Aspects of the Segre variety $\mathcal{S}_{1,1,1}(2)$

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Abstract

We consider various aspects of the Segre variety $\mathcal{S} := \mathcal{S}_{1,1,1}(2)$ in $\mathrm{PG}(7,2)$, whose stabilizer group $\mathcal{G}_{\mathcal{S}} < \mathrm{GL}(8,2)$ has the structure $\mathcal{N} \rtimes \mathrm{Sym}(3)$, where $\mathcal{N} := \mathrm{GL}(2,2) \times \mathrm{GL}(2,2) \times \mathrm{GL}(2,2)$. In particular we prove that $\mathcal{S}$ determines a distinguished Z_3-subgroup $\mathcal{Z} < \mathrm{GL}(8,2)$ such that $A\mathcal{Z}A^{-1} = \mathcal{Z}$, for all $A \in \mathcal{G}_{\mathcal{S}}$, and in consequence $\mathcal{S}$ determines a $\mathcal{G}_{\mathcal{S}}$-invariant spread of 85 lines in $\mathrm{PG}(7,2)$. Furthermore we see that Segre varieties $\mathcal{S}_{1,1,1}(2)$ in $\mathrm{PG}(7,2)$ come along in triplets $\{\mathcal{S}, \mathcal{S}', \mathcal{S}''\}$ which share the same distinguished Z_3-subgroup $\mathcal{Z} < \mathrm{GL}(8,2)$. We conclude by determining all fifteen $\mathcal{G}_{\mathcal{S}}$-invariant polynomial functions on $\mathrm{PG}(7,2)$ which have degree < 8, and their relation to the five $\mathcal{G}_{\mathcal{S}}$-orbits of points in $\mathrm{PG}(7,2)$.

1 Introduction

We work over $\mathrm{GF}(2)$, and so *we may identify the nonzero elements of a vector space $V(n+1, 2) = V_{n+1}$ with the points S_0 of the associated projective space* $\mathbb{P}V_{n+1} = \mathrm{PG}(n,2)$. Consequently we identify $\mathrm{GL}(V_{n+1}) = \mathrm{GL}(n+1,2)$ with the group $\mathrm{PGL}(n+1,2)$ of collineations of $\mathrm{PG}(n,2)$. We use $\prec u, v, \ldots \succ$ for the vector subspace spanned by vectors $u, v, \ldots$, and $\langle u, v, \ldots \rangle$ for the flat (projective subspace) generated by projective points $u, v, \ldots$. The vector space $F(S_0)$ of all functions $S_0 \to \mathrm{GF}(2)$ is of dimension $|S_0| = 2^{n+1} - 1$. Given a point-set $\psi \subset \mathrm{PG}(n,2)$ it has equation $\tilde{Q}(x) = 0$ for some polynomial $\tilde{Q}$ satisfying $\tilde{Q}(0) = 0$. Upon replacing $(x_i)^{r_i}, r_i > 1$, by x_i in any such polynomial we obtain a *uniquely determined* polynomial $Q = Q_\psi$ of the form $\sum x_{i_1} \cdots x_{i_k}$, $1 \le i_1 < \cdots < i_k \le n+1$. (This uniqueness does *not* hold for a point-set $\psi \subset \mathrm{PG}(n,q)$ for $q > 2$: see for example [13, Remark 1.2].) Briefly stated, *every point-set of $\mathrm{PG}(n,2)$ is a hypersurface.* The (reduced) degree $d = \deg Q$ of Q is the *polynomial degree* of the point-set ψ. (On account of the aforementioned uniqueness, the reduced degree d of Q is seen to be independent of

the coordinate system.) It should be noted, see [12, Section 1.2], that *if $|\psi|$ is odd then $d \leq n$ while if $|\psi|$ is even then $d = n + 1$.* Further note that if $F_d = F_d(S_0)$, $d > 0$, denotes the subspace of $F(S_0)$ which consists of functions f expressible as a polynomial function $f(x_1, x_2, \ldots, x_{n+1})$ with $\deg f \leq d$ and $f(0) = 0$, then the subspaces F_d are nested:

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset F_{n+1} = F(S_0). \tag{1}$$

Given a choice of subset $\psi \subset \mathrm{PG}(n, 2)$ a flat X of $\mathrm{PG}(n, 2)$ is termed ψ-*odd* whenever $|X \cap \psi|$ is odd and ψ-*even* whenever $|X \cap \psi|$ is even. The next lemma shows that the degree $d = \deg Q$ can be determined from the point-set ψ purely by incidence properties.

Lemma 1 *(See [14, Theorem 1.1].) If $|\psi|$ is odd then Q has polynomial degree d if and only if* (i) *every d-flat is ψ-odd and* (ii) *there exists at least one $(d-1)$-flat which is ψ-even. (Here condition* (i) $\implies \deg Q \leq d$, *and condition* (ii) $\implies \deg Q \geq d$.)

When attempting to determine the polynomial degree of a hypersurface ψ in $\mathrm{PG}(n, 2)$ it often helps to make use of the following elementary lemma.

Lemma 2 *Each $(n-d)$-flat X in $\mathrm{PG}(n, 2)$ has polynomial degree d. In detail, if X is the intersection of the d independent hyperplanes $f_1(x) = 0, \ldots, f_d(x) = 0$, where the f_i are elements of the dual V_{n+1}^* of V_{n+1}, then X has equation*

$$P_X(x) = 0, \quad where \ P_X := 1 + \Pi_{i=1}^{d}(1 + f_i), \tag{2}$$

the polynomial P_X thus having degree d.

Proof. $1 + \Pi_{i=1}^{d}(1 + f_i(x))$ equals 1 except when $f_1(x) = \cdots = f_d(x) = 0$. ∎

In fact, from now onwards, we confine our attention to projective dimension $n = 7$, and moreover will deal solely with the Segre variety $\mathcal{S}_{1,1,1}(2) \subset \mathrm{PG}(7, 2)$, see [3] or [8], and with its stabilizer $\mathcal{G}_\mathcal{S} := \mathcal{G}_{\mathcal{S}_{1,1,1}(2)} < \mathrm{GL}(V_8)$. We will be particularly concerned with various $\mathcal{G}_\mathcal{S}$-invariant attributes of the Segre variety and so will overlap at times with some of the material in the recent paper [7]. However we will work entirely over the field $\mathrm{GF}(2)$, in contrast to the frequent excursions into $\mathrm{GF}(4)$ territory undergone in [7]. There will also be neat connections to [5]. The $3 \times 3 \times 3$ grid and its Veldkamp space from [5] are in our setting the Segre variety $\mathcal{S}_{1,1,1}(2)$ (considered merely as a point-line geometry) and the dual of the ambient $\mathrm{PG}(7, 2)$ (recovered in terms of this point-line geometry). It is worth noting that the authors of [5] took their motivation from physics. They were looking for finite geometries potentially allowing physical applications, similar to the ones in [6], [9] and [10], where a class of finite symplectic polar spaces and certain finite generalized polygons were successfully linked with *quantum information theory* (commuting and non-commuting elements of *Pauli groups*) and the theory of *black holes and black strings* (symmetry properties of *entropy formulae*). The cited papers contain a wealth of further references on related

work. Another link to quantum information theory was pointed out in [4, Section 5.5]: The ambient space $\mathrm{PG}(7,2)$ of the Segre variety $\mathcal{S}_{1,1,1}(2)$ may be regarded as a finite analogue to the *state-space of a three qubit system* (over the complex numbers).

2 The Segre variety $\mathcal{S} := \mathcal{S}_{1,1,1}(2)$

If $V_2 = V(2,2)$ is a 2-dimensional vector space over $\mathrm{GF}(2)$ then the tensor product space $V_8 := V_2 \otimes V_2 \otimes V_2$ is of dimension 8 over $\mathrm{GF}(2)$ and contains $3 \times 3 \times 3 = 27$ nonzero decomposable tensors. Viewed as elements of $\mathbb{P}V_8 = \mathrm{PG}(7,2)$ these decomposable tensors constitute the Segre variety $\mathcal{S} := \mathcal{S}_{1,1,1}(2) \subset \mathrm{PG}(7,2)$. If the three points of the projective line $\mathbb{P}V_2$ are denoted $\{u_0, u_1, u_2\}$, then

$$\mathcal{S} := \mathcal{S}_{1,1,1}(2) = \{E_{ijk}, \quad i,j,k \in \{0,1,2\}\} \quad \text{where } E_{ijk} := u_i \otimes u_j \otimes u_k. \quad (3)$$

The stabilizer $\mathcal{G}_{\mathcal{S}} := \mathcal{G}_{\mathcal{S}_{1,1,1}(2)} < \mathrm{GL}(V_8)$ of $\mathcal{S}$ has the semidirect product structure

$$\mathcal{G}_{\mathcal{S}} = \mathcal{N} \rtimes \mathrm{Sym}(3), \quad \text{where } \mathcal{N} := \mathrm{GL}(V_2) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_2), \quad (4)$$

and so is of order $6^4 = 1296$. Here the element (a_0, a_1, a_2), $a_i \in \mathrm{GL}(V_2)$, of the normal subgroup $\mathcal{N}$ of $\mathcal{G}_{\mathcal{S}}$ acts upon V_8 by the tensor product map $a_0 \otimes a_1 \otimes a_2 \in \mathrm{GL}(V_8)$, and the action of $\rho \in \mathrm{Sym}(3)$ upon $V_8 = \otimes^3 V_2$ is by the linear operator (see for example [11, Section 8.8.1]) ρ_{op} given by $\rho_{\mathrm{op}}(v_1 \otimes v_2 \otimes v_3) = v_{\rho^{-1}1} \otimes v_{\rho^{-1}2} \otimes v_{\rho^{-1}3}$. (Thus $\mathcal{G}_{\mathcal{S}}$ is a wreath product $\mathcal{N} \wr \mathrm{Sym}(3)$.)

2.1 Some invariant attributes of the Segre variety $\mathcal{S}_{1,1,1}(2)$.

As well as its twenty-seven points E_{ijk} in (3), we will make use of the following $\mathcal{G}_{\mathcal{S}}$-invariant attributes of the Segre variety $\mathcal{S}_{1,1,1}(2)$.

1. The set of twenty-seven *generators* $\{L_{ij}^r, \ i,j \in \{0,1,2\}, \ r \in \{1,2,3\}\}$ of $\mathcal{S}_{1,1,1}(2)$, where the lines L_{ij}^r are defined, for $i,j \in \{0,1,2\}$, by

$$L_{ij}^1 = \{E_{kij}\}_{k \in \{0,1,2\}}, \quad L_{ij}^2 = \{E_{ikj}\}_{k \in \{0,1,2\}}, \quad L_{ij}^3 = \{E_{ijk}\}_{k \in \{0,1,2\}}. \quad (5)$$

2. The set of nine copies $\{\sigma_i^r, \ i \in \{0,1,2\}, \ r \in \{1,2,3\}\}$ of a Segre variety $\mathcal{S}_{1,1}(2)$ which are contained in the Segre variety (3), where

$$\sigma_i^1 := \{E_{ijk}\}_{j,k \in \{0,1,2\}}, \quad \sigma_i^2 := \{E_{jik}\}_{j,k \in \{0,1,2\}}, \quad \sigma_i^3 := \{E_{jki}\}_{j,k \in \{0,1,2\}}. \quad (6)$$

3. The set of nine ambient spaces $\{Y_i^r = \langle \sigma_i^r \rangle, \quad i \in \{0,1,2\}, \ r \in \{1,2,3\}\}$ of the nine $\mathcal{S}_{1,1}(2)$s in (6), each Y_i^r being of course a 3-flat.

3

4. The set of twenty-seven 3-flats

$$\{Z_{ijk} = \langle L_{jk}^1, L_{ik}^2, L_{ij}^3 \rangle, \quad i, j, k \in \{0, 1, 2\}\}, \tag{7}$$

with $Z_{ijk} = Z(p)$ being spanned by the three generators which pass through the point $p = E_{ijk}$.

5. The set of twenty-seven *distinguished tangents* $\{L(p), p \in \mathcal{S}\}$, defined by

$$L(p) := \{p, p', p''\} \tag{8}$$

where p', p'' are those two points of the 3-flat $Z(p)$, see (7), which are external to each of the three planes which are spanned by two of the generators through p.

Remark 3 *It may well help to visualize these attributes of the Segre variety $\mathcal{S}_{1,1,1}(2)$ by appealing to the '27-cube' in [5, Section 3.1] or [7, Fig. 1]. In particular the nine $\mathcal{S}_{1,1}(2)$s in (6) can be visualized as nine 'sections' of the 27-cube, three horizontal and six vertical.*

2.2 Some subgroups of $\mathcal{G}_{\mathcal{S}}$

In the following we will often choose $\mathcal{B} = \{e_1, e_2, \ldots, e_8\}$ as basis for V_8, where

$$e_1 = E_{000}, \ e_2 = E_{100}, \ e_3 = E_{110}, \ e_4 = E_{010},$$
$$e_8 = E_{111}, \ e_7 = E_{011}, \ e_6 = E_{001}, \ e_5 = E_{101}. \tag{9}$$

(So the multi-indices of e_1, e_3, e_5, e_7 have even parity and those of their 'opposites' e_8, e_6, e_4, e_2 have odd parity.) If we view these basis elements in terms of the eight vertices of the '8-cube' in Figure 1 then the two regular tetrahedra embed-
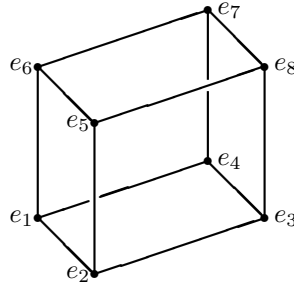

Figure 1: the '8-cube'

ded in this 8-cube have vertices $\{e_1, e_3, e_5, e_7\}$ and $\{e_2, e_4, e_6, e_8\}$. At times we will view $\langle e_1, e_2 \rangle$ as the 'x-axis', $\langle e_1, e_4 \rangle$ as the 'y-axis', $\langle e_1, e_6 \rangle$ as the 'z-axis'.

Clearly the group $\mathcal{G}_{\mathcal{B}} \cong \mathrm{Sym}(4) \times Z_2$ of 3-dimensional symmetries of the cube with vertices $\mathcal{B}$ will be a subgroup of $\mathcal{G}_{\mathcal{S}}$. The subgroup $\mathcal{G}_{\mathcal{B}} < \mathcal{G}_{\mathcal{S}}$ contains all 4! =

24 permutations of the four space diagonals $\{e_1, e_8\}, \{e_2, e_7\}, \{e_3, e_6\}, \{e_4, e_5\}$ of the cube and also the central involution

$$J : (18)(27)(36)(45) \tag{10}$$

which fixes each space diagonal. (Here, and below, we often use shorthand notation; in particular (18) is shorthand for $e_1 \rightleftarrows e_8$.) Alternatively $\mathcal{G}_\mathcal{B}$ may be defined to be that subgroup of $\mathcal{G}_\mathcal{S}$ which fixes the unit point $u := \sum_{i=1}^{8} e_i$ of the basis $\mathcal{B}$.

If $j_x \in \mathrm{GL}(V_2)$ is the element of order 2 which effects the interchange $u_0 \rightleftarrows u_1$, then, in shorthand notation, the element $J_x = j_x \otimes I \otimes I \in \mathcal{N} \lhd \mathcal{G}_\mathcal{S}$ is given in terms of the basis (9) by:

$$J_x : (12)(34)(56)(78). \tag{11}$$

Similarly we arrive at two further involutions $J_y, J_z \in \mathcal{N} \lhd \mathcal{G}_\mathcal{S}$:

$$J_y : (14)(23)(58)(67), \quad J_z : (16)(25)(38)(47). \tag{12}$$

The three involutions (11), (12) satisfy $J_x J_y J_z = J$, and are elements of $\mathcal{G}_\mathcal{B}$.

Next, if $a \in \mathrm{GL}(V_2)$ is the element of order 3 which effects the cyclic permutation $(u_0 u_1 u_2)$ then the element $A_x = a \otimes I \otimes I \in \mathcal{G}_\mathcal{S}$ is given by

$$\begin{aligned} A_x : &e_1 \mapsto e_2 \mapsto e_1 + e_2, \quad e_4 \mapsto e_3 \mapsto e_3 + e_4, \\ &e_6 \mapsto e_5 \mapsto e_5 + e_6, \quad e_7 \mapsto e_8 \mapsto e_7 + e_8. \end{aligned} \tag{13}$$

Similarly there are two further elements $A_y = I \otimes a \otimes I$ and $A_z = I \otimes I \otimes a$, each of order 3, which belong to $\mathcal{G}_\mathcal{S}$ but not to $\mathcal{G}_\mathcal{B}$.

The linear mapping $K_{12} := \rho_{\mathrm{op}}$ arising from the interchange $\rho = (12) \in \mathrm{Sym}(3)$ is another involution $\in \mathcal{G}_\mathcal{B}$ given by

$$K_{12} : (24)(57), \quad 1, 3, 6, 8 \text{ fixed.} \tag{14}$$

Other involutions $K_{13}, K_{23} \in \mathcal{G}_\mathcal{B}$ can similarly be arrived at. Observe that the product $C := J_x K_{12}$ is an element of $\mathcal{G}_\mathcal{B}$ of order 4:

$$C : (1234)(8765). \tag{15}$$

Setting $B := \rho_{\mathrm{op}}$ with the choice $\rho = (123) \in \mathrm{Sym}(3)$, then B is an element of $\mathcal{G}_\mathcal{B}$ of order 3 given by

$$B : (375)(246), \quad 1, 8 \text{ fixed.} \tag{16}$$

2.2.1 The subgroup $\mathcal{G}_\mathcal{S}^0 = \mathcal{N}_0 \rtimes \mathrm{Sym}(3)$ of $\mathcal{G}_\mathcal{S}$

The group $\mathcal{G}_\mathcal{S} = \mathcal{N} \rtimes \mathrm{Sym}(3)$ contains an obvious subgroup of index 2, namely $\mathcal{N} \rtimes \mathrm{Alt}(3)$. Of relevance to our later concerns is the fact that $\mathcal{G}_\mathcal{S}$ also contains a much less obvious subgroup $\mathcal{G}_\mathcal{S}^0$ of index 2, which arises from the special nature of the group $\mathrm{GL}(V_2)$ when the vector space V_2 is over the field $\mathrm{GF}(2)$. For then

$\mathrm{GL}(V_2)$ contains a subgroup $\mathcal{H} \cong Z_3$ of index 2. It follows that the group $\mathcal{N} := \mathrm{GL}(V_2) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_2)$ contains a subgroup $\mathcal{N}_0$ of index 2 consisting of those elements $a_1 \otimes a_2 \otimes a_3 \in \mathcal{N}$ such that an even number, 0 or 2, of the a_i belong to the coset $\mathcal{K} := \mathrm{GL}(V_2) \setminus \mathcal{H}$ (which incidentally consists of three involutions). Consequently the subgroup

$$\mathcal{G}_{\mathcal{S}}^0 := \mathcal{N}_0 \rtimes \mathrm{Sym}(3) \qquad (17)$$

has index 2 in $\mathcal{G}_{\mathcal{S}}$. Observe that the three involutions

$$J_x J_y : (13)(24)(57)(68), \quad J_x J_z : (15)(26)(37)(48), \quad J_y J_z : (17)(28)(35)(46) \qquad (18)$$

are consequently elements of $\mathcal{G}_{\mathcal{S}}^0$, while $J = J_x J_y J_z \in \mathcal{G}_{\mathcal{S}} \setminus \mathcal{G}_{\mathcal{S}}^0$.

Remark 4 *If instead we deal, for general $m > 1$, with $V_{2^m} := \otimes^m V_2$, and with $\mathcal{S} = \mathcal{S}_m(2) := \mathcal{S}_{1,1,\ldots,1}(2) \subset \mathrm{PG}(2^m - 1, 2)$, then we have $\mathcal{G}_{\mathcal{S}} = \mathcal{N} \rtimes \mathrm{Sym}(m)$ where $\mathcal{N} := \times^m \mathrm{GL}(V_2)$ has a similarly defined subgroup $\mathcal{N}_0$ giving rise to a subgroup $\mathcal{G}_{\mathcal{S}}^0 := \mathcal{N}_0 \rtimes \mathrm{Sym}(m)$ of $\mathcal{G}_{\mathcal{S}}$ of index 2. In the rather special case $m = 2$ the nine points of $\mathcal{S} := \mathcal{S}_{1,1}(2)$ are the points of a hyperbolic quadric $\mathcal{H}_3$ in $\mathrm{PG}(3, 2)$ and $\mathcal{G}_{\mathcal{S}}^0$ is seen to consist of those orthogonal transformations which fix separately each of the two external lines of the quadric.*

2.2.2 Generators for the groups $\mathcal{G}_{\mathcal{S}}$, $\mathcal{G}_{\mathcal{S}}^0$, $\mathcal{G}_{\mathcal{B}}$

It is possible to generate the group $\mathcal{G}_{\mathcal{S}}$ using just two elements M, N:

$$\mathcal{G}_{\mathcal{S}} = \langle M, N \rangle. \qquad (19)$$

One choice of generators uses the elements M, N, each of order 6, defined by:

$$M = J_x B, \qquad N = A_x K_{12}. \qquad (20)$$

This choice was checked by use of the computer algebra system Magma, see [2]. Magma was also used to check the following related choices of generators for the groups $\mathcal{G}_{\mathcal{S}}^0$ and $\mathcal{G}_{\mathcal{B}}$:

$$\mathcal{G}_{\mathcal{S}}^0 = \langle M', N \rangle, \quad \text{where } M' = JM = J J_x B \quad \text{and } N = A_x K_{12}, \qquad (21)$$

$$\mathcal{G}_{\mathcal{B}} = \langle M, K_{12} \rangle, \quad \text{where } M = J_x B. \qquad (22)$$

3 The distinguished Z_3-subgroup $\mathcal{Z} < \mathrm{GL}(8, 2)$

An intriguing aspect of the subgroup $\mathcal{G}_{\mathcal{S}}^0 < \mathrm{GL}(8, 2)$, see (17), *is that its central-izer $\mathcal{Z} := C_{\mathrm{GL}(8,2)}(\mathcal{G}_{\mathcal{S}}^0)$ in $\mathrm{GL}(8, 2)$ is non-trivial.* To see this, suppose $W \in \mathcal{Z}$, so that

$$WA = AW \quad \text{for all } A \in \mathcal{G}_{\mathcal{S}}^0. \qquad (23)$$

For each $A \in \mathcal{G}_{\mathcal{S}}^0$ it follows from this that W leaves invariant the subspace

$$\mathrm{Fix}(A) = \{x \in V_8 | \; Ax = x\} \qquad (24)$$

of V_8 consisting of all vectors fixed by A. Observe that for the first two of the involutions (18) we have

$$\text{Fix}(J_x J_y) = \langle 13, 24, 57, 68 \rangle, \quad \text{Fix}(J_x J_z) = \langle 15, 26, 37, 48 \rangle. \qquad (25)$$

(Here, and elsewhere, we use $i, ij, ijk, \ldots$ as shorthand for $e_i, e_i + e_j, e_i + e_j + e_k, \ldots$.) Noting that $\text{Fix}(J_x J_y) \cap \text{Fix}(J_x J_z) = \langle 1357, 2468 \rangle$ it follows that if $W \in \mathcal{Z}$ then W necessarily stabilizes the distinguished tangent $L(u) = \{u, 1357, 2468\}$, see (8), where $u := 12345678$ denotes the unit point of the basis $\mathcal{B}$. (Using the third involution $J_y J_z$ in (18) gives nothing further, since $\text{Fix}(J_y J_z) = \langle 17, 28, 35, 46 \rangle$ meets each of the 4-spaces (25) also in the 2-space $\langle 1357, 2468 \rangle$.)

Now $\mathcal{G}_\mathcal{S}^0$ acts transitively on the 27 points of $\mathcal{S}$, and so it acts transitively on the 27 distinguished tangents $\{L(p), p \in \mathcal{S}\}$. Consequently if $W \in \mathcal{Z}$ then W stabilizes each distinguished tangent. Moreover if $W \in \mathcal{Z}$ fixes a point on one of the distinguished tangents, then it fixes a point on each of the distinguished tangents, which can be the case only if $W = I$. So if an element $W \neq I$ exists in $\mathcal{Z}$ then W is of order 3 and cyclically permutes the three points of each distinguished tangent.

Consider the eight distinguished tangents $L_i := L(e_i)$ through the basis vectors (9):

$$
\begin{aligned}
L_1 &= \{1, 246, 1246\}, & L_8 &= \{8, 8357, 357\}, \\
L_3 &= \{3, 248, 3248\}, & L_6 &= \{6, 6157, 157\}, \\
L_5 &= \{5, 268, 5268\}, & L_4 &= \{4, 4137, 137\}, \\
L_7 &= \{7, 468, 7468\}, & L_2 &= \{2, 2135, 135\}.
\end{aligned} \qquad (26)
$$

Suppose W is that element of $\text{GL}(8, 2)$ which fixes each of these eight lines by cycling through their points in the order displayed in (26); in particular $W^3 = I$. So W is given by its effect on the basis $\mathcal{B}$ by

$$
\begin{aligned}
W : &1 \mapsto 246, \quad 2 \mapsto 2135, \quad 3 \mapsto 248, \quad 4 \mapsto 4137, \\
&5 \mapsto 268, \quad 6 \mapsto 6157, \quad 7 \mapsto 468, \quad 8 \mapsto 8357.
\end{aligned} \qquad (27)
$$

Theorem 5 *The centralizer in* $\text{GL}(8, 2)$ *of the subgroup* $\mathcal{G}_\mathcal{S}^0$ *is the subgroup* $\mathcal{Z} := \langle W \rangle \cong Z_3$, *where W, defined as in (27), fixes each line of the invariant set* $\{L(p), p \in \mathcal{S}\}$. *Moreover the orbits of $\mathcal{Z}$ in* $\text{PG}(7, 2)$ *constitute a* $\mathcal{G}_\mathcal{S}$-*invariant spread* $\mathcal{L}_{85}$ *of 85 lines.*

Proof. The fact that W centralizes the subgroup $\mathcal{G}_\mathcal{S}^0$ follows upon checking that W commutes with each of the generators in (21). The fact that $\mathcal{Z}$ fixes (even just one of) the eight distinguished tangents (26) now ensures that $\mathcal{Z}$ fixes every one of the invariant set $\{L(p), p \in \mathcal{S}\}$.

To see that $C_{\text{GL}(8,2)}(\mathcal{G}_\mathcal{S}^0)$ is no larger than $\mathcal{Z}$ suppose that $\mathcal{Z}' := \langle W' \rangle$ is another subgroup $\cong Z_3$ which fixes the 27 distinguished tangents. Consider the cyclic action of W and W' on the points of the eight distinguished tangents

(26), and observe that any four of these tangents generate $PG(7,2)$. So if there exists a subset of four of the tangents (26) upon which the action of W' agrees with that of W then $W' = W$. But if no such subset exists then there exists a subset of more than four the tangents (26) upon which the action of W' agrees with that of W^2, whence $W' = W^2$. So in either case it follows that $\mathcal{Z}' = \mathcal{Z}$.

The minimal polynomial of W is $\mu = t^2 + t + 1$, since $\mu(W)e_i = 0$ for each basis vector e_i. Consequently W is fixed-point-free on $PG(7,2)$ and so the orbits of $\mathcal{Z}$ in $PG(7,2)$ constitute a $\mathcal{G}_\mathcal{S}$-invariant spread $\mathcal{L}_{85}$ of 85 lines. $\blacksquare$

Remark 6 *Upon observing that $JWJ^{-1} = W^2$ holds for the element $J \in \mathcal{G}_\mathcal{S} \setminus \mathcal{G}_\mathcal{S}^0$ it follows that*

$$A\mathcal{Z}A^{-1} = \mathcal{Z}, \quad \text{for all } A \in \mathcal{G}_\mathcal{S}. \tag{28}$$

4 Orbits and triplets

4.1 The five $\mathcal{G}_S$-orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5$ of points

Although we will arrive at the five orbits in the order $\mathcal{O}_5, \mathcal{O}_2, \mathcal{O}_4, \mathcal{O}_3, \mathcal{O}_1$ nevertheless we have adopted the present numbering so as to be in agreement with that in [7, Proposition 5].

From the structure (4) of $\mathcal{G}_\mathcal{S}$ clearly $\mathcal{O}_5 := \mathcal{S}_{1,1,1}(2)$ is a single $\mathcal{G}_\mathcal{S}$-orbit of length 27. Secondly each of the 9 ambient spaces Y_i^r contains 6 points external to $\mathcal{S}$, and such points form an orbit $\mathcal{O}_2$ of length $9 \times 6 = 54$. Thirdly each of the 27 3-flats Z_{ijk} contains 2 points, on the distinguished tangent $L(E_{ijk})$, which are external to $\mathcal{O}_2 \cup \mathcal{O}_5$, giving rise to an orbit $\mathcal{O}_4$ of length $27 \times 2 = 54$. Consider next the lines through a point $x \in \mathcal{S}$ which meet $\mathcal{S}$ in one or more further points. Three of these lines are generators, whose union accounts for 7 points of $\mathcal{S}$. Now each of the three $\mathcal{S}_{1,1}(2)$'s which contain x contribute 4 bisecants to $\mathcal{S}$ through x, accounting for a further $3 \times 4 = 12$ points of $\mathcal{S}$. The remaining 8 points of $\mathcal{S}$ thus give rise to 8 other bisecants through x, and hence to 8 points external to $\mathcal{S}$. All together there are $\frac{1}{2}(27 \times 8) = 108$ 'other bisecants', and the external points on these bisecants form an orbit $\mathcal{O}_3$ of length 108. As we now show, the remaining $12 (= 255 - 27 - 54 - 54 - 108)$ points of $PG(7,2)$ constitute a fifth $\mathcal{G}_\mathcal{S}$-orbit $\mathcal{O}_1$.

If $\mathcal{L}_s \subset \mathcal{L}_{85}$ is a partial spread formed from s lines of the complete spread $\mathcal{L}_{85}$ in Theorem 5 then we will denote by $\mathcal{P}(\mathcal{L}_s)$ the set of $3s$ points underlying the lines of $\mathcal{L}_s$. If $\mathcal{L}_{27} \subset \mathcal{L}_{85}$ denotes the partial spread consisting of the 27 distinguished tangents then $\mathcal{P}(\mathcal{L}_{27}) = \mathcal{O}_4 \cup \mathcal{O}_5$. Next note from (27) that W maps the point $e_1 + e_3 \in \mathcal{O}_2$ to the point $e_8 + e_6 \in \mathcal{O}_2$. Hence, in view of theorem 5, W sends every point of $\mathcal{O}_2$ to another point of $\mathcal{O}_2$, So $\mathcal{O}_2 = \mathcal{P}(\mathcal{L}_{18})$ for a partial spread $\mathcal{L}_{18} \subset \mathcal{L}_{85}$. Noting also that W maps the point $e_1 + e_8 \in \mathcal{O}_3$ to the point $e_1 + u \in \mathcal{O}_3$, it similarly follows that $\mathcal{O}_3 = \mathcal{P}(\mathcal{L}_{36})$ for a partial spread $\mathcal{L}_{36} \subset \mathcal{L}_{85}$. (Alternatively, W mapping a subset of $\mathcal{O}_3$ into $\mathcal{O}_1$ would contradict $\mathcal{O}_3$ being a single $\mathcal{G}_\mathcal{S}$-orbit.) Consequently the 12-set $\mathcal{O}_1$ must be of the form $\mathcal{P}(\mathcal{L}_4)$ for a partial spread $\mathcal{L}_4 \subset \mathcal{L}_{85}$. Such a 12-set $\mathcal{O}_1$ possesses $(12 \times 9)/2 = 54$

bisecants, and so, granted the lengths $27, 54, 54, 108$ of the preceding orbits, $\mathcal{O}_1$ must be a single $\mathcal{G}_S$-orbit. Explicitly we have $\mathcal{L}_4 = \{L_a, L_b, L_c, L_d\}$ where

$$L_a = \{18u, 357u, 246u\}, \quad L_b = \{27u, 135u, 468u\},$$
$$L_c = \{36u, 157u, 248u\}, \quad L_d = \{45u, 137u, 268u\}, \tag{29}$$

and where iju and $ijku$ are shorthand for $e_i + e_j + u$ and $e_i + e_j + e_k + u$. So the points external to $\mathcal{O}_1$ on the 54 bisecants are seen to form the orbit $\mathcal{O}_2$. As an instance of the action of $\mathcal{G}_S$ on $\mathcal{O}_1$, observe that C in (15) effects the cyclic permutation $(L_a L_b L_c L_d)$. The next theorem summarizes the foregoing results.

Theorem 7 *Under the action of $\mathcal{G}_S$ the points of $\mathrm{PG}(7,2)$ fall into five orbits $\mathcal{O}_5, \mathcal{O}_2, \mathcal{O}_4, \mathcal{O}_3, \mathcal{O}_1$, of respective lengths $27, 54, 54, 108, 12$, and the lines of the invariant spread $\mathcal{L}_{85}$ fall into four orbits $\mathcal{L}_{27}, \mathcal{L}_{18}, \mathcal{L}_{36}, \mathcal{L}_4$, where*

$$\mathcal{P}(\mathcal{L}_4) = \mathcal{O}_1, \quad \mathcal{P}(\mathcal{L}_{18}) = \mathcal{O}_2, \quad \mathcal{P}(\mathcal{L}_{36}) = \mathcal{O}_3, \quad \mathcal{P}(\mathcal{L}_{27}) = \mathcal{O}_4 \cup \mathcal{O}_5. \tag{30}$$

Remark 8 *The results in the theorem agree with those of Havlicek, Odehnal and Saniga, see [7, Propositions 4, 5]. These authors arrived at their results after an excursion into $\mathrm{GF}(4)$ terrain, viewing the three points $\{u_0, u_1, u_2\}$ of the projective line $\mathbb{P}V(2,2)$ as the 'real' points of the projective line $\mathbb{P}V(2,4)$. They obtained thereby a $\mathcal{G}_S$-invariant basis for $\mathrm{PG}(7,4)$ consisting of four pairs of 'complex conjugate' points, these yielding the four real lines $\mathcal{L}_4$, and hence the orbit $\mathcal{O}_1$.*

4.2 The triplet $\{\mathcal{S}, \mathcal{S}', \mathcal{S}''\}$ of Segre varieties $\mathcal{S}_{1,1,1}(2)$

Each distinguished tangent $L(p), p \in \mathcal{S} := \mathcal{S}_{1,1,1}(2) = \mathcal{O}_5$, contains two points of the orbit $\mathcal{O}_4$, namely $p' = Wp$ and $p'' = W^2p$. Consequently we have

$$\mathcal{O}_4 = \mathcal{S}' \cup \mathcal{S}'', \quad \mathcal{S}' = W(\mathcal{S}), \quad \mathcal{S}'' = W^2(\mathcal{S}), \tag{31}$$

and so *the $\mathcal{G}_S$-orbit $\mathcal{O}_4$ splits into two copies of the Segre variety $\mathcal{S}$.* It is quite a surprise to find that the study of a single Segre variety $\mathcal{S}_{1,1,1}(2)$ inevitably leads one to deal with a triplet of such varieties which share the same 27 distinguished tangents and hence give rise to the same distinguished Z_3-subgroup $\mathcal{Z} < \mathrm{GL}(8,2)$ studied in section 3. Moreover

$$\mathcal{G}_{\mathcal{S}}^0 = \mathcal{G}_{\mathcal{S}'}^0 = \mathcal{G}_{\mathcal{S}''}^0 = \mathcal{N}_0 \rtimes \mathrm{Sym}(3), \tag{32}$$

the group $\mathcal{G}_{\mathcal{S}}^0$ having the six point-orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{S}', \mathcal{S}'', \mathcal{S}$. Further $\mathcal{G}_S$ effects the interchange $\mathcal{S}' \rightleftarrows \mathcal{S}''$, $\mathcal{G}_{\mathcal{S}'}$ effects the interchange $\mathcal{S} \rightleftarrows \mathcal{S}''$ and $\mathcal{G}_{\mathcal{S}''}$ effects the interchange $\mathcal{S} \rightleftarrows \mathcal{S}'$. In particular the involution $J \in \mathcal{G}_S$ effects $\mathcal{S}' \rightleftarrows \mathcal{S}''$, the involution $J' := WJW^{-1} = W^2J \in \mathcal{G}_{\mathcal{S}'}$ effects $\mathcal{S} \rightleftarrows \mathcal{S}''$ and the involution $J'' := W^2JW^{-2} = WJ \in \mathcal{G}_{\mathcal{S}''}$ effects $\mathcal{S} \rightleftarrows \mathcal{S}'$.

4.3 The $\mathcal{G}_\mathcal{B}$-orbits of points

Under the action of the subgroup $\mathcal{G}_\mathcal{B}$ of $\mathcal{G}_\mathcal{S}$, the orbits $\mathcal{O}_i$ decompose as in Table 1[1].

Table 1					
$\mathcal{G}_\mathcal{S}$-orbit	$\mathcal{G}_\mathcal{B}$-orbit	w	$|\mathcal{O}_{i,w}|$	$p_{i,w}$	mnemonic
$\mathcal{O}_1$	$\mathcal{O}_{1,5}$	5	8	$135u$	
	$\mathcal{O}_{1,6}$	6	4	$18u$	
$\mathcal{O}_2$	$\mathcal{O}_{2,2}$	2	12	13	face diagonals
	$\mathcal{O}_{2,3}$	3	24	123	2-arcs
	$\mathcal{O}_{2,4}$	4	6	1278	opposite edges
	$\mathcal{O}_{2,6}$	6	12	$12u$	
$\mathcal{O}_3$	$\mathcal{O}_{3,2}$	2	4	18	SD (space diagonal)
	$\mathcal{O}_{3,3}$	3	24	128	SD + point
	$\mathcal{O}_{3,4}$	4	24	1238	SD + 3-arc
	$\mathcal{O}_{3,4'}$	4	24	1248	SD + 2-arc
	$\mathcal{O}_{3,5}$	5	24	$123u$	
	$\mathcal{O}_{3,7}$	7	8	$1u$	
$\mathcal{O}_4$	$\mathcal{O}_{4,3}$	3	8	135	"tri-diagonal"
	$\mathcal{O}_{4,4}$	4	8	1246	"claw"
	$\mathcal{O}_{4,4'}$	4	2	1357	tetrahedron
	$\mathcal{O}_{4,5}$	5	24	$178u$	
	$\mathcal{O}_{4,6}$	6	12	$13u$	
$\mathcal{O}_5$	$\mathcal{O}_{5,1}$	1	8	1	vertices
$= \mathcal{S}_{1,1,1}(2)$	$\mathcal{O}_{5,2}$	2	12	12	edge midpoints
	$\mathcal{O}_{5,4}$	4	6	1234	face centres
	$\mathcal{O}_{5,8}$	8	1	u	cube centre

This information will be of help when, in Section 5, we consider $\mathcal{G}_\mathcal{S}$-invariant polynomials. In the table, $p_{i,w}$ denotes a representative point on the $\mathcal{G}_\mathcal{B}$-orbit $\mathcal{O}_{i,w}$ which consists of those points of $\mathcal{O}_i$ having weight w with respect to the basis $\mathcal{B}$. (In the entries for $p_{i,w}$ we have used shorthand notation, as after equation (29).) The 'mnemonic' entries in the final column refer to the '8-cube' of the vectors of the basis $\mathcal{B}$. Observe that the siblings of $\mathcal{S} = \mathcal{O}_5 = \mathcal{O}_{5,1} \cup \mathcal{O}_{5,2} \cup \mathcal{O}_{5,4} \cup \mathcal{O}_{5,8}$ are

$$\mathcal{S}' = W(\mathcal{S}) = \mathcal{O}_{4,3}^{\text{even}} \cup \mathcal{O}_{4,4}^{\text{odd}} \cup \mathcal{O}_{4,5}^{\text{odd}} \cup \mathcal{O}_{4,6}^{\text{even}} \cup \mathcal{O}_{4,4'}^{\text{odd}}$$
$$\mathcal{S}'' = W^2(\mathcal{S}) = \mathcal{O}_{4,3}^{\text{odd}} \cup \mathcal{O}_{4,4}^{\text{even}} \cup \mathcal{O}_{4,5}^{\text{even}} \cup \mathcal{O}_{4,6}^{\text{odd}} \cup \mathcal{O}_{4,4'}^{\text{even}}. \tag{33}$$

Here a point $p \in \mathcal{O}_4$ is classed as 'even' if its expression in terms of the basis $\mathcal{B}$ uses more vectors e_i with $i \in \{2, 4, 6, 8\}$ than with $i \in \{1, 3, 5, 7\}$; otherwise it is classed as 'odd'. Thus if $p = e_1 + e_2 \in \mathcal{O}_{5,2}$ then $Wp \in \mathcal{O}_{4,5}^{\text{odd}}$ since $Wp = e_1 + e_3 + e_4 + e_5 + e_6$.

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[1]The entries in this table are in accord with those reported in [4, Table 5.1]

5 Invariant polynomials

5.1 The $\mathcal{G}_{\mathcal{B}}$-invariant polynomials of degree ≤ 4

The permutational action of $\mathcal{G}_{\mathcal{B}}$ on the eight vectors $e_i \in \mathcal{B}$ gives rise to a corresponding permutational action of $\mathcal{G}_{\mathcal{B}}$ on the eight coordinates x_i. Consequently, by appeal to Table 1, the $\mathcal{G}_{\mathcal{B}}$-invariant polynomials $P(x_1, x_2, \ldots, x_8)$ of homogeneous degree $d \leq 4$ are as follows. For example the three $\mathcal{G}_{\mathcal{B}}$-invariant polynomials P_3, P_3', P_3'' of degree 3 arise from the coordinate orbits corresponding to the three orbits $\mathcal{O}_{2,3}, \mathcal{O}_{4,3}, \mathcal{O}_{3,3}$ of points in Table 1 which have weight 3. (*Note:* when dealing with polynomials, we use $i, ij, ijk, ijkl, \ldots$ as shorthand for $x_i, x_i x_j, x_i x_j x_k, x_i x_j x_k x_l, \ldots$.)

(i) $d = 1$

$$P_1 := \sum_{i=1}^{8} x_i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8. \tag{34}$$

(ii) $d = 2$

$$
\begin{aligned}
P_2 &:= 12 + 14 + 16 + 23 + 25 + 34 + 38 + 47 + 56 + 58 + 67 + 78 \\
P_2' &:= 13 + 15 + 17 + 35 + 37 + 57 + 24 + 26 + 28 + 46 + 48 + 68 \\
P_2'' &:= 18 + 27 + 36 + 45.
\end{aligned}
\tag{35}
$$

(iii) $d = 3$ (Here $\mathfrak{I}$ denotes the index set $\{1, 2, 3, 4, 5, 6, 7, 8\}$)

$$
\begin{aligned}
P_3 :=\;& 123 + 124 + 134 + 234 + 125 + 126 + 156 + 256 \\
&+ 146 + 147 + 167 + 467 + 235 + 238 + 258 + 358 \\
&+ 347 + 348 + 378 + 478 + 567 + 568 + 578 + 678 \\
P_3' :=\;& 135 + 137 + 157 + 357 + 246 + 248 + 268 + 468 \\
P_3'' :=\;& \sum_{i \in \mathfrak{I}\backslash\{1,8\}} 18i + \sum_{i \in \mathfrak{I}\backslash\{2,7\}} 27i + \sum_{i \in \mathfrak{I}\backslash\{3,6\}} 36i + \sum_{i \in \mathfrak{I}\backslash\{4,5\}} 45i.
\end{aligned}
\tag{36}
$$

(iv) $d = 4$

$$
\begin{aligned}
P_4 =\;& 1234 + 1256 + 1467 + 2358 + 3478 + 5678 \\
P_4' =\;& 1278 + 1368 + 1458 + 2367 + 2457 + 3456 \\
P_4'' =\;& 1246 + 1235 + 1347 + 1567 + 2348 + 2568 + 3578 + 4678 \\
P_4''' =\;& 1357 + 2468 \\
P_4^{\text{iv}} =\;& 1238 + 1258 + 1348 + 1478 + 1568 + 1678 \\
&+ 18 \text{ terms using the other 3 SDs } 27, 36, 45 \\
P_4^{\text{v}} :=\;& 1248 + 1268 + 1468 + 1358 + 1378 + 1578 \\
&+ 1237 + 1257 + 2357 + 2467 + 2478 + 2678 \\
&+ 2346 + 2368 + 3468 + 1356 + 1367 + 3567 \\
&+ 1345 + 1457 + 3457 + 2456 + 2458 + 4568.
\end{aligned}
\tag{37}
$$

Consequently we have proved the following lemma.

Lemma 9 *The most general $\mathcal{G}_\mathcal{B}$-invariant polynomial of degree ≤ 4 is, for some choice of $c_1, c_2, \ldots, c_4^{\mathrm{v}} \in \mathrm{GF}(2)$, of the form*

$$Q = c_1 P_1 + c_2 P_2 + c_2' P_2' + c_2'' P_2'' + c_3 P_3 + c_3' P_3' + c_3'' P_3''$$
$$+ c_4 P_4 + c_4' P_4' + c_4'' P_4'' + c_4''' P_4''' + c_4^{\mathrm{iv}} P_4^{\mathrm{iv}} + c_4^{\mathrm{v}} P_4^{\mathrm{v}}. \tag{38}$$

5.2 The $\mathcal{G}_\mathcal{S}$-invariant polynomial functions of degree < 8

Recall from Section 1 that, in a given coordinate system, point-sets ψ in $\mathrm{PG}(7,2)$ are in bijective correspondence with polynomial functions Q of the form $Q(x) = \sum x_{i_1} \cdots x_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq 8$. Those Q which are invariant under the coordinate transformations arising from the elements of $\mathcal{G}_\mathcal{S}$ thereby correspond to point-sets ψ_Q which are fixed under the action of $\mathcal{G}_\mathcal{S}$. So if Q is $\mathcal{G}_\mathcal{S}$-invariant the point-set $\psi_Q \subset \mathrm{PG}(7,2)$ with equation $Q(x) = 0$ must be a union of some of the five $\mathcal{G}_\mathcal{S}$-orbits $\mathcal{O}_i$. We now seek all $\mathcal{G}_\mathcal{S}$-invariant polynomial functions $Q(x), x \in V_8$, having (reduced) degree $\deg Q < 8$. But, as noted in Section 1, if $\deg Q < 8$ then $|\psi_Q|$ must be odd, and since $\mathcal{O}_5$ is the only orbit whose length is odd it follows that for some subset α of $\{1, 2, 3, 4\}$ we must have

$$\psi_Q = \mathcal{O}_5 \cup_{i \in \alpha} \mathcal{O}_i. \tag{39}$$

Consequently — leaving aside the zero polynomial arising from the choice $\alpha = \{1, 2, 3, 4\}$, and so $\psi_Q = \mathrm{PG}(7,2)$ — *there are precisely fifteen $\mathcal{G}_\mathcal{S}$-invariant polynomial functions Q of degree < 8. In fact, as we now proceed to prove, these fifteen $G_\mathcal{S}$-invariant polynomials consist of one quadratic, six quartic and eight sextic polynomials.*

5.2.1 The $\mathcal{G}_\mathcal{S}$-invariant hyperbolic quadric $\mathcal{H}_7$

The $\mathcal{G}_\mathcal{S}$-invariant tetrad $\mathcal{L}_4 = \{L_a, L_b, L_c, L_d\}$ of lines (29) gives rise to a $\mathcal{G}_\mathcal{S}$-invariant set $\mathcal{U}_4 = \{U_h\}_{h \in \{a,b,c,d\}}$ of four 5-flats, where U_h denotes the span of the three lines $\mathcal{L}_4 \setminus L_h$. If $P_h(x) = 0$ is the quadratic (Lemma 2) equation of U_h then $Q_2 := P_a + P_b + P_c + P_d$ will be a $\mathcal{G}_\mathcal{S}$-invariant polynomial of degree ≤ 2, and so, from Lemma 9, of the form

$$Q_2 = c_1 P_1 + c_2 P_2 + c_2' P_2' + c_2'' P_2''. \tag{40}$$

Consider the subset $\mathcal{P}_{81}$ consisting of those $3^4 = 81$ points $p \in \mathrm{PG}(7,2)$ of the form $p = \sum_h p_h$, $0 \neq p_h \in L_h$, which belong to none of the U_h, its complement $(\mathcal{P}_{81})^{\mathrm{c}}$ thus being the set of $255 - 81 = 174$ points which belong to the union of the U_h. By Theorem 7 we see that either $\mathcal{P}_{81} = \mathcal{O}_2 \cup \mathcal{O}_5$ or $\mathcal{P}_{81} = \mathcal{O}_4 \cup \mathcal{O}_5$. Now the point $e_1 + e_3$, which from Table 1 is on the orbit $\mathcal{O}_2$, is from (29) seen to lie in $\langle L_a, L_c \rangle$. Consequently $\mathcal{P}_{81} = \mathcal{O}_4 \cup \mathcal{O}_5$ and $(\mathcal{P}_{81})^{\mathrm{c}} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$. It follows that $Q_2(p) = 4 = 0$ for all $p \in \mathcal{O}_4 \cup \mathcal{O}_5$. Further $Q_2(p) = 2 = 0$ for all $p \in \mathcal{O}_2$; for example, since $e_1 + e_3 \in \langle L_a, L_c \rangle$, the point $e_1 + e_3 \in \mathcal{O}_2$ lies on two of the U_h, namely U_b and U_d. On the other hand each point $p \in \mathcal{O}_1$ belongs to

three of the U_h and so satisfies $Q_2(p) = 1$, while each point $p \in \mathcal{O}_3$ belongs to just one of the U_h and so satisfies $Q_2(p) = 3 = 1$. It follows that

$$\psi_{Q_2} = \mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5. \tag{41}$$

In particular, upon consulting Table 1, it follows from (41) that $Q_2(e_1) = 0$, $Q_2(e_1 + e_2) = 0$, and $Q_2(e_1 + e_3) = 0$, whence in (40) $c_1 = c_2 = c_2' = 0$. Hence $Q_2 = P_2''$. In fact, upon appeal to a later result — see theorem 17 below — in $\mathrm{PG}(7,2)$ no polynomial of degree 2 other than P_2'' is $\mathcal{G}_{\mathcal{S}}$-invariant. Consequently the following theorem holds.

Theorem 10 *In* $\mathrm{PG}(7,2)$ *the quadratic polynomial*

$$Q_2 = x_1 x_8 + x_2 x_7 + x_3 x_6 + x_4 x_5 \tag{42}$$

is the unique polynomial of degree 2 which is $\mathcal{G}_{\mathcal{S}}$-*invariant. Consequently the Segre variety* $\mathcal{O}_5 = \mathcal{S} := \mathcal{S}_{1,1,1}(2)$ *is a subset of the* $\mathcal{G}_{\mathcal{S}}$-*invariant hyperbolic quadric* $\mathcal{H}_7 = \mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5 \subset \mathrm{PG}(7,2)$ *having equation* $Q_2(x) = 0$.

Remark 11 *The Segre variety* $\mathcal{S}_{1,1,1}(2)$ *thus determines a particular orthogonal group* $\mathrm{GO}_8^+(2) \cong \mathrm{O}_8^+(2).2$ *in* $\mathrm{GL}(8,2)$, *and hence determines that particular* $\mathrm{Sp}(8,2)$ *subgroup of* $\mathrm{GL}(8,2)$ *which leaves invariant the non-degenerate symplectic form* $B(x,y) = Q_2(x+y) + Q_2(x) + Q_2(y)$. *In fact this particular* $\mathcal{G}_{\mathcal{S}}$-*invariant symplectic form can be arrived at much more simply (cf. [7, Section 2]). For since we are working over* $\mathrm{GF}(2)$ *the group* $\mathrm{GL}(V_2)$ *coincides with* $\mathrm{Sp}(V_2)$, *the space* $V_2 = V(2,2)$ *possessing a unique nonzero symplectic form. So the tensor product space* $V_8 = V_2 \otimes V_2 \otimes V_2$ *thereby inherits a particular* $\mathrm{Sp}(8,2)$ *geometry.*

Remark 12 *The orders of the simple group* $\mathrm{O}_8^+(2)$ *and of* $\mathrm{GO}_8^+(2)$, *are*

$$|\mathrm{O}_8^+(2)| = 2^{12}.3^5.5^2.7 = 174,182,400 \tag{43}$$

$$|\mathrm{GO}_8^+(2)| = 2^{13}.3^5.5^2.7 = 348,364,800 \tag{44}$$

— see [1, p. [85]], If K *denotes the involution* $e_1 \rightleftarrows e_8$, *and if* K' *denotes the involution* $e_1 \rightleftarrows e_8$, $e_2 \rightleftarrows e_7$, *we checked, using Magma [2], that*

$$\langle \mathcal{G}_{\mathcal{S}}, K \rangle = \mathrm{GO}_8^+(2), \tag{45}$$

$$\langle \mathcal{G}_{\mathcal{S}}, K' \rangle = \mathrm{O}_8^+(2). \tag{46}$$

5.2.2 The six $\mathcal{G}_{\mathcal{S}}$-invariant quartics

The ambient space Y_i^r of σ_i^r, see (6), is a 3-flat and so has a quartic equation, say $P_i^r(x) = 0$. Consider the polynomial

$$Q_4 := \sum_{i=0}^2 \sum_{r=1}^3 P_i^r. \tag{47}$$

Each point $x \in \mathcal{S} = \mathcal{O}_5$ lies on precisely three of the nine 3-flats Y_i^r, and so $Q_4(x) = 6 = 0$. Since each of the six points $x \in Y_i^r \setminus \sigma_i^r$ lies in Y_j^s only for $s = r$

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and $j = i$, we also have $Q_4(x) = 8 = 0$ for $x \in \mathcal{O}_2$. Further $Q_4(x) = 9 = 1$ for $x \notin \mathcal{O}_2 \cup \mathcal{O}_5$. So the 81-set $\mathcal{O}_2 \cup \mathcal{O}_5$ has equation $Q_4(x) = 0$ where $\deg Q_4 \le 4$. To prove that $\deg Q_4 = 4$ we may invoke part (ii) of lemma 1. For, using the basis $\mathcal{B}$, the 3-flat $X := \langle e_1, e_2, e_7, e_8 \rangle$ is seen to meet $\psi_{Q_4} = \mathcal{O}_2 \cup \mathcal{O}_5$ in an even number of points, namely the six points on the two generators $\langle e_1, e_2 \rangle$ and $\langle e_7, e_8 \rangle$ together with the two points $e_1 + e_7$ and $e_2 + e_8$. Consequently we have proved part (i) of the following theorem, the $\mathcal{G}_\mathcal{S}$ invariance following since $\mathcal{S}$ determines uniquely the nine $\mathcal{S}_{1.1}$ varieties in (6).

Theorem 13 *(i) The 81-set $\mathcal{O}_2 \cup \mathcal{O}_5$ is a quartic hypersurface $Q_4(x) = 0$ in $\mathrm{PG}(7, 2)$ which is invariant under the action of $\mathcal{G}_\mathcal{S}$.*
(ii) Using the basis (9) the quartic polynomial Q_4 has the explicit form

$$Q_4 = P_2'' + P_3' + P_4''' + P_4^v. \tag{48}$$

Proof. (ii) Since, from part (i), $\deg Q_4 = 4$, we know that Q_4 is as in (38). To determine the coefficients $c_1, c_2, \ldots, c_4^v$ in (38) we simply use $Q_4(x) = 0$ for $x \in \mathcal{O}_2 \cup \mathcal{O}_5$, and $Q_4(x) = 1$ for $x \in \mathcal{O}_1 \cup \mathcal{O}_3 \cup \mathcal{O}_4$, confining our attention to those points x in Table 1 having weight ≤ 4. From $Q_4(e_1) = 0$, $Q_4(e_1 + e_2) = 0$, $Q_4(e_1 + e_3) = 0$ and $Q_4(e_1 + e_8) = 1$, it follows that $c_1 = c_2 = c_2' = 0$ and $c_2'' = 1$. So $Q_4 = P_2'' +$ terms of degree > 2. From $Q_4(123) = 0$, $Q_4(135) = 1$ and $Q_4(128) = 1$, and since $P_2''(123) = 0, P_2''(135) = 0$ and $P_2''(128) = 1$, it follows that $c_3 = 0, c_3' = 1$ and $c_3'' = 0$. So

$$Q_4 = P_2'' + P_3' + c_4 P_4 + c_4' P_4' + c_4'' P_4'' + c_4''' P_4''' + c_4^{iv} P_4^{iv} + c_4^v P_4^v. \tag{49}$$

Next we consider in turn the six points $1234, 1278, 1246, 1357, 1238, 1248$ in Table 1 which have weight 4. Each of the points $x = 1234 \in \mathcal{O}_5$ and $x = 1278 \in \mathcal{O}_2$ satisfies $P_2''(x) = P_3'(x) = 0$, and so, from $Q_4(x) = 0$ for $x \in \mathcal{O}_5 \cup \mathcal{O}_2$, we obtain $c_4 = c_4' = 0$. The point $x = 1246 \in \mathcal{O}_4$ satisfies $P_2''(x) = 0$ and $P_3'(x) = 1$, and so from $Q_4(x) = 1$ we obtain $c_4'' = 0$. The point $x = 1357 \in \mathcal{O}_4$ satisfies $P_2''(x) = 0$ and $P_3'(x) = 4 = 0$, and so from $Q_4(x) = 1$ we obtain $c_4''' = 1$. The point $x = 1238 \in \mathcal{O}_3$ satisfies $P_2''(x) = 1$ and $P_3'(x) = 0$, and so from $Q_4(x) = 1$ we obtain $c_4^{iv} = 0$. Finally the point $x = 1248 \in \mathcal{O}_3$ satisfies $P_2''(x) = 1$ and $P_3'(x) = 1$, and so from $Q_4(x) = 1$ we obtain $c_4^v = 1$. $\blacksquare$

Remark 14 *Knowing from part (i) of the theorem that Q_4 does not contain terms of degree > 4 there was no need in the proof of part (ii) to consider points of weight greater than 4; such points will necessarily satisfy the conditions $Q_4(x) = 0$ for $x \in \mathcal{O}_2 \cup \mathcal{O}_5$, and $Q_4(x) = 1$ for $x \in \mathcal{O}_1 \cup \mathcal{O}_3 \cup \mathcal{O}_4$.*
Of course the quartic polynomial Q_4 could alternatively have been obtained directly from (47) by feeding in the explicit coordinate forms of the nine P_i^r.

Remark 15 *The polynomial Q_4 in (47) arose from the nine 3-flats Y_i^r. If instead we consider the corresponding polynomial Q arising from the twenty-seven 3-flats Z_{ijk}, see equation (7), we quickly find that $\psi_Q = \mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5$. Consequently Q is not a quartic but is in fact the quadratic Q_2 of Theorem 10.*

Theorem 16 *The 189-set $\mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5$ is a $\mathcal{G}_\mathcal{S}$-invariant quartic hypersurface in* PG$(7,2)$ *with equation $Q_4'(x) = 0$ where*

$$Q_4' = P_2' + P_3' + P_3'' + P_4'. \tag{50}$$

Proof. The $\mathcal{G}_\mathcal{S}$-invariant tetrad $\mathcal{L}_4 = \{L_a, L_b, L_c, L_d\}$ of lines (29) gives rise to a $\mathcal{G}_\mathcal{S}$-invariant set $\{U_{ab}, U_{ac}, U_{ad}, U_{bc}, U_{bd}, U_{cd}\}$ of six 3-flats, where $U_{hk} := \langle L_h, L_k \rangle$. If $P_{hk}(x) = 0$ is the quartic (Lemma 2) equation of U_{hk} then the sum Q_4' of the six P_{hk} will be a $\mathcal{G}_\mathcal{S}$-invariant polynomial of degree ≤ 4. Recalling that the points external to $\mathcal{O}_1 = \mathcal{P}(\mathcal{L}_4)$ on the bisecants of $\mathcal{O}_1$ form the orbit $\mathcal{O}_2$, we see that $Q_4'(x) = 6 = 0$ for $x \in \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5$, while $Q_4'(x) = 3 = 1$ if $x \in \mathcal{O}_1$ and $Q_4'(x) = 5 = 1$ if $x \in \mathcal{O}_2$. So $\psi_{Q_4'} = \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5$ as claimed.

Proceeding now on exactly the same lines as in the proof of part (ii) of the preceding theorem, we quickly arrive at the explicit form (50) for Q_4' (showing in particular that Q_4' indeed has degree 4). ■

Theorem 17 *There exist precisely seven $\mathcal{G}_\mathcal{S}$-invariant polynomials Q of degree ≤ 4 in the coordinates $x_1, x_2, \ldots, x_8$, as displayed in the following table:*

		$Q(p)$ if $p \in$						
Q	$\deg Q$	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$	$\mathcal{O}_5$	ψ_Q	$|\psi_Q|$
Q_2	2	1	0	1	0	0	$\mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5$	135
Q_4	4	1	0	1	1	0	$\mathcal{O}_2 \cup \mathcal{O}_5$	81
Q_4'	4	1	1	0	0	0	$\mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5$	189
$Q_4 + Q_4'$	4	0	1	1	1	0	$\mathcal{O}_1 \cup \mathcal{O}_5$	39
$Q_2 + Q_4$	4	0	0	0	1	0	$\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$	201
$Q_2 + Q_4'$	4	0	1	1	0	0	$\mathcal{O}_1 \cup \mathcal{O}_4 \cup \mathcal{O}_5$	93
$Q_2 + Q_4 + Q_4'$	4	1	1	0	1	0	$\mathcal{O}_3 \cup \mathcal{O}_5$	135

$$\tag{51}$$

Proof. We have already met Q_2, Q_4 and Q_4'; linear combinations of these three polynomials yield the further four polynomials displayed in the last four rows of the table. To prove that there are no $\mathcal{G}_\mathcal{S}$-invariant polynomials of degree ≤ 4 other than the seven in the table, recall, see after equation (39), that there exist just fifteen $\mathcal{G}_\mathcal{S}$-invariant polynomials of degree < 8. But looking ahead to Section 5.2.3, the remaining eight invariant polynomials are all of degree 6. ■

5.2.3 The eight $\mathcal{G}_\mathcal{S}$-invariant sextics

Theorem 18 *The Segre variety $\mathcal{S}_{1,1,1}(2)$ has polynomial degree 6.*

Proof. From (3) and (5) observe that $\mathcal{S}$ is the union of the nine mutually disjoint lines L_{ij}^3, $i, j \in \{0, 1, 2\}$, where by lemma 2 each line L_{ij}^3 has equation $P_{ij}(x) = 0$ where $\deg P_{ij} = 6$. Consider the polynomial $Q_6 := \sum_{i=0}^{2}\sum_{j=0}^{2} P_{ij}$. Each point $x \in \mathcal{S}$ lies on precisely one of the nine lines L_{ij}^3, and so $Q_6(x) = 8 = 0$, while if $x \in$ PG$(7,2)$ is exterior to $\mathcal{S}_{1,1,1}(2)$ then it lies on none of the nine lines L_{ij}, and so $Q_6(x) = 9 = 1$. So $\psi_{Q_6} = \mathcal{S} = \mathcal{O}_5$. (Of course in this proof we could

equally well have used instead the nine lines L_{ij}^1 or the nine lines L_{ij}^2.) From its definition the polynomial Q_6 has degree ≤ 6. To prove that $\deg Q_6 = 6$, consider the 5-flat $X := \langle e_1, e_3, e_5, e_7, e_2 + e_4 + e_6 \rangle$ and observe that X meets $\mathcal{S}$ in an even number of points, namely the 4 points $\{e_1, e_3, e_5, e_7\}$. Hence from lemma 1 it follows that $\deg Q_6 = 6$. ∎

The fact that Q_6 has degree 6 can also be shown by an explicit calculation, as in the proof of the next theorem. For this theorem, in addition to the polynomials defined in Section 5.1, we also need the polynomials P_5 and P_6 defined by

$$P_5 = x_1 x_3 x_5 x_7 (x_2 + x_4 + x_6 + x_8) + x_2 x_4 x_6 x_8 (x_1 + x_3 + x_5 + x_7),$$
$$P_6 = x_1 x_2 x_3 x_6 x_7 x_8 + x_1 x_2 x_4 x_5 x_7 x_8 + x_1 x_3 x_4 x_5 x_6 x_8 + x_2 x_3 x_4 x_5 x_6 x_7. \quad (52)$$

Observe that the polynomial P_5 can also be expressed as a product: $P_5 = P_1 P_4'''$.

Theorem 19 *The Segre variety $\mathcal{S}_{1,1,1}(2)$ is a hypersurface in $\mathrm{PG}(7,2)$ which has the sextic equation $Q_6(x) = 0$, where*

$$Q_6 = P_2' + P_2'' + P_3'' + P_4' + P_4''' + P_4^v + P_5 + P_6. \quad (53)$$

Proof. With the aid Magma, see [2], these explicit coordinate forms for $Q_6 = \sum_{i=0}^2 \sum_{j=0}^2 P_{ij}$ were obtained by use of Lemma 2. ∎

By adding Q_6 to the seven polynomials in (51) we obtain a further seven invariant polynomials of degree 6. Since we have previously obtained six invariant polynomials of degree 4 and one of degree 2, we have therefore obtained the full quota, see after equation (39), of fifteen $\mathcal{G}_\mathcal{S}$-invariant polynomials of degree < 8.

Example 20 *Consider the sextic polynomial $Q_6' = Q_6 + Q_4 + Q_4'$, which has the particularly simple form $Q_6' = P_5 + P_6$. Since $\psi_{Q_6} = \mathcal{O}_5$ and $\psi_{Q_4 + Q_4'} = \mathcal{O}_1 \cup \mathcal{O}_5$ it follows that $\psi_{Q_6'} = (\mathcal{O}_1)^c = \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5$.*

Afterthought. Consider the sextic polynomial $Q_2 Q_4'$. Observe from the table (51) that $(Q_2 Q_4')(x) \neq 0$ only for $x \in \mathcal{O}_1$, leading to an alternative derivation of the sextic equation for the Segre variety $\mathcal{S}_{1,1,1}(2) = \mathcal{O}_5$, namely in the form

$$Q_2(x) Q_4'(x) + Q_4(x) + Q_4'(x) = 0,$$

thus avoiding the computation involved in the previous proof of Theorem 19. (However we still need to sort out the $4 \times (12 + 8 + 24 + 6) = 200$ terms arising from the product of $Q_2(x)$ with $Q_4'(x)$!)

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