $\pi \circ \nu = \mathrm{id}_{\boldsymbol{V}}$. By our assumption, $Q^\uparrow \circ \gamma^\beta = Q^\uparrow$. Finally, we cancel $D^{-1} \circ \nu^{\mathrm{T}} \circ \pi^{\mathrm{T}} \circ D = \mathrm{id}_{\boldsymbol{V}}$. □

Remark 5.3. Proposition 5.1 (a) and Proposition 5.2 (with $c := 1$) are sketched without proof in [15, p. 102] using the vector space of all affine functions $\boldsymbol{V} \to F$ rather than $F \times \boldsymbol{V}^*$; see Remark 4.2. Analogues of Proposition 5.1 (b) and Proposition 5.2 appear in [9] under the extra assumption $\mathrm{Char}\, F \neq 2$. Similar results, limited to the case $\dim \boldsymbol{V} = 2$ and $\mathrm{Char}\, F \neq 2$, can be found in [1, § 8,4–§ 9,4], [26], [27], even though the approach from there does not rely on a weak orthogonal group. See [7, Sect. 1] for a detailed comparison.

The interrelation between the quadratic forms $q_{\mathcal{P}}$ and $q_{\mathcal{L}}$, as introduced in [24, Sect. 2], does not at all resemble the interrelation between our Q and $Q^\uparrow$. Nevertheless, for the Euclidean and Minkowskian planes appearing in [24], the quadratic form $q_{\mathcal{L}}$ from there admits an interpretation in terms of our $Q^\uparrow$.

Remark 5.4. Under the premises of Proposition 5.2, the faithful linear representation β maps the set of all affine Q-reflections of $\mathbb{A}(\boldsymbol{V}, Q)$ onto the set of all $Q^\uparrow$-

reflections of $(F \times \boldsymbol{V}^*, Q^\uparrow)$. To be more precise, β takes an affine Q-reflection $\xi_{\boldsymbol{p},\boldsymbol{r}}$ as in (4.8) to the $Q^\uparrow$-reflection in the direction of $(-B(\boldsymbol{r},\boldsymbol{p}), D(\boldsymbol{r})) \in F \times \boldsymbol{V}^*$. The straightforward proof, which amounts to substitutions in (4.7), is left to the reader. This bijective correspondence allows for the translation of theorems about $Q^\uparrow$-reflections into theorems about affine Q-reflections and vice versa.

A correspondence in the same spirit, even though it relies—in our terminology—on making $\mathbb{P}(F \times \boldsymbol{V})$ into a *projective metric space*, is stated in [20, pp. 164–165].

Remark 5.5. We maintain the assumptions of Proposition 5.2. The quadratic form $Q \circ \varepsilon_+^{-1} \colon \{0\} \times \boldsymbol{V} \to F \colon (0, \boldsymbol{x}) \mapsto Q(\boldsymbol{x})$, where ε is given as (4.3), defines a quadric [21, p. 964] in the hyperplane at infinity of $\mathbb{P}(F \times \boldsymbol{V})$, namely

$$\mathcal{F} := \{F(0, \boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{V} \setminus \{\boldsymbol{o}\} \text{ and } Q(\boldsymbol{x}) = 0\}.$$

This quadric is known as the *absolute quadric* or the *quadric at infinity* related with the projective embedding of $\mathbb{A}(\boldsymbol{V}, Q)$ [5, p. 267], [20, p. 96]. All points of $\mathcal{F}$ are simple [21, p. 965]. Likewise, $Q^\uparrow$ defines the quadric

$$\mathcal{F}^\uparrow := \left\{F(a_0, \boldsymbol{a}^*) \mid (a_0, \boldsymbol{a}^*) \in F \times \boldsymbol{V}^* \setminus \{(0, \boldsymbol{o}^*)\} \text{ and } Q^\uparrow(a_0, \boldsymbol{a}^*) = 0\right\}$$

in $\mathbb{P}(F \times \boldsymbol{V}^*)$. All points of $\mathcal{F}^\uparrow$ other than $F(1, \boldsymbol{o}^*)$ are simple or, in other words, $\mathcal{F}^\uparrow$ is a cone with a one-point vertex.

A theorem in [5, p. 205] allows us to describe the geometric relationship between these two quadrics when F is the field of complex numbers: If $n = \dim \boldsymbol{V} \geq 2$, then the points of $\mathcal{F}^\uparrow$ are the annihilators of those hyperplanes of $\mathbb{P}(F \times \boldsymbol{V})$ which contain a tangent space of $\mathcal{F}$ with projective dimension $n - 2$. This description remains valid in our more general setting provided that $\mathcal{F}$ is non-empty. However, the proof of the underlying theorem, as given *loc. cit.*, fails to cover the case $\operatorname{Char} F = 2$.

In the particular case where $\dim \boldsymbol{V} = 3$ and $\operatorname{Char} F \neq 2$, we have a conic $\mathcal{F}$ lying in the plane at infinity $\{0\} \times \boldsymbol{V}$ of $\mathbb{P}(F \times \boldsymbol{V})$. Furthermore, the points of $\mathcal{F}^\uparrow$ arise as annihilators of the planes containing a tangent line of $\mathcal{F}$. See Figure 1, where a few of these planes are depicted.

Remark 5.6. Let $\operatorname{Char} F \neq 2$. Then, due to $\frac{1}{2}B(\boldsymbol{x}, \boldsymbol{x}) = Q(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{V}$, it is highly common to associate with Q the bilinear form $\frac{1}{2}B$ rather than B. By doing so, the mapping $\frac{1}{2}D$ takes over the role of D. This suggests a variant form of Proposition 5.1 by considering the pullback of Q along $\left(\frac{1}{2}D\right)^{-1} \circ \nu^{\mathrm{T}}$. From $\left(\frac{1}{2}D\right)^{-1} = 2D^{-1}$, in this way the quadratic form $4Q^\uparrow$ is being linked with Q. Proposition 5.2 covers this variant by putting $c := 4$.

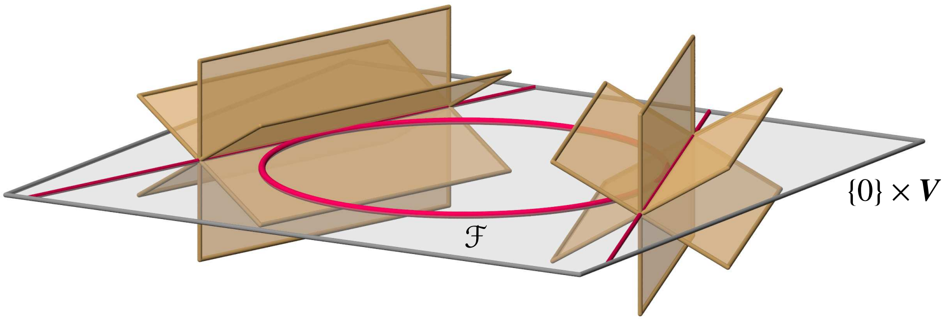


Figure 1: Planes containing a tangent line of $\mathcal{F}$, $\dim \boldsymbol{V} = 3$ and $F = \mathbb{R}$

Remark 5.7. Let us write down the transition from Q to $Q^{\uparrow}$, as in Proposition 5.1, in terms of coordinates. Upon choosing any basis of $\boldsymbol{V}$, say

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}, \tag{5.7}$$

we denote the corresponding dual basis of $\boldsymbol{V}^*$ by

$$\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \dots, \boldsymbol{e}_n^*\}. \tag{5.8}$$

Then $F \times \boldsymbol{V}$ admits the basis $\{(1, \boldsymbol{o}), (0, \boldsymbol{e}_1), (0, \boldsymbol{e}_2), \dots, (0, \boldsymbol{e}_n)\}$, which has

$$\{(1, \boldsymbol{o}^*), (0, \boldsymbol{e}_1^*), (0, \boldsymbol{e}_2^*), \dots, (0, \boldsymbol{e}_n^*)\} \tag{5.9}$$

as its dual basis. There is at least one matrix $W = (w_{ij})$ with entries in F and i, j ranging in $\{1, 2, \dots, n\}$ such that

$$Q\left(\sum_{h=1}^{n} x_h \boldsymbol{e}_h\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} x_i x_j \quad \text{for all} \quad x_1, x_2, \dots, x_n \in F. \tag{5.10}$$

The matrix of B relative to the basis (5.7) reads $W + W^{\mathrm{T}}$, where W^{T} denotes the transpose of W. From B being non-degenerate, $W + W^{\mathrm{T}}$ turns out invertible. We define the block diagonal matrix[5]

$$W^{\uparrow} := \operatorname{diag}\left(0, (W + W^{\mathrm{T}})^{-1} \cdot W \cdot (W + W^{\mathrm{T}})^{-1}\right)$$

with row and column indices of $W^{\uparrow}$ ranging in $\{0, 1, \dots, n\}$. The (i, j)-entry of $W^{\uparrow}$ will be written as $w_{ij}^{\uparrow}$. Since $(W + W^{\mathrm{T}})^{-1}$ describes D^{-1} relative to the bases (5.8) and (5.7), equations (5.3) and (5.10) yield

$$Q^{\uparrow}\left(a_0(1, \boldsymbol{o}^*) + \sum_{h=1}^{n} a_h(0, \boldsymbol{e}_h^*)\right) = \sum_{i=0}^{n} \sum_{j=0}^{n} w_{ij}^{\uparrow} a_i a_j \quad \text{for all} \quad a_0, a_1, \dots, a_n \in F.$$

[5]Note that the second block is congruent to W, since $(W + W^{\mathrm{T}})^{-1}$ is a symmetric matrix.

If $\operatorname{Char} F \neq 2$, then the above calculation can be simplified by choosing W as a *symmetric* matrix. So, $W + W^{\mathrm{T}} = 2W$ and one readily verifies[6] $4W^{\uparrow} = \operatorname{diag}(0, W^{-1})$. Moreover, we may start in (5.7) with an *orthogonal* basis of $(\boldsymbol{V}, Q)$, which makes the symmetric matrix W a diagonal matrix and simplifies the calculation of W^{-1}. In particular, over the real numbers there is a choice of (5.7) with

$$W = \operatorname{diag}(\underbrace{1, 1, \dots, 1}_{p}, \underbrace{-1, -1, \dots, -1}_{n-p}) = W^{-1},$$

that is, $\frac{1}{2}B$ (and likewise B) has signature $(p, n-p, 0)$. Then

$$4W^{\uparrow} = \operatorname{diag}(0, W) = \operatorname{diag}(0, \underbrace{1, 1, \dots, 1}_{p}, \underbrace{-1, -1, \dots, -1}_{n-p}).$$

This illustrates how the work about the real case fits into our approach; see the references at the beginning of Section 1.

If $\operatorname{Char} F = 2$, then B is a non-degenerate alternating bilinear form and so $\dim \boldsymbol{V} = n$ has to be even. We therefore are in a position to choose the basis (5.7) in such a way that the (alternating) $n \times n$ matrix of B relative to (5.7) takes the *block-diagonal form*

$$\operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Next, we may select an *upper triangular matrix* W subject to (5.10). As $W + W^{\mathrm{T}}$ equals the above-noted block diagonal matrix, we have

$$W = \operatorname{diag}\left(\begin{pmatrix} w_{11} & 1 \\ 0 & w_{22} \end{pmatrix}, \begin{pmatrix} w_{33} & 1 \\ 0 & w_{44} \end{pmatrix}, \dots, \begin{pmatrix} w_{n-1,n-1} & 1 \\ 0 & w_{nn} \end{pmatrix} \right).$$

From $W + W^{\mathrm{T}}$ being self-inverse, we end up with

$$W^{\uparrow} = \operatorname{diag}\left(0, \begin{pmatrix} w_{22} & 0 \\ 1 & w_{11} \end{pmatrix}, \begin{pmatrix} w_{44} & 0 \\ 1 & w_{33} \end{pmatrix}, \dots, \begin{pmatrix} w_{nn} & 0 \\ 1 & w_{n-1,n-1} \end{pmatrix} \right).$$

We now turn to the problem of describing *all* solutions of the problem posed at the beginning of this section. The following corollary to Lemma 3.2 will be a powerful tool, since it does not involve a quadratic form $Q \colon \boldsymbol{V} \to F$.

Corollary 5.8. *Let* $\boldsymbol{V}$ *be a vector space and let* $\tilde{Q} \colon F \times \boldsymbol{V}^* \to F$ *be a quadratic form. Then, with* β *as in* (4.6)*, the following are equivalent.*

(a) *The* β*-image of the translation group of* $\boldsymbol{V}$ *is contained in* $\mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$.

[6] As we observed in Remark 5.6, using $4Q^{\uparrow}$ simplifies matters when $\operatorname{Char} F \neq 2$.

(b) *One of the subsequent conditions holds:*

$$\tilde{Q}(1, \boldsymbol{o}^*) = 0 \ \textit{and} \ (F \times \boldsymbol{V}^*)^{\tilde{\perp}} = F(1, \boldsymbol{o}^*); \tag{5.11}$$

$$\dim(F \times \boldsymbol{V}^*) = 1; \tag{5.12}$$

$$\dim(F \times \boldsymbol{V}^*) = 2, \ \tilde{Q}(1, \boldsymbol{o}^*) \neq 0, \ \dim(F \times \boldsymbol{V}^*)^{\tilde{\perp}} = 0 \ \textit{and} \ |F| = 2. \tag{5.13}$$

Proof. Let any translation $\gamma \in \mathrm{AGL}(\boldsymbol{V})$ be given as in (4.2), that is with $\gamma_+ = \mathrm{id}_{\boldsymbol{V}}$ and $\boldsymbol{t} \in \boldsymbol{V}$. A comparison of (4.7) with (2.1) readily shows

$$\gamma^\beta = \delta_{(0,-\boldsymbol{t}),(1,\boldsymbol{o}^*)} \in \Delta(F \times \boldsymbol{V}^*, (1, \boldsymbol{o}^*));$$

see also (2.2). The annihilator $\{(1, \boldsymbol{o}^*)\}^\circ \leq F \times \boldsymbol{V}$ comprises precisely the vectors $(x_0, \boldsymbol{x})$ with $x_0 = 0$ and $\boldsymbol{x} \in \boldsymbol{V}$. So, the β-image of the translation group of $\boldsymbol{V}$ equals

$$\left\{\delta_{(x_0,\boldsymbol{x}),(1,\boldsymbol{o}^*)} \mid (x_0, \boldsymbol{x}) \in \{(1, \boldsymbol{o}^*)\}^\circ\right\}.$$

By the above, the proof is reduced to a rewording of Lemma 3.2: In the present context the "non-zero vector $\boldsymbol{f}$" and the "metric vector space $(\boldsymbol{V}, Q)$" from there have to be replaced with the "linear form $(1, \boldsymbol{o}^*)$" and the "metric vector space $(F \times \boldsymbol{V}^*, \tilde{Q})$", respectively. □

Next, we present our first main result.

Theorem 5.9. *Let* $(\boldsymbol{V}, Q)$ *and* $(F \times \boldsymbol{V}^*, \tilde{Q})$ *be metric vector spaces such that, with* β *as in* (4.6)*, one of the equations* (5.1) *or* (5.2) *is satisfied. Furthermore, suppose that none of the subsequent conditions applies:*

$$\dim \boldsymbol{V} = 0 \ \textit{and} \ \operatorname{Char} F = 2; \tag{5.14}$$

$$\dim \boldsymbol{V} = 1 \ \textit{and} \ |F| \leq 3; \tag{5.15}$$

$$\dim \boldsymbol{V} = 2 \ \textit{and} \ |F| = 2. \tag{5.16}$$

Then the following hold:

(a) *The polar form of Q is non-degenerate.*

(b) *The quadratic form $\tilde{Q}$ is a non-zero scalar multiple of that quadratic form $Q^\uparrow$ which arises from Q according to* (5.3).

Proof. *Case* 1: $\dim \boldsymbol{V} = 0$. Then (a) holds trivially, since Q is a zero quadratic form. Furthermore, $Q^\uparrow \colon F \times \boldsymbol{V}^* \to F$ is also a zero quadratic form. Since (5.14) does not apply, the weak orthogonal group of any non-zero quadratic form on $F \times \boldsymbol{V}^*$ contains $-\mathrm{id}_{F\times\boldsymbol{V}^*} \notin \mathrm{AO}(\boldsymbol{V}, Q)^\beta = \{\mathrm{id}_{F\times\boldsymbol{V}^*}\}$. Thus $\tilde{Q} = Q^\uparrow$, which verifies (b).

Case 2: $\dim \boldsymbol{V} \geq 1$. By our assumptions, the β-image of the translation group of $\boldsymbol{V}$ is contained in $\mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$. So, from Corollary 5.8 and from (5.15) being ruled out, it follows that (5.11) is satisfied. Consequently, $\tilde{Q}^{\downarrow} \colon \boldsymbol{V} \to F$ is well defined and its polar form $\tilde{B}^{\downarrow}$ is non-degenerate; see Proposition 5.1 (b). Next, we employ Proposition 5.2 on $(\boldsymbol{V}, \tilde{Q}^{\downarrow})$. Since $(\tilde{Q}^{\downarrow})^{\uparrow} = \tilde{Q}$ according to Proposition 5.1 (d), we have $\mathrm{AO}(\boldsymbol{V}, \tilde{Q}^{\downarrow})^{\beta} = \mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$. Going over to β-preimages gives, by (5.1) or (5.2), $\mathrm{AO}(\boldsymbol{V}, \tilde{Q}^{\downarrow}) = \mathrm{AO}(\boldsymbol{V}, Q)$ or $\mathrm{AO}(\boldsymbol{V}, \tilde{Q}^{\downarrow}) = \mathrm{AO}'(\boldsymbol{V}, Q)$. Consequently,

$$\mathrm{O}(\boldsymbol{V}, \tilde{Q}^{\downarrow}) = \mathrm{O}(\boldsymbol{V}, Q) \ \text{ or } \ \mathrm{O}(\boldsymbol{V}, \tilde{Q}^{\downarrow}) = \mathrm{O}'(\boldsymbol{V}, Q).$$

From the above statement and due to the exclusion of the cases appearing in (5.15) and (5.16), we are now in a position to make use of Lemma 3.4 with $\tilde{Q}^{\downarrow}$ and Q playing the roles of the quadratic forms Q_1 and Q_2, respectively. So, there is a $c \in F^{\times}$ with $cQ = \tilde{Q}^{\downarrow}$, which establishes (a). The last equation together with Proposition 5.1 gives $cQ^{\uparrow} = (cQ)^{\uparrow} = (\tilde{Q}^{\downarrow})^{\uparrow} = \tilde{Q}$, that is, (b) is satisfied too. □

Theorem 5.9 shows—loosely speaking—that in general $\tilde{Q}$ is determined by Q up to a non-zero scalar factor.

Remark 5.10. In Tables 1–4, we summarise all dyads of metric vector spaces $(\boldsymbol{V}, Q)$ and $(F \times \boldsymbol{V}^*, \tilde{Q})$ satisfying (5.1) or (5.2) *without* being covered by Theorem 5.9. We thereby make use of a fixed basis of $\boldsymbol{V}$ as in (5.7) together with its dual basis (5.8), the basis (5.9) of $F \times \boldsymbol{V}^*$ and coordinates $x_1, x_2, \ldots, x_n, a_0, a_1, \ldots, a_n \in F$.

Each table is to be read in two ways: First, quadratic forms Q and $\tilde{Q}$ appear in the *same block*, that is to mean between two adjacent horizontal lines, precisely when (5.1) or (5.2) applies. Take notice that in all these instances equation (5.1) is satisfied. So, all quadratic forms Q from within the same block share a common orthogonal group and all quadratic forms $\tilde{Q}$ from within the same block share a common weak orthogonal group. Second, quadratic forms Q and $\tilde{Q}$ appear in the *same row* if, and only if, $\tilde{Q} = Q^{\uparrow}$ or, equivalently, $Q = \tilde{Q}^{\downarrow}$. A quadratic form appears in a *row with one entry left in blank* if, and only if, it fails to meet the corresponding hypotheses of Proposition 5.1 (a) or (b).

There are but a few cases, where $\mathrm{AO}'(\boldsymbol{V}, Q)$ is a *proper subgroup* of $\mathrm{AO}(\boldsymbol{V}, Q)$. This happens precisely when Q is given as in the last row of Table 3 or as in the second, fourth, sixth or eighth row of Table 4.

Let $\dim \boldsymbol{V} = 0$ and $\operatorname{Char} F = 2$. The first row of Table 1 arises from that solution of (5.1), where both Q and $\tilde{Q} = Q^{\uparrow}$ are zero quadratic forms. Due to $\operatorname{Char} F = 2$, any non-zero quadratic form on $F \times \boldsymbol{V}^*$ has $\{\mathrm{id}_{F \times \boldsymbol{V}^*}\}$ as its weak orthogonal group. Thus, together with the zero quadratic form on $\boldsymbol{V}$, it provides a solution of (5.1). Therefore, *all* non-zero quadratic forms on $F \times \boldsymbol{V}^*$ are listed on the right hand side of the table's second row.

$Q(\boldsymbol{o})$	$\tilde{Q}(a_0, \boldsymbol{o}^*)$
$= 0$	$= 0$ $= \tilde{w}_{00} a_0^2$ (with $\tilde{w}_{00} \in F^\times$)

Table 1: $\dim \boldsymbol{V} = 0$ and $\operatorname{Char} F = 2$

Let $\dim \boldsymbol{V} = 1$ and $|F| = 2$. If Q and $\tilde{Q}$ satisfy (5.1) or (5.2), then $\tilde{Q}$ meets the requirements of Corollary 5.8 (a). Consequently, at least one of the three conditions of Corollary 5.8 (b) applies. Take notice that (5.11) cannot be met, since $\dim(F \times \boldsymbol{V}^*) = 2$ and $\operatorname{Char} F = 2$ forces $\dim(F \times \boldsymbol{V}^*)^{\tilde{\perp}}$ to be even. Equation (5.12) fails obviously. Thus we are led to (5.13), which is satisfied by precisely two quadratic forms. The first one appears in Table 2 on the right hand side of the first row, since it provides a solution of (5.1) together with the quadratic forms on $\boldsymbol{V}$, as listed on the left hand side of the second and third row. The second one is given by $(a_0, a_1\boldsymbol{e}_1^*) \mapsto a_0^2 + a_0a_1 + a_1^2$ for all $a_0, a_1 \in F$. But its weak orthogonal group, which equals $\mathrm{GL}(F \times \boldsymbol{V}^*)$, is not contained in $\mathrm{AGL}(\boldsymbol{V})^\beta$ due to $6 = \left|\mathrm{GL}(F \times \boldsymbol{V}^*)\right| > \left|\mathrm{AGL}(\boldsymbol{V})^\beta\right| = 2$. So, this second quadratic form does not appear in Table 2.

$Q(x_1\boldsymbol{e}_1)$	$\tilde{Q}(a_0, a_1\boldsymbol{e}_1^*)$
	$= a_0^2 + a_0a_1$
$= 0$	
$= x_1^2$	

Table 2: $\dim \boldsymbol{V} = 1$ and $|F| = 2$

Let $\dim \boldsymbol{V} = 1$ and $|F| = 3$. The first two rows of Table 3 contain all solutions $Q, \tilde{Q}$ of (5.1) with $\tilde{Q} = Q^\uparrow$. The zero quadratic form on $\boldsymbol{V}$ appears on the left hand side of the third row, since it satisfies (5.1) together with each of the two quadratic forms on the right hand side. So, the left column of Table 3 comprises *all* quadratic forms on $\boldsymbol{V}$. As we noted above, a quadratic form $\tilde{Q}$ may only appear in the right column if it meets one of the three conditions of Corollary 5.8 (b). Since neither (5.12) nor (5.13) can be satisfied, there remains (5.11). All $\tilde{Q}$ subject to (5.11) are listed in the right column of the table, as follows from Proposition 5.1. Consequently, the right column already has been filled up completely.

Let $\dim \boldsymbol{V} = 2$ and $|F| = 2$. There are four solutions $Q, \tilde{Q}$ of (5.1) with $\tilde{Q} = Q^\uparrow$. They can be read off from the first, third, fifth and seventh row of Table 4. We put these rows into four different blocks, since the corresponding quadratic forms on $\boldsymbol{V}$ have mutually distinct orthogonal groups; see below. There are no more entries

$Q(x_1\boldsymbol{e}_1)$	$\tilde{Q}(a_0, a_1\boldsymbol{e}_1^*)$
$= x_1^2$ $= -x_1^2$ $= 0$	$= a_1^2$ $= -a_1^2$

Table 3: $\dim \boldsymbol{V} = 1$ and $|F| = 3$

in the right column: This follows, as in the previous cases, since no quadratic form on $F \times \boldsymbol{V}^*$ satisfies one of the conditions (5.12) or (5.13) of Corollary 5.8 (b). However, it turns out that in each block there is precisely one more entry on the left hand side. The two quadratic forms on $\boldsymbol{V}$ from the first block have $\mathrm{GL}(\boldsymbol{V})$ as their common orthogonal group. The two quadratic forms on $\boldsymbol{V}$ from any of the remaining blocks also share a common orthogonal group, namely the stabiliser in $\mathrm{GL}(\boldsymbol{V})$ of a particular non-zero vector. More precisely, for the second, third and fourth block, this vector reads $\boldsymbol{e}_1 + \boldsymbol{e}_2$, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, respectively.

$Q(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2)$	$\tilde{Q}(a_0, a_1\boldsymbol{e}_1^* + a_2\boldsymbol{e}_2^*)$
$= x_1^2 + x_1x_2 + x_2^2$ $= 0$	$= a_1^2 + a_1a_2 + a_2^2$
$= x_1x_2$ $= x_1^2 + x_2^2$	$= a_1a_2$
$= x_1^2 + x_1x_2$ $= x_2^2$	$= a_1^2 + a_1a_2$
$= x_1x_2 + x_2^2$ $= x_1^2$	$= a_1a_2 + a_2^2$

Table 4: $\dim \boldsymbol{V} = 2$ and $|F| = 2$

To close this paper, let us address another question: Is there a way to describe, with the techniques at our disposal, the motion group or the weak motion group of additional affine metric spaces $\mathbb{A}(\boldsymbol{V}, Q)$ by changing over to the projective space $\mathbb{P}(F \times \boldsymbol{V}^*)$? The idea behind is that in this way one gains an additional "degree of freedom", since a projective collineation of $\mathbb{P}(F \times \boldsymbol{V}^*)$ onto itself is induced by $|F^\times|$ linear bijections differing by non-zero scalar factors. Furthermore, nothing is lost by the transition to $\mathbb{P}(F \times \boldsymbol{V}^*)$, since the linear representation β of $\mathrm{AGL}(\boldsymbol{V})$, as in (4.6), has a property that goes beyond its being faithful: The β-images of distinct affinities are not proportional and so they act differently on $\mathbb{P}(F \times \boldsymbol{V}^*)$.

We proceed by writing up a version of Lemma 3.3 in the same way as Corollary 5.8 resembles Lemma 3.2. Then we answer the raised question.

Corollary 5.11. *Let $\boldsymbol{V}$ be a vector space and let $\tilde{Q}\colon F \times \boldsymbol{V}^* \to F$ be a quadratic form. Then, with β as in* (4.6)*, for all $s \in F \setminus \{0, 1\}$ and all non-identical translations $\gamma \in \mathrm{AGL}(\boldsymbol{V})$, the mapping $s\gamma^\beta$ does not belong to the weak orthogonal group $\mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$.*

Theorem 5.12. *Let $(\boldsymbol{V}, Q)$ and $(F \times \boldsymbol{V}^*, \tilde{Q})$ be metric vector spaces such that $\mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$ induces the same collineation group on the projective space $\mathbb{P}(F \times \boldsymbol{V}^*)$ as one of the groups $\mathrm{AO}(\boldsymbol{V}, Q)^\beta$ or $\mathrm{AO}'(\boldsymbol{V}, Q)^\beta$, where β is given as in* (4.6)*. Then $\mathrm{AO}(\boldsymbol{V}, Q)^\beta = \mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$, unless $\dim \boldsymbol{V} = 0$, $\mathrm{Char}\, F \neq 2$ and $\tilde{Q}(F \times \boldsymbol{V}^*) \neq \{0\}$.*

Proof. *Case* 1: $\dim \boldsymbol{V} = 0$. Then $\mathrm{AO}(\boldsymbol{V}, Q)^\beta = \mathrm{AO}'(\boldsymbol{V}, Q)^\beta = \{\mathrm{id}_{\boldsymbol{V}}\}^\beta = \{\mathrm{id}_{F\times\boldsymbol{V}^*}\}$. If $\mathrm{Char}\, F = 2$ or $\tilde{Q}(F \times \boldsymbol{V}^*) = \{0\}$, then the radical of $\tilde{B}$ equals $F \times \boldsymbol{V}^*$ and so $\mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q}) = \{\mathrm{id}_{F\times\boldsymbol{V}^*}\}$. If $\mathrm{Char}\, F \neq 2$ and $\tilde{Q}(F \times \boldsymbol{V}^*) \neq \{0\}$, then the groups $\mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q}) = \{\pm \mathrm{id}_{F\times\boldsymbol{V}^*}\}$ and $\{\mathrm{id}_{F\times\boldsymbol{V}^*}\}$ are distinct, even though they determine the same (trivial) group of collineations on $\mathbb{P}(F \times \boldsymbol{V}^*)$.

Case 2: $\dim \boldsymbol{V} \geq 1$. By our assumptions, for each $\gamma \in \mathrm{AO}(\boldsymbol{V}, Q)$ or for each $\gamma \in \mathrm{AO}'(\boldsymbol{V}, Q)$ there is at least one scalar $s_\gamma \in F^\times$ such that $s_\gamma\gamma^\beta \in \mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$. We claim that any such s_γ has to be $1 \in F$. This is obvious when $|F| = 2$. Up to the end of the current paragraph, we therefore assume $|F| \geq 3$. If γ is an arbitrary non-trivial translation of $\boldsymbol{V}$, then $\gamma^\beta \in \mathrm{O}'(F \times \boldsymbol{V}^*)$ follows from Corollary 5.11. This implies, together with $\mathrm{id}_{\boldsymbol{V}}^\beta \in \mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$, that condition (a) of Corollary 5.8 is satisfied. Hence the equivalent condition (b) from there is satisfied too. Since (5.12) and (5.13) are false in the present setting, we read off from (5.11) that the radical of $\tilde{B}$ equals $F(1, \boldsymbol{o}^*)$. Now, returning to an arbitrary γ as described above, Lemma 4.1 gives $s_\gamma\gamma^\beta(1, \boldsymbol{o}^*) = (s_\gamma, \boldsymbol{o}^*)$, whereas $s_\gamma\gamma^\beta \in \mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$ forces $s_\gamma\gamma^\beta(1, \boldsymbol{o}^*) = (1, \boldsymbol{o}^*)$. Therefore $s_\gamma = 1$.

By the above, the hypotheses of Theorem 5.9 are fulfilled provided that neither (5.15) nor (5.16) is satisfied. So, up to these cases, Theorem 5.9 (a) gives the even stronger result $\mathrm{AO}(\boldsymbol{V}, Q)^\beta = \mathrm{AO}'(\boldsymbol{V}, Q)^\beta = \mathrm{O}'(F \times \boldsymbol{V}^*, \tilde{Q})$. Otherwise, the claim follows from Tables 2, 3 and 4 in Remark 5.10. □

All things considered, up to a single trivial case, adopting the projective point of view fails to significantly amplify the scope of our approach.

Remark 5.13. Tables 1 and 3 in [17] (and likewise Tables 3.1 and 3.3 in [16, pp. 192–195]) about metric vector spaces over $\mathbb{R}$ appear to be partially incorrect. The reason is that these tables contain—using our terminology—several dyads of metric vector spaces (V, Q) and $(F \times V^*, \tilde{Q})$, where the polar form of Q is degenerate and $F = \mathbb{R}$. Moreover, it is claimed (without giving a formal proof) that these dyads satisfy the assumptions of Theorem 5.12, which seems impossible by the above proof and Theorem 5.9.

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