Automorphisms of a Clifford-like parallelism

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Abstract

In this paper we focus on the description of the automorphism group $\Gamma_{\|}$ of a Clifford-like parallelism $\|$ on a 3-dimensional projective double space $(\mathbb{P}(H_F), \|_\ell, \|_r)$ over a quaternion skew field H (with centre a field F of any characteristic). We compare $\Gamma_{\|}$ with the automorphism group Γ_ℓ of the left parallelism $\|_\ell$, which is strictly related to $\mathrm{Aut}(H)$. We build up and discuss several examples showing that over certain quaternion skew fields it is possible to choose $\|$ in such a way that $\Gamma_{\|}$ is either properly contained in Γ_ℓ or coincides with Γ_ℓ even though $\| \neq \|_\ell$.

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1 Introduction

As a far-reaching generalisation of the situation in 3-dimensional real elliptic geometry, H. Karzel, H.-J. Kroll and K. Sörensen coined the notion of a *projective double space*, that is, a projective space $\mathbb{P}$ together with a *left parallelism* $\|_\ell$ and a *right parallelism* $\|_r$ on the line set of $\mathbb{P}$ such that—loosely speaking—all "mixed parallelograms" are closed [21], [22]. It is common to address the given parallelisms as the *Clifford parallelisms* of the projective double space. We shall not be concerned with the particular case where $\|_\ell = \|_r$, which can only happen over a ground field of characteristic two. All other projective double spaces are three-dimensional and they can be obtained algebraically in terms of a quaternion skew field H with centre F by considering the projective space $\mathbb{P}(H_F)$ on the vector space H over the field F and defining $\|_\ell$ and $\|_r$ via left and right multiplication

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in H. (See [6], [10], [11], [20, pp. 75–76] and the references given there.) In their work [6] about generalisations of Clifford parallelism, A. Blunck, S. Pianta and S. Pasotti pointed out that a projective double space $(\mathbb{P}(H_F), \|_\ell, \|_r)$ may be equipped in a natural way with so-called *Clifford-like* parallelisms, namely parallelisms for which each equivalence class is either a class of left parallel lines or a class of right parallel lines. The exposition of this topic in [12] serves as major basis for this article.

Our main objective is to describe the group of all collineations that preserve a given Clifford-like parallelism $\|$ of a projective double space $(\mathbb{P}(H_F), \|_\ell, \|_r)$. Since we work most of the time in terms of vector spaces, we shall consider instead the underlying group $\Gamma_\|$ of all $\|$-preserving semilinear transformations of the vector space H_F, which we call the *automorphism group* of the given parallelism. In a first step we focus on the *linear automorphisms* of $\|$. We establish in Theorem 3.5 that the group of all these linear automorphism does not depend on the choice of $\|$ among all Clifford-like parallelisms of $(\mathbb{P}(H_F), \|_\ell, \|_r)$. Since $\|_\ell$ and $\|_r$ are also Clifford-like, it is impossible to characterise Clifford parallelism in terms of its linear automorphism group in our general setting of an arbitrary quaternion skew field. On the other hand, there are projective double spaces in which there are no Clifford-like parallelisms other than its Clifford parallelisms. This happens, for instance, if H is chosen to be the skew field of Hamilton's quaternions over the real numbers. (It is worth noting that D. Betten, R. Löwen and R. Riesinger characterised Clifford parallelism among the topological parallelisms of the 3-dimensional real projective space by its (linear) automorphism group in [2], [4], [25], [26], [27].) The next step is to consider the (full) automorphism group $\Gamma_\|$. Here the situation is more intricate, since in general the group depends on the underlying quaternion skew field as well as the choice of $\|$. We know from previous work of S. Pianta and E. Zizioli (see [29] and [30]) that the left and right Clifford parallelism of $(\mathbb{P}(H_F), \|_\ell, \|_r)$ share the same automorphism group, say Γ_ℓ. According to Corollary 3.7, Γ_ℓ cannot be a proper subgroup of $\Gamma_\|$. In Section 4, we construct a series of examples showing that over certain quaternion skew fields it is possible to choose $\|$ in such a way that $\Gamma_\|$ is either properly contained in Γ_ℓ or coincides with Γ_ℓ even though $\| \neq \|_\ell, \|_r$.

One open problem remains: Is there a projective double space $(\mathbb{P}(H_F), \|_\ell, \|_r)$ that admits a Clifford-like parallelism $\|$ for which none of the groups $\Gamma_\|$ and Γ_ℓ is contained in the other one?

2 Basic notions and results

Let $\mathbb{P}$ be a projective space with line set $\mathcal{L}$. We recall that a *parallelism* on $\mathbb{P}$ is an equivalence relation on $\mathcal{L}$ such that each point of $\mathbb{P}$ is incident with precisely one line from each equivalence class. We usually denote a parallelism by the symbol $\parallel$. For each line $M \in \mathcal{L}$ we then write $\mathcal{S}(M)$ for the equivalence class of M, which is also addressed as the *parallel class* of M. Any such parallel class is a spread (of lines) of $\mathbb{P}$, that is, a partition of the point set of $\mathbb{P}$ by lines. When dealing with several parallelisms at the same time we add some subscript or superscript to the symbols $\parallel$ and $\mathcal{S}$. The seminal book [18] covers the literature about parallelisms up to the year 2010. For the state of the art, various applications, connections with other areas of geometry and historical remarks, we refer also to [1], [3], [7], [13], [20], [26], [32] and the references therein.

The following simple observation, which seems to be part of the folklore, will be useful.

Lemma 2.1. *Let $\mathbb{P}$ and $\mathbb{P}'$ be projective spaces with parallelisms $\parallel$ and $\parallel'$, respectively. Suppose that κ is a collineation of $\mathbb{P}$ to $\mathbb{P}'$ such that any two $\parallel$-parallel lines go over to $\parallel'$-parallel lines. Then κ takes any $\parallel$-class to a $\parallel'$-class.*

Proof. In $\mathbb{P}'$, the κ-image of any $\parallel$-class is a spread that is contained in a spread, namely some $\parallel'$-class. Any proper subset of a spread fails to be a spread, whence the assertion follows. $\square$

Let H be a quaternion skew field with centre F; see, for example, [8, pp. 103–105] or [31, pp. 46–48]. If E is a subfield of H then H is a left vector space and a right vector space over E. These spaces are written as ${}_EH$ and H_E, respectively. We do not distinguish between ${}_EH$ and H_E whenever $E \subseteq F$. Given any $x \in H$ we denote by $\overline{x}$ the *conjugate quaternion* of x. Then $x = \overline{x}$ holds precisely when $x \in F$. We write $\operatorname{tr}(x) = x + \overline{x} \in F$ for the *trace* of x and $N(x) = \overline{x}x = x\overline{x} \in F$ for the *norm* of x. We have the identity

$$x^2 - \operatorname{tr}(x)x + N(x) = 0. \tag{1}$$

In H_F, the symmetric bilinear form associated to the quadratic form $N\colon H \to F$ is

$$\langle\cdot,\cdot\rangle\colon H \times H \to F\colon (x, y) \mapsto \langle x, y\rangle = \operatorname{tr}(x\overline{y}) = x\overline{y} + y\overline{x}. \tag{2}$$

Let α be an automorphism of the quaternion skew field H. Then $\alpha(F) = F$ and so α is a semilinear transformation of the vector space H_F with $\alpha_{|F}\colon F \to F$ being its accompanying automorphism. Furthermore,

$$\forall\, x \in H\colon\ \operatorname{tr}(\alpha(x)) = \alpha(\operatorname{tr}(x)),\ N(\alpha(x)) = \alpha(N(x)),\ \overline{\alpha(x)} = \alpha(\overline{x}). \tag{3}$$

This is immediate for all $x \in F$, since here $\operatorname{tr}(x) = 2x$, $N(x) = x^2$, and $\overline{x} = x$. For all $x \in H \setminus F$ the equations in (3) follow by applying α to (1) and by taking into account that $\alpha(x^2) = \alpha(x)^2$ can be written in a unique way as an F-linear combination of $\alpha(x)$ and 1.

The *projective space* $\mathbb{P}(H_F)$ is understood to be the set of all subspaces of H_F with *incidence* being symmetrised inclusion. We adopt the usual geometric terms: *points*, *lines* and *planes* are the subspaces of H_F with vector dimension one, two, and three, respectively. We write $\mathcal{L}(H_F)$ for the line set of $\mathbb{P}(H_F)$. The *left parallelism* $\parallel_\ell$ on $\mathcal{L}(H_F)$ is defined by letting $M_1 \parallel_\ell M_2$ precisely when there is a $g \in H^* := H \setminus \{0\}$ with $gM_1 = M_2$. The *right parallelism* $\parallel_r$ is defined in the same fashion via $M_1 g = M_2$. Then $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ is a *projective double space* with $\parallel_\ell$ and $\parallel_r$ being its *Clifford parallelisms* (see [21], [22], [20, pp. 75–76]).

A parallelism $\parallel$ of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ is *Clifford-like*, if each $\parallel$-class is a left or a right parallel class (see Def. 3.2. of [12] where the construction of Clifford-like parallelisms appears frequently in the more general framework of "blending"; this point of view will be disregarded here). Any Clifford-like parallelism $\parallel$ of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ admits the following explicit description:

Theorem 2.2 (see [12, Thm. 4.10])**.** *In $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$, let $\mathcal{A}(H_F) \subset \mathcal{L}(H_F)$ denote the star of lines with centre $F1$, let $\mathcal{F}$ be any subset of $\mathcal{A}(H_F)$, and define a relation $\parallel$ on $\mathcal{L}(H_F)$ by taking the left parallel classes of all lines in $\mathcal{F}$ and the right parallel classes of all lines in $\mathcal{A}(H_F) \setminus \mathcal{F}$. This will be an equivalence relation (and hence, a parallelism) if, and only if, the defining set $\mathcal{F}$ is invariant under the inner automorphisms of H.*

We note that —from an algebraic point of view— the lines from $\mathcal{A}(H_F)$ are precisely the maximal subfields of the quaternion skew field H.

Let $\parallel$ be any parallelism on $\mathbb{P}(H_F)$. We denote by $\Gamma_\parallel$ the set of all mappings from $\Gamma\mathrm{L}(H_F)$ that act on $\mathbb{P}(H_F)$ as $\parallel$-preserving collineations. By Lemma 2.1, $\Gamma_\parallel$ is a subgroup of $\Gamma\mathrm{L}(H_F)$ and we shall call it the *automorphism group* of the parallelism $\parallel$. Even though we are primarily interested in the group of all $\parallel$-preserving collineations of $\mathbb{P}(H_F)$, which is a subgroup of $\mathrm{P}\Gamma\mathrm{L}(H_F)$, we investigate instead the corresponding group $\Gamma_\parallel$. The straightforward task of rephrasing our findings about $\Gamma_\parallel$ in projective terms is usually left to the reader.

The Clifford parallelisms of the projective double space $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ give rise to automorphism groups $\Gamma_{\parallel_\ell} =: \Gamma_\ell$ and $\Gamma_{\parallel_r} =: \Gamma_r$. We recall from [29, p. 166] that

$$\Gamma_\ell = \Gamma_r. \tag{4}$$

Equation (4) is based on the following noteworthy geometric result. In $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$, the right (left) parallelism can be defined in terms of incidence, non-incidence and left (right) parallelism. See, for example, [20, pp. 75–76] or make use of the (much more general) findings in [15, §6], which are partly summarised in [14] and [16]. In order to describe the group Γ_ℓ more explicitly, we consider several other groups. First, the group of all *left translations* $\lambda_g \colon H \to H \colon x \mapsto gx$, $g \in H^*$, is precisely the group $\mathrm{GL}(H_H)$. The group $\mathrm{GL}(H_H)$

is contained in $\mathrm{GL}(H_F)$ and it acts regularly on H^*. Next, the automorphism group $\mathrm{Aut}(H)$ of the skew field H is a subgroup of $\Gamma\mathrm{L}(H_F)$. Finally, we write $\widetilde{H^*}$ for the group of all inner automorphisms $\tilde{h}\colon H \to H\colon x \mapsto h^{-1}xh$, $h \in H^*$, and so $\widetilde{H^*}$ is a subgroup of $\mathrm{GL}(H_F)$. According to [29, Thm. 1] and [30, Prop. 4.1 and 4.2][1],

$$\Gamma_\ell = \mathrm{GL}(H_H) \rtimes \mathrm{Aut}(H) = \Gamma\mathrm{L}(H_H). \tag{5}$$

By symmetry of 'left' and 'right', (5) implies $\Gamma_r = \mathrm{GL}({}_HH) \rtimes \mathrm{Aut}(H) = \Gamma\mathrm{L}({}_HH)$, where $\mathrm{GL}({}_HH)$ is the group of *right translations*. Note that $\Gamma\mathrm{L}(H_H) = \Gamma\mathrm{L}({}_HH)$. From this fact (4) follows once more and in an algebraic way. By virtue of the Skolem-Noether theorem [17, Thm. 4.9], the F-linear skew field automorphisms of H are precisely the inner automorphisms. We therefore obtain from (5) that

$$\Gamma_\ell \cap \mathrm{GL}(H_F) = \mathrm{GL}(H_H) \rtimes \widetilde{H^*}. \tag{6}$$

The subgroups of Γ_ℓ and $\Gamma_\ell \cap \mathrm{GL}(H_F)$ that stabilise $1 \in H$ are the groups $\mathrm{Aut}(H)$ and $\widetilde{H^*}$, respectively.

Remark 2.3. The natural homomorphism $\mathrm{GL}(H_F) \to \mathrm{PGL}(H_F)$ sends the group from (6) to the group of all $\parallel_\ell$-preserving projective collineations of $\mathbb{P}(H_F)$. This collineation group can be written as the *direct product* of two (isomorphic) subgroups, namely the image of the group of left translations $\mathrm{GL}(H_H)$ and the image of the group of right translations $\mathrm{GL}({}_HH)$ under the natural homomorphism.

If $\alpha\colon H \to H$ is an antiautomorphism of the quaternion skew field H, then $\alpha \in \Gamma\mathrm{L}(H_F)$ and α takes left (right) parallel lines to right (left) parallel lines. In particular, the conjugation $\overline{(\cdot)}\colon H \to H$ is an F-linear antiautomorphism of H. Therefore, the set

$$(\mathrm{GL}(H_H) \rtimes \mathrm{Aut}(H)) \circ \overline{(\cdot)} \tag{7}$$

comprises precisely those mappings in $\Gamma\mathrm{L}(H_F)$ that interchange the left with the right Clifford parallelism. The analogous subset of $\mathrm{GL}(H_F)$ is given by

$$(\mathrm{GL}(H_H) \rtimes \widetilde{H^*}) \circ \overline{(\cdot)}.$$

Alternative proofs of the previous results can be found in [5, Sect. 4].

3 Automorphisms

Throughout this section, we always assume $\parallel$ to be a Clifford-like parallelism of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ as described in Section 2. Our aim is to determine the group $\Gamma_\parallel$ of automorphisms of $\parallel$. In a first step we focus on the transformations appearing in (5) and (7).

[1]We wish to note here that Prop. 4.3 of [30] is not correct, since the group $\overleftarrow{K}$ from there in general is not a subgroup of $\mathrm{Aut}(H)$.

Proposition 3.1. *Let $\|$ be a Clifford-like parallelism of $(\mathbb{P}(H_F), \|_\ell, \|_r)$. Then the following assertions hold.*

(a) *An automorphism $\alpha \in \mathrm{Aut}(H)$ preserves $\|$ if, and only if, $\alpha(\mathcal{F}) = \mathcal{F}$.*

(b) *An antiautomorphism α of the quaternion skew field H preserves $\|$ if, and only if, $\alpha(\mathcal{F}) = \mathcal{A}(H_F) \setminus \mathcal{F}$.*

(c) *For all $h \in H^*$, the inner automorphism $\tilde{h}$ preserves $\|$.*

(d) *For all $g \in H^*$, the left translation λ_g preserves $\|$.*

(e) *If $\beta \in \mathrm{GL}(H_F)$ preserves $\|_\ell$, then β preserves also $\|$.*

Proof. (a) We read off from $\alpha(1) = 1$ that $\alpha(\mathcal{A}(H_F)) = \mathcal{A}(H_F)$ and from (5) that $\alpha \in \mathrm{Aut}(H) \subset \Gamma_\ell$. The assertion now is an immediate consequence of Theorem 2.2.

(b) The proof follows the lines of (a) taking into account that α interchanges the left with the right parallelism.

(c) [12, Thm. 4.10] establishes $\tilde{h}(\mathcal{F}) = \mathcal{F}$. Applying (a) we get $\tilde{h} \in \Gamma_\|$.

(d) Choose any $\|$-class, say $\mathcal{S}(L)$ with $L \in \mathcal{A}(H_F)$. In order to verify that $\lambda_g(\mathcal{S}(L))$ is also a $\|$-class, we first observe that (5) gives $\lambda_g \in \mathrm{GL}(H_H) \subset \Gamma_\ell$. Next, we distinguish two cases. If $L \in \mathcal{F}$, then, by Theorem 2.2, $\mathcal{S}(L) = \mathcal{S}_\ell(L)$ and so $\lambda_g(\mathcal{S}(L)) = \lambda_g(\mathcal{S}_\ell(L)) = \mathcal{S}_\ell(gL) = \mathcal{S}_\ell(L) = \mathcal{S}(L)$. If $L \in \mathcal{A}(H_F) \setminus \mathcal{F}$, then, by Theorem 2.2, $\mathcal{S}(L) = \mathcal{S}_r(L)$. Furthermore, (c) gives $gLg^{-1} \in \mathcal{A}(H_F) \setminus \mathcal{F}$. By virtue of these results and (4), we obtain $\lambda_g(\mathcal{S}(L)) = \lambda_g(\mathcal{S}_r(L)) = \mathcal{S}_r(gL) = \mathcal{S}_r(gLg^{-1}) = \mathcal{S}(gLg^{-1})$.

(e) By (6), there exist $g, h \in H^*$ such that $\beta = \lambda_g \circ \tilde{h}$. We established already in (d) and (c) that $\lambda_g, \tilde{h} \in \Gamma_\|$, which entails $\beta \in \Gamma_\|$. □

We proceed with a lemma that, apart from the quaternion formalism, follows easily from [28, Thm. 1.10, Thm 1.11]; those theorems are about spreads, their kernels and their corresponding translation planes. We follow instead the idea of proof used in [5, Thm. 4.3].

Lemma 3.2. *Let $L \in \mathcal{A}(H_F)$ and $\alpha \in \Gamma\mathrm{L}(H_F)$ be given such that $\alpha(1) = 1$ and such that α takes one of the two parallel classes $\mathcal{S}_\ell(L)$, $\mathcal{S}_r(L)$ to one of the two parallel classes $\mathcal{S}_\ell(\alpha(L))$, $\mathcal{S}_r(\alpha(L))$. Then*

$$\forall\, x \in H,\ z \in L\colon \left\{ \begin{array}{l} \alpha(xz) = \left\{ \begin{array}{ll} \alpha(x)\alpha(z) & \text{if } \alpha(\mathcal{S}_\ell(L)) = \mathcal{S}_\ell(\alpha(L)); \\ \alpha(z)\alpha(x) & \text{if } \alpha(\mathcal{S}_\ell(L)) = \mathcal{S}_r(\alpha(L)); \end{array} \right. \\ \alpha(zx) = \left\{ \begin{array}{ll} \alpha(x)\alpha(z) & \text{if } \alpha(\mathcal{S}_r(L)) = \mathcal{S}_\ell(\alpha(L)); \\ \alpha(z)\alpha(x) & \text{if } \alpha(\mathcal{S}_r(L)) = \mathcal{S}_r(\alpha(L)). \end{array} \right. \end{array} \right. \tag{8}$$

Proof. First, let us suppose that α takes the *left* parallel class $\mathcal{S}_\ell(L)$ to the *left* parallel class $\mathcal{S}_\ell(\alpha(L))$. We consider H, on the one hand, as a 2-dimensional *right* vector space H_L and, on the other hand, as a 2-dimensional *right* vector space $H_{\alpha(L)}$. By our assumption, α takes $\mathcal{S}_\ell(L) = \{gL \mid g \in H^*\}$ to $\mathcal{S}_\ell(\alpha(L)) = \{g'\alpha(L) \mid g' \in H^*\}$, *i.e.*, the set of one-dimensional subspaces of H_L goes over to the set of one-dimensional subspaces of $H_{\alpha(L)}$. Since α is additive, it is a collineation of the affine plane on H_L to the affine plane on $H_{\alpha(L)}$. From $\alpha(0) = 0$ and the Fundamental Theorem of Affine Geometry, α is a semilinear transformation of H_L to $H_{\alpha(L)}$. Let $\varphi_L\colon L \to \alpha(L)$ be its accompanying isomorphism of fields. From $\alpha(1) = 1$, we obtain $\alpha(z) = \alpha(1z) = \alpha(1)\varphi_L(z) = \varphi_L(z)$ for all $z \in L$, whence the φ_L-semilinearity of α can be rewritten as

$$\forall\, x \in H,\ z \in L\colon \alpha(xz) = \alpha(x)\alpha(z). \tag{9}$$

Next, suppose that α takes the *left* parallel class $\mathcal{S}_\ell(L)$ to the *right* parallel class $\mathcal{S}_r(\alpha(L))$. We proceed as above except for $H_{\alpha(L)}$, which is replaced by the 2-dimensional *left* vector space ${}_{\alpha(L)}H$. In this way all products of α-images have to be rewritten in reverse order so that the equation in (9) changes to $\alpha(xz) = \alpha(z)\alpha(x)$.

There remain the cases when α takes $\mathcal{S}_r(L)$ to $\mathcal{S}_\ell(\alpha(L))$ or $\mathcal{S}_r(\alpha(L))$. Accordingly, the equation in (9) takes the form $\alpha(zx) = \alpha(x)\alpha(z)$ or $\alpha(zx) = \alpha(z)\alpha(x)$. □

We now establish that any $\alpha \in \Gamma_\parallel$ fixing 1 satisfies precisely one of the two properties concerning $\alpha(\mathcal{F})$, as appearing in Proposition 3.1 (a) and (b). Afterwards, we will be able to show that any such α is actually an automorphism or antiautomorphism of the skew field H.

Proposition 3.3. *Let $\alpha \in \Gamma_\parallel$ be such that $\alpha(1) = 1$. If there exists a line $L \in \mathcal{A}(H_F)$ such that $\mathcal{S}(L)$ and $\alpha(\mathcal{S}(L))$ are of the same kind, that is, both are left or both are right parallel classes, then $\alpha(\mathcal{F}) = \mathcal{F}$. Similarly, if there exists a line $L \in \mathcal{A}(H_F)$ such that $\mathcal{S}(L)$ and $\alpha(\mathcal{S}(L))$ are of different kind, then $\alpha(\mathcal{F}) = \mathcal{A}(H_F) \setminus \mathcal{F}$.*

Proof. First, let us suppose that $\mathcal{S}(L) = \mathcal{S}_\ell(L)$ and $\alpha(\mathcal{S}(L)) = \mathcal{S}_\ell(\alpha(L))$. This means that L and $\alpha(L)$ are in $\mathcal{F}$. We proceed by showing $\alpha(\mathcal{F}) \subseteq \mathcal{F}$. If this were not the case, then a line $L' \in \mathcal{F}$ would exist such that $\alpha(L') \in \mathcal{A}(H_F) \setminus \mathcal{F}$, that is, $\mathcal{S}(\alpha(L')) = \mathcal{S}_r(\alpha(L'))$. Furthermore, there would exist quaternions $e \in L \setminus F$, $e' \in L' \setminus F$ and we would have $e'e \neq ee'$. By Lemma 3.2, applied to L and also to L', we would finally obtain $\alpha(e'e) = \alpha(e')\alpha(e) = \alpha(ee')$, which is absurd due to α being injective. The same kind of reasoning can be applied to $\alpha^{-1} \in \Gamma_\parallel$, whence $\alpha^{-1}(\mathcal{F}) \subseteq \mathcal{F}$. Summing up, we have shown $\alpha(\mathcal{F}) = \mathcal{F}$ in our first case.

The case when $\mathcal{S}(L) = \mathcal{S}_r(L)$ and $\alpha(\mathcal{S}(L)) = \mathcal{S}_r(\alpha(L))$ can be treated in an analogous way and leads us to $\alpha(\mathcal{A}(H_F) \setminus \mathcal{F}) = \mathcal{A}(H_F) \setminus \mathcal{F}$. Clearly, this is equivalent to $\alpha(\mathcal{F}) = \mathcal{F}$.

Let us now suppose that $\mathcal{S}(L)$ and $\alpha(\mathcal{S}(L))$ are of different kind, that is, one of them is a left and the other one is a right parallel class. Then, by making the appropriate changes in the reasoning above, we obtain $\alpha(\mathcal{F}) = \mathcal{A}(H_F) \setminus \mathcal{F}$. $\square$

On the basis of our previous results, we now establish our two main theorems.

Theorem 3.4. *Let $\parallel$ be a Clifford-like parallelism of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$. Then a semilinear transformation $\beta \in \Gamma\mathrm{L}(H_F)$ preserves $\parallel$ if, and only if, it can be written in the form*

$$\beta = \lambda_{\beta(1)} \circ \alpha, \tag{10}$$

where $\lambda_{\beta(1)}$ denotes the left translation of H by $\beta(1)$ and α either is an automorphism of the quaternion skew field H satisfying $\alpha(\mathcal{F}) = \mathcal{F}$ or an antiautomorphism of H satisfying $\alpha(\mathcal{F}) = \mathcal{A}(H_F) \setminus \mathcal{F}$.

Proof. If β can be factorised as in (10), then $\beta \in \Gamma_{\parallel}$ follows from Proposition 3.1 (a), (b), and (d).

In order to verify the converse, we define $\alpha := \lambda_{\beta(1)}^{-1} \circ \beta$. Then $\alpha(1) = 1$ and $\alpha \in \Gamma_{\parallel}$ by Proposition 3.1 (d). We now distinguish two cases.

Case (i). There exists a line $L \in \mathcal{A}(H_F)$ such that $\mathcal{S}(L)$ and $\alpha(\mathcal{S}(L))$ are of the same kind. We claim that under these circumstances $\alpha \in \mathrm{Aut}(H)$.

First, we confine ourselves to the subcase $\mathcal{S}(L) = \mathcal{S}_\ell(L)$. By the theorem of Cartan-Brauer-Hua [24, (13.17)], there is an $h \in H^*$ such that $L' := h^{-1}Lh \neq L$. From Proposition 3.1 (c), $\mathcal{S}(L') = \tilde{h}(\mathcal{S}(L))$ is a left parallel class and, from Proposition 3.3, the same holds for $\alpha(\mathcal{S}(L'))$. There exists an $e' \in L' \setminus L$ and, consequently, the elements $1, e'$ constitute a basis of H_L. Given arbitrary quaternions x, y we may write $y = z_0 + e'z_1$ with $z_0, z_1 \in L$. By virtue of Lemma 3.2, we obtain the intermediate result

$$\forall\, x \in H,\ z \in L\colon \alpha(xz) = \alpha(x)\alpha(z),\ \alpha(xe') = \alpha(x)\alpha(e'). \tag{11}$$

Using repeatedly the additivity of α and (11) gives

$$\begin{aligned} \alpha(xy) &= \alpha(xz_0) + \alpha\big((xe')z_1\big) = \alpha(x)\alpha(z_0) + \alpha(xe')\alpha(z_1) \\ &= \alpha(x)\big(\alpha(z_0) + \alpha(e')\alpha(z_1)\big) = \alpha(x)\big(\alpha(z_0) + \alpha(e'z_1)\big) = \alpha(x)\alpha(y). \end{aligned} \tag{12}$$

Thus α is an automorphism of H.

The subcase $\mathcal{S}(L) = \mathcal{S}_r(L)$ can be treated in an analogous way. It suffices to replace H_L with ${}_LH$ and to revert the order of the factors in all products appearing in (11) and (12).

Case (ii). There exists a line $L \in \mathcal{A}(H_F)$ such that $\mathcal{S}(L)$ and $\alpha(\mathcal{S}(L))$ are of different kind. Then, by reordering certain factors appearing in Case (i) in the appropriate way, the mapping α turns out to be an antiautomorphism of H.

Altogether, since there exists a line in $\mathcal{A}(H_F)$, α is an automorphism or an antiautomorphism of H. Accordingly, from Proposition 3.1 (a) or (b), $\alpha(\mathcal{F}) = \mathcal{F}$ or $\alpha(\mathcal{F}) = \mathcal{A}(H_F) \setminus \mathcal{F}$. □

Theorem 3.5. *Let* $\|$ *be a Clifford-like parallelism of* $(\mathbb{P}(H_F), \|_\ell, \|_r)$. *Then the group* $\Gamma_\| \cap \mathrm{GL}(H_F)$ *of linear transformations preserving* $\|$ *coincides with the group* $\Gamma_\ell \cap \mathrm{GL}(H_F)$ *of linear transformations preserving the left Clifford parallelism* $\|_\ell$.

Proof. In view of Proposition 3.1 (e) it remains to show that any $\beta \in \Gamma_\| \cap \mathrm{GL}(H_F)$ is contained in $\Gamma_\ell \cap \mathrm{GL}(H_F)$. From (10), we deduce $\beta = \lambda_{\beta(1)} \circ \alpha$, where $\alpha \in \mathrm{GL}(H_F)$ is an automorphism of H such that $\alpha(\mathcal{F}) = \mathcal{F}$ or an antiautomorphism of H such that $\alpha(\mathcal{F}) = \mathcal{A}(H_F) \setminus \mathcal{F}$. There are two possibilities.

Case (i). α is an automorphism. By the Skolem-Noether theorem, α is inner. Consequently, (4) and (6) give $\beta \in \Gamma_\ell \cap \mathrm{GL}(H_F)$.

Case (ii). α is an antiautomorphism. Again by Skolem-Noether, the product $\alpha' := \alpha \circ \overline{(\cdot)}$ of the given α and the conjugation is in $\widetilde{H^*}$. The conjugation fixes 1 and sends any $x \in H$ to $\overline{x} = \mathrm{tr}(x) - x \in F1 + Fx$. Therefore, all lines of the star $\mathcal{A}(H_F)$ remain fixed under conjugation. The inner automorphism α' fixes $\mathcal{F}$ as a set [12, Thm. 4.10]. This gives $\alpha(\mathcal{F}) = \mathcal{F}$ and contradicts $\alpha(\mathcal{F}) = \mathcal{A}(H_F) \setminus \mathcal{F}$. So, the second case does not occur. □

Theorem 3.5 may be rephrased in the language of projective geometry as follows: if a *projective collineation* of $\mathbb{P}(H_F)$ preserves a *single* Clifford-like parallelism $\|$ of $(\mathbb{P}(H_F), \|_\ell, \|_r)$, then *all* Clifford-like parallelisms of $(\mathbb{P}(H_F), \|_\ell, \|_r)$ (including $\|_\ell$ and $\|_r$) are preserved. This means that a characterisation of the Clifford-parallelisms of $(\mathbb{P}(H_F), \|_\ell, \|_r)$ by their common group of linear automorphisms (or by the corresponding subgroup of the projective group $\mathrm{PGL}(H_F)$) is out of reach whenever there exist Clifford-like parallelisms of $(\mathbb{P}(H_F), \|_\ell, \|_r)$ other than $\|_\ell$ and $\|_r$. (Cf. the beginning of Section 4.) Indeed, by [12, Thm. 4.15], any Clifford-like parallelism of this kind is not Clifford with respect to any projective double space structure on $\mathbb{P}(H_F)$.

Corollary 3.6. *Let* $\alpha_1 \in \Gamma_\|$ *be a fixed automorphism of* H. *Then the following assertions hold.*

(a) *All automorphisms* α *of the skew field* H *satisfying* $(\alpha_1)_{|F} = \alpha_{|F}$ *are in the group* $\Gamma_\|$.

(b) *All antiautomorphisms* α *of the skew field* H *satisfying* $(\alpha_1)_{|F} = \alpha_{|F}$ *are not in the group* $\Gamma_\|$.

The whole statement remains true if the words "automorphism" and "antiautomorphism" are switched.

Proof. (a) By the Skolem-Noether theorem, $\alpha^{-1} \circ \alpha_1$ is an inner automorphism of H. Thus, from Proposition 3.1 (c), $\alpha^{-1} \circ \alpha_1 \in \Gamma_{\|}$, which implies $\alpha \in \Gamma_{\|}$.

(b) The conjugation $\overline{(\cdot)}$ is F-linear. We therefore can apply (a) to $\alpha \circ \overline{(\cdot)}$ and in this way we obtain $\alpha \circ \overline{(\cdot)} \in \Gamma_{\|}$. The proof of Theorem 3.5, Case (ii), gives $\overline{(\cdot)} \notin \Gamma_{\|}$. Hence $\alpha \notin \Gamma_{\|}$ as well. □

Theorem 3.4 and Corollary 3.6 (with $\alpha_1 :=$ id) together entail that

$$\{\alpha \in \Gamma_{\|} \mid \alpha(1) = 1\} \subset \mathrm{Aut}(H) \circ \{\mathrm{id}_H, \overline{(\cdot)}\}.$$

In particular, for all $h \in H^*$, the inner automorphism $\tilde{h}$ is in $\Gamma_{\|}$, whereas the antiautomorphism $\tilde{h} \circ \overline{(\cdot)}$ of the skew field H does not belong to $\Gamma_{\|}$.

Theorem 3.5 motivates to compare the automorphism groups $\Gamma_{\|}$ and Γ_ℓ with respect to inclusion. This leads to four (mutually exclusive) possibilities as follows:

$$\Gamma_{\|} = \Gamma_\ell, \tag{13}$$

$$\Gamma_{\|} \subset \Gamma_\ell, \tag{14}$$

$$\Gamma_{\|} \supset \Gamma_\ell, \tag{15}$$

$$\Gamma_{\|} \not\subseteq \Gamma_\ell \text{ and } \Gamma_{\|} \not\supseteq \Gamma_\ell. \tag{16}$$

In Section 4, it will be shown, by giving illustrative examples, that each of (13) and (14) is satisfied by some Clifford-like parallelisms. The situation in (15) does not occur due to Corollary 3.7 below. Whether or not there exists a Clifford-like parallelism subject to (16) remains as an open problem.

Corollary 3.7. *In* $(\mathbb{P}(H_F), \|_\ell, \|_r)$*, there exists no Clifford-like parallelism* $\|$ *satisfying* (15).

Proof. If (15) holds for some Clifford-like parallelism $\|$, then, by Theorem 3.4, there exists an antiautomorphism α_1 of H such that $\alpha_1 \in \Gamma_{\|}$. Corollary 3.6 (b) shows $\alpha_1 \circ \overline{(\cdot)} \in \mathrm{Aut}(H) \setminus \Gamma_{\|}$. But (5) and (15) force $\alpha_1 \circ \overline{(\cdot)} \in \mathrm{Aut}(H) \subset \Gamma_\ell \subset \Gamma_{\|}$, an absurdity. □

Remark 3.8. For any Clifford-like parallelism $\|$ of $(\mathbb{P}(H_F), \|_\ell, \|_r)$ there are also correlations that preserve $\|$. We just give one example. The orthogonality relation $\perp$ that stems from the non-degenerate symmetric bilinear form (2) determines a projective polarity of $\mathbb{P}(H_F)$ by sending any subspace S of H_F to its orthogonal space $S^\perp$. Using [12, Cor. 4.4] or [19, (2.6)] one obtains that $\mathcal{S}_\ell(M) \cap \mathcal{S}_r(M) = \{M, M^\perp\}$ for all lines $M \in \mathcal{L}(H_F)$. So, for all $M \in \mathcal{L}(H_F)$, we have $M \|_\ell M^\perp$ and $M \|_r M^\perp$, which implies $M \| M^\perp$. In other words, the polarity $\perp$ fixes all parallel classes of the parallelisms $\|_\ell$, $\|_r$ and $\|$. Consequently, each of the parallelisms $\|_\ell$, $\|_r$ and $\|$ is preserved under the action of $\perp$ on the line set $\mathcal{L}(H_F)$.

4 Examples

We first turn to equation (13), that is, $\Gamma_{\parallel} = \Gamma_{\ell}$. In any projective double space $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$, this equation has two trivial solutions, namely $\parallel = \parallel_\ell$ and, by (4), $\parallel = \parallel_r$. According to [12, Thm. 4.12], which relies on [9], a projective double space $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ admits no Clifford-like parallelisms other than $\parallel_\ell$ and $\parallel_r$ precisely when F is a formally real Pythagorean field and H is the ordinary quaternion skew field over F. (See also [5, Thm. 9.1].) Thus, when looking for non-trivial solutions of (13), we have to avoid this particular class of quaternion skew fields.

Example 4.1. Let H be any quaternion skew field of characteristic two. From [12, Ex. 4.13], there exists a Clifford-like parallelism $\parallel$ of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ such that $\mathcal{F}$ comprises *all* lines $L \in \mathcal{A}(H_F)$ that are—in an algebraic language—separable extensions of F. The set $\mathcal{F}$ is fixed under all automorphisms of H, since any $L' \in \mathcal{A}(H_F) \setminus \mathcal{F}$ is an inseparable extension of F. Equation (5) and Theorem 3.4 together give $\Gamma_\ell \subseteq \Gamma_{\parallel}$. As (15) cannot apply, we get $\Gamma_\ell = \Gamma_{\parallel}$. Each of the sets $\mathcal{F}$ and $\mathcal{A}(H_F) \setminus \mathcal{F}$ is non-empty; see, for example, [8, pp. 103–104] or [31, pp. 46–48]. Hence $\parallel$ does not coincide with $\parallel_\ell$ or $\parallel_r$.

Example 4.2. Let H be a quaternion skew field that admits only inner automorphisms. Then all automorphisms and all antiautomorphisms of H are in $\mathrm{GL}(H_F)$. By Theorem 3.5, Γ_ℓ is the common automorphism group of all Clifford-like parallelisms of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$.

In particular, any quaternion skew field H with centre $\mathbb{Q}$ admits only inner automorphisms by the Skolem-Noether theorem. Since $\mathbb{Q}$ is not Pythagorean, we may infer from [12, Thm. 4.12] that any $(\mathbb{P}(H_{\mathbb{Q}}), \parallel_\ell, \parallel_r)$ possesses Clifford-like parallelisms other than $\parallel_\ell$ and $\parallel_r$. (See [12, Ex. 4.14] for detailed examples.)

In order to establish the existence of Clifford-like parallelisms $\parallel$ that satisfy (14), we shall consider certain quaternion skew fields admitting an outer automorphism of order two. The idea to use this kind of automorphism stems from the theory of *involutions of the second kind* [23, §2, 2.B.]. Indeed, for each of the automorphisms α from Examples 4.4, 4.5, 4.6, 4.7 and 4.8 the product $\alpha \circ \overline{(\cdot)}$ is such an involution. Also, we shall use the following auxiliary result.

Lemma 4.3. *Let L be a maximal commutative subfield of H, let $\alpha \in \mathrm{Aut}(H)$, and let $h \in H^*$. Furthermore, assume that $\alpha(L) = h^{-1}Lh$.*

(a) *If $\mathrm{Char}\, H \neq 2$, then for each $q \in L \setminus \{0\}$ with $\mathrm{tr}(q) = 0$ there exists an element $c \in F^*$ such that*

$$c^2 = N(\alpha(q))\, N(q)^{-1}. \tag{17}$$

(b) *If $\mathrm{Char}\, H = 2$ and L is separable over F, then for each $q \in L$ with $\mathrm{tr}(q) = 1$ there exists an element $d \in F$ such that*

$$d^2 + d = N(\alpha(q)) + N(q). \tag{18}$$

(c) *If* $\operatorname{Char} H = 2$ *and* L *is inseparable over* F*, then for each* $q \in L \setminus F$ *there exist elements* $c \in F^*$*,* $d \in F$ *such that*

$$d^2 = N(\alpha(q)) + c^2 N(q). \tag{19}$$

Proof. (a) From (3), applied first to α and then to the inner automorphism $\tilde{h}$, we obtain $\operatorname{tr}(\alpha(q)) = 0 = \operatorname{tr}(h^{-1}qh)$. The elements of $\alpha(L)$ with trace zero constitute a one-dimensional F-subspace of $\alpha(L)$. Hence there exists an element $c \in F^*$ with $\alpha(q) = c(h^{-1}qh)$. Application of the norm function N establishes (17).

(b) Like before, (3) implies $\operatorname{tr}(\alpha(q)) = 1 = \operatorname{tr}(h^{-1}qh)$. The elements of $\alpha(L)$ with trace 1 constitute the set $\alpha(q)+F \subset \alpha(L)$. Hence there exists an element $d \in F$ with $\alpha(q) + d = h^{-1}qh$. Taking the norm on both sides gives $N(d + \alpha(q)) = N(q)$. This equation can be rewritten as in (18), which follows from $N(\alpha(q) + d) = (\alpha(q) + d)\overline{(\alpha(q) + d)} = (\alpha(q) + d)(\alpha(q) + d + 1)$.

(c) Since both L and $\alpha(L)$ are inseparable over F, for any $x \in L \cup \alpha(L)$ it follows $\operatorname{tr}(x) = 0$ and, by (1), $N(x) = x^2$. Thus, in particular, $\operatorname{tr}(\alpha(q)) = 0 = \operatorname{tr}(h^{-1}qh)$. Since $\alpha(q)$ belongs to $\alpha(L)$, which is a 2-dimensional F-vector space spanned by $h^{-1}qh$ and 1, there exist $c, d \in F$ such that $\alpha(q) = c(h^{-1}qh) + d$. Note that $c \neq 0$ since $\alpha(q) \notin F$. Taking the norm on both sides of the previous equation gives $N(\alpha(q)) = N(c(h^{-1}qh) + d) = (c(h^{-1}qh) + d)^2 = c^2 N(q) + d^2$, which entails (19). □

Example 4.4. Let $F = \mathbb{Q}(\sqrt{3})$ and denote by H the ordinary quaternions over F with the usual F-basis $\{1, i, j, k\}$. The mapping $v + w\sqrt{3} \mapsto v - w\sqrt{3}$, $v, w \in \mathbb{Q}$, is an automorphism of F. It can be extended to a unique F-semilinear transformation, say $\alpha\colon H \to H$, such that $\{1, i, j, k\}$ is fixed elementwise. This α is an automorphism of the skew field H, since all structure constants of H with respect to the given basis are in $\mathbb{Q}$, and so all of them are fixed under α.

Following Lemma 4.3, we define $q := i + (1 + \sqrt{3})j$ and $L := F1 \oplus Fq$. Then $\operatorname{tr}(q) = q + \overline{q} = 0$,

$$N(q) = 1 + (1 + \sqrt{3})^2 = 5 + 2\sqrt{3}, \quad N(\alpha(q)) = \alpha(N(q)) = 5 - 2\sqrt{3}$$

and $N(\alpha(q))\, N(q)^{-1} = (5 - 2\sqrt{3})^2/13 \neq c^2$ for all $c \in F^*$, since 13 is not a square in F. By Lemma 4.3 (a), there is no $h \in H^*$ such that $\alpha(L) = h^{-1}Lh$.

We now apply the construction from [12, Thm. 4.10 (a)] to the set $\mathcal{D} := \{L\}$. This gives a Clifford-like parallelism $\|$ with the property $\mathcal{F} = \{h^{-1}Lh \mid h \in H^*\}$. Under the action of the group of inner automorphisms, $\widetilde{H^*}$, the star $\mathcal{A}(H_F)$ splits into orbits of the form $\{h^{-1}L'h \mid h \in H^*\}$ with $L' \in \mathcal{A}(H_F)$. One such orbit is $\mathcal{F}$ and, due to $\alpha(L) \notin \mathcal{F}$, another one is $\alpha(\mathcal{F})$. The automorphism α interchanges these two distinct orbits, but it fixes the $\widetilde{H^*}$-orbit of the line $F1 \oplus Fi$. Therefore, $\mathcal{A}(H_F) \setminus \mathcal{F}$ contains at least two distinct $\widetilde{H^*}$-orbits. Consequently, there is no antiautomorphism of H taking $\mathcal{F}$ to $\mathcal{A}(H_F) \setminus \mathcal{F}$. So, by Theorem 3.4, $\Gamma_{\|} \subseteq \Gamma_\ell$. From (5), Theorem 3.4 and $\alpha(L) \notin \mathcal{F}$, follows $\alpha \in \Gamma_\ell \setminus \Gamma_{\|}$. Summing up, we have $\Gamma_{\|} \subset \Gamma_\ell$, as required.

Example 4.5. Let $\mathbb{F}_2$ be the Galois field with two elements, and let $F = \mathbb{F}_2(t, u)$, where t and u denote independent indeterminates over $\mathbb{F}_2$.

First, we collect some facts about the polynomial algebra $\mathbb{F}_2[t, u]$ over $\mathbb{F}_2$. Let $\mathbb{N}$ denote the set of non-negative integers. The monomials of the form

$$t^\gamma u^\delta \text{ with } (\gamma, \delta) \in \mathbb{N} \times \mathbb{N} \tag{20}$$

constitute a basis of the $\mathbb{F}_2$-vector space $\mathbb{F}_2[t, u]$. Each non-zero polynomial $p \in \mathbb{F}_2[t, u]$ can be written in a unique way as a non-empty sum of basis elements from (20). Among the elements in this sum there is a unique one, say $t^m u^n$, such that (m, n) is maximal w.r.t. the lexicographical order on $\mathbb{N} \times \mathbb{N}$. We shall refer to (m, n) as the *t-leading pair* of p. (In this definition the indeterminates t and u play different roles, because of the lexicographical order. Due to this lack of symmetry the degree of p can be strictly larger than $m + n$.) If $p_1, p_2 \in \mathbb{F}_2[t, u]$ are non-zero polynomials with t-leading pairs (m_1, n_1) and (m_2, n_2), then $p_1 p_2$ is immediately seen to have the t-leading pair $(m_1 + m_2, n_1 + n_2)$.

Next, we construct a quaternion algebra with centre F. We follow the notation from [6] and [11, Rem. 3.1]. Let $K := F(i)$ be a separable quadratic extension of F with defining relation $i^2 + i + 1 = 0$. Furthermore, we define $b := t + u$. The quaternion algebra $(K/F, b)$ has a basis $\{1, i, j, k\}$ such that its multiplication is given by the following table:

$\cdot$	i	j	k
i	$1+i$	k	$j+k$
j	$j+k$	$t+u$	$(t+u)(1+i)$
k	j	$(t+u)i$	$t+u$

The conjugation $\overline{(\cdot)}\colon H \to H$ sends $i \mapsto \bar{i} = i + 1$ and fixes both j and k.

In order to show that $(K/F, b)$ is a skew field we have to verify $b \notin N(K)$. Assume to the contrary that there are polynomials p_1, $p_2 \neq 0$, p_3, and $p_4 \neq 0$ in $\mathbb{F}_2[t, u]$ such that

$$\begin{aligned} N(p_1/p_2 + (p_3/p_4)i) &= (p_1/p_2 + (p_3/p_4)i)(p_1/p_2 + (p_3/p_4)(i+1)) \\ &= p_1^2/p_2^2 + (p_1p_3)/(p_2p_4) + p_3^2/p_4^2 \\ &= t + u. \end{aligned}$$

Consequently,

$$(p_1p_4)^2 + p_1p_2p_3p_4 + (p_2p_3)^2 + (t+u)(p_2p_4)^2 = 0. \tag{21}$$

We cannot have $p_1 = 0$ or $p_3 = 0$, since then the left hand side of (21) would reduce to a sum of two terms, with one being a square in $\mathbb{F}_2[t,u]$ and the other being a non-square. We define (m_s, n_s) to be the t-leading pair of p_s, $s \in \{1,2,3,4\}$. So, the t-leading pairs of the first three summands on the left hand side of (21) are

$$(2(m_1+m_4), 2(n_1+n_4)),\ (m_1+m_2+m_3+m_4, n_1+n_2+n_3+n_4),$$
$$(2(m_2+m_3), 2(n_2+n_3)).$$

Let us expand each of the four summands on the left hand side of (21) in terms of the monomial basis (20). All monomials in the fourth expansion have odd degree. There are three possibilities.

Case (i). $m_1 + m_4 \neq m_2 + m_3$. Then, for example, $m_1 + m_4 > m_2 + m_3$. From

$$2(m_1+m_4) > m_1+m_2+m_3+m_4 > 2(m_2+m_3), \tag{22}$$

the monomial $t^{2(m_1+m_4)}u^{2(n_1+n_4)}$ appears in the expansion of $(p_1p_4)^2$, but not in the expansions of $p_1p_2p_3p_4$ and $(p_2p_3)^2$. This monomial remains unused in the expansion of $(t+u)(p_2p_4)^2$, since both of its exponents are even numbers. So, the left hand side of (21) does not vanish, whence this case cannot occur.

Case (ii). $m_1 + m_4 = m_2 + m_3$ and $n_1 + n_4 \neq n_2 + n_3$. Then, for example, $n_1 + n_4 > n_2 + n_3$. Formula (22) remains true when replacing m_s by n_s, $s \in \{1,2,3,4\}$. We now can deduce, as in Case (i), that the monomial $t^{2(m_1+m_4)}u^{2(n_1+n_4)}$ appears precisely once when expanding each of the four summands the left hand side of (21) in the monomial basis. So, this case is impossible.

Case (iii). $m_1 + m_4 = m_2 + m_3$ and $n_1 + n_4 = n_2 + n_3$. Then $t^{2(m_1+m_4)}u^{2(n_1+n_4)}$ appears precisely three times when expanding the four summands on the left hand side of (21) and, due to $1 + 1 + 1 \neq 0$, this case cannot happen either.

Since none of the Cases (i)–(iii) applies, we end up with a contradiction.

There is a unique automorphism of F that interchanges the indeterminates t and u. It can be extended to a unique F-semilinear transformation, say $\alpha\colon H \to H$, such that $\{1, i, j, k\}$ is fixed elementwise. This α is an automorphism of H, because $\alpha(t+u) = u + t = t + u$.

Following Lemma 4.3, we define $q := i + uj$ and $L := F1 \oplus Fq$. Then $\mathrm{tr}(q) = 1$, $N(q) = 1 + u^2(t+u)$, and $N(\alpha(q)) = 1 + t^2(u+t)$.

We claim that

$$N(\alpha(q)) + N(q) = (u+t)^3 \neq d^2 + d \ \text{ for all } \ d \in F.$$

Let us assume, by way of contraction, that there are polynomials d_1 and $d_2 \neq 0$ in $\mathbb{F}_2[t,u]$ satisfying $(u+t)^3 = d_1^2/d_2^2 + d_1/d_2$. Hence $d_1 \neq 0$ and

$$(u+t)^3 d_2^2 + d_1^2 + d_1 d_2 = 0. \tag{23}$$

We expand the first summand in (23) in terms of the monomial basis (20). This gives a sum of monomials all of which have odd degree. Likewise, the expansion of the second summand in (23) results in a sum of monomials all of which have even degree. Let us also expand the third summand in (23) to a sum of monomials and let us then collect all monomials with odd (resp. even) degree. In this way we get precisely the monomials appearing in the first (resp. second) sum from above. Thus, with $n_1 := \deg d_1$, $n_2 := \deg d_2$ we obtain that the degrees of the summands in (23) satisfy the inequalities

$$3 + 2n_2 \leq n_1 + n_2, \; 2n_1 \leq n_1 + n_2.$$

These inequalities imply $3 + 2n_2 \leq n_1 + n_2 \leq n_2 + n_2$, which is absurd.

By Lemma 4.3 (b), there is no $h \in H^*$ such that $\alpha(L) = h^{-1}Lh$.

We now repeat the reasoning from the end of Example 4.4. This shows that the Clifford-like parallelism $\parallel$ that arises from $\mathcal{D} := \{L\}$ satisfies $\Gamma_{\parallel} \subset \Gamma_{\ell}$.

Example 4.6. Let $H = (K/F, b)$, $\alpha \in \mathrm{Aut}(H)$ and L be given as in Example 4.5. We know from Example 4.1 that

$$\mathcal{E}_{\text{insep}} := \{L' \in \mathcal{A}(H_F) \mid L'/F \text{ is inseparable}\} \neq \emptyset. \tag{24}$$

In contrast to Example 4.5, we adopt an alternative definition of $\mathcal{D}$, namely $\mathcal{D} := \{L\} \cup \mathcal{E}_{\text{insep}}$. The construction from [12, Thm. 4.10 (a)] applied to this $\mathcal{D}$ gives a Clifford-like parallelism $\parallel$ with the property $\mathcal{F} = \{h^{-1}Lh \mid h \in H^*\} \cup \mathcal{E}_{\text{insep}}$. The set $\mathcal{E}_{\text{insep}}$ remains fixed under any antiautomorphism of H. Consequently, there is no antiautomorphism of H taking $\mathcal{F}$ to $\mathcal{A}(H_F) \setminus \mathcal{F}$. So, by Theorem 3.4, $\Gamma_{\parallel} \subseteq \Gamma_{\ell}$. From (5), Theorem 3.4 and $\alpha(L) \notin \mathcal{F}$, follows $\alpha \in \Gamma_{\ell} \setminus \Gamma_{\parallel}$. Summing up, we have $\Gamma_{\parallel} \subset \Gamma_{\ell}$, as required.

Example 4.7. Consider the same quaternion skew field $H = (K/F, b)$ and the same automorphism $\alpha \in \mathrm{Aut}(H)$ as in Example 4.5. However, now we define $q := j + uk$ and $L := F1 \oplus Fq$. Then L is inseparable over F, $\mathrm{tr}(q) = 0$, $N(q) = (j + uk)^2 = (u + t)(1 + u + u^2)$ and $N(\alpha(q)) = (u + t)(1 + t + t^2)$. Equation (19) of Lemma 4.3 is

$$(u + t)(1 + t + t^2) + c^2(u + t)(1 + u + u^2) = d^2$$

which, upon fixing $c = c_1/c_2$ and $d = d_1/d_2$ with $c_1, c_2, d_1, d_2 \in \mathbb{F}_2[t, u]$ and $c_1, c_2, d_2 \neq 0$, is equivalent to

$$d_2^2(u + t)(c_2^2 + c_2^2t^2 + c_1^2 + c_1^2u^2) + d_2^2(u + t)(c_2^2t + uc_1^2) = d_1^2c_2^2. \tag{25}$$

All the monomials in the first summand of the left hand side of equation (25) are of odd degree, while the monomials in the second one are of even degree. Since $d_1^2c_2^2$ is a sum of monomials of even degree this entails

$$\begin{cases} d_2^2(u + t)(c_2^2 + c_2^2t^2 + c_1^2 + c_1^2u^2) = 0, \\ d_2^2(u + t)(c_2^2t + uc_1^2) = d_1^2c_2^2. \end{cases} \tag{26}$$

The second equation in (26) yields $ut(1+c)^2 = (d+t+uc)^2$, and since $c = 1$ (*i.e.*, $c_1 = c_2$) is not a solution of the first equation in (26), we can assume $1 + c \neq 0$, thus $ut = \left((d+t+uc)/(1+c)\right)^2$. This equation, after all, cannot be satisfied for any choice of $c \in F^*$ since ut is not a square in F. Thus we can conclude by Lemma 4.3 (c) that there exists no $h \in H^*$ such that $\alpha(L) = h^{-1}qh$.

The final step is to define a Clifford-like parallelism subject to (14). This can be done as in Example 4.5 using $\mathcal{D} := \{L\}$.

Example 4.8. Let $H = (K/F, b)$, $\alpha \in \mathrm{Aut}(H)$ and L be given as in Example 4.7. Then a Clifford-like parallelism that satisfies (14) can be obtained along the lines of Example 4.6 by replacing everywhere the set $\mathcal{E}_{\mathrm{insep}}$ from (24) with $\mathcal{E}_{\mathrm{sep}} := \{L' \in \mathcal{A}(H_F) \mid L'/F \text{ is separable}\}$.

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