Affine Spaces within Projective Spaces

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Abstract

We endow the set of complements of a fixed subspace of a projective space with the structure of an affine space, and show that certain lines of such an affine space are affine reguli or cones over affine reguli. Moreover, we apply our concepts to the problem of describing dual spreads. We do not assume that the projective space is finite-dimensional or pappian.

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1 Introduction

The aim of this paper is to equip the set of complements of a distinguished subspace in an arbitrary projective space with the structure of an affine space. In this introductory section we first formulate the problem more exactly and then describe certain special cases that have already been treated in the literature.

Let K be a not necessarily commutative field, and let V be a left vector space over K with arbitrary — not necessarily finite — dimension. Moreover, fix a subspace W of V. We are interested in the set

$$\mathcal{S} := \{S \leq W \mid V = W \oplus S\} \tag{1}$$

of all complements of W.

Note that often we shall take the projective point of view; then we identify $\mathcal{S}$ with the set of all those projective subspaces of the projective space $\mathbb{P}(K, V)$ that are complementary to the fixed projective subspace induced by W.

In the special case that W is a hyperplane, the set $\mathcal{S}$ obviously is the point set of an affine space, namely, of the affine derivation of $\mathbb{P}(K, V)$ w.r.t. this hyperplane.

This can be generalized to arbitrary W: In Section 2, we are going to introduce on $\mathcal{S}$ the structure of a left vector space over K, and hence the structure of an affine space. In general, this affine space with point set $\mathcal{S}$ will not be determined uniquely.

In Sections 3 and 4 we study the lines of these affine spaces with point set $\mathcal{S}$ in terms of the projective space $\mathbb{P}(K, V)$. Finally, in Section 5 we use our results in order to describe dual spreads.

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For the case that K is commutative and V is finite-dimensional over K, the construction of the affine space on $\mathcal{S}$ seems to be due to J.G. Semple [14] (where $K = \mathbb{R}, \mathbb{C}$). Independently, in [11] R. Metz shows that then $\mathcal{S}$ can be identified with the set of $(n-k) \times k$-matrices over K (where $n = \dim V$ and $k = \dim W$) and thus has the structure of a vector space over K in a natural way. The affine lines in $\mathcal{S}$ are "generalized reguli", i.e., reguli in quotient spaces within $\mathbb{P}(K, V)$ (compare [11]). In certain special cases this is true in our more general situation as well.

Metz's construction is also used by A. Herzer in [7]. Moreover, in [7] and in [14] a representation of the affine space as a "stereographic projection" of the corresponding Grassmann variety is given. For applications in differential geometry see [1], [12], [15].

In [5], the case of lines skew to a fixed one in projective 3-space is generalized to not necessarily pappian spaces via an explicit description of the lines (cf. also [6]). In [5] it is also indicated how to generalize the used approach to higher (but finite) dimensions. We will not follow these ideas here but generalize Metz's coordinatization method and results on reguli obtained in [3].

Finally we should mention that one could also interpret some results of this paper in terms of the extended concept of chain geometry introduced in [4].

2 Affine spaces with point set $\mathcal{S}$

Let W be a subspace of the left vector space V, and let $\mathcal{S}$ be the set of complements of W as introduced in (1). For our subsequent considerations we exclude the trivial cases that $W = \{0\}$ or $W = V$. In both cases $\mathcal{S}$ has only one element and thus is a trivial affine space consisting of one single point.

Let $U \in \mathcal{S}$ be a fixed complement of W. We are going to coordinatize the set $\mathcal{S}$ by the set $\mathrm{Hom}(U, W)$ of all K-linear mappings $U \to W$, thus generalizing Metz's coordinatization by matrices. Note that what follows is independent of the choice of U.

Since $V = W \oplus U$, each endomorphism $\varphi \in \mathrm{End}(V)$ has the form

$$v = w + u \mapsto (w^\alpha + u^\gamma) + (w^\beta + u^\delta),$$

$(w \in W, u \in U)$ with linear mappings $\alpha : W \to W$, $\beta : W \to U$, $\gamma : U \to W$, $\delta : U \to U$. So the endomorphism ring $\mathrm{End}(V)$ is isomorphic to the ring

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha \in \mathrm{End}(W), \beta \in \mathrm{Hom}(W, U), \gamma \in \mathrm{Hom}(U, W), \delta \in \mathrm{End}(U) \right\},$$

equipped with the usual matrix addition and multiplication (compare [10], p. 643). We shall frequently identify $\mathrm{End}(V)$ with this matrix ring.

The stabilizer of our distinguished subspace W in the group $\mathrm{Aut}(V)$ is

$$\left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \mid \alpha \in \mathrm{Aut}(W), \gamma \in \mathrm{Hom}(U, W), \delta \in \mathrm{Aut}(U) \right\}, \tag{2}$$

where 0 denotes the zero mapping $W \to U$.

Now consider an arbitrary complement of W, i.e., an element $S \in \mathcal{S}$. Because of $V = W \oplus U = W \oplus S$ there is a linear bijection $\varphi : V \to V$ fixing W elementwise and mapping U to S. This means $\varphi = \begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix}$, where $\gamma \in \mathrm{Hom}(U, W)$, $\delta \in \mathrm{Aut}(U)$, and $1 = 1_W$ is the identity mapping on W. So

$$S = U^\varphi = \{u^\gamma + u^\delta \mid u \in U\} =: U^{(\gamma,\delta)}.$$

However, this description is not unique: We have $U^{(\gamma,\delta)} = U^{(\gamma',\delta')}$ exactly if $(\gamma', \delta') = (\rho\gamma, \rho\delta)$ for some $\rho \in \mathrm{Aut}(U)$. In particular, since δ is invertible, we may always assume $\delta = 1 = 1_U$. So we have

$$\mathcal{S} = \{U^{(\gamma,1)} \mid \gamma \in \mathrm{Hom}(U, W)\}$$

and we can identify $\mathcal{S}$ with $\mathrm{Hom}(U, W)$ via

$$U^{(\gamma,1)} \mapsto \gamma. \tag{3}$$

Note that instead of (3), we could also use the identification

$$\mathcal{S} \ni U^{(\gamma,1)} = U^{(\rho\gamma,\rho)} \mapsto \rho\gamma \in \mathrm{Hom}(U, W),$$

for any fixed $\rho \in \mathrm{Aut}(U)$. One can easily check that this approach would yield essentially the same.

The abelian group $(\mathrm{Hom}(U, W), +)$ is a faithful left module over the ring $\mathrm{End}(U)$ w.r.t. composition of mappings. Hence any embedding of some not necessarily commutative field F into $\mathrm{End}(U)$ makes $\mathrm{Hom}(U, W)$ a left vector space over F. Analogously, $\mathrm{Hom}(U, W)$ is a faithful right module over $\mathrm{End}(W)$, and any embedding $F \hookrightarrow \mathrm{End}(W)$ makes $\mathrm{Hom}(U, W)$ a right vector space over F.

We will consider only the case where $\mathrm{Hom}(U, W)$ becomes a left vector space over K. So we have to embed the field K into the endomorphism ring $\mathrm{End}(U)$. In general, there are many possibilities for such embeddings. We will restrict ourselves to the following type:

Lemma 2.1 *(a) Let $(b_i)_{i\in I}$ be a basis of U. For each $k \in K$ we define $\lambda_k \in \mathrm{End}(U)$ by $b_i \mapsto kb_i$ ($i \in I$). The mapping $\lambda : K \to \mathrm{End}(U) : k \mapsto \lambda_k$ is an injective homomorphism of rings, and hence embeds the field K into the ring $\mathrm{End}(U)$. The associated left scalar multiplication on $\mathrm{Hom}(U, W)$ is given by $k\cdot\gamma := \lambda_k\gamma \in \mathrm{Hom}(U, W)$ (for $k \in K$, $\gamma \in \mathrm{Hom}(U, W)$).*

(b) Let $(b'_i)_{i\in I}$ be another basis of U. Then the associated embedding $\lambda' : K \to \mathrm{End}(U)$ is given by $\lambda' : k \mapsto \rho^{-1}\lambda_k\rho$, where $\rho \in \mathrm{Aut}(U)$ is the unique linear bijection with ${b_i}^\rho = b'_i$.

Note that if K is commutative, then each λ_k is central in $\mathrm{End}(U)$, and hence any two embeddings of this type coincide. So in this case $\mathrm{Hom}(U, W)$ is a (left and right) vector space over K in a canonical way. The same holds if we consider $\mathrm{Hom}(U, W)$ as a vector space over the center Z of K.

The vector space structures on $\mathrm{Hom}(U, W)$ obtained in Lemma 2.1 can be carried over to the set $\mathcal{S}$ by using the identification (3):

Proposition 2.2 *Let* $(b_i)_{i\in I}$ *be a basis of* U*, and let* $\lambda : K \to \mathrm{End}(U)$ *be the embedding associated to* $(b_i)_{i\in I}$ *according to Lemma 2.1(a).*
Then $\mathcal{S}$ *is a left vector space over* K*, denoted by* $(\mathcal{S}, (b_i))$*, with addition*

$$U^{(\gamma,1)} + U^{(\eta,1)} = U^{(\gamma+\eta,1)}$$

and scalar multiplication

$$k \cdot U^{(\gamma,1)} = U^{(\lambda_k\gamma,1)}$$

(where $\gamma, \eta \in \mathrm{Hom}(U,W)$*,* $k \in K$*).*

Each vector space $(\mathcal{S}, (b_i))$ gives rise to an affine space with point set $\mathcal{S}$. This affine space will be denoted by $\mathbb{A}(\mathcal{S}, (b_i))$.
If the dimension of V is finite, say n, and $\dim W = k$, then the affine space $\mathbb{A}(\mathcal{S}, (b_i))$ is $k(n-k)$-dimensional. If $\dim V = \infty$, then at least one of the subspaces W, U, and thus also $\mathbb{A}(\mathcal{S}, (b_i))$, is infinite-dimensional (compare Proposition 5.1 below).
Now we consider the action of certain elements of the stabilizer of W in $\mathrm{Aut}(V)$ (i.e., the group (2)) on the affine spaces $\mathbb{A}(\mathcal{S}, (b_i))$.

Lemma 2.3 *(a) The group of all* $\begin{pmatrix}1&0\\ \eta&1\end{pmatrix}$ *(with* $\eta \in \mathrm{Hom}(U,W)$*) induces on* $\mathcal{S}$ *the common translation group of all the affine spaces* $\mathbb{A}(\mathcal{S}, (b_i))$*. This translation group is isomorphic to* $(\mathrm{Hom}(U,W), +)$*, each translation has the form* $U^{(\gamma,1)} \mapsto U^{(\gamma+\eta,1)}$*.*

(b) The group of all $\begin{pmatrix}\alpha&0\\ 0&1\end{pmatrix}$ *(with* $\alpha \in \mathrm{Aut}(W)$*) induces linear automorphisms of all vector spaces* $(\mathcal{S}, (b_i))$*. So all these mappings are collineations of all* $\mathbb{A}(\mathcal{S}, (b_i))$*, they have the form* $U^{(\gamma,1)} \mapsto U^{(\gamma\alpha,1)}$*.*

(c) The group of all $\begin{pmatrix}1&0\\ 0&\rho\end{pmatrix}$ *(with* $\rho \in \mathrm{Aut}(U)$*) induces linear bijections* $(\mathcal{S}, (b_i)) \to (\mathcal{S}, ({b_i}^\rho))$ *and hence isomorphisms between the associated affine spaces. In particular, this group acts transitively on the set of all affine spaces* $\mathbb{A}(\mathcal{S}, (b_i))$*.*

Proof: The mapping induced on $\mathcal{S}$ by the matrix $\begin{pmatrix}\alpha&0\\ \eta&\rho\end{pmatrix}$ is given by $U^{(\gamma,1)} \mapsto U^{(\gamma\alpha+\eta,\rho)} = U^{(\rho^{-1}(\gamma\alpha+\eta),1)}$. So assertions (a) and (b) are obvious. Moreover, a closer look at Lemma 2.1(b) yields assertion (c). □

Note that the statements of Lemma 2.3 (a) and (b) are valid for arbitrary affine spaces with point set $\mathcal{S}$ that are based upon the abelian group $(\mathrm{Hom}(U,W), +)$, equipped with a scalar multiplication by embedding K into $\mathrm{End}(U)$: Of course all affine spaces of this type have $(\mathrm{Hom}(U,W), +)$ as translation group. Moreover, $\gamma \mapsto \gamma\alpha$ ($\alpha \in \mathrm{Aut}(W)$) is an endomorphism of the left $\mathrm{End}(U)$-module $\mathrm{Hom}(U,W)$ and hence induces a K-linear bijection w.r.t. all scalar multiplications under consideration.
The permutations of $\mathcal{S}$ that appear in Lemma 2.3 are restrictions of projective collineations of $\mathbb{P}(K,V)$ fixing the subspace W. This will become important later when studying the affine spaces $\mathbb{A}(\mathcal{S}, (b_i))$ in terms of $\mathbb{P}(K,V)$.
In particular, Lemma 2.3(c) implies that any two of the affine spaces $\mathbb{A}(\mathcal{S}, (b_i))$ are projectively equivalent. Next we study under which conditions two of these affine spaces coincide.

Lemma 2.4 *Let (b_i) and (b'_i) be bases of U. Then $\mathbb{A}(\mathcal{S},(b_i)) = \mathbb{A}(\mathcal{S},(b'_i))$ if, and only if, the unique element $\rho \in \mathrm{Aut}(U)$ with $b'_i = {b_i}^\rho$ belongs to*

$$N := \{\nu \in \mathrm{Aut}(U) \mid \forall k \in K^* \; \exists l \in K^* : \nu\lambda_k = \lambda_l \nu\},$$

which is the normalizer of the image $(K^*)^\lambda$ of the multiplicative group K^* under the embedding λ in the group $\mathrm{Aut}(U)$.

Proof: If the two affine spaces are affine lines, then they obviously coincide. In this case, we have $\dim W = \dim U = 1$, and hence in particular $\mathrm{Aut}(U) = (K^*)^\lambda = N$.

Otherwise, the two affine spaces coincide exactly if the identity on $\mathcal{S}$ is a K-semilinear bijection $(\mathcal{S},(b_i)) \to (\mathcal{S},(b'_i))$. Since both vector spaces have the same additive group, this is equivalent to the following:

$$\exists \varphi \in \mathrm{Aut}(K) \; \forall \gamma \in \mathrm{Hom}(U,W) \; \forall k \in K : \lambda_k \gamma = \rho^{-1} \lambda_{k^\varphi} \rho \gamma \tag{4}$$

The $\mathrm{End}(U)$-module $\mathrm{Hom}(U,W)$ is faithful, whence (4) is equivalent to

$$\exists \varphi \in \mathrm{Aut}(K) \; \forall k \in K : \lambda_k = \rho^{-1} \lambda_{k^\varphi} \rho.$$

This proves the assertion, because for each $\nu \in N$ the mapping $k \mapsto l$ (where $\lambda_l = \nu\lambda_k\nu^{-1}$) is an automorphism of K. □

We want to describe the group N in terms of the projective space. This will yield a projective characterization of the coincidence of two affine spaces of type $\mathbb{A}(\mathcal{S},(b_i))$.

We need the following two concepts:

Let X be a left vector space over K, and let $(x_j)_{j\in J}$ be a basis of X. The set of all linear combinations of the x_j's with coefficients in Z is called the *Z-subspace of X with respect to (x_j)*. The associated set of points of the projective space $\mathbb{P}(K,X)$ is the *(projective) Z-subspace of $\mathbb{P}(K,X)$ with respect to (x_j)*.

Now let Y be another left vector space over K, with basis $(y_i)_{i\in I}$. We call a linear mapping $\zeta : X \to Y$ *central w.r.t. (x_j) and (y_i)*, if it maps each x_j into the Z-subspace of Y w.r.t. (y_i). If in particular $X = Y$ and $(x_j) = (y_i)$, then we say that ζ is *central w.r.t. (x_j)*.

Now we can formulate our next lemma.

Lemma 2.5 *The group N consists exactly of all products $\lambda_m\zeta$, where $m \in K^*$ and $\zeta \in \mathrm{Aut}(U)$ is central w.r.t. (b_i).*

This means that N consists of exactly those automorphisms of U that induce projective collineations of $\mathbb{P}(K,U)$ leaving the projective Z-subspace w.r.t. (b_i) invariant.

Proof: We only prove the first assertion, because the second one is just a re-formulation. Obviously, all products $\lambda_m\zeta$ belong to N. Conversely, consider $\nu \in N$. Fix one $k \in K^*$. Then there is an $l \in k^*$ such that

$$\forall i \in I : {b_i}^{\nu\lambda_k} = l{b_i}^\nu. \tag{5}$$

Let x be a coordinate (w.r.t. (b_i)) of any ${b_j}^\nu$. Then (5) implies $xk = lx$. Thus if we fix one such non-zero coordinate, say x_0, then $x = x_0 z_k$, where z_k centralizes k. But this holds true for all $k \in K^*$, so that $x \in x_0 Z$ for all coordinates x of all vectors ${b_j}^\nu$. Hence $\nu = \lambda_m \zeta$, where $m = x_0$ and ζ is central with ${b_i}^\zeta = x_0^{-1} {b_i}^\nu$. $\square$

The following is a direct consequence of Lemmas 2.4 and 2.5:

Theorem 2.6 *Let (b_i) and (b'_i) be bases of U. Then $\mathbb{A}(\mathcal{S}, (b_i)) = \mathbb{A}(\mathcal{S}, (b'_i))$ if, and only if, the projective Z-subspaces of $\mathbb{P}(K, U)$ w.r.t. (b_i) and (b'_i) coincide.*

This means that actually the affine space $\mathbb{A}(\mathcal{S}, (b_i))$ does not depend on the basis (b_i) but only on the associated projective Z-subspace of $\mathbb{P}(K, U)$.

A special case is the following: If $\dim U = 1$, then there is exactly one projective Z-subspace in $\mathbb{P}(K, U)$, namely, the point U itself, and hence there is a unique affine space of type $\mathbb{A}(\mathcal{S}, (b_i))$. Of course this is the affine derivation w.r.t. the hyperplane W.

3 The symmetric case

As mentioned in the introduction, we want to find out how the lines in the affine spaces $\mathbb{A}(\mathcal{S}, (b_i))$ look like in $\mathbb{P}(K, V)$. From now on we fix a basis (b_i) and write $\mathbb{A} := \mathbb{A}(\mathcal{S}, (b_i))$ for short. An arbitrary line of $\mathbb{A}$ has the form

$$\ell(\alpha, \beta) := \{U^{(\lambda_k \alpha + \beta, 1)} \mid k \in K\} \tag{6}$$

where $\alpha, \beta \in \mathrm{Hom}(U, W)$, $\alpha \neq 0$.

In this section we study *symmetric case*, i.e., the case that W and its complement U are isomorphic. In this special situation we can use results obtained in [3]. In analogy with [3], we assume w.l.o.g. that $V = U \times U$, $W = U \times \{0\}$, and identify U with $\{0\} \times U$.

In [3] the concept of a *regulus* in $\mathbb{P}(K, V)$ is introduced, generalizing the well known definition for the pappian case (see, e.g., [8]) and also the definition for not necessarily pappian 3-space (cf. [13]). It is shown in [3] that each regulus is the image of the *standard regulus*

$$\mathcal{R}_0 = \{W\} \cup \{U^{(\lambda_k, 1)} \mid k \in K\}$$

under a projective collineation. In particular, any two elements of a regulus are complementary.

Obviously, the standard regulus equals the line $\ell(1, 0)$ of $\mathbb{A}$, extended by W. We call such a set $\mathcal{R} \setminus \{W\}$, where $\mathcal{R}$ is a regulus containing W, an *affine regulus* (w.r.t. W). Each affine regulus is a subset of $\mathcal{S}$.

Moreover, we say that the line $\ell(\alpha, \beta)$ of $\mathbb{A}$ is *regular*, if $\alpha \in \mathrm{End}(U)$ is invertible. Since two elements $U^{(\gamma_1, 1)}$ and $U^{(\gamma_2, 1)}$ are complementary if, and only if, $\gamma_1 - \gamma_2$ is invertible (compare [2]), the regular lines are exactly the lines joining two complementary elements of $\mathcal{S}$, and any two elements of such a line are complementary.

The following can be verified easily:

Proposition 3.1 *A line of $\mathbb{A}$ is regular, exactly if it is an affine regulus in $\mathbb{P}(K, V)$. In particular, the regular line $\ell(\alpha, \beta)$ is the image of $\mathcal{R}_0 \setminus \{W\}$ under the projective collineation induced by $\begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix}$.*

Proposition 3.1 implies that the affine space $\mathbb{A}$ possesses a group of collineations that acts transitively on the set of regular lines, because by Lemma 2.3 the matrix $\left(\begin{smallmatrix}\alpha & 0\\ \beta & 1\end{smallmatrix}\right)$ induces a collineation of $\mathbb{A}$.

Now we are looking for an explicit geometric description of the regular lines as affine reguli. First we need some more information on reguli.

Recall that a line of $\mathbb{P}(K,V)$ is called a *transversal* of a regulus $\mathcal{R}$, if it meets each element of $\mathcal{R}$ in exactly one point. Through each point on a transversal of $\mathcal{R}$ there is a unique element of $\mathcal{R}$. The transversals of the standard regulus can be specified easily:

Remark 3.2 The set $\mathcal{T}_0$ of transversals of $\mathcal{R}_0$ consists exactly of the lines $K(z,0) \oplus K(0,z)$, where $z \neq 0$ belongs to the Z-subspace of U w.r.t. (b_i).

Recall, moreover, that for any three elements U_1, U_2, U_3 of a regulus $\mathcal{R}$ there is a perspectivity $\pi : U_1 \to U_2$ with center U_3 (compare [3] for our general case).

Lemma 3.3 *Let $\mathcal{R}$ be a regulus in $\mathbb{P}(K,V)$, let $U_1, U_2, U_3 \in \mathcal{R}$ be pairwise different, and let $\mathcal{T}$ be the set of all transversals of $\mathcal{R}$. Then the following holds:*

(a) The set $\mathcal{T} \cap U_1 := \{T \cap U_1 \mid T \in \mathcal{T}\}$ is a projective Z-subspace $\mathcal{Z}(U_1)$ of U_1.

(b) If $\pi : U_1 \to U_2$ is the perspectivity with center U_3, then $\mathcal{Z}(U_2) = \mathcal{Z}(U_1)^\pi$. In particular, $\mathcal{T} = \{P \oplus P^\pi \mid P \in \mathcal{Z}(U_1)\}$.

(c) The regulus $\mathcal{R}$ is the set of all subspaces X satisfying the following conditions:

(i) Each $T \in \mathcal{T}$ meets X in exactly one point, and X is spanned by $\mathcal{T} \cap X$.

(ii) If $T_1 \leq T_2 + T_3$ holds for $T_1, T_2, T_3 \in \mathcal{T}$, then the points $T_1 \cap X$, $T_2 \cap X$, $T_3 \cap X$ are collinear.

Proof: The assertions (a) and (b) follow directly from 3.2, because by [3] the group $\mathrm{Aut}(V)$ acts transitively on the set of triples $(\mathcal{R}, U_1, U_2)$ where $\mathcal{R}$ is a regulus and U_1, U_2 are different elements of $\mathcal{R}$.

(c): Choose $T_1 \in \mathcal{T}$ and a point $P_1 \in T_1$. Then there is a unique $Y \in \mathcal{R}$ through P_1. On the other hand, let X be any subspace through P_1 satisfying (i) and (ii). We have to show that $X = Y$.

We first observe that for each $T_2 \in \mathcal{T} \setminus \{T_1\}$ there is a $T_3 \in \mathcal{T} \setminus \{T_1, T_2\}$ such that the point $T_3 \cap Y$ lies on the line L joining $T_1 \cap Y$ and $T_2 \cap Y$, since $\mathcal{T}$ meets Y in a projective Z-subspace by (a). Then the transversals T_1, T_2, T_3 all belong to the 3-space spanned by L and L^π, where π is the perspectivity mapping Y to another element of $\mathcal{R}$, according to (b). Moreover, also by (b), any two transversals of $\mathcal{R}$ are skew, and hence we have $T_1 \leq T_2 \oplus T_3$. Thus the points $T_i \cap X$ $(i = 1,2,3)$, that exist by (i), are collinear by (ii). Since there is a unique line through $P_1 = T_1 \cap X$ meeting T_2 and T_3 (namely, the line L joining P_1 and $T_2 \cap Y = (P_1 \oplus T_3) \cap T_2$), we obtain $T_i \cap X = T_i \cap Y$. Applying this to each $T_2 \in \mathcal{T} \setminus \{T_1\}$ yields $X = Y$, because by (i) X is spanned by $\mathcal{T} \cap X$. $\square$

Using Lemma 3.3(c) one can reconstruct the regulus $\mathcal{R}$ from its set of transversals.

Corollary 3.4 *Every regulus is uniquely determined by the set of its transversals. More exactly: If $\mathcal{R}$ and $\mathcal{R}'$ are reguli with the same set of transversals, then $\mathcal{R} = \mathcal{R}'$.*

Now consider a regulus $\mathcal{R}$ containing W, with transversal set $\mathcal{T}$. We associate to it the set

$$W + \mathcal{T} := \{W + T \mid T \in \mathcal{T}\}. \tag{7}$$

Corollary 3.5 *Let U_1 and U_2 be two different elements of $\mathcal{S}$. Let $\mathcal{R}$ and $\mathcal{R}'$ be reguli containing W, U_1, U_2, and let $\mathcal{T}$ and $\mathcal{T}'$ be their sets of transversals, respectively. Then $W + \mathcal{T} = W + \mathcal{T}'$ implies $\mathcal{R} = \mathcal{R}'$.*

Proof: Each transversal $T \in \mathcal{T}$ is the line joining the points $(W+T) \cap U_1$ and $(W+T) \cap U_2$. So our assumptions imply $\mathcal{T} = \mathcal{T}'$ and hence $\mathcal{R} = \mathcal{R}'$, by Corollary 3.4. $\square$

From now on, let $\mathcal{Z}(U)$ be the projective Z-subspace of U w.r.t. (b_i). Recall that by Theorem 2.6 it is $\mathcal{Z}(U)$ rather than (b_i) that determines the affine space $\mathbb{A}$. We set

$$W + \mathcal{Z}(U) := \{W + P \mid P \in \mathcal{Z}(U)\}.$$

The following is clear by 3.2:

Lemma 3.6 *Let $\mathcal{T}_0$ be the set of transversals of $\mathcal{R}_0$. Then $W + \mathcal{T}_0 = W + \mathcal{Z}(U)$.*

Now we turn back to the affine reguli that are lines of $\mathbb{A}$. Note that by Proposition 3.1 such a line is necessarily regular.

Theorem 3.7 *Let $\mathcal{R}$ be a regulus containing W, and let $\mathcal{T}$ be its set of transversals. Then $\mathcal{R} \setminus \{W\}$ is a regular line of $\mathbb{A}$, exactly if*

$$W + \mathcal{T} = W + \mathcal{Z}(U). \tag{8}$$

Proof: Let $\mathcal{R} \setminus \{W\}$ be the regular line $\ell(\alpha, \beta)$. Then Proposition 3.1 implies $\mathcal{R} = \mathcal{R}_0{}^{\varphi}$, with φ induced by $\begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix}$. Hence also $\mathcal{T} = \mathcal{T}_0{}^{\varphi}$, and we compute $W + \mathcal{T} = W + \mathcal{T}_0{}^{\delta} = (W + \mathcal{T}_0)^{\delta} = W + \mathcal{T}_0$. Now Lemma 3.6 yields (8).
Conversely, let $\mathcal{R}$ be a regulus as in the assertion. Choose complementary $U_1, U_2 \in \mathcal{R} \setminus \{W\}$. They are joined by a regular line $\ell(\alpha, \beta)$, which is an affine regulus $\mathcal{R}' \setminus \{W\}$ with transversal set satisfying (8). So Corollary 3.5 yields $\mathcal{R} = \mathcal{R}'$ and thus $\mathcal{R} \setminus \{W\} = \ell(\alpha, \beta)$. $\square$

Corollary 3.8 *Let $U_1, U_2 \in \mathcal{S}$ be complementary. Then the regular line in $\mathbb{A}$ joining U_1 and U_2 is $\mathcal{R} \setminus \{W\}$, where $\mathcal{R}$ is the unique regulus containing W, U_1, U_2 that has a transversal set $\mathcal{T}$ satisfying $W + \mathcal{T} = W + \mathcal{Z}(U)$.*

4 The general case

Now we turn back to the general case that $V = W \oplus U$ for arbitrary W and U. We want to investigate the lines in $\mathbb{A} = \mathbb{A}(\mathcal{S}, (b_i))$. They all have the form $\ell(\alpha, \beta)$ as introduced in (6). It is sufficient to consider the lines containing $U = U^{(0,1)}$, i.e., the lines of type $\ell(\alpha, 0)$, because the other lines are images of these under the translation group introduced in Lemma 2.3(a).

More exactly, the line $\ell(\alpha, 0)$ is mapped by $\left(\begin{smallmatrix} 1 & 0 \\ \beta & 1 \end{smallmatrix}\right)$ to the line $\ell(\alpha, \beta)$. These translations can also be considered as collineations of the projective space $\mathbb{P}(K, V)$ that fix all points of and all subspaces through the projective subspace induced by W. On the other hand, each projective collineation fixing all points of and all subspaces through W and mapping U to $U^{(\beta,1)}$ is induced by a matrix $\left(\begin{smallmatrix} z & 0 \\ \beta & 1 \end{smallmatrix}\right)$ with $z \in Z^*$ and hence maps $\ell(\alpha, 0)$ to $\ell(\alpha, \beta)$. Thus we have obtained:

Lemma 4.1 *The line $\ell(\alpha, \beta)$ of $\mathbb{A}$ is the image of $\ell(\alpha, 0)$ under each projective collineation of $\mathbb{P}(K, V)$ that fixes all points of and all subspaces through W and maps U to $U^{(\beta,1)}$.*

We need the following slight generalization of Lemma 2.3(b):

Lemma 4.2 *Let $V_1 = W_1 \oplus U$ and $V_2 = W_2 \oplus U$ be left vector spaces over K. Let $\mathcal{S}_j$ be the set of complements of W_j in V_j, and let $\mathbb{A}_j = \mathbb{A}(\mathcal{S}_j, (b_i))$ be the associated affine space ($j = 1, 2$). If $\alpha : W_1 \to W_2$ is linear, then also*

$$\hat{\alpha} : V_1 \to V_2 : w_1 + u \mapsto w_1^\alpha + u$$

is linear. The induced mapping on $\mathcal{S}_1$, given by

$$U^{(\gamma,1)} \mapsto U^{(\gamma\alpha,1)}, \tag{9}$$

and also denoted by $\hat{\alpha}$, is a linear mapping $(\mathcal{S}_1, (b_j)) \to (\mathcal{S}_2, (b_j))$ and hence a homomorphism of affine spaces *$\mathbb{A}_1 \to \mathbb{A}_2$, i.e., it maps lines to lines or points and preserves parallelity.*

We apply this to $V_1 = U \times U$ with $W_1 = U \times \{0\}$, $U = \{0\} \times U$, and $V_2 = V = W \oplus U$, and thus obtain from Proposition 3.1:

Proposition 4.3 *The line $\ell(\alpha, 0)$ is the image of the standard affine regulus $\mathcal{R}_0 \setminus \{(U \times \{0\})\}$ in $\mathbb{P}(K, U \times U)$ under the linear mapping $\hat{\alpha} : U^{(\gamma,1)} \mapsto U^{(\gamma\alpha,1)}$.*

Since the linear mapping $\hat{\alpha}$ is defined on the whole vector space $V_1 = U \times U$, and thus on $\mathbb{P}(K, U \times U)$, we can make a statement on the image of the transversals of $\mathcal{R}_0$ under $\hat{\alpha}$.

Proposition 4.4 *Let $\ell(\alpha, 0)$ be a line of $\mathbb{A}$ and let $\mathcal{R}_0$ be the standard regulus in $\mathbb{P}(K, U \times U)$. Then the set $\mathcal{T}_0$ of transversals of $\mathcal{R}_0$ is mapped by $\hat{\alpha}$ onto a set $\mathcal{T} = \mathcal{T}_0^{\hat{\alpha}}$ of points and lines in $\mathbb{P}(K, V)$, such that the following holds:*

(a) If $T \in \mathcal{T}$ is a point, then each element of $\ell(\alpha, 0)$ contains T.

(b) If $T \in \mathcal{T}$ is a line, then T is a transversal *of $\ell(\alpha, 0)$, i.e., T meets W and the mapping $\ell(\alpha, 0) \cup \{W\} \to T : X \mapsto X \cap T$ is a bijection.*

Proof: This is a direct consequence of the fact that $\hat{\alpha}$ is linear and each restriction $\hat{\alpha}|_{U^{(\gamma,1)}} : U^{(\gamma,1)} \to U^{(\gamma\alpha,1)}$ is a bijection. □

One can also compute the set $\mathcal{T} = \mathcal{T}_0^{\hat{\alpha}}$ explicitly: By 3.2, each transversal $T \in \mathcal{T}_0$ has the form $T = K(z, 0) \oplus K(0, z)$ for some $z \neq 0$ in the Z-subspace of U w.r.t. (b_i). Its image is $T^{\hat{\alpha}} = Kz^\alpha + Kz$, which is a point exactly if $z \in \ker(\alpha)$.

The points of the line $\ell(\alpha,0)$ are the subspaces $U^{(\lambda_k\alpha,1)} = \{u^{\lambda_k\alpha} + u \mid u \in U\}$ with $k \in K$. In particular, they all belong to the projective space over $\mathrm{im}(\alpha) \oplus U \leq V$. Using Lemma 4.2, we can interpret this as follows:

Remark 4.5 The line $\ell(\alpha,0)$ is a line in the affine space $\mathbb{A}_0 = \mathbb{A}(\mathcal{S}_0,(b_i))$ of complements of $\mathrm{im}(\alpha)$ in $\mathrm{im}(\alpha) \oplus U$. The affine space $\mathbb{A}_0$ is embedded into $\mathbb{A}$ via $\hat{\iota}$, where ι is the inclusion mapping $\mathrm{im}(\alpha) \hookrightarrow W$. We will not distinguish between $\ell(\alpha,0)$ as a line in $\mathbb{A}_0$ and $\ell(\alpha,0)$ as a line in $\mathbb{A}$.

If α is injective, then the mappings $\alpha : U \to \mathrm{im}(\alpha)$ and $\hat{\alpha} : U \times U \to \mathrm{im}(\alpha) \oplus U$ are linear bijections. In particular, $\hat{\alpha}$ induces an isomorphism of projective spaces $\mathbb{P}(K, U \times U) \to \mathbb{P}(K, \mathrm{im}(\alpha) \oplus U)$, and $\ell(\alpha,0)$ is the image of the standard affine regulus in $\mathbb{P}(K, U \times U)$ under this isomorphism. This, together with Corollary 3.8, yields the following:

Proposition 4.6 *If $\alpha \in \mathrm{Hom}(U,W)$ is injective, then $\ell(\alpha,0)$ is an affine regulus in the projective subspace $\mathbb{P}(K, \mathrm{im}(\alpha) \oplus U)$ of $\mathbb{P}(K,V)$.*
The associated regulus $\ell(\alpha,0) \cup \{\mathrm{im}(\alpha)\}$ is the unique regulus in $\mathbb{P}(K, \mathrm{im}(\alpha) \oplus U)$ containing $\mathrm{im}(\alpha)$, U, and $U^{(\alpha,1)}$ that has a transversal set $\mathcal{T}$ satisfying $\mathrm{im}(\alpha) + \mathcal{T} = \mathrm{im}(\alpha) + \mathcal{Z}(U)$.

We need another type of homomorphism between affine spaces $\mathbb{A}(\mathcal{S},(b_i))$:

Lemma 4.7 *Let $V_1 = W \oplus U_1$ and $V_2 = W \oplus U_2$ be left vector spaces over K. Let $\mathcal{S}_j$ be the set of complements of W in V_j, let $(b_i^{(j)})$ be a basis of U_j, and let $\mathbb{A}_j = \mathbb{A}(\mathcal{S}_j,(b_i^{(j)}))$ be the associated affine space $(j = 1,2)$.*
If $\delta : U_1 \to U_2$ is central w.r.t. $(b_j^{(1)})$ and $(b_j^{(2)})$, then

$$\delta^* : \mathcal{S}_2 \to \mathcal{S}_1 : U_2^{(\eta,1)} \mapsto U_1^{(\delta\eta,1)}$$

is a linear mapping $(\mathcal{S}_2,(b_i^{(2)})) \to (\mathcal{S}_1,(b_i^{(1)}))$ and hence a homomorphism of affine spaces $\mathbb{A}_2 \to \mathbb{A}_1$.

Proof: Let $\lambda^{(j)}$ be the embedding of K into $\mathrm{End}(U_j)$ w.r.t. $(b_i^{(j)})$. Using that δ is central, we obtain $\lambda_k^{(2)}\delta = \delta\lambda_k^{(1)}$ for each $k \in K$. This implies that δ^* is linear. □
If $\delta : U_1 \to U_2$ and $\rho : U_2 \to U_3$ are central linear mappings w.r.t. given bases, then the associated homomorphisms of affine spaces obviously satisfy the condition

$$(\delta\rho)^* = \rho^*\delta^*. \tag{10}$$

We apply Lemma 4.7 and formula (10) to central subspaces of U. A subspace $U' \leq U$ is called *central* (w.r.t. the fixed basis (b_i)), if it possesses a basis of Z-linear combinations of the b_i's. For the investigation of $\mathbb{A}$ it suffices (by Theorem 2.6) to consider central subspaces with basis $(b_j)_{j\in J}$, where $J \subseteq I$. Note that for each subspace of U one can find a complement that is central in U.

Lemma 4.8 *Let U' be a central subspace of U with basis $(b_j)_{j\in J}$, let $\mathcal{S}'$ be the set of complements of W in $W \oplus U'$, and let $\mathbb{A}' = \mathbb{A}(\mathcal{S}',(b_j))$ be the associated affine space. Then the following statements hold:*

(a) *The inclusion mapping* $\iota : U' \hookrightarrow U$ *is central (w.r.t.* $(b_j)_{j\in J}$ *and* $(b_i)_{i\in I}$*). The associated homomorphism* $\iota^* : \mathbb{A} \to \mathbb{A}'$ *is the "intersection mapping"*

$$U^{(\eta,1)} \mapsto U^{(\eta,1)} \cap (W \oplus U').$$

(b) *For any central complement* C *of* U' *the projection* $\pi : U \to U'$ *with kernel* C *is central. The associated homomorphism* $\pi^* : \mathbb{A}' \to \mathbb{A}$ *is the "join mapping"*

$$U'^{(\eta',1)} \mapsto U'^{(\eta',1)} \oplus C.$$

(c) *The mapping* $\pi^*\iota^*$ *is the identity on* $\mathbb{A}'$*. So* π^* *is injective and* ι^* *is surjective.*

Proof: (a): The definition of ι^* yields $(U^{(\eta,1)})^{\iota^*} = \{(u'^{\eta}, u') \mid u' \in U'\} = U^{(\eta,1)} \cap (W \oplus U')$.
(b): We compute $(U'^{(\eta',1)})^{\pi^*} = \{(u'^{\eta'}, u' + c) \mid u' \in U', c \in C\} = U'^{(\eta',1)} \oplus C$.
(c): This follows from (10), since $\iota\pi = 1_{U'}$. □

The following corollary will be important later:

Corollary 4.9 *Let* C *be a central subspace of* U*. Then the set*

$$\mathcal{S}/C := \{S \in \mathcal{S} \mid C \leq S\}$$

is an affine subspace of $\mathbb{A}$.

Proof: Let $U' \leq U$ be any central complement of C, let $\mathbb{A}'$ be the affine space associated to $W \oplus U'$, and let π be the projection onto U' with kernel C. By Lemma 4.8, the linear injection π^* embeds the affine space $\mathbb{A}'$ into $\mathbb{A}$, and the image $\mathbb{A}'^{\pi^*}$ equals $\mathcal{S}/C$. □

Lemma 4.8(a) enables us to investigate the lines of $\mathbb{A}$ by means of their intersections with subspaces $W \oplus U'$, where U' is central in U. For every $\ell(\alpha, 0)$ we choose an appropriate maximal central U' and investigate

$$\ell(\alpha, 0) \cap (W \oplus U') := \{X \cap (W \oplus U') \mid X \in \ell(\alpha, 0)\}.$$

By $\mathcal{Z}(U')$ we denote the projective Z-subspace of $\mathbb{P}(K, U')$ w.r.t. the central basis of U'.

Theorem 4.10 *Let* $\ell(\alpha, 0)$ *be a line of* $\mathbb{A}$*, and let* $U' \leq U$ *be a central complement of* $\ker(\alpha)$*. Then the intersection*

$$\ell(\alpha, 0) \cap (W \oplus U') = \ell(\alpha, 0) \cap (\mathrm{im}(\alpha) \oplus U')$$

is an affine regulus in $\mathbb{P}(K, \mathrm{im}(\alpha) \oplus U') \cong \mathbb{P}(K, U' \times U')$*. The associated regulus* $\mathcal{R}$ *is the unique one containing* $\mathrm{im}(\alpha)$, U', $U'^{(\alpha,1)}$ *that has a transversal set* $\mathcal{T}$ *satisfying* $\mathrm{im}(\alpha) + \mathcal{T} = \mathrm{im}(\alpha) + \mathcal{Z}(U')$.

Proof: By Lemma 4.8(a) and Remark 4.5, the set $\ell(\alpha, 0) \cap (\mathrm{im}(\alpha) \oplus U')$ is a line in the affine space $\mathbb{A}'$ associated to $\mathrm{im}(\alpha) \oplus U'$, namely $\ell(\iota\alpha, 0)$, where $\iota : U' \hookrightarrow U$ is the inclusion. Since U' is complementary to $\ker(\alpha)$, we have that $\iota\alpha : U' \to \mathrm{im}(\alpha)$ is a bijection, whence $\ell(\iota\alpha, 0)$ is a regular line in $\mathbb{A}'$. The rest follows from Corollary 3.8. □

Of course a similar statement holds for each central subspace U' that is skew to $\ker(\alpha)$. Then the intersection of $\ell(\alpha,0)$ with $U'^{\alpha} \oplus U'$ is an affine regulus in $\mathbb{P}(K, U'^{\alpha} \oplus U')$. A special case is $U' = Kz$, with central $z \in U \setminus \ker(\alpha)$, which leads to a transversal of $\ell(\alpha,0)$ as described in Proposition 4.4(b).

Now we consider the kernel of α. Note that $\ker(\alpha)$ — like any other subspace of U — contains a unique maximal central subspace.

Theorem 4.11 *Let $\ell(\alpha,0)$ be a line of $\mathbb{A}$, and let M be the maximal central subspace of* $\ker(\alpha)$*. Moreover, let U' be a central complement of* $\ker(\alpha)$ *and let $\mathcal{R}$ be the regulus in* $\mathbb{P}(K, \operatorname{im}(\alpha) \oplus U')$ *appearing in Theorem 4.10. Then the following statements hold:*

(a) The set $\ell(\alpha,0)$ is entirely contained in

$$\mathcal{S}/M = \{S \in \mathcal{S} \mid M \leq S\}.$$

(b) The intersection $\ell(\alpha,0) \cap (\operatorname{im}(\alpha) \oplus U' \oplus M)$ is the cone with vertex M over *the affine regulus $\mathcal{R} \setminus \{\operatorname{im}(\alpha)\}$, i.e., the set*

$$\{X \oplus M \mid X \in \mathcal{R}, X \neq \operatorname{im}(\alpha)\}.$$

Proof: (a): By Corollary 4.9 the set $\mathcal{S}/M$ is an affine subspace of $\mathbb{A}$. Since $M \leq \ker(\alpha) = U \cap U^{(\alpha,1)}$, the line $\ell(\alpha,0)$ joining U and $U^{(\alpha,1)}$ belongs to $\mathcal{S}/M$.
(b): This follows directly from (a), Lemma 4.8(a), and Theorem 4.10. □

The bigger the maximal subspace M of $\ker(\alpha)$ is, the better we know $\ell(\alpha,0)$. In particular, if $\ker(\alpha)$ itself is central, we obtain the following:

Corollary 4.12 *Let $\ell(\alpha,0)$ be a line of $\mathbb{A}$, and let $\mathcal{R}$ be the regulus of Theorem 4.10. Assume that* $\ker(\alpha)$ *is a central subspace of U. Then*

$$\ell(\alpha,0) = \{X \oplus \ker(\alpha) \mid X \in \mathcal{R}, X \neq \operatorname{im}(\alpha)\},$$

i.e., $\ell(\alpha,0)$ is the cone with vertex $\ker(\alpha)$ *over the affine regulus $\mathcal{R} \setminus \{\operatorname{im}(\alpha)\}$.*

The statement of Corollary 4.12 means in other words that "modulo $\ker(\alpha)$" the line $\ell(\alpha,0)$ is an affine regulus. This is what Metz in [11] called a generalized regulus. Since Metz considers only commutative fields, in his case all lines of $\mathbb{A}$ are such generalized reguli.

The special case that $\dim W = \dim U = 2$ was treated in [5]. Then each $\alpha \in \operatorname{Hom}(U,W) \setminus \{0\}$ that is not injective has a one-dimensional kernel. So there are only two types of non-regular lines $\ell(\alpha,0)$, namely, those where $\ker(\alpha) = Kz$ for some central $z \neq 0$, and the others. In the first case we have a line pencil with carrier Kz and one line removed. This is the cone with vertex Kz over an affine regulus, which here is the set of points of an affine line.

5 An application: Dual spreads

In this section we aim at a description of dual spreads.

In a first step, where $V = W \oplus U$ is arbitrary, we show that $(\mathcal{S}, (b_i)_{i \in I})$ is isomorphic to the left vector space W^I of all I-families in W.

Proposition 5.1 *The mapping*

$$\psi : W^I \to \mathcal{S} : (w_i)_{i\in I} \mapsto U^{(\gamma,1)}, \tag{11}$$

where $\gamma \in \mathrm{Hom}(U, W)$ is given by $b_i \mapsto w_i$, is a K-linear bijection.

Proof: Let $(w_i)^\psi = U^{(\gamma,1)}$ and $(w_i')^\psi = U^{(\gamma',1)}$. Since ${b_i}^{\gamma+\gamma'} = w_i + w_i'$ and ${b_i}^{\lambda_k\gamma} = kw_i$ hold for all $i \in I$ and all $k \in K$, we have that ψ is linear. The bijectivity of ψ is obvious. □

This mapping ψ has also a simple geometric interpretation: For each $i \in I$ put

$$E_i = W \oplus Kb_i.$$

Then $\mathbb{P}(K, E_i) \setminus \mathbb{P}(K, W)$ is an affine space isomorphic to $\mathbb{A}(K, W)$, as follows from the isomorphism

$$\psi_i : W \to \mathbb{P}(K, E_i) \setminus \mathbb{P}(K, W) : w \mapsto K(w + b_i). \tag{12}$$

Now $(w_i)^\psi$ is simply the element of $\mathcal{S}$ that is spanned by the points $K(w_i + b_i)$, where i ranges in I.

Another tool for our investigation of dual spreads is the following:

Lemma 5.2 *Let $(c_i) \in W^I$ and let $H \le W$ be a hyperplane. Then the ψ-image of the affine subspace $(c_i) + H^I$ of W^I equals*

$$\mathcal{S}(X) := \{S \in \mathcal{S} \mid S \le X\}$$

where $X := (c_i)^\psi \oplus H$ is a hyperplane of V with $W \not\le X$.
Conversely, each hyperplane $X \le V$ with $W \not\le X$ arises in this way from exactly one affine subspace of W^I.

Proof: Given (c_i) and H as above, then $X := (c_i)^\psi \oplus H \le V$ is in fact a hyperplane with $W \not\le X$, and $((c_i) + H^I)^\psi = \mathcal{S}(X)$ follows from (12).
Conversely, if $X \le V$ is a hyperplane with $W \not\le X$, then X contains an element of $\mathcal{S}$, say $(c_i)^\psi$ with $(c_i) \in W^I$, and $X \cap W =: H$ is a hyperplane of W. It is easily seen that $((c_i) + H^I)^\psi = \mathcal{S}(X)$. □

In what follows we restrict ourselves to the symmetric case and we adopt the settings of Section 3 ($V = U \times U$, $W = U \times \{0\}$, $U = \{0\} \times U$). Recall that a line $\ell(\alpha, \beta)$ of $\mathbb{A}$ is regular if $\alpha \in \mathrm{Aut}(U)$. Now a subspace of $\mathbb{A}$ will be called *singular*, if none of its lines is regular. We describe one family of maximal singular subspaces:

Lemma 5.3 *Let $X \le V$ be a hyperplane with $W \not\le X$. Then $\mathcal{S}(X) = \{S \in \mathcal{S} \mid S \le X\}$ is a maximal singular subspace of $\mathbb{A}$.*

Proof: By Lemma 5.2, the set $\mathcal{S}(X)$ is a subspace of $\mathbb{A}$. Obviously, any two elements of $\mathcal{S}(X)$ are not complementary, whence $\mathcal{S}(X)$ is singular by Proposition 3.1.
Given $U_1 \in \mathcal{S} \setminus \mathcal{S}(X)$ there exists a linear bijection $\beta : W \to U_1$ such that $(W \cap X)^\beta = U_1 \cap X$. Then $Y := \{w + w^\beta \mid w \in W \cap X\}$ is skew to $W \cap X$ and $U_1 \cap X$. There exists a complement $U_2 \supseteq Y$ of $W \cap X$ relative to X. By construction, W, U_1, and U_2 are pairwise complementary in V. So the affine subspace spanned by $\mathcal{S}(X)$ and U_1 cannot be singular. □

A *dual spread* of $\mathbb{P}(K, V)$ is a set of pairwise complementary subspaces such that each hyperplane contains one of its elements.

Proposition 5.4 *A subset $\mathcal{B} \subseteq \mathcal{S}$ together with W is a dual spread if, and only if, the following conditions hold true:*

(DS1) *Distinct elements of $\mathcal{B}$ are joined by a regular affine line.*

(DS2) *Each maximal singular subspace $\mathcal{S}(X)$, where $X \leq V$ is a hyperplane with $W \not\leq X$, contains an element of $\mathcal{B}$,*

Proof: Let $\mathcal{B} \subseteq \mathcal{S}$. All elements of $\mathcal{B}$ are complements of W, and obviously every hyperplane of V through W contains $W \in \mathcal{B} \cup \{W\}$.
By Proposition 3.1, the elements of $\mathcal{B}$ are pairwise complementary exactly if (DS1) holds true. Moreover, (DS2) is just a re-formulation of the condition that there is an element of $\mathcal{B}$ in each hyperplane $X \leq V$ with $W \not\leq X$. $\square$
For another description of dual spreads containing W we generalize the concept of $*$-transversal mappings described by N. Knarr in [9], p.29. Moreover, we use Lemma 5.2, where $W = U \times \{0\}$ can be replaced by the isomorphic vector space U.
A family $(\tau_i)_{i\in I}$ of mappings $\tau_i : D \to U$, where $D \subseteq U$, is called $*$-*transversal*, if the following conditions hold true:

(T1*) Given distinct $u, u' \in D$, then $(u^{\tau_i} - u'^{\tau_i})_{i\in I}$ is a basis of U.

(T2*) Given a family $(c_i)_{i\in I} \in U^I$ and a hyperplane H of U, there is a $u \in D$ such that $(u^{\tau_i}) \in (c_i) + H^I$.

By (T1*) each τ_i is an injective mapping. So, after an appropriate change of the domain D, one may always assume that for one index $i_0 \in I$ the mapping τ_{i_0} is the canonical inclusion $D \hookrightarrow U$. Therefore, in case $|I| = 2$ it is enough to have one more mapping $\tau_{i_1} : D \to U$. This is in fact the approach in [9].

Theorem 5.5 *Let $(\tau_i)_{i\in I}$ be a $*$-transversal family of mappings $\tau_i : D \to U$. Then*

$$\mathcal{D} := \{(u^{\tau_i})^{\psi} \mid u \in D\} \cup \{W\} = \{\textstyle\bigoplus_{i\in I} K(u^{\tau_i}, b_i) \mid u \in D\} \cup \{W\}$$

is a dual spread of $\mathbb{P}(K, V)$. Conversely, each dual spread of $\mathbb{P}(K, V)$ containing W can be obtained in this way.

Proof: Let $(\tau_i)_{i\in I}$ be a $*$-transversal family. Choose distinct elements $(u^{\tau_i})^{\psi} = U^{(\gamma,1)}$ and $(u'^{\tau_i})^{\psi} = U^{(\gamma',1)}$, where $u, u' \in D$ and $\gamma, \gamma' \in \mathrm{End}(U)$. Then $b_i^{\gamma-\gamma'} = u^{\tau_i} - u'^{\tau_i}$ for all $i \in I$, so that $\gamma - \gamma' \in \mathrm{Aut}(U)$ by (T1*). Hence $\mathcal{D} \setminus \{W\}$ satisfies (DS1). By (T2*) and Lemma 5.2, $\mathcal{D} \setminus \{W\}$ satisfies (DS2). So Proposition 5.4 shows that $\mathcal{D}$ is a dual spread.
On the other hand, let $\mathcal{D}$ be a dual spread containing W. If $(s_i), (s'_i) \in (\mathcal{D}\setminus\{W\})^{\psi^{-1}}$ coincide in one entry, say $s_j = s'_j$, then $K(s_j, b_j)$ is a common point of their ψ-images by (12). Hence in this case $(s_i) = (s'_i)$.
We fix one index $i_0 \in I$ and put $D := \{s_{i_0} \mid (s_i) \in (\mathcal{D}\setminus\{W\})^{\psi^{-1}}\} \subseteq U$. By the above, for each $u \in D$ there is a unique $(s_i) \in (\mathcal{D} \setminus \{W\})^{\psi^{-1}}$ with $u = s_{i_0}$. Moreover, $\tau_i : D \to U : u \mapsto s_i$ is well defined for all $i \in I$. By reversing the arguments of the first direction, it is easily seen that the family $(\tau_i)_{i\in I}$ is $*$-transversal. $\square$

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