Baer subspaces within Segre manifolds

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1. Introduction

Baer subspaces of projective spaces as well as Segre manifolds of pappian projective spaces are very well known. But seemingly they are unrelated topics, apart from the (more or less formal) fact that both of them may be described in terms of tensor products of vector spaces.

Baer subspaces of a desarguesian projective space with an underlying (not necessarily commutative) field L arise from subfields K of right and left degree 2 over L. (Recall that the right and left degree of a field extension may be different; cf. [4,123ff].) If $\mathfrak{W}$ is a right vector space over K, then the tensor product $\mathfrak{W} \otimes_K L$ is a right vector space over L. With $1 \in L$ we have the canonical embedding $\mathfrak{w} \mapsto \mathfrak{w} \otimes 1$ of $\mathfrak{W}$ in $\mathfrak{W} \otimes_K L$. This yields an embedding of the projective space on $\mathfrak{W}$ in the projective space on $\mathfrak{W} \otimes_K L$ as a Baer subspace.

When $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are vector spaces over the same commutative field L, then the set of all non-zero pure bivectors of $\mathfrak{V}_1 \otimes_L \mathfrak{V}_2$ determines a Segre manifold in the projective space on $\mathfrak{V}_1 \otimes_L \mathfrak{V}_2$. Following geometric ideas in [3] and [10], a definition of Segre manifolds will be given when the ground field L is arbitrary. However, by following this definition, the connection to tensor products of vector spaces seems to be lost when L is a skew field, since forming $\mathfrak{V}_1 \otimes_L \mathfrak{V}_2$ requires a *right* vector space $\mathfrak{V}_1$ and a *left* vector space $\mathfrak{V}_2$ over L. And this is not in accordance with the geometric approach.

The following construction of a Baer subspace within a Segre manifold essentially depends on the existence of an element $a \in L$ which has degree two over the centre of

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L. So it will not work in pappian spaces. Furthermore we are going to use that cross-ratios in a skew field L are conjugacy classes of L rather than single elements. Moreover we shall show that the generators of the given Segre manifold yield a 1-spread of the Baer subspace. Any transversal subspace of the Segre manifold may be regarded as an indicator set of this 1-spread.

2. Segre manifolds

Let $\mathcal{P}$ be a projective space. Given complementary subspaces $\mathcal{U}$ and $\mathcal{U}'$ of $\mathcal{P}$ and a projective collineation $\kappa : \mathcal{U} \to \mathcal{U}'$, where $\mathcal{U},\mathcal{U}'$ are regarded as sets of points, the set

$$\mathbf{S} := \{X \in Y \vee Y^{\kappa} \mid Y \in \mathcal{U}\} \subset \mathcal{P}$$

is called a **Segre manifold**. Every line belonging to

$$\mathbf{S}_{\mathrm{I}} := \{Y \vee Y^{\kappa} \mid Y \in \mathcal{U}\}$$

is called a **generator** of $\mathbf{S}$. A subspace $\mathcal{T} \subset \mathcal{P}$ is named a **transversal subspace** of $\mathbf{S}_{\mathrm{I}}$, if

$$\ell \ (\in \mathbf{S}_{\mathrm{I}}) \ \mapsto \ \ell \cap \mathcal{T}$$

defines a bijection of $\mathbf{S}_{\mathrm{I}}$ onto $\mathcal{T}$.

When $\mathcal{P}$ is pappian and $(2n+1)$-dimensional, then every Segre manifold according to this definition is a Corrado Segre manifold $\mathbf{S}_{1;n}$ in the notation of W. Burau [3,133]. Cf. also [6,189 ff]. One could also ask for generalizations of Segre manifolds whose generators are subspaces of higher dimension. But we shall not be concerned with this possibility.

If $\dim\mathcal{P} = 3$, then $\mathbf{S}_{\mathrm{I}}$ is a regulus in the sense of Beniamino Segre [10,319], and conversely every regulus is the set of generators of a Segre manifold. Reguli often are defined in such a way that their existence forces $\mathcal{P}$ to be pappian. But we shall stick to the more general terminology introduced by B. Segre.

In the sequel let $\mathfrak{V}$ be a right vector space over a field L. When $\mathfrak{U}$ is a subspace of $\mathfrak{V}$, then $\mathcal{P}(\mathfrak{U})$ stands for the projective space on $\mathfrak{U}$. Given $M \subset L$, we denote by $Z(M)$ the centralizer of M in L.

Suppose that $\mathbf{S}$ is a Segre manifold in $\mathcal{P}(\mathfrak{V})$ with $\operatorname{span}\mathbf{S} = \mathcal{P}(\mathfrak{V})$. Then $\mathfrak{V}$ is the direct sum of two subspaces $\mathfrak{U},\mathfrak{U}'$, say, and there is a linear bijection

$$' : \mathfrak{U} \to \mathfrak{U}', \ \mathfrak{u} \ \mapsto \ \mathfrak{u}' \tag{1}$$

such that the defining projective collineation κ for $\mathbf{S}$ may be written as

$$\kappa : \ \mathcal{P}(\mathfrak{U}) \to \mathcal{P}(\mathfrak{U}'), \ \mathfrak{x}L \ \mapsto \ \mathfrak{x}'L. \tag{2}$$

We use this to determine all lines of $\mathcal{P}(\mathfrak{B})$ within the manifold $\mathbf{S}$:

THEOREM 1. *Let* $\mathbf{S}_{\mathrm{I}}$ *be the family of generators of a Segre manifold* $\mathbf{S}$ *which is spanning* $\mathcal{P}(\mathfrak{B})$*. Then the family of all transversal subspaces of* $\mathbf{S}_{\mathrm{I}}$ *is given by*

$$\mathbf{S}_{\mathrm{II}} := \{\mathcal{T}(x,y) \mid (0,0) \neq (x,y) \in Z(L)^2\}$$

where

$$\mathcal{T}(x,y) := \mathcal{P}(\{\mathfrak{u}x+\mathfrak{u}'y\}) \mid \mathfrak{u} \in \mathfrak{U}\setminus\{\mathfrak{o}\}).$$

If $g \subset \mathbf{S}$ *is a line of* $\mathcal{P}(\mathfrak{B})$*, then either* $g \in \mathbf{S}_{\mathrm{I}}$ *or* g *is contained in a transversal subspace.*

Proof. A straightforward calculation shows that every $\mathcal{T}(x,y) \in \mathbf{S}_{\mathrm{II}}$ is a transversal subspace of $\mathbf{S}_{\mathrm{I}}$.

On the other hand, if $\mathcal{T}$ is transversal, then pick different lines $\ell_0,\ell_1 \in \mathbf{S}_{\mathrm{I}}$. Put $Q := \ell_0 \vee \ell_1$, whence $\dim Q = 3$. All lines of $\mathbf{S}_{\mathrm{I}}$ within Q form a regulus with $\mathcal{T}\cap Q$ being a transversal line of this regulus. We infer from [10,319] that $\ell_0\cap\mathcal{T}$ and $\ell_1\cap\mathcal{T}$ have the form

$$(\mathfrak{u}_0x+\mathfrak{u}_0'y)L,\ (\mathfrak{v}_0x+\mathfrak{v}_0'y)L \text{ with } (0,0) \neq (x,y) \in Z(L)^2,$$

respectively. Fixing ℓ_0 and varying ℓ_1 in $\mathbf{S}_{\mathrm{I}}\setminus\{\ell_0\}$ yields $\mathcal{T} \subset \mathcal{T}(x,y)$ which forces $\mathcal{T} = \mathcal{T}(x,y)$.

Now let $g \subset \mathbf{S}$ be a line. It will be sufficient to show that $g \notin \mathbf{S}_{\mathrm{I}}$ implies $g \subset \mathcal{T} \in \mathbf{S}_{\mathrm{II}}$. Hence we have points $X_0, X_1 \in g$ which are incident with different lines $\ell_0,\ell_1 \in \mathbf{S}_{\mathrm{I}}$. Repeating the arguments just used, we deduce that $g = \mathcal{T}\cap Q$ for some transversal subspace $\mathcal{T} \in \mathbf{S}_{\mathrm{II}}$.■

On every line $\ell \in \mathbf{S}_{\mathrm{I}}$ the set $\ell\cap\mathbf{S}_{\mathrm{II}}$ is a subline over the centre $Z(L)$ of L, so the cross-ratio (CR) of any four different points of $\ell\cap\mathbf{S}_{\mathrm{II}}$ lies in $Z(L)$; cf. [10,321], [8]. Given any transversal subspace $\mathcal{T}(x,y)$, the mapping

$$\mathcal{P}(\mathfrak{U}) \to \mathcal{T}(x,y),\ \mathfrak{u}L \mapsto (\mathfrak{u}x+\mathfrak{u}'y)L$$

is a projective collineation, so that

$$\mathcal{T}(x_0,y_0) \to \mathcal{T}(x_1,y_1),\ \ell\cap\mathcal{T}(x_0,y_0) \mapsto \ell\cap\mathcal{T}(x_1,y_1) \text{ with } \ell \in \mathbf{S}_{\mathrm{I}}$$

is again a projective collineation for any two transversal subspaces $\mathcal{T}(x_0,y_0)$ and $\mathcal{T}(x_1,y_1)$. This mapping in turn may be used to generate the given Segre manifold $\mathbf{S}$.

Every Segre manifold $\mathbf{S}$ has at least three different transversal subspaces and it is easily shown that $\mathbf{S}$ equals the union over all lines which intersect three of its transversal subspaces. It is immediate from theorem 1 that $\mathbf{S}_{\mathrm{I}}\cup\mathbf{S}_{\mathrm{II}}$ is the set of all maximal subspaces which are contained in $\mathbf{S}$. The rôle of $\mathbf{S}_{\mathrm{I}}$ and $\mathbf{S}_{\mathrm{II}}$ cannot be interchanged unless $\mathcal{P}(\mathfrak{B})$ is 3-dimensional and pappian; cf. e.g. [10,319-321].

3. Baer subspaces

Let K be a subfield of L whose left and right degree over L equals 2. If $\mathfrak{B}$ is a right vector space over K, then the projective space $\mathcal{P}(\mathfrak{B})$ yields a Baer subspace $\tilde{\mathcal{P}}$ of $\mathcal{P}(\mathfrak{B}\otimes_K L)$, as has been sketched in section 1. Given a subspace $\tilde{\mathcal{M}}$ of $\tilde{\mathcal{P}}$, there is a unique subspace $\mathcal{M}$ of $\mathcal{P}(\mathfrak{B})$ with $\tilde{\mathcal{M}} = \tilde{\mathcal{P}}\cap\mathcal{M}$. We shall also say that $\mathcal{M}$ is a subspace of $\tilde{\mathcal{P}}$. Since the *left* degree[1] of L over K is 2, every point X of $\mathcal{P}(\mathfrak{B}\otimes_K L)\setminus\tilde{\mathcal{P}}$, say

$$X = \Big(\sum_{\mathfrak{w}\in\mathfrak{B}} \mathfrak{w}\otimes(\xi_{\mathfrak{w}}+\eta_{\mathfrak{w}}i)\Big)L \quad (\xi,\eta \in K,\ i \in L\setminus K),$$

is incident with a unique line of $\tilde{\mathcal{P}}$ which is spanned by

$$\Big(\sum_{\mathfrak{w}\in\mathfrak{B}} \mathfrak{w}\xi_{\mathfrak{w}}\otimes 1\Big)L \text{ and } \Big(\sum_{\mathfrak{w}\in\mathfrak{B}} \mathfrak{w}\eta_{\mathfrak{w}}\otimes 1\Big)L.$$

It follows in the same fashion that every hyperplane of $\mathcal{P}(\mathfrak{B}\otimes_K L)$, which is not a hyperplane of $\tilde{\mathcal{P}}$, contains a unique co-line of $\tilde{\mathcal{P}}$, since 2 is also the *right* degree of L over K.

So far there was no restriction on the ground field L. From now on, however, we assume that L is a non-commutative field. The conjugacy class of any $a\in L$ will be written as $\hat{a}$. The main result of this paper is

THEOREM 2. *Let* $\mathbf{S}$ *be a Segre manifold spanning* $\mathcal{P}(\mathfrak{B})$ *and denote by* $\mathcal{U},\mathcal{U}',\mathcal{T} \in \mathbf{S}_{\mathrm{II}}$ *three different transversal subspaces of* $\mathbf{S}_{\mathrm{I}}$. *Assume that* $a\in L$ *is quadratic over the centre of* L. *Then the set of all points* $X\in\mathbf{S}$ *satisfying*

$$X\in\ell_X\in\mathbf{S}_{\mathrm{I}} \Rightarrow \mathrm{CR}(X,\ell_X\cap\mathcal{T},\ell_X\cap\mathcal{U},\ell_X\cap\mathcal{U}') = \hat{a}$$

is a Baer subspace $\tilde{\mathcal{P}}$ *of* $\mathcal{P}(\mathfrak{B})$. *The centralizer of* a *in* L *is an underlying field of* $\tilde{\mathcal{P}}$.

Proof. (a) Let $\mathbf{S}$ be given by (2) and suppose $\mathcal{U} = \mathcal{P}(\mathfrak{U})$, $\mathcal{U}' = \mathcal{P}(\mathfrak{U}')$, $\mathcal{T} = \mathcal{T}(1,1)$. The element $a \in L$ is a zero of its minimal polynomial

$$X^2-m_1X-m_0\in Z(L)[X]. \tag{3}$$

Denote by A the commutative subfield of L spanned by $Z(L)\cup\{a\}$. Hence the centralizers of $\{a\}$ and A in L are the same. We obtain

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[1]The following calculation runs in a well-known manner. The only reason for writing it down is to emphasize the significance of left and right degrees in geometric terms.

$$|L : Z(A)|_{\text{left}} = |A : Z(L)|_{\text{left}} = 2 = |A : Z(L)|_{\text{right}} = |L : Z(A)|_{\text{right}},$$

where the first and the last sign of equality follows from the centralizer theorem [4,49; Corollary 2], while the others are obvious. Every point $X \in \widetilde{\mathcal{P}}$ has the form $X = \mathfrak{w}L$ with $\mathfrak{w}$ being an element of

$$\mathfrak{W} := \{\mathfrak{u}+\mathfrak{u}'a \mid \mathfrak{u}\in\mathfrak{U}\} \subset \mathfrak{V}. \tag{4}$$

By construction $(\mathfrak{W},+)$ is a subgroup of $(\mathfrak{V},+)$ which is closed under multiplication with scalars of $Z(A)$. Therefore $\mathfrak{W}$ is a right vector space over $Z(A)$. We shall emphasize this by writing $\mathfrak{W}_{Z(A)}$. If we regard $\mathfrak{U}$ as a right vector space $\mathfrak{U}_{Z(A)}$ over $Z(A)$, then

$$\alpha : \mathfrak{U}_{Z(A)} \to \mathfrak{W}_{Z(A)},\ \mathfrak{u} \mapsto \mathfrak{u}+\mathfrak{u}'a \tag{5}$$

is a $Z(A)$-linear bijection of vector spaces.

(b) In order to show that $\mathfrak{W}$ gives rise to a Baer subspace of $\mathcal{P}(\mathfrak{V})$, we establish that the mapping

$$f : \mathfrak{W} \otimes_{Z(A)} L \to \mathfrak{V},\ \sum_{\mathfrak{w}\in\mathfrak{W}} \mathfrak{w}\otimes x_{\mathfrak{w}} \mapsto \sum_{\mathfrak{w}\in\mathfrak{W}} \mathfrak{w}x_{\mathfrak{w}} \quad (x_{\mathfrak{w}} \in L)$$

is an L-linear bijection.

This f is well defined and L-linear. Let $\mathfrak{B}$ be a basis of $\mathfrak{U}_{Z(A)}$ and let d be any element of $L\setminus Z(A)$. Then every $\mathfrak{x} \in \mathfrak{W} \otimes_{Z(A)} L$ can be written as

$$\mathfrak{x} = \sum_{\mathfrak{b}\in\mathfrak{B}} (\mathfrak{b}+\mathfrak{b}'a)\otimes(\xi_{\mathfrak{b}}+d\eta_{\mathfrak{b}}) \text{ with } \xi_{\mathfrak{b}},\eta_{\mathfrak{b}} \in Z(A).$$

Suppose that

$$f(\mathfrak{x}) = \sum_{\mathfrak{b}\in\mathfrak{B}} (\mathfrak{b}+\mathfrak{b}'a)(\xi_{\mathfrak{b}}+d\eta_{\mathfrak{b}}) = \mathfrak{o}$$

whence, by $\mathfrak{V} = \mathfrak{U}\oplus\mathfrak{U}'$ and the inverse of the mapping (1),

$$\sum_{\mathfrak{b}\in\mathfrak{B}} \mathfrak{b}(\xi_{\mathfrak{b}}+d\eta_{\mathfrak{b}}) = \mathfrak{o},\ \sum_{\mathfrak{b}\in\mathfrak{B}} (\mathfrak{b}a)(\xi_{\mathfrak{b}}+d\eta_{\mathfrak{b}}) = \mathfrak{o}.$$

Multiplying the second equation by $-a^{-1}$ and adding the first equation yields

$$\sum_{\mathfrak{b}\in\mathfrak{B}} \mathfrak{b}d\eta_{\mathfrak{b}} = \sum_{\mathfrak{b}\in\mathfrak{B}} \mathfrak{b}(ada^{-1})\eta_{\mathfrak{b}}. \tag{6}$$

Case 1: a is separable over $Z(L)$. Hence

$$\bar{a} := (m_1-a) \in A\setminus\{a\}$$

is a zero of the polynomial (3). There is an automorphism of A which fixes $Z(L)$ elementwise and takes a to $\bar{a}$. By the Skolem Noether theorem (cf. e.g. the corrollary in [4,46]), that automorphism of A can be extended to an inner automorphism of L. So there is an element $c \in L\setminus Z(A)$ such that $c^{-1}ac = \bar{a}$. Since $d \in L\setminus Z(A)$ has been chosen arbitrarily, we may assume that $d = c$. Hence

$$ad = d\bar{a} = d(m_1 - a). \tag{7}$$

We deduce from $d^{-1}Ad = A$ that $d^{-1}Z(A)d = Z(A)$, so that

$$z\ (\in Z(A)) \mapsto dzd^{-1}$$

is an automorphism of $Z(A)$. Therefore $\mathfrak{u} \mapsto \mathfrak{u}d$ is a $Z(A)$-semilinear bijection of $\mathfrak{U}$ and $\{\mathfrak{b}d \mid \mathfrak{b}\in\mathfrak{B}\}$ is a basis of $\mathfrak{U}_{Z(A)}$. Now, by (7), equation (6) becomes

$$\sum_{\mathfrak{b}\in\mathfrak{B}} \mathfrak{b}d\eta_{\mathfrak{b}} = \sum_{\mathfrak{b}\in\mathfrak{B}} \mathfrak{b}d(m_1-a)a^{-1}\eta_{\mathfrak{b}}$$

which forces

$$\eta_{\mathfrak{b}} = a^{-1}(m_1-a)\eta_{\mathfrak{b}} \text{ for all } \mathfrak{b}\in\mathfrak{B}.$$

Assume that $\eta_{\mathfrak{b}} \neq 0$ for some $\mathfrak{b}\in\mathfrak{B}$. Hence $2a = m_1$, a contradiction. So $\eta_{\mathfrak{b}} = 0$ for all $\mathfrak{b}\in\mathfrak{B}$.

Case 2: a is inseparable over $Z(L)$. Consequently Char$L = 2$ and $m_1 = 0$. We read off from $a^2 = m_0 \in Z(L)$ that the inner automorphism

$$x \mapsto a^{-1}xa$$

of L has order 2. Since $x + a^{-1}xa \in Z(A)$ for all $x \in L$, the element d can be chosen such that $a^{-1}da = d+1$. We obtain

$$ad = da+a. \tag{8}$$

By (8) equation (6) can be written as

$$\sum_{\mathfrak{b}\in\mathfrak{B}} \mathfrak{b}d\eta_{\mathfrak{b}} = \sum_{\mathfrak{b}\in\mathfrak{B}} \mathfrak{b}(da+a)a^{-1}\eta_{\mathfrak{b}} = \sum_{\mathfrak{b}\in\mathfrak{B}} \mathfrak{b}d\eta_{\mathfrak{b}} + \sum_{\mathfrak{b}\in\mathfrak{B}} \mathfrak{b}\eta_{\mathfrak{b}}$$

which implies $\eta_{\mathfrak{b}} = 0$ for all $\mathfrak{b}\in\mathfrak{B}$.

In either case $\xi_{\mathfrak{b}} = 0$ for all $\mathfrak{b}\in\mathfrak{B}$ and, finally, $\mathfrak{x} = \mathfrak{o}$. So f is injective. Furthermore f is surjective, since $\mathfrak{B}$ spans all of $\mathfrak{B}$.■

We remark that this proof can be reduced drastically if $\dim\mathcal{U} < \infty$, since then f being surjective already implies that f is a bijection.

Now we are in a position to show how the generators of the given Segre manifold are "seen" from within the Baer subspace. Recall that a set $\mathcal{S}$ of mutually skew lines of a projective space $\mathcal{P}$ is called a 1-spread, if $\mathcal{S}$ is covering $\mathcal{P}$. When $\dim\mathcal{P} = 3$, we shall use the term spread rather than 1-spread and a dual spread is to mean a set of mutually skew lines such that every plane of $\mathcal{P}$ contains one line of $\mathcal{S}$.

THEOREM 3. *Under the assumptions of theorem 2, the set of lines*

$$\tilde{\mathcal{S}} := \{\ell\cap\tilde{\mathcal{P}} \mid \ell\in \mathbf{S}_I\}$$

is a 1-spread of $\tilde{\mathcal{P}}$. *If* $q \subset \mathcal{P}$ *is a line of* $\tilde{\mathcal{P}}$ *which carries one point of* $\mathcal{U}$, *then* $\tilde{q} \in \tilde{\mathcal{S}}$. *If furthermore* $\dim\mathcal{P}(\mathfrak{B}) = 3$, *then the spread* $\tilde{\mathcal{S}}$ *is desarguesian and its kernel is isomorphic to L. Moreover* $\tilde{\mathcal{S}}$ *is a dual spread.*

Proof. If $\ell \in \mathbf{S}_{\mathrm{I}}$, then $\ell \cap \tilde{\mathcal{P}}$ is a line of $\tilde{\mathcal{P}}$. Any two different lines $\ell_0, \ell_1 \in \mathbf{S}_{\mathrm{I}}$ are skew. By theorem 2 every point of $\tilde{\mathcal{P}}$ is incident with an element of $\tilde{\mathcal{S}}$. Thus $\tilde{\mathcal{S}}$ is a 1-spread.

Suppose that q carries different points

$$(\mathfrak{u}+\mathfrak{u}'a)L,\ (\mathfrak{v}+\mathfrak{v}'a)L \in \tilde{\mathcal{P}},\ \mathfrak{t}L \in \mathcal{U}.$$

So $\mathfrak{t}$ is a linear combination (over L) of $\mathfrak{u}+\mathfrak{u}'a$, $\mathfrak{v}+\mathfrak{v}'a$. We deduce from $\mathfrak{B} = \mathfrak{U}\oplus\mathfrak{U}'$ that $\mathfrak{v}'L = \mathfrak{u}'L$ and therefore $\mathfrak{u}L = \mathfrak{v}L = \mathfrak{t}L$ which implies $\tilde{q} \in \tilde{\mathcal{S}}$, as required.

Letting $\dim\mathcal{P}(\mathfrak{B}) = 3$, we see that $\mathfrak{B}_{Z(A)}$ is an underlying vector space of $\tilde{\mathcal{P}}$. Define a multiplication

$$\mathfrak{B}\times L \to \mathfrak{B},\ (\mathfrak{u}+\mathfrak{u}'a, x) \mapsto \mathfrak{u}x+\mathfrak{u}'xa.$$

Then $\mathfrak{B}$ becomes a right vector space $\mathfrak{B}_L$ over L. The 1-dimensional subspaces of $\mathfrak{B}_L$ are exactly the elements of the partition of $\mathfrak{B}$ induced by $\tilde{\mathcal{S}}$. Now the mapping α, as is given by (5), is an L-linear bijection of $\mathfrak{U}_L$ onto $\mathfrak{B}_L$. Hence $\dim\mathfrak{B}_L = 2$ which in turn shows that the kernel of $\tilde{\mathcal{S}}$ is isomorphic to L.

In order to show that $\tilde{\mathcal{S}}$ is also a dual spread, take any plane $\mathcal{E}$ of $\tilde{\mathcal{P}}$. The line $\mathcal{U}$ cannot be contained in $\mathcal{E}$, since then $\tilde{\mathcal{E}}$ being a Baer subplane of $\mathcal{E}$ would imply $\mathcal{U}\cap\tilde{\mathcal{E}} = \mathcal{U}\cap\tilde{\mathcal{P}}$ to be non-empty. Hence $\mathcal{E}\cap\mathcal{U}$ is a point off $\tilde{\mathcal{P}}$ and the only line of $\tilde{\mathcal{P}}$ passing through it has to be in $\mathcal{E}$.■

We close with some remarks:

A well known example, where this theorem can be applied, is the skew field $\mathbb{H}$ of real quaternions. Here the centralizer of *any* $a \in \mathbb{H}\setminus Z(\mathbb{H})$ is - up to isomorphism - a field of complex numbers.

Results similar to theorem 3 on spreads of Baer subspaces of pappian projective spaces can be found in [1], [9]: A. Beutelspacher and J. Ueberberg [1] show that *every* t-dimensional subspace which is skew to a Baer subspace $\tilde{\mathcal{P}}$ of $\mathcal{P} = \mathrm{PG}(2t+1,L)$ yields a 1-spread of $\tilde{\mathcal{P}}$. Theorem 3 states that *some* subspaces of $\mathcal{P}(\mathfrak{B})$ share this property. But on the other hand there is no restriction on the dimension of $\mathfrak{B}$.

Now let $\dim\mathcal{P}(\mathfrak{B}) = 3$. In [7] (Definition 2.4) N. Knarr generalizes the concept of "indicator set" (due to A. Bruen [2]) to the infinite case. Moreover it is shown how to obtain a dual spread from such an indicator set. Our construction of $\tilde{\mathcal{S}}$ and the proof that $\tilde{\mathcal{S}}$ is a dual spread fit into Knarr's general concept: Denote by $\mathcal{I} \in \mathbf{S}_{\mathrm{II}}$ any transversal line and fix any line $\ell \in \mathbf{S}_I$. Then $\mathcal{I}\setminus\ell$ is an indicator set within the affine plane $(\mathcal{I}\vee\ell)\setminus\ell$.

Still assuming $\dim\mathcal{P}(\mathbb{B}) = 3$, take two different generators $\ell_0,\ell_1 \in \mathbf{S}_I$ and a plane $\mathcal{E} \supset \ell_0$ which does not contain any transversal line. By [10,325-329], $(\mathcal{E}\cap\mathbf{S})\setminus\ell_0$ is the proper part of a degenerate conic or, in Segre's terminology, a C-configuration. Suppose that

$$\mathrm{CR}(\mathcal{E}\cap\ell_1, \mathcal{I}\cap\ell_1, \mathcal{U}\cap\ell_1, \mathcal{U}'\cap\ell_1) = \hat{b}$$

for some $b \in L$, whence $b \notin Z(L)$. When $\mathcal{E}$ is a plane of $\tilde{\mathcal{P}}$, then $\hat{a} = \hat{b}$ and $(\mathcal{E}\cap\mathbf{S})\setminus\ell_0$ is an affine part of the Baer subplane $\mathcal{E}\cap\tilde{\mathcal{P}} \subset \mathcal{E}$. However, when $\mathcal{E}$ does not belong to $\tilde{\mathcal{P}}$, then $\hat{a} \neq \hat{b}$ and $(\mathcal{E}\cap\mathbf{S})\setminus\ell_0$ is an indicator set of $\tilde{\mathcal{I}}$ within the plane $\mathcal{E}$. This indicator set again may be again be an affine Baer subplane: By [5] the set $(\mathcal{E}\cap\mathbf{S})\setminus\ell_0$ is an affine Baer subplane of the affine plane $\mathcal{E}\setminus\ell_0$ if, and only if, the parameter b is quadratic over the centre of L.

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