Germán Ancochea's work on projectivities, harmonicity preservers and semi-homomorphisms

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1 Introduction

Our main aim is to analyse three articles of Germán Ancochea ([5], [6], [7]) and to describe their impact in algebra and geometry. Even though most basic mathematical concepts remained unchanged ever since, terminology, notation, and the style of exposition has undergone substantial changes. We decided to write our note in today's mathematical language and to add remarks on the original wording in footnotes. Furthermore, we provide short expositions of topics that constitute the basis of Ancochea's research. The surveys [41] and [133] contain a wealth of references to the material presented there.

2 Ancochea's contributions

2.1 Projective spaces

One major theme of Ancochea's contributions is the interplay between synthetic projective geometry and algebra. Recall that a *projective space* $(\mathcal{P}, \mathcal{L})$ consists of a set $\mathcal{P}$ of *points* and a set $\mathcal{L}$ of subsets of $\mathcal{P}$ called *lines*. In such a space it is common to introduce the following terminology. A collection of points is said to be *collinear*, if there exists a line containing all of them. A *triangle* is a set of three non-collinear points. A *side* of a triangle is a line that contains two distinct points of the given triangle. In terms of these notions the axioms of a projective space $(\mathcal{P}, \mathcal{L})$ read as follows:

1. Any two distinct points lie on a unique line.

2. If a line meets two distinct sides of a triangle, not at a point of the triangle, then it meets the third side also.
3. Any line contains at least three distinct points.

The above definition of a projective space contains no restrictions on its dimension, which may be finite or infinite.

We now assume, until further notice, $(\mathcal{P}, \mathcal{L})$ to be a projective space of dimension > 2, that is, it contains at least two disjoint lines. Such a space has a remarkable property. Upon choosing a line L and three distinct *points of reference* on L, which are labelled as $0, 1, \infty$, the set $K := L \setminus \{\infty\}$ can be made into a field[1] by defining the sum $x + y$ and the product $x \cdot y$ of $x, y \in K$ in a purely geometric manner. Starting with x and y one has to draw several auxiliary points and lines in some (projective) plane[2] containing L according to Fig. 1.[3] The distinguished

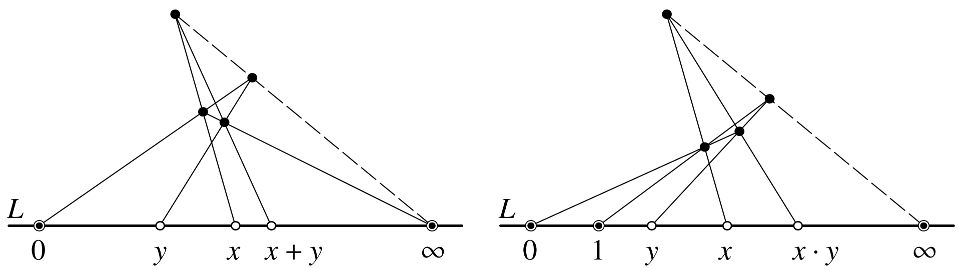

Figure 1: Addition and multiplication

points 0 and 1 are the neutral elements with respect to addition and multiplication in the field K, which will be called an *underlying field* of $(\mathcal{P}, \mathcal{L})$. All underlying fields of a given projective space $(\mathcal{P}, \mathcal{L})$ of dimension > 2 are isomorphic.

One crucial point is, of course, that the definition of the sum and the product of points does not depend on the choice of the auxiliary elements. This is due to the fact that $(\mathcal{P}, \mathcal{L})$ is a *desarguesian* projective space, that is, in any of its planes the following *Theorem of Desargues* (Fig. 2) holds. *If two triangles a_1, a_2, a_3 and b_1, b_2, b_3 of a projective plane are in perspective from a point p, then the intersections of their corresponding sides are collinear.*

Now let $(\mathcal{P}, \mathcal{L})$ be a projective plane. Then the construction of an underlying field can be carried out like before provided that $(\mathcal{P}, \mathcal{L})$ is desarguesian. Otherwise

[1] As is customary among geometers, multiplication in a field is not assumed to be commutative.

[2] A *projective plane* is a two-dimensional projective space. When speaking of subspaces of a projective space, we usually drop the adjective “projective”.

[3] The classical example of a three-dimensional projective space arises from the three-dimensional Euclidean space by adding “points and lines at infinity” in an appropriate way. If the dashed line in Fig. 1 coincides with a line at infinity and 0 is not at infinity, then there is an elementary interpretation. The sum $x + y$ is the image of x under the translation taking 0 to y. The product $x \cdot y$ is the image of x under the homothety with centre 0 taking 1 to y.

one arrives at the problem of coordinatisation of a non-desarguesian projective plane [82], [131], [132], which is beyond our scope.

Given a projective space $(\mathcal{P}, \mathcal{L})$ of dimension ≥ 2 a mapping $\pi\colon L_1 \to L_2\colon x \mapsto x^\pi$ of a line L_1 to a line L_2 is called a *perspectivity*, if there exists a point $p \notin L_1 \cup L_2$ (called the *centre* of π) such that p, x, x^π are collinear for all $x \in L$ (Fig. 3). Any

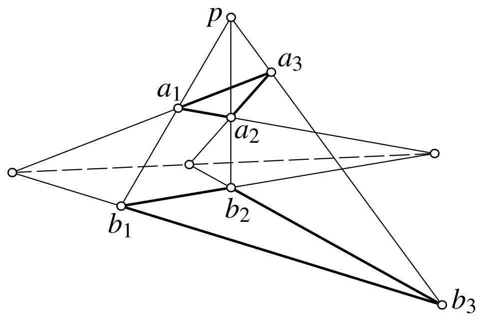

Figure 2: Theorem of Desargues

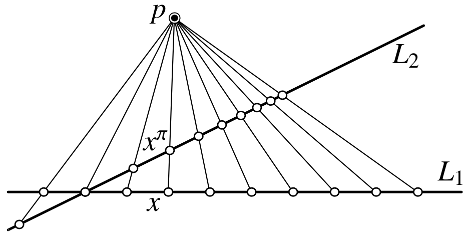

Figure 3: Perspectivity

product of finitely many perspectivities is called a *projectivity*[4]. Projectivities are bijective. If (p_1, p_2, p_3) and (q_1, q_2, q_3) are triples of distinct points on lines L and M, respectively, then there is at least one projectivity ψ sending $p_1 \mapsto q_1$, $p_2 \mapsto q_2$, $p_3 \mapsto q_3$. In order to obtain *all* projectivities with this property it is therefore enough to look for all projectivities of L onto itself such that p_1, p_2, p_3 remain fixed.

In the classical case, where $(\mathcal{P}, \mathcal{L})$ is three-dimensional and any underlying field is isomorphic to the field of real numbers, the name *Fundamental Theorem of Projective Geometry* has been given to following result. *The identity is the only projectivity of a line onto itself with three distinct fixed points.* If $(\mathcal{P}, \mathcal{L})$ is a desarguesian projective space of dimension ≥ 2, then this fundamental theorem remains valid if, and only if, one underlying field of $(\mathcal{P}, \mathcal{L})$ is commutative.

2.2 Sobre el teorema fundamental de la geometria proyectiva

In this article ([5]) from 1941, Ancochea considered a projective space $(\mathcal{P}, \mathcal{L})$ of dimension > 2 and referred to the books of A. N. Whitehead [164, p. 15] and L. Bieberbach [28, p. 5] for the notion of a projective space.[5] Also, he quoted [28, Kap. 1, § 4] for the construction of a field K from a line of $(\mathcal{P}, \mathcal{L})$ and recalled the definition and some properties of a projectivity.

[4]This definition goes back to J.-V. Poncelet. Ancochea used the name *projectivity in the sense of Poncelet* for such a mapping.

[5]In both books there are also extra axioms that force the dimension of a projective space to be three. The formalism used in [28] is different from ours.

After these preliminaries, the main question of the article was posed. *What can be said about a projectivity of a line L onto itself that fixes three distinct points?* Ancochea immediately derived a partial answer from a note by O. Veblen [157]. *If these three points are labelled as* $0, 1, \infty$ *and are used to construct a field* $K := L \setminus \{\infty\}$*, then such a projectivity restricts to an automorphism of* K. (Let us add that the reasoning in [157] relies on two observations. First, any projectivity can be extended to a *collineation* of $(\mathcal{P}, \mathcal{L})$, i. e. a bijection $\mathcal{P} \to \mathcal{P}$ that takes any line onto a line. Second, any collineation of $(\mathcal{P}, \mathcal{L})$ fixing the points 0, 1, ∞ induces an automorphism of K.) Next, he stated the following complete answer to his question.

Theorem ([5, Teorema fundamental])**.** *Let ψ be a projectivity of a line L with three distinct fixed points. If these points are chosen as the points of reference* $(0, 1, \infty)$*, then the image of any point $x \in L$ other than ∞ arises by applying an inner automorphism of $K = L \setminus \{\infty\}$. More precisely, there is an $a \in K \setminus \{0\}$, which is fixed under ψ, such that $x^{\psi} = a^{-1}xa$ for all $x \in K$. Conversely, any inner automorphism of K extends to a projectivity of L that fixes the points of reference.*

Let us sketch Ancochea's elegant proof. One may assume without loss of generality that ψ is given as a product $\pi_1\pi_2 \cdots \pi_{n+1}$ of perspectivities and, moreover, that π_{n+1} takes the form $\pi_{n+1} \colon \overline{L} \to L$ with 0 being the only common point of L and $\overline{L}$. By a theorem of F. Schur [142], the projectivity $\overline{\psi} := \pi_1\pi_2 \cdots \pi_n \colon L \to \overline{L}$ can also be written as a product of *at most two* perspectivities. If $\overline{\psi}$ is a single perspectivity, then 0, 1, ∞ being fixed yields that $\psi = \overline{\psi}\pi_{n+1}$ is the identity. Otherwise, ψ is a product of three perspectivities. The rest of the proof is contained in Fig. 4, which is a reproduction[6] from [5]. A carefully explained way of how to

To be included in the final version only.

Figure 4: Copy of Ancochea's drawing

[6]The author is grateful to the editors of "Revista Matemática Iberoamericana" for granting permission to include a copy of Ancochea's drawing.

read this figure[7] in terms of the field K yields that $x^\psi = axa^{-1}$ for an appropriate $a \in K$. By reversing these arguments the converse can be established.

Ancochea closed with the observation that the "classical" Fundamental Theorem of Projective Geometry (see page 3) appears as an immediate corollary of his more general version. Indeed, the projectivity ψ reduces to the identity if, and only if, a belongs to the centre of K.

In a subsequent article [6], Ancochea expressed his thanks to F. Bachmann for pointing out that his *Teorema fundamental* has been obtained already in 1930, using different methods, by K. Reidemeister [136, p. 137, Satz 4] (reprint [137]).

2.3 Harmonic quadruples

Let $(\mathcal{P}, \mathcal{L})$ be a projective space of dimension ≥ 2. In $(\mathcal{P}, \mathcal{L})$, a *quadrangle* means a set of four points of a fixed plane no three of which are collinear. A quadruple (p_1, p_2, p_3, p_4) of points on a line is said to be *harmonic*, in symbols $\mathrm{H}(p_1, p_2, p_3, p_4)$, if there exists a quadrangle, say q_1, q_2, q_3, q_4 with $p_1 = q_1q_2 \cap q_3q_4$, $p_2 = q_2q_3 \cap q_4q_1$, $p_3 \in q_1q_3$, and $p_4 \in q_2q_4$. (Here we denote the unique line joining p and q by pq etc.) The first three points p_1, p_2, p_3 of a harmonic quadruple are distinct. Likewise, the fourth point p_4 cannot coincide with p_1 or p_2. In general, nothing can be said about p_3 and p_4 being different or not. However, if $(\mathcal{P}, \mathcal{L})$ is desarguesian, then harmonic quadruples are well understood. First of all, the theorem about the *uniqueness of the fourth harmonic point* holds. *Given three distinct points* p_1, p_2, p_3 *on a line* L *there is a unique point* $p_4 \in L$ *such that* $\mathrm{H}(p_1, p_2, p_3, p_4)$. Furthermore, an underlying field K has characteristic two if, and only if, $p_3 = p_4$.

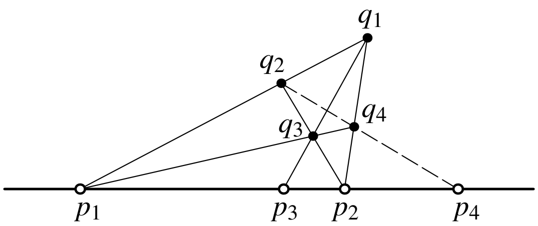

Figure 5: Harmonic quadruple

Let us now switch from synthetic to analytic projective geometry. Given any left vector space V over a field K the *projective space on* V, which will be written as $\mathbb{P}(V)$, has $\{Kv \mid v \in V \setminus \{0\}\}$, i. e. the set of one-dimensional subspaces of V, as its set of points. A line of $\mathbb{P}(V)$ is defined as the set of all points that are contained

[7]The points of reference are labelled as A_0, A_1, A_∞. In order to illustrate the case of a non-commutative field K, Ancochea decided to "bend" one line so that the point x and its image under ψ (written as x') turn out different.

in some fixed subspace of V with (vector) dimension two. We shall also use the name *projective space over* K for $\mathbb{P}(V)$ in order to emphasise the ground field. A projective space $\mathbb{P}(V)$ has dimension $\dim V - 1$, and it is desarguesian.[8]

If (p_1, p_2, p_3) is a triple of distinct points on a line L of $\mathbb{P}(V)$, then there exists at least one pair of linearly independent vectors (w_1, w_2) such that $p_1 = Kw_1$, $p_2 = Kw_2$, $p_3 = K(w_1 + w_2)$. The mapping

$$L \setminus \{p_1\} \to K\colon K(x_1w_1 + x_2w_2) \mapsto x_2^{-1}x_1 \ \text{ with } x_1, x_2 \in K \text{ and } x_2 \neq 0 \qquad (1)$$

is bijective. If K is non-commutative, then this bijection depends not only on the given triple of points but also on the choice of (w_1, w_2). Indeed, all pairs of vectors with the required property comprise the set $\{(cw_1, cw_2) \mid c \in K \setminus \{0\}\}$. When fixing any point $p_4 = K(x_1w_1 + x_2w_2) \in L \setminus \{p_1\}$ and calculating the corresponding element of K with respect to all these pairs in analogy to (1), one obtains the set

$$\mathrm{CR}(p_1, p_2, p_3, p_4) := \left\{c(x_2^{-1}x_1)c^{-1} \mid c \in K \setminus \{0\}\right\}, \qquad (2)$$

which is called the *cross ratio* of the quadruple (p_1, p_2, p_3, p_4). This cross ratio is a *conjugacy class* in K and it depends only on the given points. Precisely when $x_2^{-1}x_1$ is in the centre of K, the above cross ratio may be considered as the single element $x_2^{-1}x_1 \in K$ rather than a subset of K.[9]

If $\dim V > 2$, then harmonic quadruples can be characterised as follows:

$$\mathrm{CR}(p_1, p_2, p_3, p_4) = -1 \quad \Leftrightarrow \quad \mathrm{H}(p_1, p_2, p_3, p_4). \qquad (3)$$

Also, with $(0, 1, \infty) := (p_2, p_3, p_1)$ the bijection in (1) turns into an isomorphism of the underlying field $L \setminus \{\infty\}$ onto K. So, all underlying fields of $\mathbb{P}(V)$ are isomorphic to K. If $\dim V = 2$ or, in other words, if $\mathbb{P}(V)$ is a projective line, then (3) can be used as definition of harmonic quadruples. An abstract projective line, however, has in general not enough "intrinsic structure" to define harmonic quadruples.

The seminal book *Geometrie der Lage* by K. G. C. von Staudt contains a most remarkable result, namely a characterisation of projectivities in terms of harmonic quadruples [151, pp. 49ff.]. In its original version, which is confined to projective spaces of dimension three over the real numbers $\mathbb{R}$, it reads as follows:

Theorem (von Staudt's Theorem)**.** *Let lines L_1 and L_2 be given. Then a bijective mapping $\lambda\colon L_1 \to L_2$ is a projectivity precisely when it takes any harmonic quadruple on L_1 to a harmonic quadruple on L_2.*

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[8] The projective spaces $\mathbb{P}(V)$ with $\dim V \geq 3$ are, up to isomorphism, precisely the desarguesian projective spaces of dimension ≥ 2.

[9] Ancochea never mentions "cross ratios" in [5], [6], and [7]. When he wrote these articles, the notion of "cross ratio" has been established for commutative fields only. The definition in (2) was given in 1952 by R. Baer [10, pp. 71–72] and, in a slightly different setting, in 1948–1949 by E. Sperner [149, p. 149] and [150, p. 425].

For this result to be true, two properties of the field $\mathbb{R}$ are essential. First, the characteristic $\operatorname{Char}\mathbb{R}$ is different from two and, second, the field $\mathbb{R}$ admits no automorphism other than the identity. This was pointed out in a note by M. G. Darboux [47] from 1870, where also a small gap in von Staudt's original proof was closed.

In order to generalise von Staudt's Theorem we adopt the following terminology. Given lines L_1 and L_2 in a desarguesian projective space $(\mathcal{P}, \mathcal{L})$ of dimension ≥ 2 a mapping $\lambda\colon L_1 \to L_2$ is called a *harmonicity preserver*[10] if it takes any harmonic quadruple on L_1 to a harmonic quadruple on L_2.

Harmonicity preservers do not deserve interest if an underlying field of $(\mathcal{P}, \mathcal{L})$ is of characteristic two, since then a quadruple (p_1, p_2, p_3, p_4) of points is harmonic if, and only if, p_1, p_2, p_3 are distinct collinear points and $p_3 = p_4$. So, in case of characteristic two, a mapping $\lambda\colon L_1 \to L_2$ is a harmonicity preserver precisely when it is injective.

Suppose now that $(\mathcal{P}, \mathcal{L})$ has an underlying field of characteristic $\neq 2$. Projectivities are obvious examples of *bijective* harmonicity preservers. In order to determine all bijective harmonicity preservers between lines L_1 and L_2 of $(\mathcal{P}, \mathcal{L})$, it is therefore enough to consider all mappings λ of this kind that fix three arbitrarily chosen points on some line L. When labelling the three fixed points as 0, 1, ∞, the points of $K := L \setminus \{\infty\}$ provide an underlying field of $(\mathcal{P}, \mathcal{L})$, and λ restricts to a bijection σ of K; see page 2. O. Schreier and E. Sperner [141, pp. 191–194] used these ideas and extended the findings of von Staudt and Darboux in the year 1935. *If the field K is commutative,* $\operatorname{Char} K \neq 2$, *then any bijective harmonicity preserver $\lambda\colon L \to L$ fixing the points* 0, 1, ∞ *restricts to an automorphism of K and, conversely, any automorphism of K extends to a unique bijective harmonicity preserver of the line L.*

2.4 Le théorème de von Staudt en géométrie projective quaternionienne

Ancochea started his article [6] by quoting the result of O. Schreier and E. Sperner that we encountered above. He observed that an extension thereof to the case of a non-commutative field K seemed to be unknown. In order to state such an extension he introduced an entirely new algebraic concept.

Definition ([6, p. 193])**.** A bijective mapping σ of a field K onto itself is called a *semi-automorphism* if for all $x, y \in K$ the following conditions hold:

$$(x + y)^\sigma = x^\sigma + y^\sigma, \tag{4}$$

$$(xy)^\sigma + (yx)^\sigma = x^\sigma y^\sigma + y^\sigma x^\sigma. \tag{5}$$

[10] Some authors use the name *harmonic mapping* instead. We refrain from doing so in order to avoid confusion with the "harmonic mappings" known from differential geometry.

This allowed him to state the following theorem.

Theorem ([6, p. 193]). *In a projective space over a field of characteristic $\neq 2$, any bijective harmonicity preserver*[11] *of a line L onto itself with three distinct fixed points* 0, 1, ∞ *restricts to a semi-automorphism of the field $L \setminus \{\infty\}$.*

In order to establish the theorem, Ancochea considered the projective space $\mathbb{P}(V)$ on some left vector space V over a field K of characteristic $\neq 2$. The given line L therefore corresponds to a two-dimensional subspace W of V. There exists a basis (w_1, w_2) of W such that the distinguished points $0, 1, \infty$ are Kw_2, $K(w_1 + w_2)$, Kw_1, respectively.[12] The isomorphism (1) allowed him to *identify* the points of the underlying field $L \setminus \{\infty\}$ with the elements of the field K. Next, he addressed the problem of calculating the fourth harmonic point, say p of three distinct points $a_1, a_2, a_3 \in K$. Based on calculations, which in most cases were omitted by Ancochea, he arrived at the following results: $p = \infty$ is equivalent to $a_3 = \frac{1}{2}(a_1 + a_2)$. If $p \neq \infty$, letting $p =: a_4 \in K$ gives

$$a_4 = \left((a_1 - a_3)^{-1} + (a_2 - a_3)^{-1}\right)^{-1} \left((a_1 - a_3)^{-1} a_1 + (a_2 - a_3)^{-1} a_2\right). \tag{6}$$

Equation (6) takes the form

$$a_4 = 2a_1(a_1 + a_2)^{-1} a_2 \tag{7}$$

provided that $a_3 = 0$ and $a_1 + a_2 \neq 0$. Likewise, the remaining exposition is extremely brief. The reader merely is invited to repeat the proof from [141, pp. 192–194] in order to verify that the given harmonicity preserver of L gives rise to a bijection $\sigma \colon K \to K$ satisfying equations (4) and (5).

We present here a more detailed description, because we want to illustrate how the definition of a semi-automorphism arises by following Ancochea's advise. First, a proof of (7) can be done using

$$a_4 = \left(a_1^{-1} + a_2^{-1}\right)^{-1} \left(a_1^{-1} a_1 + a_2^{-1} a_2\right) = 2\left(a_2^{-1}(a_2 + a_1) a_1^{-1}\right)^{-1}.$$

Next, in order to establish (4), it is enough to start on page 192 of [141] before formula (15) and to repeat the reasoning from there up to equation (20). Thereby, one has to reinterpret the cross ratios appearing in [141]. Any cross ratio of the form $\mathrm{CR}(\infty, 0, 1, p)$ with p in $L \setminus \{\infty\}$ is to be understood as the corresponding element $x_2^{-1} x_1 \in K$ according to (1); cross ratios equal to $-1 \in K$ characterise harmonic quadruples. The transfer from [141] of the proof for (5) appeared at the

[11] Ancochea used instead the name *projectivity in the sense of von Staudt*.

[12] Ancochea actually used the left vector space K^2 and its canonical basis instead of our W and (w_1, w_2).

first sight impossible to the author, since the commutativity of the ground field is used at the very beginning of the reasoning in the middle of page 193. Fortunately, it suffices to start reading on that page a few lines further down at equation (21) taking into account that the fraction

$$\frac{2ab}{a+b}$$

appearing there has to be replaced by the right hand side of our equation (7). In the same way several other fractions, which do not make sense over a skew field, have to be rewritten appropriately. The next formula that needs to be altered is

$$ab = \frac{1}{2}\left((a+b)^2 - a^2 - b^2\right),$$

since this requires commuting elements a, b. In our not necessarily commutative field K this formula has to be replaced with

$$a_1 a_2 + a_2 a_1 = (a_1 + a_2)^2 - a_1^2 - a_2^2 \text{ for all } a_1, a_2 \in K.$$

In this way one readily obtains the condition in (5) rather than the multiplicativity of σ as in [141].

After the proof, Ancochea gave examples of harmonicity preservers. Remarkably enough, he did not use formula (6) for this purpose. Instead, a result of S. Wachs [158, p. 82] was quoted. It states that, for all $a_1, a_2, a_3, a_4 \in K$, the condition $\mathrm{H}(a_1, a_2, a_3, a_4)$ is equivalent to

$$(a_2 - a_4)^{-1}(a_2 - a_3)(a_1 - a_3)^{-1}(a_1 - a_4) = -1. \tag{8}$$

By virtue of this characterisation it turned out that *any automorphism and any anti-automorphism*[13] *of K extends to a bijective harmonicity preserver of L fixing the points* 0, 1, ∞.

In [6, § 2–3], Ancochea dealt with an arbitrary quaternion skew field Q of characteristic $\neq 2$. He showed that *any semi-automorphism of Q is either an automorphism or an anti-automorphism of Q.* We do not enter a detailed analysis here, since in his subsequent article [7] a more general result can be found. The geometric significance is, of course, that *any semi-automorphism of Q gives rise to a bijective harmonicity preserver.*

The article [6] closes with some remarks on the case when the centre of a quaternion skew field Q, $\operatorname{Char} Q \neq 2$, admits only the identical automorphism. Ancochea recalled that, by what nowadays is called the *Theorem of Skolem-Noether*, all automorphisms of Q are inner [128] (or see [89, Thm. 4.9]). Making

[13] Ancochea's terminology reads *direct automorphism* and *reciprocal automorphism*, respectively.

use of [5, Teorema fundamental] or the analogous result in [136] he derived that over such a quaternion skew field *any bijective harmonicity preserver either is a projectivity or—loosely speaking—the product of some projectivity and the mapping that takes any quaternion in Q to its conjugate*. He also referred to the description of all *continuous* bijective harmonicity preservers of the projective line over the real quaternions given by S. Wachs [158, pp. 108–109]; he thereby stressed that his reasoning was based exclusively on algebraic tools. Finally, he emphasised the difference with the situation over the field of complex numbers. Indeed, in order to obtain there a "similar" result, one has to assume continuity due to the existence of discontinuous automorphisms of the complex number field [95] (or see [98], [127], [143]).

Summing up, Ancochea generalised the first part of von Staudt's theorem to the general case of an arbitrary field K, commutative or not. The second part, namely the question whether or not all semi-automorphism of K arise in this way, remained as an open problem in [6].

2.5 On semi-automorphisms of division algebras

The article [7], which is the third and last in this series, commences with a short summary of the basic notions and results from [6]. In doing so, a *semi-automorphism* of a ring R is defined as a bijection $\sigma\colon R \to R$ possessing the properties (4) and (5) up to a change of notation. Further below, *semi-isomorphisms* of rings are introduced in the same fashion. Like Ancochea, we tacitly assume from now on any *algebra* to be associative and finite-dimensional over its centre (unless explicitly stated otherwise). So, *division algebra* means a field with finite dimension over its centre. The main results read as follows:

Theorem ([7, Principal Theorem])**.** *Let D be a division algebra of characteristic $\neq 2$. Then any semi-automorphism σ of D is an automorphism or an anti-automorphism of D.*

Theorem ([7, von Staudt's Theorem])**.** *In a projective space over a division algebra of characteristic $\neq 2$, any bijective harmonicity preserver of a line L onto itself with three distinct fixed points 0, 1, ∞ restricts to a semi-automorphism of the division algebra $D := L \setminus \{\infty\}$. Conversely, any semi-automorphism of D extends to a bijective harmonicity preserver of L that fixes the points of reference.*

Ancochea based the proof of his Principal Theorem upon several auxiliary results, which are stated below.

Theorem ([7, Theorem 1])**.** *Under any semi-automorphisms σ of a division algebra D (of arbitrary characteristic) the centre F of D is invariant.*

In order to show his Theorem 1, Ancochea followed the lines of [6] by concluding that for all $\gamma \in F$ the image γ^σ commutes with all elements of the form

$$ab - ba \text{ with arbitrary } a, b \in D. \tag{9}$$

Thus, it was enough to establish:

Lemma ([7, Lemma]). *If $c \in D$ commutes with all elements $ab - ba$ as in* (9)*, then c belongs to the centre F of D.*

In [7], there are two different proofs of this Lemma. The first one is subject to the extra assumption $\operatorname{Char} D \neq 2$, the second one applies for arbitrary characteristic of D. As main tool in both proofs, Ancochea considered for any fixed element $x \in D \setminus F$ the mapping sending a variable element $a \in D$ to $ax - xa$. This mapping is an F-endomorphism of the F-vector space D and its image is therefore an F-subspace M_x of D. Taking into account the dimension of M_x and using results about the structure of division algebras from the books of A. A. Albert [3] (reprint [4]) and B. van der Waerden [156] (various reprints and translations), Ancochea verified that for any element $c \in D \setminus F$ there is at least one non-commuting element of the form (9).

Returning to the proof of the Principal Theorem (taking into account $\operatorname{Char} D \neq 2$), Ancochea first observed that the given semi-automorphism σ restricts to an automorphism of the commutative field F. Furthermore, he obtained $(\gamma a)^\sigma = \gamma^\sigma a^\sigma$ for all $\gamma \in F$, $a \in D$, i. e., σ is a semilinear mapping of the F-vector space D. Next, an auxiliary F-algebra D' was defined and an F-semilinear isomorphism $\iota\colon D \to D'$ was explicitly established. He noted that $\sigma_1 := \sigma\iota$ is a semi-isomorphism of D onto D'. Also, due to the particular choice of ι, this σ_1 is *F-linear*.[14] Ancochea's conclusion says that *σ will be an (anti-)automorphism of D precisely when σ_1 is an (anti-)isomorphism $D \to D'$.*

After these technical preliminaries, Ancochea considered a commutative field K that is a finite algebraic extension of F and a splitting field of D and D'. Then he extended σ_1 to a K-linear semi-isomorphism $\sigma'\colon D_K \to D'_K$, where $D_K := K \otimes_F D$ and $D'_K := K \otimes_F D'$. From D_K and D'_K being full matrix algebras of the same K-dimension, followed their being K-isomorphic. Consequently, Ancochea continued by assuming σ_1 to be a K-linear semi-automorphism of the algebra $(\mathcal{M}_n)_K$ of all $n \times n$ matrices over K (for some positive integer n). *Evidently, σ will be an (anti-)automorphism of D if σ' is an (anti-)automorphism.* In this way the proof of the Principal Theorem was reduced to showing Theorem 2 below, which is important for its own sake.

[14]In [7] such a mapping is called a semi-isomorphism *over F*.

Theorem ([7, Theorem 2])**.** *Every K-linear semi-automorphism σ of a full matrix algebra $(\mathcal{M}_n)_K$, where K is a commutative field and n is a positive integer, is an automorphism or an anti-automorphism.*

Ancochea's proof of Theorem 2 uses that σ preserves orthogonal idempotents of rank 1. Let us write $e_{ij} \in (\mathcal{M}_n)_K$ for the matrix whose (i, j)-entry equals 1, whereas all other entries vanish. Then it can be assumed, without loss of generality, that σ fixes the idempotents e_{ii}, $i = 1, 2, \ldots, n$. The images of the matrices e_{ij}, where $i \neq j$, turned out to satisfy $e_{ij}^{\sigma} = \alpha_{ij}e_{ij} + \beta_{ij}e_{ji}$ with coefficients $\alpha_{ij}, \beta_{ij} \in K$ subject to $\alpha_{ij}\beta_{ij} = 0$. This intermediate result led to two mutually exclusive cases: either all $\beta_{ij} = 0$ vanish or all α_{ij} vanish. In the first case, a short calculation showed that σ is an automorphism. In the second case, σ turned out to be an anti-automorphism, as required.

The version of von Staudt's Theorem in [7] was then an immediate consequence of [7, Principal Theorem] and the results from [6].

Ancochea closed his article with two important observations. On the one hand, he sketched that the Principal Theorem remains true if D is a *simple algebra of characteristic* $\neq 2$. He also claimed that the Principal Theorem remained valid for semisimple algebras of characteristic $\neq 2$. However, the last statement needs to be altered. We shall come across a correct version on page 13. On the other hand, he pointed out neat links to the work of P. Jordan, J. von Neumann, and E. Wigner [93], [94]. Given an associative, but not commutative F-algebra D, $\operatorname{Char} F \neq 2$, one obtains a commutative, but not associative algebra D^+ by maintaining the addition in D and defining a new product $a \circ b := \frac{1}{2}(ab + ba)$ for all $a, b \in D$. Obviously, *the semi-automorphisms of D are precisely the automorphisms of D^+*. (Such an algebra D^+ has been given the name *special Jordan algebra* [33, p. 178].)

3 Rounding off Ancochea's results (1947–1953)

In 1947, I. Kaplansky [96] mentioned that Ancochea's theorem on the semi-automorphisms of a simple algebra [7] fails for characteristic two, because condition (5) looses most of its strength.[15] In order to overcome this phenomenon, Kaplansky modified the definition of a semi-isomorphism. We stick here to the slightly different version that was given three years later by N. I. Jacobson and C. E. Rickart [90]. Given (associative) rings R and R' a mapping $\sigma\colon R \to R'$ is

[15]For example, if K is a commutative field of characteristic two and $\sigma\colon K \to K$ is an additive automorphism of K, then (5) is trivially satisfied for all $x, y \in K$ due to $(xy)^\sigma + (yx)^\sigma = 0 = x^\sigma y^\sigma + y^\sigma x^\sigma$. If, furthermore, K has more than two elements, then there is a choice of σ that does not fix $1 \in K$ and therefore cannot be a multiplicative automorphism.

called a *Jordan homomorphism* or, in other words, a *semi-homomorphism*[16] if it satisfies for all $x, y \in R$ the following three conditions:

$$(x + y)^\sigma = x^\sigma + y^\sigma, \tag{10}$$

$$(x^2)^\sigma = (x^\sigma)^2, \tag{11}$$

$$(xyx)^\sigma = x^\sigma y^\sigma x^\sigma. \tag{12}$$

Conditions (4) and (10) are identical. Any semi-homomorphism $\sigma\colon R \to R'$ (in the above sense) satisfies Ancochea's condition (5) for all $x, y \in R$. This is immediately seen, using (11), from $xy + yx = (x + y)^2 - x^2 - y^2$. Conversely, any mapping $\sigma\colon R \to R'$ satisfying Ancochea's conditions (4) and (5) for all $x, y \in R$ is a semi-homomorphism provided that R' is a 2-*torsion free ring*, i. e., R' has no elements of additive order two; see [96, Lemma 1], where Kaplansky used the identity

$$2xyx = 4(x + y)^3 - (x + 2y)^3 - 3x^3 + 4y^3 - 2(x^2y + yx^2).$$

When dealing with *unital rings* R and R' it is common to replace (11) with

$$1^\sigma = 1' \tag{13}$$

in the definition of a semi-homomorphism. As a consequence, with $y := 1$ in (12), one obtains (11) for all $x \in R$.

Based on these observations and some lemmas, [96] comprises three main results. In Theorem 1, any semi-isomorphism $\sigma\colon R \to R'$ is considered, where R and R' are semisimple rings with unity. It is shown that *the restriction of σ to the centre of R yields an isomorphism onto the centre of R'*. Theorem 2 describes the semi-isomorphisms between simple algebras. *Any mapping of this form turns out to be either an isomorphism or an anti-isomorphism.* In Theorem 3, the erroneous statement from [7] about semi-automorphisms of semisimple algebras is corrected and extended to a wider class of rings. *Let any semi-isomorphism $\sigma\colon R \to R'$ be given, where R and R' are direct sums of simple algebras. Then the simple components of R and R' may be paired off in such a way that the given mapping is an isomorphism or an anti-isomorphism of each pair.*[17] The proofs of these theorems follow in part Ancochea's approach from [7].

Next, the article of L.-K. Hua [77] closed gaps that were left open by Ancochea. Hua adopted the above definition of a Jordan homomorphism for unital

[16]Both names are currently used in the literature. We decided to switch freely between these names at our own discretion.

[17]Theorem 3 provides an easy way of constructing semi-homomorphisms that are neither homomorphisms nor anti-homomorphisms. Take, for example, the ring $\mathcal{M}_n(F)$ of $n \times n$ matrices, $n \geq 2$, over any commutative field F and consider the mapping of the ring $\mathcal{M}_n(F) \oplus \mathcal{M}_n(F)$ onto itself that sends any matrix pair (A_1, A_2) to $(A_1, A_2^\top)$, where $A_2^\top$ denotes the transpose of A_2.

rings. His Theorem 1 states that *any Jordan homomorphism of a field K onto itself is either an automorphism or an anti-automorphism of K*. The way of proof differs considerably from Ancochea's, as it depends on a series of subtle algebraic manipulations involving inverses rather than any structure theory. Theorem 2 in [77] provides an improvement and the missing converse of Ancochea's theorem from [6]. *Over a field K of characteristic ≠ 2, any bijective harmonicity preserver of a projective line L onto itself with three distinct fixed points determines an automorphism or an anti-automorphism of K. Conversely, any surjective Jordan endomorphism of K gives rise to a bijective harmonicity preserver of the line L.* A detailed exposition of the conclusions from [77] was published by Hua in [78]. Furthermore, he sketched there ties to the so-called *geometry of matrices*, where it is also possible to characterise harmonicity preserves by entirely different methods.[18]

Let us return to N. I. Jacobson and C. E. Rickart, whose article [90] contains the following theorem. *Any Jordan homomorphism of an arbitrary ring into an integral domain is either a homomorphism or an anti-homomorphism*. Other results in [90] provide necessary conditions for a Jordan homomorphism $\sigma\colon R \to R'$ to be the *sum of a homomorphism and an anti-homomorphism*. This is to mean that there are ideals R'_1 and R'_2 of R' with $R' = R'_1 \oplus R'_2$, a homomorphism $\sigma_1\colon R \to R'_1$ and an anti-homomorphism $\sigma_2\colon R \to R'_2$ such that $\sigma = \sigma_1 + \sigma_2$. (The example in footnote 17 is of this kind.) This entails the existence of Jordan homomorphisms that are neither a homomorphism nor an anti-homomorphism.

The short communications [64] and [65] authored by I. N. Herstein contain an alternative proof of Ancochea's Lemma from [7] on the elements of a division algebra D commuting with all elements as in (9).

4 The impact of Ancochea's work

4.1 Applications in algebra and geometry

There is a wealth of articles deploying results of Ancochea (primarily those from [7]) in order to solve a variety of problems in algebra. The following list, which is in chronological order, comprises publications from 1959 up to 2005: M. Gerstenhaber [56], N. Jacobson [86], H.-J. Hoehnke [75], M. Raïs [134], C. V. Devapakkiam [49], M.-A. Knus [100], H. F. de Groote [48], M. Raïs [135], M. O'Ryan and D. B. Shapiro [129], R. Parimala, R. Sridharan, and M. L. Thakur [130], L. Grunenfelder, T. Košir, M. Omladič and H. Radjavi [58], M. A. Chebotar, W.-F. Ke and P.-H. Lee [42], M. A. Chebotar, W.-F. Ke, P.-H. Lee and L.-S. Shiao [43].

[18] We refer to [159] and [80] for the further development in this area.

Ancochea's article [6] is one of the main sources in J. Bilo's monograph [29] about *projective geometry over the real quaternions*. Also, when dealing there with projectivities, a short comment about the findings and methods used in [5] is made on page 60. The characterisation of harmonicity preservers (in Hua's version [77]), was used by W. Benz [25], [26, p. 175], [26, p. 346] in order to determine all isomorphisms of certain *Möbius geometries*. K. List [114] made use of the same result when characterising *orthogonality preserving transformations* on the line set of a three-dimensional *hyperbolic space*.

4.2 Semi-homomorphisms and their generalisations

The outcomes of Ancochea together with the contributions by others (from 1947–1953) are at the beginning of a long series of articles. The results proved there frequently read that—under certain extra conditions on the rings R and R'—*any Jordan homomorphism* $\sigma\colon R \to R'$ *is either a homomorphism or an anti-homomorphism.* By relaxing these extra conditions, the conclusion often says that σ is *the sum of a homomorphism and an anti-homomorphism*. We present a short overview and sketch various generalisations.

The result from [90], saying that any Jordan homomorphism into an integral domain is either a homomorphism or an anti-homomorphism, was shown independently by L.-K. Hua (under slightly stronger assumptions) in [78] and [79]. In 1956, I. N. Herstein [66] extended these results to *surjective* Jordan homomorphisms onto *prime rings*[19] with characteristic greater than three. K. Yamaguti [166] established essentially the same theorem one year later. M. F. Smiley [145] gave a new proof covering also the case of characteristic three and, by adopting Kaplansky's definition of a Jordan homomorphism [96], extended Herstein's theorem to the missing case of characteristic two. The next steps were taken between 1979 and 1989 by W. E. Baxter and W. S. Martindale [15], M. Brešar [35], [36], who went over from prime to *semiprime rings* [105, p. 158].

Let us go back to the year 1948, when N. Jacobson [85] took up a remark of Ancochea by saying that the statements from [7] can be seen as results about the isomorphisms of the non-associative special Jordan rings determined by the given rings. Based on this observation, Jacobson initiated the study of isomorphisms between various *Jordan subsystems* of (associative) rings. In this way he gave also new proofs for some results from [7]. He also noted that, by replacing the plus sign with a minus in Ancochea's condition (5), one arrives at another kind of "semi-isomorphism" of associative rings, which is related to the theory of *Lie rings*.[20] His work led to a series of results stating that *any Jordan homo-*

[19] A prime ring R is one in which $aRb = 0$ implies that $a = 0$ or $b = 0$.

[20] Several of our bibliographical items deal also with this topic. Further information may be

morphism between certain Jordan subsystems can be lifted to a homomorphism or an anti-homomorphism of the ambient rings. In 1949, F. D. Jacobson and N. Jacobson [84] determined all embeddings of a special Jordan algebra in associative algebras over a commutative field of characteristic $\neq 2$, thereby regaining a theorem from [7]. N. Jacobson and C. E. Rickart [91] considered the Jordan subsystem of symmetric elements of a ring with involution. Their work was later extended by W. S. Martindale [118], L. A. Lagutina [103], K. McCrimmon [123], W. S. Martindale [120], K. I. Beidar and M. A. Chebotar [19]. For summaries and extensive bibliographies we refer to the work of I. N. Herstein [67], N. Jacobson [87], I. N. Herstein [70], N. Jacobson [88], W. S. Martindale [119], R. P. Sullivan [152], K. McCrimmon [123].

A detailed analysis of the Jordan homomorphisms of the ring $\mathcal{T}_r(R)$ of *upper triangular* $r \times r$*-matrices* with entries in a ring R commenced in 1998 with the article [126] of L. Molnár and P. Šemrl on *linear preserver problems*. They described all *linear* Jordan automorphism of $\mathcal{T}_r(\mathbb{C})$, where $\mathbb{C}$ denotes the field of complex numbers. Shortly afterwards, K. I. Beidar, M. Brešar, M. A. Chebotar [16] switched to triangular matrices over a unital commutative ring R. One of their results is in the spirit of Ancochea and illustrates the extra features arising from idempotent ring elements. *Let R be a 2-torsion-free commutative ring with identity. Then R contains no idempotents except 0 and 1 if, and only if, every Jordan isomorphism of $\mathcal{T}_r(R)$, $r \geq 2$, onto an arbitrary algebra over R is either an isomorphism or an anti-isomorphism.* Further contributions (with varying assumptions on the ring R) have been given between 2001 and 2014: X. M. Tang, C. G. Cao and X. Zhang [154], X. T. Wang and H. You [162], D. Benkovič [24], T. L. Wong [165], C.-K. Liu and W.-Y. Tsai [115], X. T. Wang [160], H. M. Yao, C. G. Cao and X. Zhang [169], L.-P. Huang [81], X. T. Wang and Y. M. Li [161], Y. Wang and Y. Wang [163] (erratum by Y. Du, Y. Wang and Y. Wang [52]), Y. Du and Y. Wang [51].

Jordan homomorphisms between other kinds of rings have been thoroughly investigated. We refer to J. H. Zhang [170], K. I. Beidar, M. Brešar and M. A. Chebotar [17], F. Lu [116], F. Lu and T.-L. Wong [165], A. L. Yang and J. H. Zhang [168], F. Kuzucuoğlu and V. M. Levchuk [102], J. Alaminos, J. Extremera and A. R. Villena [2], E. Akkurt, M. Akkurt and G. P. Barker [1], A. L. Yang [167], R. Brusamarello, E. Z. Fornaroli and M. Khrypchenko [39].

There are many articles dealing with mappings between rings that satisfy a *relaxed version* of the properties defining a Jordan homomorphism $\sigma \colon R \to R'$ or some "similar" functional equations. Two types have found particular interest. First, *n-Jordan mappings* are additive and satisfy $(x^n)^\sigma = (x^\sigma)^n$ for all $x \in R$ and some fixed integer $n \geq 2$. These mappings have been investigated by I. N. Herstein

retrieved from [20] and [23].

[66], [67], [70], M. Brešar, W. S. Martindale and C. R. Miers [38]. Second, we mention *Jordan triple homomorphisms*, which are characterised by the conditions in (10) and (12) and are the topic of articles by K. Yamaguti [166], I. N. Herstein [68], M. Brešar [35], [36], L. A. Lagutina [104], F. Lu [117]. Other contributions in this spirit came between 1953 and 1999 from J. K. Goldhaber [57], E. Artin [8, pp. 37–40], I. N. Herstein and E. Kleinfeld [71], M. F. Smiley [146], D. W. Barnes [11], K. McCrimmon [122], S. A. Huq [83], R. Awtar [9], K. I. Beidar and Y. Fong [21], and after the year 2000 from K. I. Beidar, S.-C. Chang, M. A. Chebotar and Y. Fong [18], M. Brešar [37], M. A. Chebotar, F.-W. Ke, P.-H. Lee and L.-S. Shiao [43], M. Brešar, M. A. Chebotar and W. S. Martindale [34], A. K. Faraj, A. H. Majeed, C. Haetinger and N. u. Rehman [53].

Yet another way of generalising the notion of semi-homomorphism consists in considering algebraic structures other than rings. Already in 1951, F. Dinkines [50] introduced *semi-homomorphisms of groups* by following condition (12). Her results were extended by I. N. Herstein and M. F. Ruchte [72], I. N. Herstein [69], K. I. Beidar, Y. Fong, W.-F. Ke, W.-R. Wu [22]. In the 1980s, R. P. Sullivan [152], [153] considered also *semigroups* and introduced even more general *half-automorphisms*. Other generalisations deal with mappings between *alternative division rings*, *near rings* and *semirings*; see M. F. Smiley [147], K. C. Smith and L. van Wyk [148], K. I. Beidar, Y. Fong, W.-F. Ke and W.-R. Wu [22], S. Shafiq and M. Aslam [144], B. L. M. Ferreira and R. N. Ferreira [55].

4.3 Harmonicity preservers

The complete description of all bijective harmonicity preservers of the projective line L over a field K with characteristic $\neq 2$ has become a standard topic in textbooks on projective geometry. We confine ourselves to quoting several books from the 1950s.[21] The first exposition appears to be the one of R. Baer [10, p. 78]. He considered a more general setting, namely bijections of L that preserve an arbitrary cross ratio $d \neq 0, 1$ lying in the centre of K without any restriction on the characteristic of K. Again, it was enough to treat the case of such a bijection with three distinct fixed points 0, 1, ∞. Baer showed that such a bijection gives rise to an automorphism or an anti-automorphism of $K = L \setminus \{\infty\}$ fixing the element $d \in K$ and vice versa. He thereby generalised also an outcome of A. J. Hoffman [76], who had obtained the same kind of result for a commutative field K. An alternative proof of Baer's theorem can be found in the book of G. Pickert [131, p. 121] from 1955 (second edition [132]). E. Artin [8, pp. 84–85] reproved the original result about harmonicity preservers in an elegant way using an alternative characterisation of semi-automorphisms [8, pp. 37–40].

[21] Further references and historical remarks are contained in [97, pp. 57–58].

The uniqueness of the fourth harmonic point (see page 5) holds more generally in projective *Moufang planes*, which satisfy a weaker form of Desargues' Theorem [125], [131], [132]. These projective planes admit coordinates from an *alternative division ring* and the bijective harmonicity preservers between their lines admit an algebraic description in terms of Jordan isomorphisms. This investigation was initiated in the 1950s by V. Havel [59], [60], [61] and continued by M. F. Smiley [147], J. van Buggenhout [155], J. C. Ferrar [54]. Generalisations (in terms of other cross ratios) were given by A. Schleiermacher [140], H. Schaeffer [139], A. Blunck [30].

In [62], V. Havel considered harmonicity preservers in certain *translation planes*. The work of W. Bertram [27] on harmonicity preserving mappings of some other geometric structures also deserves mention.

An area of ongoing research is the study of harmonicity preservers between *projective lines over unital rings* (with $1 \neq 0$). Given such a ring R one considers any free left R-module M that has at least one basis with two elements. The point set of the projective line $\mathbb{P}(M)$ is the set of all cyclic submodules Ru of M for which there is a $v \in M$ such that (u, v) is a basis of M [32], [74]. Points p_1, p_2 of $\mathbb{P}(M)$ are called *distant* (or *non-neighbouring*) if there exists a basis (u_1, u_2) of M with $p_1 = Ru_1$ and $p_2 = Ru_2$. This distant relation is symmetric and anti-reflexive, and it turns the point set of $\mathbb{P}(M)$ into the so-called *distant graph* of $\mathbb{P}(M)$.[22]

Notions like *harmonic quadruple* and *cross ratio* are defined on $\mathbb{P}(M)$ in almost same way as on $\mathbb{P}(V)$ when V is a left K-vector space. But, instead of "distinct points" of $\mathbb{P}(V)$ one has to consider "distant points" of $\mathbb{P}(M)$. Furthermore, upon choosing a basis (w_1, w_2) of M, one obtains $0 := Rw_2$, $1 := R(w_1 + w_2)$ and $\infty := Rw_1$ as three *points of reference* on $\mathbb{P}(M)$. It is straightforward then to identify the points of $\mathbb{P}(M)$ that are distant to ∞ with the elements of R as Ancochea did in [6]. For doing so, it is enough to replace in (1) the field K by the ring R and to assume that $x_2 \in R$ is invertible. However, unless R is a field, the "rest" of the projective line contains apart from ∞ many other "points at infinity".

There is a widespread literature on *harmonicity preservers*, which are defined in the same manner as on page 7, under varying assumptions on the underlying unital rings R and R'. Below we collect the relevant contributions. A common feature in all of them is that $2 = 1 + 1$ *has to be invertible in* R.

Let a harmonicity preserver $\lambda \colon \mathbb{P}(M) \to \mathbb{P}(M')$ be given. Then, after the identification of a subset of $\mathbb{P}(M)$ with R and an analogous identification in $\mathbb{P}(M')$, λ *restricts to a Jordan homomorphism* $R \to R'$ *provided that* R *contains "sufficiently many" invertible elements.* A proof of this result can be derived from B. V. Limaye and N. B. Limaye [110], despite the fact that their work from 1977 is mainly about *commutative* rings. A formal proof under slightly weaker assumptions was

[22] Precisely when R is a field, "being distant" means the same as "being distinct".

given by the author [63]. Already in 1971, N. B. Limaye [112], [113] proved a version for *commutative local* rings [105, pp. 280f.] and then for *commutative semilocal* rings [105, p. 296]; H. Schaeffer [138], B. V. Limaye and N. B. Limaye [110], B. R. McDonald [124] treated also the case of *commutative* rings.

The converse problem, like before, is to decide whether or not *any Jordan homomorphism* $\sigma\colon R \to R'$ *gives rise to a harmonicity preserver*. After choosing bases w_1, w_2 of M and w_1', w_2' of M' it is tempting to define a mapping

$$M \to M' \colon x_1 w_1 + x_2 w_2 \mapsto x_1^\sigma w_1' + x_2^\sigma w_2' \text{ with } x_1, x_2 \in R. \tag{14}$$

However, this mapping will in general not give rise to a mapping $\mathbb{P}(M) \to \mathbb{P}(M')$, let alone its being a harmonicity preserver. Nevertheless, for commutative rings the approach in (14) does work, since a Jordan homomorphism of R in R' is nothing but a homomorphism. For *not necessarily commutative* rings the situation is much more involved, due to the possibly large number of "points at infinity". B. V. Limaye and N. B. Limaye [111] (erratum [109]) gave an affirmative answer to the problem for *local* rings. They actually determined all bijections of the projective line $\mathbb{P}(M)$ onto itself such that all quadruples with a given cross ratio d go over to quadruples with a given cross ratio d', where d, d' are elements in the centre of R other that $0, 1$. A. Herzer [73] showed how to obtain well-defined point-to-point mappings from certain Jordan homomorphisms. A breakthrough is due to C. Bartolone [12], who defined the desired mapping $\mathbb{P}(M) \to \mathbb{P}(M')$ under the extra condition that R is a ring of *stable rank two* [32, p. 24]. In terms of the bases used in (14), his solution[23] reads

$$R(xw_1 + (1 + xy)w_2) \mapsto R'(x^\sigma w_1' + (1' + x^\sigma y^\sigma)w_2') \text{ with } x, y \in R. \tag{15}$$

A. Blunck and the author treated in [31] the general case taking into account that the distant graph on $\mathbb{P}(M)$ need not be connected. It turned out that *a Jordan homomorphism σ determines a harmonicity preserving mapping only on that connected component of the distant graph on $\mathbb{P}(M)$ which contains the chosen points of reference*. (The formulas used for this purpose arise from (15) in a recursive way.) As a consequence, one may select arbitrarily *one Jordan homomorphism $R \to R'$ per component* in order to create a harmonicity preserver $\mathbb{P}(M) \to \mathbb{P}(M')$.

The material from the last two paragraphs forms the foundation for the version of von Staudt's Theorem in [63, Thm. 1]. When dropping the assumption on the existence of "sufficiently many units" in R, this theorem fails. A lucid counterexample in terms of the polynomial ring in one indeterminate over the real numbers was given by C. Bartolone and F. Di Franco [14] already in 1979. They therefore initiated the study of mappings that preserve *generalised harmonic quadruples* and succeeded in describing all such mappings for commutative rings; see

[23]The authors of [73] and [12] had different aims and did not exhibit the preservation of harmonicity in their publications.

also M. Kulkarni [101], C. Bartolone and F. Bartolozzi [13], L. Cirlincione and M. Enea [45], A. A. Lashkhi [106], [107], [108], D. Chkhatarashvili [44]. Closely related is the work of F. Buekenhout [40], St. P. Cojan [46], D. G. James [92], B. Klotzek [99], who characterised mappings that satisfy a rather weak form of cross ratio preservation between projective lines over fields. The algebraic background of their work is a re-coordinatisation of the domain projective line in terms of a *valuation ring* [121, p. 1].

All things considered, Ancochea's semi-homomorphisms keep going strong. They constitute the indispensable algebraic tool for describing harmonicity preservers between projective lines over unital rings.

References

[1] E. Akkurt, M. Akkurt, G. P. Barker, Jordan homomorphisms of the structural matrix algebras. *Linear Multilinear Algebra* **63** (2015), 2518–2525.

[2] J. Alaminos, J. Extremera, A. R. Villena, Metric versions of Herstein's theorems on Jordan maps. *Linear Multilinear Algebra* **60** (2012), 941–954.

[3] A. A. Albert, *Structure of Algebras*. American Mathematical Society, New York 1939.

[4] A. A. Albert, *Structure of Algebras*. American Mathematical Society, Providence, RI 1961.

[5] G. Ancochea, Sobre el teorema fundamental de la geometria proyectiva. *Revista Mat. Hisp.-Amer. (4)* **1** (1941), 37–42.

[6] G. Ancochea, Le théorème de von Staudt en géométrie projective quaternionienne. *J. Reine Angew. Math.* **184** (1942), 193–198.

[7] G. Ancochea, On semi-automorphisms of division algebras. *Ann. of Math. (2)* **48** (1947), 147–153.

[8] E. Artin, *Geometric Algebra*. Interscience Publishers, Inc., New York London Sydney 1957.

[9] R. Awtar, Jordan derivations and Jordan homomorphisms on prime rings of characteristic 2. *Acta Math. Hungar.* **51** (1988), 61–63.

[10] R. Baer, *Linear Algebra and Projective Geometry*. Academic Press, New York 1952.

[11] D. W. Barnes, Lattice isomorphisms of associative algebras. *J. Austral. Math. Soc.* **6** (1966), 106–121.

[12] C. Bartolone, Jordan homomorphisms, chain geometries and the fundamental theorem. *Abh. Math. Sem. Univ. Hamburg* **59** (1989), 93–99.

[13] C. Bartolone, F. Bartolozzi, Topics in geometric algebra over rings. In: R. Kaya, P. Plaumann, K. Strambach, editors, *Rings and Geometry*, 353–389, Reidel, Dordrecht 1985.

[14] C. Bartolone, F. Di Franco, A remark on the projectivities of the projective line over a commutative ring. *Math. Z.* **169** (1979), 23–29.

[15] W. E. Baxter, W. S. Martindale, III, Jordan homomorphisms of semiprime rings. *J. Algebra* **56** (1979), 457–471.

[16] K. I. Beidar, M. Brešar, M. A. Chebotar, Jordan isomorphisms of triangular matrix algebras over a connected commutative ring. *Linear Algebra Appl.* **312** (2000), 197–201.

[17] K. I. Beidar, M. Brešar, M. A. Chebotar, Jordan superhomomorphisms. *Comm. Algebra* **31** (2003), 633–644.

[18] K. I. Beidar, S.-C. Chang, M. A. Chebotar, Y. Fong, On functional identities in left ideals of prime rings. *Comm. Algebra* **28** (2000), 3041–3058.

[19] K. I. Beidar, M. A. Chebotar, On Lie-admissible algebras whose commutator Lie algebras are Lie subalgebras of prime associative algebras. *J. Algebra* **233** (2000), 675–703.

[20] K. I. Beidar, M. A. Chebotar, On Lie derivations of Lie ideals of prime algebras. *Israel J. Math.* **123** (2001), 131–148.

[21] K. I. Beidar, Y. Fong, On additive isomorphisms of prime rings preserving polynomials. *J. Algebra* **217** (1999), 650–667.

[22] K. I. Beidar, Y. Fong, W.-F. Ke, W.-R. Wu, On semi-endomorphisms of groups. *Comm. Algebra* **27** (1999), 2193–2205.

[23] K. I. Beidar, A. V. Mikhalev, M. A. Chebotar, Functional identities in rings and their applications. *Russian Math. Surveys* **59** (2004), 403–428.

[24] D. Benkovič, Jordan homomorphisms on triangular matrices. *Linear Multilinear Algebra* **53** (2005), 345–356.

[25] W. Benz, Zur Geometrie der Körpererweiterungen. *Canad. J. Math.* **21** (1969), 1097–1122.

[26] W. Benz, *Vorlesungen über Geometrie der Algebren*. Springer, Berlin 1973.

[27] W. Bertram, The geometry of null systems, Jordan algebras and von Staudt's theorem. *Ann. Inst. Fourier (Grenoble)* **53** (2003), 193–225.

[28] L. Bieberbach, *Einleitung in die höhere Geometrie*. B. G. Teubner, Leipzig 1933.

[29] J. Bilo, *Onderzoekingen betreffende de meetkundige grondslagen van de projectieve quaternionenmeetkunde*. A. Vanderlinden, Brussel 1949.

[30] A. Blunck, Cross-ratios in Moufang planes. *J. Geom.* **40** (1991), 20–25.

[31] A. Blunck, H. Havlicek, Jordan homomorphisms and harmonic mappings. *Monatsh. Math.* **139** (2003), 111–127.

[32] A. Blunck, A. Herzer, *Kettengeometrien – Eine Einführung*. Shaker Verlag, Aachen 2005.

[33] H. Braun, M. Koecher, *Jordan-Algebren*. Springer-Verlag, Berlin New York 1966.

[34] M. Brešar, M. A. Chebotar, W. S. Martindale, III, *Functional Identities*. Birkhäuser Verlag, Basel 2007.

[35] M. Brešar, Jordan mappings of semiprime rings. *J. Algebra* **127** (1989), 218–228.

[36] M. Brešar, Jordan mappings of semiprime rings. II. *Bull. Austral. Math. Soc.* **44** (1991), 233–238.

[37] M. Brešar, Functional identities: a survey. In: *Algebra and its Applications (Athens, OH, 1999)*, volume 259 of *Contemp. Math.*, 93–109, Amer. Math. Soc., Providence, RI 2000.

[38] M. Brešar, W. S. Martindale, 3rd, C. R. Miers, Maps preserving nth powers. *Comm. Algebra* **26** (1998), 117–138.

[39] R. Brusamarello, É. Z. Fornaroli, M. Khrypchenko, Jordan isomorphisms of finitary incidence algebras. *Linear Multilinear Algebra* **66** (2018), 565–579.

[40] F. Buekenhout, Une généralisation du théorème de von Staudt-Hua. *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **51** (1965), 1282–1293.

[41] F. Buekenhout, P. J. Cameron, Projective and affine geometry over division rings. In: F. Buekenhout, editor, *Handbook of Incidence Geometry*, 27–62, North-Holland, Amsterdam 1995.

[42] M. A. Chebotar, W.-F. Ke, P.-H. Lee, Maps preserving zero Jordan products on Hermitian operators. *Illinois J. Math.* **49** (2005), 445–452.

[43] M. A. Chebotar, W.-F. Ke, P.-H. Lee, L.-S. Shiao, On maps preserving products. *Canad. Math. Bull.* **48** (2005), 355–369.

[44] D. Chkhatarashvili, K. von Staudt's theorem over Ore domains. *Bull. Georgian Acad. Sci.* **158** (1998), 18–20.

[45] L. Cirlincione, M. R. Enea, Una generalizzazione del birapporto sopra un anello. *Rend. Circ. Mat. Palermo (II)* **39** (1990), 271–280.

[46] S. P. Cojan, A generalization of the theorem of von Staudt-Hua-Buekenhout. *Mathematica (Cluj)* **27 (52)** (1985), 93–96.

[47] M. G. Darboux, Sur le théorème fondamental de la géométrie projective (Extrait d'une lettre à M. Klein). *Math. Ann.* **17** (1880), 55–61.

[48] H. F. de Groote, On varieties of optimal algorithms for the computation of bilinear mappings. I. The isotropy group of a bilinear mapping. *Theoret. Comput. Sci.* **7** (1978), 1–24.

[49] C. V. Devapakkiam, P. S. Rema, Hilbert space methods in the theory of Jordan algebras. II. *Math. Proc. Cambridge Philos. Soc.* **79** (1976), 307–319.

[50] F. Dinkines, Semi-automorphisms of symmetric and alternating groups. *Proc. Amer. Math. Soc.* **2** (1951), 478–486.

[51] Y. Du, Y. Wang, Jordan homomorphisms of upper triangular matrix rings over a prime ring. *Linear Algebra Appl.* **458** (2014), 197–206.

[52] Y. Du, Y. Wang, Y. Wang, Corrigendum to "Jordan homomorphisms of upper triangular matrix rings" [Linear Algebra Appl. 439 (12) (2013) 4063–4069]. *Linear Algebra Appl.* **452** (2014), 345–350.

[53] A. K. Faraj, A. H. Majeed, C. Haetinger, N. u. Rehman, On generalized Jordan higher homomorphisms. *Palest. J. Math.* **3** (2014), 406–421.

[54] J. C. Ferrar, Cross-ratios in projective and affine planes. In: P. Plaumann, K. Strambach, editors, *Geometry – von Staudt's Point of View*, 101–125, Reidel, Dordrecht 1981.

[55] B. L. M. Ferreira, R. N. Ferreira, Automorphisms on the alternative division ring. *Rocky Mountain J. Math.* **49** (2019), 73–78.

[56] M. Gerstenhaber, Note on a theorem of Wielandt. *Math. Z.* **71** (1959), 141–142.

[57] J. K. Goldhaber, Special types of linear mappings of algebras. *Amer. J. Math.* **75** (1953), 91–97.

[58] L. Grunenfelder, T. Košir, M. Omladič, H. Radjavi, Maximal Jordan algebras of matrices with bounded number of eigenvalues. *Israel J. Math.* **128** (2002), 53–75.

[59] V. Havel, Harmonical quadruplet in Moufang plane. *Czechoslovak Math. J.* **5 (80)** (1955), 76–82.

[60] V. Havel, On the theorem of Staudt in Moufang plane. *Czechoslovak Math. J.* **5 (80)** (1955), 83–90.

[61] V. Havel, Eine Bemerkung zum Staudtschen Satz in der Moufang-Ebene. *Czechoslovak Math. J.* **7 (82)** (1957), 314–317.

[62] V. Havel, One generalization of the fourth harmonic point. *Czechoslovak Math. J.* **18 (93)** (1968), 294–300.

[63] H. Havlicek, Von Staudt's theorem revisited. *Aequationes Math.* **89** (2015), 459–472.

[64] I. N. Herstein, A note on a commutativity theorem. *Kōdai Math. Sem. Rep.* **5** (1953), 119–120.

[65] I. N. Herstein, Une note sur un article de M. Tumuraru. *Portugaliae Math.* **12** (1953), 113–114.

[66] I. N. Herstein, Jordan homomorphisms. *Trans. Amer. Math. Soc.* **81** (1956), 331–341.

[67] I. N. Herstein, Lie and Jordan structures in simple, associative rings. *Bull. Amer. Math. Soc.* **67** (1961), 517–531.

[68] I. N. Herstein, On a type of Jordan mapping. *An. Acad. Brasil. Ci.* **39** (1967), 357–360.

[69] I. N. Herstein, Semi-homomorphisms of groups. *Canadian J. Math.* **20** (1968), 384–388.

[70] I. N. Herstein, *Topics in Ring Theory*. The University of Chicago Press, Chicago London 1969.

[71] I. N. Herstein, E. Kleinfeld, Lie mappings in characteristic 2. *Pacific J. Math.* **10** (1960), 843–852.

[72] I. N. Herstein, M. F. Ruchte, Semi-automorphisms of groups. *Proc. Amer. Math. Soc.* **9** (1958), 145–150.

[73] A. Herzer, On isomorphisms of chain geometries. *Note Mat.* **7** (1987), 251–270.

[74] A. Herzer, Chain geometries. In: F. Buekenhout, editor, *Handbook of Incidence Geometry*, 781–842, North-Holland, Amsterdam 1995.

[75] H.-J. Hoehnke, Über Beziehungen zwischen Problemen von H. Brandt aus der Theorie der Algebren und den Automorphismen der Normenform. *Math. Nachr.* **34** (1967), 229–255.

[76] A. J. Hoffman, A note on cross ratio. *Amer. Math. Monthly* **58** (1951), 613–614.

[77] L.-K. Hua, On the automorphisms of a sfield. *Proc. Nat. Acad. Sci. U. S. A.* **35** (1949), 386–389.

[78] L.-K. Hua, Fundamental theorem of the projective geometry on a line and geometry of matrices. In: *Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950*, 317–325, Akadémiai Kiadó, Budapest 1952.

[79] L.-K. Hua, On semi-homomorphisms of rings and their application in projective geometry (Russian). *Uspekhi Matem. Nauk. (N.S.)* **8** (1953), 143–148.

[80] L.-P. Huang, *Geometry of Matrices over Ring*. Science Press, Beijing 2006.

[81] L.-P. Huang, Adjacency preserving bijective maps on triangular matrices over any division ring. *Linear Multilinear Algebra* **58** (2010), 815–846.

[82] D. R. Hughes, F. C. Piper, *Projective Planes*. Springer, New York 1973.

[83] S. A. Huq, Semi-homomorphisms of rings. *Bull. Austral. Math. Soc.* **36** (1987), 121–125.

[84] F. D. Jacobson, N. Jacobson, Classification and representation of semi-simple Jordan algebras. *Trans. Amer. Math. Soc.* **65** (1949), 141–169.

[85] N. Jacobson, Isomorphisms of Jordan rings. *Amer. J. Math.* **70** (1948), 317–326.

[86] N. Jacobson, Some groups of transformations defined by Jordan algebras. I. *J. Reine Angew. Math.* **201** (1959), 178–195.

[87] N. Jacobson, *Structure and Representation of Jordan Algebras*. Amer. Math. Soc., Providence, RI 1968.

[88] N. Jacobson, Structure groups and Lie algebras of Jordan algebras of symmetric elements of associative algebras with involution. *Advances in Math.* **20** (1976), 106–150.

[89] N. Jacobson, *Basic Algebra II*. Freeman, New York 1989.

[90] N. Jacobson, C. E. Rickart, Jordan homomorphisms of rings. *Trans. Amer. Math. Soc.* **69** (1950), 479–502.

[91] N. Jacobson, C. E. Rickart, Homomorphisms of Jordan rings of self-adjoint elements. *Trans. Amer. Math. Soc.* **72** (1952), 310–322.

[92] D. G. James, Projective homomorphisms and von Staudt's theorem. *Geom. Dedicata* **13** (1982), 291–294.

[93] P. Jordan, Über die Multiplikation quantenmechanischer Größen. *Z. Phys.* **80** (1933), 285–291.

[94] P. Jordan, J. von Neumann, E. Wigner, On an algebraic generalization of the quantum mechanical formalism. *Ann. of Math. (2)* **35** (1934), 29–64.

[95] E. Kamke, Zur Definition der affinen Abbildung. *Jahresbericht D. M. V.* **36** (1927), 145–156.

[96] I. Kaplansky, Semi-automorphisms of rings. *Duke Math. J.* **14** (1947), 521–525.

[97] H. Karzel, H.-J. Kroll, *Geschichte der Geometrie seit Hilbert*. Wissenschaftliche Buchgesellschaft, Darmstadt 1988.

[98] H. Kestelman, Automorphisms of the field of complex numbers. *Proc. London Math. Soc. (2)* **53** (1951), 1–12.

[99] B. Klotzek, Eine Verallgemeinerung des Satzes von v. Staudt-Hua. *Wiss. Z. Pädagog. Hochsch. "Karl Liebknecht" Potsdam* **32** (1988), 169–172.

[100] M.-A. Knus, M. Ojanguren, Modules and quadratic forms over polynomial algebras. *Proc. Amer. Math. Soc.* **66** (1977), 223–226.

[101] M. Kulkarni, Fundamental theorem of projective geometry over a commutative ring. *Indian J. Pure Appl. Math.* **11** (1980), 1561–1565.

[102] F. Kuzucuoğlu, V. M. Levchuk, Jordan isomorphisms of radical finitary matrix rings. *J. Algebra Appl.* **9** (2010), 659–667.

[103] L. A. Lagutina, Jordan homomorphism of associative algebras with involution. *Algebra and Logic* **27** (1988), 250–260.

[104] L. A. Lagutina, Jordan triple homomorphisms of associative rings with involution. *Algebra and Logic* **30** (1991), 199–208.

[105] T. Y. Lam, *A First Course in Noncommutative Rings*, volume 131 of *Graduate Texts in Mathematics*. Springer-Verlag, New York 2001.

[106] A. A. Lashkhi, Harmonic mappings and collineations of modules (Russian). *Soobshch. Akad. Nauk Gruzin. SSR* **133** (1989), 497–500.

[107] A. A. Lashkhi, Harmonic mappings of modules (Russian). *Mat. Zametki* **47** (1990), 161–163.

[108] A. A. Lashkhi, Harmonic maps over rings. *Georgian Math. J.* **4** (1997), 41–64.

[109] B. V. Limaye, N. B. Limaye, Correction to: Fundamental theorem for the projective line over non-commutative local rings. *Arch. Math. (Basel)* **29** (1977), 672.

[110] B. V. Limaye, N. B. Limaye, The fundamental theorem for the projective line over commutative rings. *Aequationes Math.* **16** (1977), 275–281.

[111] B. V. Limaye, N. B. Limaye, Fundamental theorem for the projective line over non-commutative local rings. *Arch. Math. (Basel)* **28** (1977), 102–109.

[112] N. B. Limaye, Projectivities over local rings. *Math. Z.* **121** (1971), 175–180.

[113] N. B. Limaye, Cross-ratios and projectivities of a line. *Math. Z.* **129** (1972), 49–53.

[114] K. List, Harmonic mappings and hyperbolic Plücker transformations. *J. Geom.* **70** (2001), 108–116.

[115] C.-K. Liu, W.-Y. Tsai, Jordan isomorphisms of upper triangular matrix rings. *Linear Algebra Appl.* **426** (2007), 143–148.

[116] F. Lu, Additive Jordan isomorphisms of nest algebras on normed spaces. *J. Math. Anal. Appl.* **284** (2003), 127–143.

[117] F. Lu, Jordan triple isomorphisms of nest algebras and applications. *Houston J. Math.* **34** (2008), 249–267.

[118] W. S. Martindale, III, Jordan homomorphisms of the symmetric elements of a ring with involution. *J. Algebra* **5** (1967), 232–249.

[119] W. S. Martindale, III, Lie and Jordan mappings in associative rings. In: *Ring theory (Proc. Conf., Ohio Univ., Athens, Ohio, 1976)*, volume 25 of *Lecture Notes in Pure and Appl. Math.*, 71–84, Dekker, New York 1977.

[120] W. S. Martindale, III, Jordan homomorphisms onto nondegenerate Jordan algebras. *J. Algebra* **133** (1990), 500–511.

[121] K. Mathiak, *Valuations of Skew Fields and Projective Hjelmslev Spaces*, volume 1175 of *Lecture Notes in Mathematics*. Springer, Berlin 1986.

[122] K. McCrimmon, Quadratic Jordan algebras and cubing operations. *Trans. Amer. Math. Soc.* **153** (1971), 265–278.

[123] K. McCrimmon, The Zel′manov approach to Jordan homomorphisms of associative algebras. *J. Algebra* **123** (1989), 457–477.

[124] B. R. McDonald, Projectivities over rings with many units. *Comm. Algebra* **9** (1981), 195–204.

[125] N. S. Mendelsohn, Non-Desarguesian projective plane geometries which satisfy the harmonic point axiom. *Canadian J. Math.* **8** (1956), 532–562.

[126] L. Molnár, P. Šemrl, Some linear preserver problems on upper triangular matrices. *Linear and Multilinear Algebra* **45** (1998), 189–206.

[127] U. Morin, Gli automorfismi del corpo complesso ed il teorema fondamentale della geometria proiettiva. *Atti Ist. Veneto Sci. Lett. Arti, Parte II, Sci. Mat. Nat.* **94** (1935), 285–296.

[128] E. Noether, Nichtkommutative Algebra. *Math. Z.* **37** (1933), 514–541.

[129] M. O'Ryan, D. B. Shapiro, On trace forms of higher degree. *Linear Algebra Appl.* **246** (1996), 313–333.

[130] R. Parimala, R. Sridharan, M. L. Thakur, A classification theorem for Albert algebras. *Trans. Amer. Math. Soc.* **350** (1998), 1277–1284.

[131] G. Pickert, *Projektive Ebenen*, volume Band 80 of *Die Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin Göttingen Heidelberg 1955.

[132] G. Pickert, *Projektive Ebenen*, volume Band 80 of *Die Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin New York 1975.

[133] G. Pickert, Projectivities in projective planes. In: P. Plaumann, K. Strambach, editors, *Geometry – von Staudt's Point of View*, 1–49, Reidel, Dordrecht 1981.

[134] M. Raïs. Distributions homogènes sur des espaces de matrices, 1972. Thèse Sci. Math., Paris, 1970, Bull. Soc. Math. France Mém., No. 30, Supplément au Bull. Soc. Math. France, Tome 100, no. 2.

[135] M. Raïs, The unitary group preserving maps (the infinite-dimensional case). *Linear Multilinear Algebra* **20** (1987), 337–345.

[136] K. Reidemeister, *Vorlesungen über Grundlagen der Geometrie*. Springer, Berlin 1930.

[137] K. Reidemeister, *Vorlesungen über Grundlagen der Geometrie*. Springer, Berlin Heidelberg New York 1968.

[138] H. Schaeffer, Das von Staudtsche Theorem in der Geometrie der Algebren. *J. Reine Angew. Math.* **267** (1974), 133–142.

[139] H. Schaeffer, Zur Möbiusgeometrie über Alternativkörpern. *Geom. Dedicata* **10** (1981), 183–189.

[140] A. Schleiermacher, Doppelverhältnisse auf einer Geraden in einer Moufangebene (Char. $\neq$ 2). *Nederl. Akad. Wetensch. Proc. Ser. A* **68** (1965), 482–496.

[141] O. Schreier, E. Sperner, *Einführung in die analytische Geometrie und Algebra. Bd. II*. B. G. Teubner, Leipzig 1935.

[142] F. Schur, Ueber den Fundamentalsatz der projectiven Geometrie. *Math. Ann.* **51** (1898), 401–409.

[143] B. Segre, Gli automorfismi del corpo complesso, ed un problema di Corrado Segre. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8)* **3** (1947), 414–420.

[144] S. Shafiq, M. Aslam, On Jordan mappings of inverse semirings. *Open Math.* **15** (2017), 1123–1131.

[145] M. F. Smiley, Jordan homomorphisms onto prime rings. *Trans. Amer. Math. Soc.* **84** (1957), 426–429.

[146] M. F. Smiley, A remark on the definition of Jordan homomorphisms. *Portugal. Math.* **20** (1961), 147–148.

[147] M. F. Smiley, Von Staudt projectivities of Moufang planes. In: *Algebraical and Topological Foundations of Geometry (Proc. Colloq., Utrecht, 1959)*, 165–166, Pergamon Press, Oxford London New York Paris 1962.

[148] K. C. Smith, L. van Wyk, Semiendomorphisms of simple near-rings. *Proc. Amer. Math. Soc.* **115** (1992), 613–627.

[149] E. Sperner, Beziehungen zwischen geometrischer und algebraischer Anordnung. *Arch. Math.* **1** (1948), 148–153.

[150] E. Sperner, Beziehungen zwischen geometrischer und algebraischer Anordnung. *S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl.* **1949** (1949), 413–448.

[151] K. G. C. v. Staudt, *Geometrie der Lage*. Bauer und Raspe (Julius Merz), Nürnberg 1847.

[152] R. P. Sullivan, Semi-automorphisms of transformation semigroups. *Czechoslovak Math. J.* **33 (108)** (1983), 548–554.

[153] R. P. Sullivan, Half-automorphisms of transformation semigroups. *Czechoslovak Math. J.* **35 (110)** (1985), 415–418.

[154] X. M. Tang, C. G. Cao, X. Zhang, Modular automorphisms preserving idempotence and Jordan isomorphisms of triangular matrices over commutative rings. *Linear Algebra Appl.* **338** (2001), 145–152.

[155] J. van Buggenhaut, Les droites projectives d'un plan de Moufang. *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **55** (1969), 687–692.

[156] B. L. van der Waerden, *Moderne Algebra. Band II*, volume 34 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin 1940.

[157] O. Veblen, Collineations in a finite projective geometry. *Trans. Amer. Math. Soc.* **8** (1907), 366–368.

[158] S. Wachs, Essai sur la géométrie projective quaternionnienne. *Acad. Roy. Belg. Cl. Sci. Mém. Collect. 8°* **15** (1936), 135 pp.

[159] Z.-X. Wan, *Geometry of Matrices*. World Scientific, Singapore 1996.

[160] X. T. Wang, Decomposition of Jordan automorphisms of strictly upper triangular matrix algebra over commutative rings. *Comm. Algebra* **35** (2007), 1133–1140.

[161] X. T. Wang, Y. M. Li, Decomposition of Jordan automorphisms of triangular matrix algebra over commutative rings. *Glasg. Math. J.* **52** (2010), 529–536.

[162] X. T. Wang, H. You, Decomposition of Jordan automorphisms of strictly triangular matrix algebra over local rings. *Linear Algebra Appl.* **392** (2004), 183–193.

[163] Y. Wang, Y. Wang, Jordan homomorphisms of upper triangular matrix rings. *Linear Algebra Appl.* **439** (2013), 4063–4069.

[164] A. N. Whitehead, *The Axioms of Projective Geometry*. Cambridge University Press, Cambridge 1906.

[165] T.-L. Wong, Jordan isomorphisms of triangular rings. *Proc. Amer. Math. Soc.* **133** (2005), 3381–3388.

[166] K. Yamaguti, Jordan and Jordan triple isomorphisms of rings. *J. Sci. Hiroshima Univ. Ser. A* **20** (1957), 107–110.

[167] A. Yang, Jordan isomorphisms on nest subalgebras. *Adv. Math. Phys.* **2015** (2015), Art. ID 864792, 6 pp.

[168] A. L. Yang, J. H. Zhang, Jordan isomorphisms on nest subalgebras of factor von Neumann algebras (Chinese). *Acta Math. Sinica (Chin. Ser.)* **51** (2008), 219–224.

[169] H. M. Yao, C. G. Cao, X. Zhang, Additive preservers of idempotence and Jordan homomorphisms between rings of square matrices. *Acta Math. Sin. (Engl. Ser.)* **25** (2009), 639–648.

[170] J. H. Zhang, Jordan isomorphisms of nest algebras (Chinese). *Acta Math. Sinica (Chin. Ser.)* **45** (2002), 819–824.

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