Isomorphisms of Affine Plücker Spaces

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Abstract

All isomorphisms of Plücker spaces on affine spaces with dimensions ≥ 3 arise from collineations of the underlying affine spaces.

1 Introduction

Let L be a set and $\sim$ a reflexive and symmetric binary relation on L such that $(L, \sim)$ is connected, i.e., for any $a, b \in L$ there exists a finite sequence $a = a_1 \sim a_2 \sim \cdots \sim a_n = b$. Following W. Benz the pair $(L, \sim)$ is called a *Plücker space* [1, p. 199]. Elements $a, b \in L$ are said to be *related* if $a \sim b$. *Adjacent* elements ($a \approx b$) are characterized by $a \sim b$ and $a \neq b$.

The relation $\not\approx$ is reflexive and symmetric. However, $(L, \not\approx)$ is not necessarily a Plücker space, since it need not be connected. Nevertheless, L splits into a family of *connected components* with respect to $\not\approx$, say $(L_i)_{i \in I}$. Each component L_j ($j \in I$) gives rise to the Plücker space $(L_j, \not\approx_j)$, where $\not\approx_j$ denotes the restriction of $\not\approx$ to $L_j \times L_j$. On the other hand, L_j is not necessarily connected with respect to $\sim_j$, i.e., the restriction of $\sim$ to $L_j \times L_j$. Hence $(L_j, \sim_j)$ need not be a Plücker space.

Given two Plücker spaces $(L, \sim)$ and $(L', \sim')$ a bijection $\varphi : L \to L'$ is called an *isomorphism* if

$$a \sim b \Longleftrightarrow a^\varphi \sim' b^\varphi \text{ for all } a, b \in L. \tag{1}$$

Obviously, (1) and

$$a \not\approx b \Longleftrightarrow a^\varphi \not\approx' b^\varphi \text{ for all } a, b \in L \tag{2}$$

are equivalent conditions.

All automorphisms of $(L, \sim)$ form its so-called *Plücker group.* Write, as above, $(L_i)_{i \in I}$ for the connected components of L with respect to $\not\approx$. Then each automorphism of $(L_j, \not\approx_j)$ ($j \in I$) extends to an automorphism of $(L, \sim)$ by setting $x \mapsto x$ for all $x \in L \setminus L_j$.

Let $\mathsf{A} = (\mathcal{P}, \mathcal{L}, \parallel)$ be an affine space, where $\mathcal{P}$, $\mathcal{L}$ and $\parallel$ denotes the set of points, the set of lines and the parallelism, respectively. Lines $a, b \in \mathcal{L}$ are called *related* ($a \sim b$), if $a \cap b \neq \emptyset$. The pair $(\mathcal{L}, \sim)$ is satisfying the conditions mentioned before and will be called an *affine Plücker space*. We remark that for $\dim \mathsf{A} \geq 3$ the set $\mathcal{L}$ is the set of 'points' of partial linear space, the *affine Grassmann space* on $\mathcal{L}$; cf. [4], [5], [6], [16] and [18]. However, the relation $\sim$ is not the same as the binary relation of 'collinearity' used in those papers, since 'collinear points' are represented by lines that are related or parallel.

If $\dim \mathsf{A} \neq 2$, then $(\mathcal{L}, \not\approx)$ is a Plücker space. If A is an affine plane, then $(\mathcal{L}, \not\approx)$ is not a Plücker space. The connected components $(\mathcal{L}_i)_{i \in I}$ with respect to $\not\approx$ are the pencils of parallel lines, since the relations $\not\approx$ and $\parallel$ are coinciding now. If $\mathcal{L}_j$ $(j \in I)$ is a fixed pencil of parallel lines, then the relation $\not\approx_j$ is the coarsest relation on $\mathcal{L}_j$. Thus Plücker spaces on affine planes have indeed a very poor structure. The case $\dim \mathsf{A} \leq 1$ cannot deserve interest at all.

We shall determine all isomorphisms of Plücker spaces on affine spaces A, A' with dimensions ≥ 3: Any collineation yields an isomorphism of the associated affine Plücker spaces and vice versa. If we impose additional assumptions on A, A' (cf. Theorem 4), then any bijection $\varphi : \mathcal{L} \to \mathcal{L}'$ is already an isomorphism of Plücker spaces whenever (1) is satisfied with an implication $(\Longrightarrow)$ rather than an equivalence $(\Longleftrightarrow)$.

Similar theorems for Plücker spaces on projective spaces are due to W.L. Chow [9], H. Brauner [7] and the author [12]. For further results and references on Plücker spaces see, among others, [1], [2] and [13].

2 Isomorphisms

Let $\mathsf{A} = (\mathcal{P}, \mathcal{L}, \parallel)$ and $\mathsf{A}' = (\mathcal{P}', \mathcal{L}', \parallel')$ be affine spaces. If $\kappa : \mathcal{P} \to \mathcal{P}'$ is a collineation, i.e., a bijection preserving collinearity and non-collinearity of points, then κ gives rise to a bijection

$$\varphi : \mathcal{L} \to \mathcal{L}',\ Q \vee R \mapsto Q^\kappa \vee R^\kappa\ (Q, R \in \mathcal{P},\ Q \neq R) \tag{3}$$

taking related lines to related lines in both directions.

We shall prove the following converse:

Theorem 1 *Let* $\mathsf{A} = (\mathcal{P}, \mathcal{L}, \parallel)$ *and* $\mathsf{A}' = (\mathcal{P}', \mathcal{L}', \parallel')$ *be affine spaces with* $\dim \mathsf{A}' \geq 3$. *Suppose that* $\varphi : \mathcal{L} \to \mathcal{L}'$ *is an isomorphism of the Plücker space* $(\mathcal{L}, \sim)$ *onto the Plücker space* $(\mathcal{L}', \sim')$. *Then*

$$\kappa : \mathcal{P} \to \mathcal{P}',\ a \cap b \mapsto a^\varphi \cap b^\varphi\ (a, b \in \mathcal{L},\ a \approx b) \tag{4}$$

is a well-defined collineation.

Proof. *(a)* We infer from $\dim \mathsf{A}' \geq 3$ and the bijectivity of φ that $\#\mathcal{L} > 1$. Therefore $\dim \mathsf{A} \geq 2$. With $Q \in \mathcal{P}$ write $\mathcal{L}(Q)$ for the star of lines with centre Q, i.e., the set of all lines in $\mathcal{L}$ running through Q. Any star of lines is a maximal set of mutually related lines.

Suppose that $(\mathcal{L}(Q))^\varphi$ contains a trilateral spanning a plane $\mathcal{E}' \subset \mathcal{P}'$, say. All lines of $(\mathcal{L}(Q))^\varphi$ are mutually related. Therefore they are all contained in $\mathcal{E}'$. By $\dim \mathsf{A}' \geq 3$, there exists a line $a \in \mathcal{L}$ with $a^\varphi \cap \mathcal{E}' = \emptyset$. Thus

$$a^\varphi \not\sim' x^\varphi \text{ for all } x \in \mathcal{L}(Q) \tag{5}$$

and therefore

$$a \not\sim x \text{ for all } x \in \mathcal{L}(Q). \tag{6}$$

On the other hand, there exists a line joining Q with an arbitrarily chosen point of the line a. This contradicts (6).

Thus we have established that $(\mathcal{L}(Q))^\varphi$ is a subset of a star of lines for any $Q \in \mathcal{P}$. It is obvious now that (4) is a well-defined mapping.

(b) Given a point $Q' \in \mathcal{P}'$ one may show as above that $(\mathcal{L}'(Q'))^{\varphi^{-1}}$ is a subset of a star of lines. Therefore κ is a surjection and under φ stars of lines go over to stars of lines in both directions.

If points $Q, R \in \mathcal{P}$ are distinct, then

$$\#(\mathcal{L}(Q) \cap \mathcal{L}(R)) = \#((\mathcal{L}(Q))^\varphi \cap (\mathcal{L}(R))^\varphi) = 1 \tag{7}$$

and $Q^\kappa \neq R^\kappa$, whence κ is injective.

Three mutually distinct points $Q, R, S \in \mathcal{P}$ are collinear if, and only if,

$$\#(\mathcal{L}(Q) \cap \mathcal{L}(R) \cap \mathcal{L}(S)) = 1. \tag{8}$$

This in turn is equivalent to the collinearity of $Q^\kappa, R^\kappa, S^\kappa \in \mathcal{P}'$. Hence κ is a collineation. □

Remark 1 If the order of A' is greater than two or if $\dim \mathsf{A}' \leq 2$, then any collineation $\mathcal{P} \to \mathcal{P}'$ is even an *affinity*, i.e. a collineation preserving parallelism in both directions. Otherwise, the existence of a collineation $\mathcal{P} \to \mathcal{P}'$ implies the existence of an affinity $\mathcal{P} \to \mathcal{P}'$; see [17, 32.5 and 40.4]. Hence for $\dim \mathsf{A}' \geq 3$ we obtain all isomorphisms of $(\mathcal{L}, \sim)$ onto $(\mathcal{L}', \sim')$ via the Plücker group of $(\mathcal{L}, \sim)$ and a single affinity $\mathcal{P} \to \mathcal{P}'$.

Remark 2 If $\mathsf{A} = \mathsf{A}'$, then Theorem 1 describes the Plücker group for affine spaces with dimension ≥ 3. This generalizes a result in [1, p. 205] for real affine spaces[1].

Remark 3 Suppose that $\dim \mathsf{A} \geq 2$. The following construction yields all maximal sets of mutually related lines that are different from stars: Choose a point Q, an incident line a and a plane $\mathcal{E}$ containing a. Write $\mathcal{L}(Q, \mathcal{E})$ for the pencil of lines in $\mathcal{E}$ running through Q. Next define a family $(\tau_x)_{x \in \mathcal{L}(Q,\mathcal{E})}$ of translations $\tau_x : \mathcal{E} \to \mathcal{E}$ such that $\bigcap_{x \in \mathcal{L}(Q,\mathcal{E})} x^{\tau_x} = \emptyset$. Then $\{x^{\tau_x} \mid x \in \mathcal{L}(Q, \mathcal{E})\}$ is a maximal set of mutually related lines other than a star.

[1] The proof given there fails to work in case of characteristic two.

Remark 4 If $\varphi : \mathcal{L} \to \mathcal{L}'$ is an isomorphism and if $\dim \mathsf{A}' = 2$, then $\dim \mathsf{A} = 2$ according to Theorem 1. By virtue of (2) it is easy to establish the following result: Plücker spaces on affine planes A and A' are isomorphic if, and only if, A and A' have equipotent pencils of parallel lines or, in other words, if the order of A equals the order of A'.

The transposition of two distinct parallel lines of an affine plane is an example of a Plücker transformation that does not stem from a collineation.

We are now going to weaken the assumptions on φ in Theorem 1.

Theorem 2 *Let* $\mathsf{A} = (\mathcal{P}, \mathcal{L}, \parallel)$ *and* $\mathsf{A}' = (\mathcal{P}', \mathcal{L}', \parallel')$ *be affine spaces with* $\dim \mathsf{A}' \geq 3$. *Suppose that* $\varphi : \mathcal{L} \to \mathcal{L}'$ *is a bijection satisfying*

$$a \sim b \Longrightarrow a^\varphi \sim' b^\varphi \text{ for all } a, b \in \mathcal{L}. \tag{9}$$

Then

$$\lambda : \mathcal{P} \to \mathcal{P}',\ a \cap b \mapsto a^\varphi \cap b^\varphi\ (a, b \in \mathcal{L},\ a \approx b) \tag{10}$$

is a well-defined injection that preserves collinearity and non-collinearity of points. Moreover,

$$\mathcal{L}(Q)^\varphi = \mathcal{L}'(Q^\lambda) \text{ for all } Q \in \mathcal{P}. \tag{11}$$

Proof. *(a)* By the proof of Theorem 1, part (a), the following assertions have been already verified: The dimension of A is ≥ 2. For all $Q \in \mathcal{P}$ the set $(\mathcal{L}(Q))^\varphi$ is a subset of a star of lines, whence (10) is a well-defined mapping.

(b) Let $Q, R \in \mathcal{P}$ be distinct and assume that $Q^\lambda = R^\lambda$. Choose a line $c \in \mathcal{L} \setminus (\mathcal{L}(Q) \cup \mathcal{L}(R))$ and a point $S \in c$ such that Q, R, S are not collinear. Therefore $Q \vee S$, $R \vee S$ and c are three distinct concurrent lines. We deduce from (10) that

$$S^\lambda = (Q \vee S)^\varphi \cap (R \vee S)^\varphi = Q^\lambda = R^\lambda. \tag{12}$$

Consequently, $Q^\lambda \in x^\varphi$ for all $x \in \mathcal{L}$. This is impossible due to the surjectivity of φ. Hence λ is injective.

If $\{Q, R, S\} \subset \mathcal{P}$ is a triangle, then $Q \vee R$, $R \vee S$ and $S \vee Q$ are three distinct lines. The injectivity of φ and the injectivity of λ force that $\{Q^\lambda, R^\lambda, S^\lambda\} \subset \mathcal{P}'$ is a triangle. By definition, λ is a collinearity-preserving mapping.

(c) Finally, we establish (11). Assume to the contrary that there exists a point $Q \in \mathcal{P}$ and a line $b \in \mathcal{L} \setminus \mathcal{L}(Q)$ with $Q^\lambda \in b^\varphi$. Choose two distinct points $R_1, R_2 \in b$. Then $\{Q, R_1, R_2\}$ is a triangle, but $\{Q^\lambda, R_1^\lambda, R_2^\lambda\} \subset b^\varphi$ is a collinear set, an absurdity. □

The aim of the following discussion is to give sufficient conditions for λ to be a collineation or, equivalently, a surjection. We could apply results on injective mappings of affine spaces preserving collinearity of points; see [1, 3.1–3.3], [19] and the references in [11]. However, we proceed instead in close analogy with

[12]. Any star of lines in an affine space, for example, a star $\mathcal{L}(Q)$ in A, carries in a natural way the structure of a projective space, viz.

$$\mathsf{A}/Q := (\mathcal{L}(Q), \{\mathcal{L}(Q,\mathcal{E}) \mid Q \in \mathcal{E}, \mathcal{E} \text{ a plane}\}). \tag{13}$$

Recall the following concept due to P.V. Ceccherini [8]: A *semicollineation* of projective spaces is a bijection taking any three collinear points to collinear points. The existence of *proper* semicollineations (different from collineations) of Desarguesian projective spaces seems to be an open problem; see also [3], [14] and [15]. Semicollineations fall within the wider class of *weak linear mappings* that have been characterized independently in [10] and [11].

Now (11) can be improved as follows:

Theorem 3 *Let A, A' and φ be given as in Theorem 2. Then $\dim \mathsf{A} \geq 3$. If moreover the order of A is not two and if $Q \in \mathcal{P}$, then the restricted mapping*

$$\varphi|\mathcal{L}(Q) \,:\, \mathcal{L}(Q) \to \mathcal{L}'(Q^\lambda) \tag{14}$$

is a semicollineation of A/Q onto A'/Q^λ.

Proof. A cannot be the affine plane of order two, since $6 < \#\mathcal{L}'$.

Suppose that the order of A is not two. By (11), the mapping (14) is bijective. Let $a, b, c \in \mathcal{L}(Q)$ be 'collinear points' of A/Q. There exists a line $d \in \mathcal{L} \setminus \mathcal{L}(Q)$ that is adjacent to a, b and c. Hence $a^\varphi, b^\varphi, c^\varphi \in \mathcal{L}(Q^\lambda, Q^\lambda \vee d^\varphi)$ represent 'collinear points' in A'/Q^λ so that (14) is a semicollineation. There are non-coplanar lines through Q^λ representing 'non-collinear' points of A'/Q^λ. Their pre-images under (14) are distinct and non-coplanar, so that $\dim \mathsf{A} \geq 3$. □

If A is of order two, then (14) is in general merely a bijection.

Theorem 4 *With the settings of Theorem 2, each of the following conditions is sufficient for λ to be a collineation:*

1. *A or A' is a finite affine space.*
2. *$\dim \mathsf{A} \leq \dim \mathsf{A}' < \infty$.*
3. *The order of A is different from two and every monomorphism of an underlying field F of A in an underlying field F' of A' is surjective.*
4. *A and A' are affine spaces of order two.*

Proof. Ad 1. Since φ is bijective, both A and A' are finite affine spaces.

Let $\mathsf{A} \cong \mathrm{AG}(n,2)$ and $\mathsf{A}' \cong \mathrm{AG}(m,p^h)$, where $n \geq 3$, $m \geq 3$, $h \geq 1$ are integers and p is a prime. Choose $Q \in \mathcal{P}$. We deduce from (11) that

$$2^{n-1}\#\mathcal{L}(Q) = \#\mathcal{L} = \#\mathcal{L}' = p^{h(m-1)}\#\mathcal{L}(Q^\lambda) = p^{h(m-1)}\#\mathcal{L}(Q). \tag{15}$$

Consequently, $p=2$, $n-1=h(m-1)$ and, by $\#\mathcal{L}(Q)=\#\mathcal{L}(Q^\lambda)$,

$$\sum_{i=0}^{n-1} 2^i = \sum_{i=0}^{m-1} 2^{hi}. \tag{16}$$

We infer that each summand on the right hand side of (16) appears exactly once on the left hand side, whence $h=1$ and $n=m$. This implies that λ is surjective.

If the order of A is greater than two, then (14) is a semicollineation and, by [8, 14.2], even a collineation. Therefore A and A' are of equal order. Now λ turns out to be surjective, because of

$$\dim \mathsf{A} = \dim(\mathsf{A}/Q)+1 = \dim(\mathsf{A}'/Q^\lambda)+1 = \dim \mathsf{A}' < \infty. \tag{17}$$

Ad 2. By virtue of the previous result, we may exclude affine spaces A, A' of order two from the following discussion.

Choose any point $R' \in \mathcal{P}'$. As φ is surjective, there exists a line $d \in \mathcal{L}$ with $R' \in d^\varphi$. Let $Q \in \mathcal{P}$ be off the line d and put $\mathcal{E} := Q \vee d$. We observe that

$$\dim(\mathsf{A}/Q) = \dim \mathsf{A} - 1 \le \dim \mathsf{A}' - 1 = \dim(\mathsf{A}'/Q^\lambda) < \infty. \tag{18}$$

By [8, 8.4] or [14, Theorem 2.2] the semicollineation (14) turns out to be a collineation. Consequently, $\mathcal{L}(Q,\mathcal{E})^\varphi$ is a pencil of lines $\mathcal{L}'(Q^\lambda,\mathcal{E}')$, say. The line $d^\varphi \not\ni Q^\lambda$ is related to all lines of this pencil with at most one exception. This implies that the point-set d^λ is equal to the affine line d^φ. Thus

$$R' \in d^\varphi = d^\lambda \subset \operatorname{im}\lambda. \tag{19}$$

Ad 3. Choose any point $Q \in \mathcal{P}$. By Theorem 3, the mapping (14) is a semicollineation. This implies the existence of a monomorphism $F \to F'$; cf. [8, 5.1], [10, Theorem 5.4.1] or [11, Theorem 2]. By [8, 5.3], the mapping (14) is a collineation. From this the surjectivity of λ is established as above.

Ad 4. Choose any point $R' \in \mathcal{P}'$. Since φ is surjective, there exists a line $\{R_1,R_2\} \in \mathcal{L}$ with $R' \in \{R_1,R_2\}^\varphi = \{R_1^\lambda, R_2^\lambda\}$. □

Remark 5 Let A be an affine space over GF(2) with a countable basis and let A' be an m-dimensional affine space ($3 \le m \le \aleph_0$) over a countable field F' of arbitrary characteristic. Hence there is either no monomorphism or no surjective monomorphism of GF(2) in F'. As $\#\mathcal{P} = \#\mathcal{L}' = \aleph_0$, we can index all points of $\mathcal{P}$ as $Q_1, Q_2, \ldots$ and all lines of $\mathcal{L}'$ as $a_1', a_2', \ldots$ such that there are no repeated elements.

Let us define, by recursion, an injective sequence $\{1,2,\ldots\} \to \mathcal{P}'$, $s \mapsto R_s'$ such that each line of $\mathcal{L}'$ contains exactly two points: We start with a point $R_1' \in a_1'$ and put $\mathcal{B}_1' := \{R_1'\}$. Next assume that we are already given a set $\mathcal{B}_i' = \{R_1', \ldots, R_i'\}$ formed by $i \ge 1$ mutually distinct points no three of which are collinear. Write $\mathcal{N}_i'$ for the set of all lines that arise by joining distinct points of $\mathcal{B}'_i$. Then let $j \in \{1, 2, \ldots\}$ be the least element such that the line a'_j is not in $\mathcal{N}'_i$. Since a'_j carries an infinite number of points, we can choose such a point $R'_{i+1} \in a'_j \setminus \mathcal{B}'_i$ that no three elements of the set $\mathcal{B}'_{i+1} := \mathcal{B}'_i \cup \{R'_{i+1}\}$ are collinear.

Put $\mathcal{B}' := \bigcup_{s=1}^{\infty} \mathcal{B}'_s$. By construction, no three distinct points of $\mathcal{B}'$ are collinear. Furthermore, given a line $a'_k \in \mathcal{L}'$ we obtain that $a'_k \in \mathcal{N}'_{2k}$, as required.

The mapping

$$\varphi : \mathcal{L} \to \mathcal{L}',\ \{Q_s, Q_t\} \mapsto R'_s \vee R'_t, \quad (s, t \in \{1, 2, \ldots\},\ s \neq t) \tag{20}$$

is a bijection satisfying (9). The associated injection λ (see (10)) takes Q_s to R'_s ($s \in \{1, 2, \ldots\}$). Only two points of a'_1 belong to $\operatorname{im} \lambda$, whence λ is not surjective.

Remark 6 Let A, A' be affine spaces with equal infinite order and $2 = \dim \mathsf{A}' < \dim \mathsf{A} \leq \aleph_0$. There exists a bijection $\varphi : \mathcal{L} \to \mathcal{L}'$ such that any class of parallel lines in A is mapped onto a pencil of parallel lines in A'. Such a φ is satisfying (9) without being an isomorphism of Plücker spaces.

Remark 7 Let $\mathsf{A} = \mathsf{A}'$ be an affine plane of infinite order. Choose two non-parallel lines $a, b \in \mathcal{L}$. There exists a bijection $\varphi : \mathcal{L} \to \mathcal{L}$ such that the parallel class of a is mapped onto the union of the parallel classes of a and b, whereas any other pencil of parallel lines is mapped onto a pencil of parallel lines. Then φ is satisfying (9) without being a Plücker transformation.

References

[1] Benz, W.: *Geometrische Transformationen*. B.I. Wissenschaftsverlag, Mannheim Leipzig Wien Zürich, 1992.

[2] Benz, W.: *Real Geometries*. BI-Wissenschaftsverlag, Mannheim Leipzig Wien Zürich, 1994.

[3] Bernardi, M.P., Torre, A.: *Alcune questioni di esistenza e continuità per (m, n)-fibrazioni e semicollineazioni*. Boll. U.M.I. (6) **3**-B (1984), 611–622.

[4] Bichara, A., Mazzocca, F.: *On a characterization of Grassmann space representing the lines in an affine space*. Simon Stevin **56** (1982), 129–141.

[5] Bichara, A., Mazzocca, F.: *On the independence of the axioms defining the affine and projective Grassmann spaces*. Ann. Discr. Math. **14** (1982), 123–128.

[6] Bichara, A., Mazzocca, F.: *On a characterization of the Grassmann spaces associated with an affine space*. Ann. Discr. Math. **18** (1983), 95–112.

[7] Brauner, H.: *Über die von Kollineationen projektiver Räume induzierten Geradenabbildungen*. Sitz. Ber. österr. Akad. Wiss., math.-naturw. Kl. Abt. II, **197** (1988), 327–332.

[8] Ceccherini, P.V.: *Collineazioni e semicollineazioni tra spazi affini o proiettivi*. Rend. Mat. (5) **26** (1967), 309–348.

[9] Chow, W.L.: *On the geometry of algebraic homogeneous spaces*. Ann. of Math. **50** (1949), 32–67.

[10] Faure C.-A., Frölicher, A.: *Morphisms of Projective Geometries and Semilinear Maps*. Geom. Dedicata **53** (1994), 237–262.

[11] Havlicek, H.: *A Generalization of Brauner's Theorem on Linear Mappings*. Mitt. Math. Sem. Univ. Gießen **215** (1994), 27–41.

[12] Havlicek, H.: *On Isomorphisms of Grassmann Spaces*. Mitt. Math. Ges. Hamburg **14** (1995), 117–120.

[13] Havlicek, H.: *Symplectic Plücker Transformations*. Math. Pannonica **6** (1995), 145–153.

[14] Kreuzer, A.: *On the Definition of Isomorphisms of linear spaces*. Geom. Dedicata (to appear).

[15] Maroscia, P.: *Semicollineazioni e semicorrelazioni tra spazi lineari*. Rend. Mat. (6) **3** (1970), 507–521.

[16] Tallini, G.: *Partial Line Spaces and Algebraic Varieties*. Symp. Math. **28** (1986), 203–217.

[17] Tamaschke, O.: *Projektive Geometrie II*. Bibliographisches Institut, Mannheim Wien Zürich, 1969.

[18] Zanella, C.: *Spazi di Grassmann affini topologici*. Rend. Mat. Appl. (7) **8** (1988), 315–328.

[19] Zick, W.: *Der Satz von Martin in Desargues'schen affinen Räumen*. Inst. f. Mathematik, Universität Hannover, Preprint No. **134** (1981), 1–20.

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