

# Blocking sets in chain geometries

*Andrea Blunck*<sup>1</sup>     *Hans Havlicek*<sup>2</sup>     *Corrado Zanella*<sup>3</sup>

## Contents

1. Introduction (3).
2. Examples of chain geometries (4).
3. Finite chain geometries (5).
4. Blocking sets (6).

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<sup>1</sup>Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany. e-mail [andrea.blunck@math.uni-hamburg.de](mailto:andrea.blunck@math.uni-hamburg.de)

<sup>2</sup>Hans Havlicek, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8–10, A-1040 Wien, Austria. e-mail [havlicek@geometrie.tuwien.ac.at](mailto:havlicek@geometrie.tuwien.ac.at)

<sup>3</sup>Corrado Zanella, Dipartimento di Tecnica e Gestione dei Sistemi Industriali, Università di Padova, Stradella S. Nicola, 3, I-36100 Vicenza, Italy. e-mail [corrado.zanella@unipd.it](mailto:corrado.zanella@unipd.it)



## 1. Introduction

Let  $R$  be a ring<sup>1</sup>. The *projective line over  $R$* , denoted by  $\mathbb{P}(R)$ , is the set of all submodules of  $R^2$  of type  $R(a, b)$ , where  $(a \ b)$  is the first row of some invertible  $2 \times 2$  matrix over  $R$ .

Suppose now that a field  $K$  is contained in  $R$ , as a subring. The (*generalized*) *chain geometry* associated with  $K$  and  $R$ , denoted by  $\Sigma(K, R)$ , is the structure whose *points* are the elements of  $\mathbb{P}(R)$  and whose *blocks* (called *chains*) are the sets  $\mathbb{P}(K)^g$  with  $g \in \text{GL}_2(R)$ . Here  $\mathbb{P}(K)$  is embedded in  $\mathbb{P}(R)$  by means of  $K(a, b) \hookrightarrow R(a, b)$ . Roughly speaking, the chains are projective lines over  $K$  contained in the projective line over  $R$ .

The classical example of a chain geometry is  $\Sigma(\mathbb{R}, \mathbb{C})$ , or, by generalizing a little,  $\Sigma(K, R)$  where  $R$  is a field and  $[R : K] = 2$ . In this case  $\Sigma(K, R)$  is usually called *Miquelian Möbius plane*.

Two points  $R(a, b)$  and  $R(c, d)$  in  $\mathbb{P}(R)$  are called *distant*, in symbols  $R(a, b) \triangle R(c, d)$ , when  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R)$ . We have  $R(a, b) \triangle R(c, d)$  if, and only if, both points are on a common chain. The group  $\text{GL}_2(R)$  acts transitively on the set of all triples of mutually distant points.

A *blocking set* in a geometry of points and blocks is a set, say  $B$ , of points, such that every block contains at least one element of  $B$ . The most investigated question regarding the blocking sets concerns their minimum size. In this paper we give some basic results on this problem for a finite chain geometry. More precisely, in Section 2, two examples of chain geometries are given. Section 3 is concerned with the number  $\lambda_i$  of blocks containing  $i$  given mutually distant points,  $i = 0, 1, 2, 3$ . In Section 4, lower bounds for the size of a blocking set in  $\Sigma(K, R)$  are given both in the general case (see (4.2)) and in case  $R$  is a local ring (Theorem 4.1). Two examples attaining the general lower bound are exhibited. It is also shown that it is possible to construct blocking sets in chain geometries, starting from a blocking set in a Möbius geometry (Theorem 4.3).

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<sup>1</sup>The ring  $R$  is assumed to be associative, with a unit element 1, which is inherited by subrings. The trivial case  $1 = 0$  is excluded.

## 2. Examples of chain geometries

We give a short description of two classes of chain geometries we will deal with in this paper.

*Example 2.1* - Let  $R$  be the direct product  $K \times K$ . Then  $R$  has precisely two nontrivial ideals:  $K(1, 0)$  and  $K(0, 1)$ . The ring  $R$  becomes a  $K$ -algebra via the embedding  $x \mapsto (x, x)$  of  $K$  into  $R$ . A submodule  $R(a, b)$  of  $R^2$  is a point if and only if  $a$  and  $b$  do not belong to a common nontrivial ideal. Let  $x_0, x_1, x_2$  and  $x_3$  denote the homogeneous coordinates in  $\text{PG}(3, K)$ . For  $a, b \in R$  set  $a = (a_1, a_2), b = (b_1, b_2)$ . The map  $\psi : \mathbb{P}(R) \rightarrow \text{PG}(3, K)$  defined by

$$R(a, b)^\psi = K(a_1b_2, a_2b_1, a_1a_2, b_1b_2)$$

is a bijection between  $\mathbb{P}(R)$  and the hyperbolic quadric  $\mathcal{Q}$  in  $\text{PG}(3, K)$  of equation  $x_0x_1 - x_2x_3 = 0$ . The image of  $\mathbb{P}(K)$  under  $\psi$  is the intersection of  $\mathcal{Q}$  with the plane  $x_0 - x_1 = 0$ . Since a plane of equation  $u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0$  is tangent to  $\mathcal{Q}$  if and only if  $u_0u_1 - u_2u_3 = 0$ , we see that  $\mathbb{P}(K)^\psi$  is a nondegenerate conic. On the other hand, the points which are non-distant from  $R(1, 0)$  are precisely those on the tangent plane section given by  $x_3 = 0$ . Next, the mapping  $\varphi : (\text{GL}_2(K))^2 \rightarrow \text{GL}_2(R)$  defined by setting, for every  $M, N \in \text{GL}_2(K)$ ,

$$((a_1, a_2) (b_1, b_2)) (M, N)^\varphi = ((\alpha_1, \alpha_2) (\beta_1, \beta_2)),$$

$$(\alpha_1 \ \beta_1) = (a_1 \ b_1)M, \quad (\alpha_2 \ \beta_2) = (a_2 \ b_2)N,$$

is an isomorphism between the direct product of  $\text{GL}_2(K)$  with itself and  $\text{GL}_2(R)$ .

For  $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \in \text{GL}_2(K)$ , the action of  $(M, 1)^\varphi$  on  $\mathcal{Q}$  is the restriction of the projectivity of  $\text{PG}(3, K)$  defined by

$$(x_0 \ x_1 \ x_2 \ x_3) \mapsto (x_0 \ x_1 \ x_2 \ x_3) \begin{pmatrix} m_1 & 0 & 0 & m_2 \\ 0 & m_4 & m_3 & 0 \\ 0 & m_2 & m_1 & 0 \\ m_3 & 0 & 0 & m_4 \end{pmatrix},$$

that fixes  $\mathcal{Q}$ . A similar property is satisfied by  $(1, N)^\varphi$  where  $N \in \text{GL}_2(K)$ . It easily follows that (i) the action of each element of  $\text{GL}_2(R)$  on  $\mathcal{Q}$  can be

represented by an element of  $\text{PGL}(4, K)$  fixing  $\mathcal{Q}$ , (ii) the images of the chains under the embedding  $\psi$  are precisely the nondegenerate conics contained in  $\mathcal{Q}$ , and (iii) two distinct points  $p, q \in \mathbb{P}(R)$  are distant if, and only if, the line through  $p^\psi$  and  $q^\psi$  is not contained in  $\mathcal{Q}$ .

More generally, if  $R$  is a kinematic algebra, i.e. for each  $x \in R$  two elements  $k, l \in K$  exist such that  $x^2 = kx + l$ , and  $K \neq \mathbb{F}_2$ , then the points of  $\Sigma(K, R)$  can be represented as points of a quadric  $\mathcal{Q}'$  in a projective space over  $K$  of suitable dimension, and distant points correspond to points that are not conjugate with respect to  $\mathcal{Q}'$  [6]. See also [5, Section 6.2].

*Example 2.2* - Let  $R$  be a local ring, and let  $R^*$  be the set of all units in  $R$ . Each point, say  $R(a, b)$ , of the projective line  $\mathbb{P}(R)$  has the property that at least one of the two elements  $a, b$  is invertible. Because since  $R \setminus R^*$  is an ideal the existence of an inverse matrix  $\begin{pmatrix} x & * \\ y & * \end{pmatrix}$  would otherwise lead to the contradiction  $1 = ax + by \in R \setminus R^*$ . So  $\mathbb{P}(R)$  is the disjoint union

$$(2.1) \quad \mathbb{P}(R) = \{R(x, 1) \mid x \in R\} \cup \{R(1, z) \mid z \in R \setminus R^*\}.$$

In this case the complementary relation of  $\Delta$ , which we will denote by  $\parallel$  (*parallelism*), is an equivalence relation. More explicitly, this means for arbitrary  $x, y \in R, z, w \in R \setminus R^*$ :

$$(2.2) \quad R(1, z) \parallel R(1, w); \quad R(x, 1) \not\parallel R(1, z); \quad (R(x, 1) \parallel R(y, 1) \Leftrightarrow x - y \in R \setminus R^*).$$

Using the description in (2.2) one can easily see that  $\parallel$  in fact is an equivalence relation.

### 3. Finite chain geometries

From now on we assume that  $R$  is finite. So,  $K = \mathbb{F}_q$ ,  $q$  a prime power, and  $R$  is in a natural way a left vector space over  $\mathbb{F}_q$ . Define  $d = \dim_{\mathbb{F}_q} R$ . Since  $\text{GL}_2(R)$  acts transitively on the triples of mutually distant points, the number of chains containing  $i$  given mutually distant points,  $i = 0, 1, 2, 3$ , is a constant, say  $\lambda_i$ . The problem to determine the numbers  $\lambda_0, \dots, \lambda_3$  is intricate. However, to our purposes it is enough to describe their ratios.

**Proposition 3.1.** *Let  $v$  be the number of points of  $\Sigma(\mathbb{F}_q, R)$ . Denote by  $R^*$  the set of units of  $R$ , and let  $\#R = q^d$ ,  $\#R^* = r^*$ . Then*

$$(3.1) \quad \lambda_0 = \frac{vq^{d-1}r^*}{q^2 - 1} \lambda_3;$$

$$(3.2) \quad \lambda_1 = \frac{q^{d-1}r^*}{q - 1} \lambda_3;$$

$$(3.3) \quad \lambda_2 = \frac{r^*}{q - 1} \lambda_3.$$

PROOF – The points which are distant from  $R(1, 0)$  are precisely those in the form  $R(a, 1)$ ,  $a \in R$ ; since they are all distinct and  $\text{GL}_2(R)$  acts transitively on  $\mathbb{P}(R)$ , we have that each point in  $\mathbb{P}(R)$  is distant from precisely  $q^d$  points. Similarly, the points which are distant from both  $R(1, 0)$  and  $R(0, 1)$  form the set  $\{R(1, a) \mid a \in R^*\}$ . There are  $r^*$  of such points.

Assume now that  $p_1$  and  $p_2$  are two distant points. Each chain has precisely  $q + 1$  points. So, by counting in two ways the number  $M$  of pairs  $(p, C)$ , where  $p$  is a point distant from both  $p_1$  and  $p_2$ , and  $C$  is a chain through  $p$ ,  $p_1$  and  $p_2$ , we obtain  $M = r^* \lambda_3 = (q - 1) \lambda_2$  and this gives (3.3). A similar argument yields (3.2) and (3.1).  $\square$

For sake of completeness we mention that  $\lambda_3 = r^*/\#N$ , where  $N = \{n \in R^* \mid n^{-1}K^*n = K^*\}$  is the normalizer of  $K^*$  in  $R^*$ . See e.g. [4].

Since all points described in (2.1) are in  $\mathbb{P}(R)$ , even if  $R$  is not local, we see that in general

$$(3.4) \quad v \geq 2q^d - r^*.$$

#### 4. Blocking sets

A *blocking set* in  $\Sigma(\mathbb{F}_q, R)$  is a set  $B$  of points, such that every chain contains at least one element of  $B$ . A trivial lower bound for the size of  $B$ , holding in each geometry where the number of blocks through a point is a constant, is

$$(4.1) \quad \#B \geq \frac{\lambda_0}{\lambda_1},$$

whence, by (3.1), (3.2), and (3.4),

$$(4.2) \quad \#B \geq \left\lceil \frac{2q^d - r^*}{q + 1} \right\rceil.$$

The question arises, whether (4.2) can be improved for all  $\Sigma(\mathbb{F}_q, R)$  due to its algebraic definition in terms of  $\mathbb{F}_q$  and  $R$ . The answer is negative: take the geometric model  $Q^+(3, q)$  of  $\Sigma(\mathbb{F}_q, \mathbb{F}_q \times \mathbb{F}_q)$ . Since each line of  $Q^+(3, q)$  is a blocking set, we see that (4.2) is sharp. A further example of a blocking set for which in (4.2) the equality holds will be dealt with in case (i) of theorem 4.1. For this reason we have to investigate blocking sets in particular chain geometries.

The case in which  $\Sigma(\mathbb{F}_q, R)$  is a Möbius plane has been dealt with in [1, 2, 7] (actually, the results in these papers hold for arbitrary 3- $(q^2 + 1, q + 1, 1)$ -designs). In the quoted papers it is proved that if  $B$  is a blocking set in a Möbius plane of order  $q$ , then

$$(4.3) \quad \#B \geq 2q - 1;$$

furthermore,  $\#B \geq 2q$  for  $q \geq 4$ . Examples of blocking sets attaining the lower bounds are known only for  $q \leq 5$  and were found and classified by means of a computer search [7]. If more generally  $R$  is a local ring we can give a generalization of (4.3) for sufficiently large  $q$ . To this end we use a polynomial of constant sign introduced in [2].

Let  $\theta_n = (q^{n+1} - 1)/(q - 1)$  for  $n \in \mathbb{N} \cup \{-1\}$ .

**Theorem 4.1.** *Let  $B$  be a blocking set in the chain geometry  $\Sigma(\mathbb{F}_q, R)$ , where  $R$  is a local ring, and let*

$$(4.4) \quad d = \dim_{\mathbb{F}_q} R, \quad \delta = \dim_{\mathbb{F}_q} (R \setminus R^*),$$

where  $R^*$  denotes the set of units of  $R$ . Then

(i) if  $\delta = d - 1$ , then  $\#B \geq q^{d-1}$ ; the equation  $\#B = q^{d-1}$  holds if, and only if,  $B$  is a parallel class;

(ii) if  $d > 2$ ,  $\delta = d - 2$  and  $\varepsilon > 0$ , then  $\#B > 2q^{d-1} - (\frac{7}{2} + \varepsilon)q^{d-2}$  for  $q$  sufficiently large;

(iii) if  $d > 2$ ,  $\delta < d - 2$  and  $\varepsilon > 0$ , then  $\#B > 2q^{d-1} - (1 + \varepsilon)q^{d-2}$  for  $q$  sufficiently large.

PROOF – For each point  $p$ , let  $[p]$  denote the related parallel class. We have  $\#[p] = q^\delta$ .

(i) The first assertion follows from (4.2).

Now assume that  $B$  is a blocking set such that  $\#B = q^{d-1}$ . Let  $p$  be a point outside  $B$ . Every chain through  $p$  intersects  $B$ , so  $B \setminus [p]$  contains at least  $\lambda_1/\lambda_2 = q^{d-1}$  points. Therefore  $[p] \cap B = \emptyset$ . This holds for any point not in  $B$ , so  $B$  is a parallel class.

(ii), (iii) Let  $x = \#B$ . Denote by  $n_i$  the number of chains meeting  $B$  in exactly  $i$  points. Since  $B$  is a blocking set,  $n_0 = 0$ . By (3.1), taking into account  $v = q^d + q^\delta$ , we have

$$(4.5) \quad \sum_{i \geq 1} n_i = \lambda_0 = \left( q^{2(d-\delta-1)} + q^{2(d-\delta-2)} + \dots + q^2 + 1 \right) q^{d+2\delta-1} \lambda_3.$$

Computing in two ways the number of the ordered pairs  $(p, C)$ ,  $C$  a chain,  $p \in B \cap C$ , we obtain

$$(4.6) \quad \sum_{i \geq 1} i n_i = x \lambda_1 = x q^{d+\delta-1} \theta_{d-\delta-1} \lambda_3.$$

Analogously, by taking into account the ordered triples  $(p_1, p_2, C)$  and quadruples  $(p_1, p_2, p_3, C)$ , where the  $p_i$ s are distinct points of  $B$  incident with  $C$ , we have

$$(4.7) \quad \sum_{i \geq 1} i(i-1) n_i \geq x(x - q^\delta) \lambda_2 = x(x - q^\delta) \theta_{d-\delta-1} q^\delta \lambda_3;$$

$$(4.8) \quad \sum_{i \geq 1} i(i-1)(i-2) n_i \leq x(x-1)(x-2) \lambda_3.$$

The polynomial

$$P(i) = (i-1)(i-3)(i-4) = i(i-1)(i-2) - 5i(i-1) + 12i - 12,$$

introduced in [2], is non-negative for all positive integers  $i$ . From (4.5)–(4.8), it follows

$$(4.9) \quad \begin{aligned} 0 &\leq \frac{1}{\lambda_3} \sum_{i \geq 1} n_i P(i) \leq \\ &\leq x(x-1)(x-2) - 5x(x - q^\delta) \theta_{d-\delta-1} q^\delta + 12x q^{d+\delta-1} \theta_{d-\delta-1} \\ &\quad - 12 \left( q^{2(d-\delta-1)} + q^{2(d-\delta-2)} + \dots + q^2 + 1 \right) q^{d+2\delta-1}. \end{aligned}$$



Assume

$$(4.10) \quad x = 2q^{d-1} - kq^{d-2}.$$

Since  $d > 2$ , by (4.9) and (4.10) we obtain

$$(4.11) \quad 0 \leq (4 - 4k)q^{3d-4} + 10q^{2d+\delta-2} + (\text{terms of degree } < 3d - 4)$$

and this implies (ii) and (iii).  $\square$

*Remark 4.1* - In the previous proof, actually only the combinatorial structure of  $\Sigma(\mathbb{F}_q, R)$  is essential, and such structure is a  $3-(q^\delta, q+1, \lambda_3)$ -divisible design with  $q^d + q^\delta$  points. See [4] for generalities on divisible designs.

*Remark 4.2* - A proposition like (i), characterizing some geometric configurations as blocking sets of minimum size, is often called a *Bose-Burton type theorem*.

*Remark 4.3* - If  $K$  and  $F$  are fields with  $K \subseteq F$  and  $[F : K] = d$ , then  $\Sigma(K, F)$  is called a *d-dimensional Möbius geometry over  $K$* . Theorem 4.1 gives in particular a lower bound for the blocking sets in the finite Möbius geometries.

*Remark 4.4* - In case the term of degree  $3d - 4$  in (4.11) vanishes, the term of degree  $3d - 5$  always turns out to be positive, with one exception given by  $d = 3$  and  $\delta = 0$ . By substituting  $x = 2q^2 - q + t$  in (4.9) we obtain

$$(4.12) \quad 0 \leq (-1 + 4t)q^4 + (19 - 10t)q^3 + (-11 - 2t + t^2)q^2 \\ + (-7 + 21t - 8t^2)q + (7t - 8t^2 + t^3),$$

whence

**Theorem 4.2.** *Let  $B$  be a blocking set in the three-dimensional Möbius geometry over  $\mathbb{F}_q$ . Then  $\#B \geq 2q^2 - q - 2$ . Furthermore,  $\#B \geq 2q^2 - q - 1$  for  $q \geq 4$ ,  $\#B \geq 2q^2 - q$  for  $q \geq 7$ , and  $\#B \geq 2q^2 - q + 1$  for  $q \geq 19$ .*

It is not clear whether there exist blocking sets of size near to the lower bounds given in theorem 4.1. In [3, 8] the existence of blocking sets in the Möbius planes of size  $O(q \log q)$  is proved. The following theorem allows to construct blocking sets in generalized chain geometries, starting from blocking sets in Möbius geometries.

**Theorem 4.3.** *Let  $R$  be a local ring, and  $F = R/(R \setminus R^*)$ . If  $\Sigma(\mathbb{F}_q, F)$  contains a blocking set of size  $x$ , then  $\Sigma(\mathbb{F}_q, R)$  contains a blocking set of size  $xq^\delta$  ( $\delta$  as in (4.4)).*

PROOF – Let  $I = R \setminus R^*$  and, for  $R(a, b) \in \mathbb{P}(R)$ ,  $R(a, b)^\varphi = F(a+I, b+I)$ . We obtain a well-defined map  $\varphi : \mathbb{P}(R) \rightarrow \mathbb{P}(F)$  such that (i) if  $C$  is a chain in  $\Sigma(\mathbb{F}_q, R)$ , then  $C^\varphi$  is a chain in  $\Sigma(\mathbb{F}_q, F)$ , (ii) for  $p, q \in \mathbb{P}(R)$ , it holds  $p^\varphi = q^\varphi$  if and only if  $p \parallel q$ . By such properties, if  $B$  is a blocking set in  $\Sigma(\mathbb{F}_q, F)$  with  $\#B = x$ , then  $B = B_0^\varphi$ , where  $B_0$  can be chosen as the union of exactly  $x$  parallel classes, each of size  $q^\delta$ . This  $B_0$  is a blocking set of size  $xq^\delta$  in  $\Sigma(\mathbb{F}_q, R)$ .  $\square$

**Corollary 4.1.** *If  $R$  is a local ring and  $d = \dim_{\mathbb{F}_q} R = 2 + \delta$ , then  $\Sigma(\mathbb{F}_q, R)$  contains a blocking set of size  $O(q^{d-1} \log q)$ .*

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