A generalization of Brauner's theorem on linear mappings

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ABSTRACT. We introduce and characterize weak linear mappings of Desarguesian projective spaces. This gives a unified treatment of several classes of mappings like linear mappings, injective lineations and embeddings that have been discussed separately before.

1. Introduction

The second fundamental theorem of projective geometry says that collineations of Desarguesian projective spaces $\mathcal{P},\mathcal{P}'$ (of dimension ≥ 2) are exactly those mappings that are induced by semilinear bijections of underlying vector spaces. This well-known result permits several generalizations.

H. BRAUNER [4] gives a geometric characterization of *linear mappings*, i.e. those mappings that are induced by (not necessarily injective) semilinear mappings of vector spaces. Cf. also [5], [6], [7], [8], the books of N. BOURBAKI [3], O. GIERING [17] and papers by R. FRANK [16], J. HARTL [18], [19], H. LENZ [28], K. SÖRENSEN [38], H. TIMMERMANN [40] and the author [20].

Another generalization is concerned with injective *lineations* (preserving collinearity of points). We refer to the book [1] of W. BENZ and articles by A. BREZULEANU, D.C. RADULESCU [9], [10] and D.S. CARTER, A. VOGT [12], [13]. Those injective lineations that are also preserving non-collinearity of points are called *embeddings* and were characterized by M. LIMBOS [30], [31], [32] for finite Desarguesian projective spaces. See also the examples of embeddings and injective lineations of projective spaces given by J.M.N. BROWN [11], J.A. THAS [39] and the author [21], [22]. Since the pre-image of every line of

$\mathcal{P}'$ is a subspace of $\mathcal{P}$, there is also a relationship to *weak projective spaces* studied by H. KARZEL and M. MARCHI in [27].

In the present paper we introduce the concept of *weak linear mapping* of projective spaces as a generalization of the various mappings mentioned above. Our definition is in terms of vector spaces, but we shall give a geometric characterization of those weak linear mappings whose image set contains a triangle. The crucial condition of this characterization is just a slight modification of the initial result of H. BRAUNER. What is not covered by our approach are, e.g., those non-injective lineations of Desarguesian projective spaces that arise from valuations. See [9], [10], [12], [13], [33], [34] and [41] for results and references on those mappings.

Finally, we refer to the master's report of T. PFEIFFER [35] on mappings of more general projective geometries defined via lattices (based upon papers by S.E. SCHMIDT [36], [37]); see also articles by H.H. CRAPO, C.G. ROTA [14, ch. 9], U. FAIGLE [15] and D.A. HIGGS [24], [25].

2. Weak linear mappings

2.1. Suppose that A and B are sets. We shall write

$$\varphi : A \rightarrowtail B,\ x \mapsto x^\varphi$$

if φ is a mapping with domain[1] $\operatorname{dom}\varphi \subset A$; when writing x^φ we always assume that $x \in \operatorname{dom}\varphi$. If $\operatorname{dom}\varphi = A$ then instead of "$\rightarrowtail$" also an ordinary arrow "$\to$" is used. Given any subset $M \subset A$ then

$$M^\varphi := \{x^\varphi \mid x \in M \cap \operatorname{dom}\varphi\} \subset B$$

is well-defined[2]. Thus $M \subset A\backslash\operatorname{dom}\varphi$ if, and only if, $M^\varphi = \emptyset$.

2.2. Let $\mathbf{V}$ be a right vector space over a field[3] K and $\mathbf{W}$ a right vector space over a field L. If $\zeta : K \to L$ and $f : \mathbf{V} \to \mathbf{W}$ are mappings such that

$$(\mathbf{x}+\mathbf{y})^f = \mathbf{x}^f+\mathbf{y}^f,\ (\mathbf{x}a)^f = \mathbf{x}^f a^\zeta \ \text{ for all } \mathbf{x},\mathbf{y} \in \mathbf{V},\ a \in K$$

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[1]The symbol "$\subset$" does not mean strict inclusion.

[2]To make the definition unambiguous, assume that $M \subset A$ never is equivalent to $M \in A$.

[3]In this paper *field* always stands for *not necessarily commutative field*.

then f is called a *weak semilinear mapping* with respect to ζ.

The zero mapping $\mathbf{V} \to \mathbf{W}$ is weak semilinear with respect to every mapping $K \to L$. If f is weak semilinear and if $\mathbf{V}^f \neq \{\mathbf{o}\}$ then ζ is a ring homomorphism as follows in a well known way by calculating $\mathbf{x}(a+b)$ and $\mathbf{x}(ab)$ $(\mathbf{x}\in\mathbf{V},\ \mathbf{x}^f\neq\mathbf{o},\ a,b\in K)$ in two different ways. Moreover ζ is injective, since $K^\zeta \neq \{0\}$.

We obtain the usual definition of a semilinear (or linear) mapping by assuming that ζ is surjective (or that $K = L$, $\zeta = \mathrm{id}_K$).

Given a weak semilinear mapping f with respect to a monomorphism ζ then there exists a right vector space $\mathbf{Y}$ over the field $F := K^\zeta$ and a semilinear bijection $i : \mathbf{V} \to \mathbf{Y}$ with respect to ζ (regarded as isomorphism $K \to F$). Define

$$\gamma : \mathbf{Y}\times L \to \mathbf{W},\ (\mathbf{y},z) \mapsto \mathbf{y}^{i^{-1}f}z.$$

This γ is biadditive and $(\mathbf{y}c,z)^\gamma = (\mathbf{y},cz)^\gamma = (\mathbf{y},1)^\gamma cz$ for all $\mathbf{y} \in \mathbf{Y}$, $c \in F$, $z \in L$. We deduce from the universal property of the tensor product $\mathbf{Y}\otimes_F L$ (cf., e.g., [3,§3]) that there exists an L-linear mapping

$$h : (\mathbf{Y}\otimes_F L) \to \mathbf{W} \text{ such that } \mathbf{x}^f = (\mathbf{x}^i\otimes 1)^h \text{ for all } \mathbf{x} \in \mathbf{V}.$$

Reversing this construction gives a procedure to construct weak semilinear mappings.

The following result, due to W. Zick [42][4], will be used later: An additive mapping $f : \mathbf{V} \to \mathbf{W}$ is weak semilinear provided that, firstly,

$$(\mathbf{x}K)^f \subset (\mathbf{x}^f)L \text{ for all } \mathbf{x} \in \mathbf{V}\setminus\{\mathbf{o}\}$$

and, secondly, the image of f contains two linearly independent vectors.

2.3. Write $\mathcal{P}(\mathbf{X})$ for the projective space on any right vector space $\mathbf{X}$. The elements of $\mathcal{P}(\mathbf{X})$ are the 1-dimensional subspaces of $\mathbf{X}$ and are called points. Subspaces of $\mathcal{P}(\mathbf{X})$ are regarded as sets of points and have the form $\mathcal{P}(\mathbf{X}_1)$ with $\mathbf{X}_1$ being a subspace of the vector space $\mathbf{X}$. Every weak semilinear mapping $f : \mathbf{V} \to \mathbf{W}$ (cf. 2.2) gives rise to a *weak linear mapping*

$$\varphi : \mathcal{P}(\mathbf{V}) \rightarrowtail \mathcal{P}(\mathbf{W}),\ \mathbf{x}K \mapsto (\mathbf{x}^f)L \text{ for all } \mathbf{x} \in \mathbf{V}\setminus\ker f.$$

The domain of φ is $\mathcal{P}(\mathbf{V})\setminus\mathcal{P}(\ker f)$. Given subspaces $\mathcal{U},\mathcal{T} \subset \mathcal{P}(\mathbf{V})$ then

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[4]Cf. the proof of the theorem in that paper rather than the theorem itself.

$$(\mathcal{U}\vee\mathcal{T})^\varphi \subset \operatorname{span}\big((\mathcal{U}\vee\mathcal{T})^\varphi\big) = \operatorname{span}(\mathcal{U}^\varphi)\vee\operatorname{span}(\mathcal{T}^\varphi), \tag{1}$$

where $\vee$ stands for the operation of join in the lattice of subspaces of $\mathcal{P}(\mathbf{V})$ and $\mathcal{P}(\mathbf{W})$, respectively. If f is semilinear then φ is a *linear mapping*[5] and in formula (1) inclusion can be replaced by equality.

By 2.2 every weak linear mapping $\varphi : \mathcal{P}(\mathbf{V}) \rightarrowtail \mathcal{P}(\mathbf{W})$ permits a factorization into a collineation $\mathcal{P}(\mathbf{V}) \to \mathcal{P}(\mathbf{Y})$, a *strong embedding* (preserving independence; cf. [31]) $\mathcal{P}(\mathbf{Y}) \to \mathcal{P}(\mathbf{Y} \otimes_F L)$ and a linear mapping $\mathcal{P}(\mathbf{Y} \otimes_F L) \rightarrowtail \mathcal{P}(\mathbf{W})$.

3. A characterization of weak linear mappings

3.1. Suppose that $\mathcal{P}$ and $\mathcal{P}'$ are projective spaces and that $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ is any mapping. If $\mathcal{M} \subset \mathcal{P}$ then put

$$\mathcal{M}_\varphi := \operatorname{span}(\mathcal{M}^\varphi) \subset \mathcal{P}'.$$

Thus to every subspace $\mathcal{U} \subset \mathcal{P}$ there corresponds the subspace $\mathcal{U}_\varphi \subset \mathcal{P}'$. If $X \in \mathcal{P}$ is a point then either $X \in \operatorname{dom}\varphi$, whence[6] $\{X\}_\varphi = \{X^\varphi\}$, or $X \notin \operatorname{dom}\varphi$, whence $\{X\}_\varphi = \varnothing$. In [35], where mappings of more general geometric structurs are under discussion from a lattice-theoretical point of view, the term "point" is being used in different way. For example, in a projective geometry on an unitary module every cyclic submodule is called a point, whence there may be points of "different sizes". Within the context of this paper this would mean to consider the empty set as an "extra point" that is contained in every "ordinary point". Cf. also [4].

In 3.2 - 3.4 we shall suppose that φ satisfies the geometric condition

(WL1) $\quad (\{X\}\vee\{Y\})_\varphi = \{X\}_\varphi\vee\{Y\}_\varphi$ for all $X,Y \in \mathcal{P}$ with $X \neq Y$

motivated by (1). Later we shall impose the additional condition

(WL2) $\quad \mathcal{P}^\varphi$ contains a triangle.

But at this moment we refrain from assuming (WL2).

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[5]Other names are *collinear mapping* and *regular or singular collineation*. But this list is far from being complete. If f belongs to an inner automorphism of $K = L$ then φ is also attributed to be *projective*.

[6]In writing $\{X\}^\varphi$ we are not allowed to mix up (as is usually done) a point $X \in \mathcal{P}$ and the subspace $\{X\} \subset \mathcal{P}$. We shall, however, frequently write X_φ or $X\vee Y$ rather than $\{X\}_\varphi$ or $\{X\}\vee\{Y\}$, respectively, since no confusion can occur.

3.2. Let $A,B \in \mathcal{P}$ be distinct points and set $\ell := A \vee B$. There are four possibilities for $\ell^\varphi \subset \ell_\varphi$ and the restriction of φ to $\ell \cap \operatorname{dom}\varphi$:

(I) If $\ell \subset \operatorname{dom}\varphi$ and $A^\varphi = B^\varphi$ then $\varphi|\ell$ is a constant mapping.

(II) If $\ell \subset \operatorname{dom}\varphi$ and $A^\varphi \neq B^\varphi$ then $\varphi|\ell$ is injective, since otherwise, by (I), ℓ^φ would be a single point.

(III) If $A,B \notin \operatorname{dom}\varphi$ then $\ell^\varphi \subset \varnothing \vee \varnothing = \varnothing$, whence $\ell \subset \mathcal{P}\setminus\operatorname{dom}\varphi$.

(IV) If $A \in \operatorname{dom}\varphi$, $B \notin \operatorname{dom}\varphi$ then $A^\varphi \in \ell^\varphi \subset A^\varphi \vee \varnothing$, whence $\ell^\varphi = A^\varphi$. We infer from case (III) that $\ell \cap \operatorname{dom}\varphi = \ell\setminus\{B\}$.

According to this list we shall refer to a line of $\mathcal{P}$ as being of type (I) - (IV) (with respect to φ).

3.3. It is immediate from 3.2 that $\mathcal{P}\setminus\operatorname{dom}\varphi$ is a subspace of $\mathcal{P}$. Hence $\operatorname{dom}\varphi$ is a slit space. Given subspaces $\mathcal{U},\mathcal{T} \subset \mathcal{P}$ then

$$(\mathcal{U}\vee\mathcal{T})_\varphi = \operatorname{span}\big((\mathcal{U}\vee\mathcal{T})^\varphi\big) = \operatorname{span}\big((\mathcal{U}\cup\mathcal{T})^\varphi\big) = \operatorname{span}(\mathcal{U}^\varphi\cup\mathcal{T}^\varphi) = \operatorname{span}(\mathcal{U}_\varphi\cup\mathcal{T}_\varphi) = \mathcal{U}_\varphi\vee\mathcal{T}_\varphi.$$

The second sign of equality follows from 3.2, the others are obvious. If $\mathcal{U}'$ is a subspace of $\mathcal{P}'$ then its *extended pre-image*, viz. the set $\{X \in \mathcal{P} \mid X_\varphi \subset \mathcal{U}'\}$, is a subspace of $\mathcal{P}$.

Write $\mathcal{R}$ for a complement of $\mathcal{P}\setminus\operatorname{dom}\varphi$ with respect to $\mathcal{P}$ and

$$\pi : \mathcal{P} \rightarrowtail \mathcal{R},\ X \mapsto \big(X\vee(\mathcal{P}\setminus\operatorname{dom}\varphi)\big)\cap\mathcal{R} \qquad (X \in \operatorname{dom}\varphi)$$

for the projection with centre $\mathcal{P}\setminus\operatorname{dom}\varphi$ onto the subspace $\mathcal{R}$. Given a point $X \in \operatorname{dom}\varphi = \operatorname{dom}\pi$ then either $X \in \mathcal{R}$, whence $X = X^\pi$, or the line $X \vee X^\pi$ meets $\mathcal{P}\setminus\operatorname{dom}\varphi$ at a unique point, whence $X^\varphi = X^{\pi\varphi}$. Thus

$$\varphi = \pi(\varphi|\mathcal{R}) \text{ with } \mathcal{R} = \operatorname{dom}(\varphi|\mathcal{R}). \tag{2}$$

By virtue of this factorization we may restrict our attention to $\varphi|\mathcal{R} : \mathcal{R} \to \mathcal{P}'$. Every line in $\mathcal{R}$ is of type (I) or (II).

3.4. Let $\{A,B,C\} \subset \mathcal{R}$ be a triangle[7] and set $\mathcal{E} := A\vee B\vee C$. There are four possibilities for $\mathcal{E}^\varphi \subset \mathcal{E}_\varphi = \{A,B,C\}_\varphi$ and the restriction of φ to $\mathcal{E}$:

(I) If $A^\varphi = B^\varphi = C^\varphi$ then $\varphi|\mathcal{E}$ is a constant mapping.

(II) If $\{A,B,C\}_\varphi$ is a line of $\mathcal{P}'$ and if $\varphi|\mathcal{E}$ is not injective then assume

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[7]It would be sufficient to assume $\mathcal{E} \subset \operatorname{dom}\varphi$. The listing of possible cases for a plane that is not part of $\operatorname{dom}\varphi$ is left to the reader.

$A^\varphi = B^\varphi$. Hence $C^\varphi \neq A^\varphi$ and $A \vee B$ is a line of type (I). We claim that $\varphi|\big(\mathcal{E}\setminus(A\vee B)\big)$ is injective: Assume to the contrary that $X^\varphi = Y^\varphi$ for distinct points $X,Y \in \mathcal{E}\setminus(A\vee B)$. Then $X\vee Y$ meets $A\vee B$ at a point U, say, and $(X\vee Y)^\varphi = X^\varphi$ implies that $A^\varphi = U^\varphi = X^\varphi$. But $C\vee X$ or $C\vee Y$ meets $A\vee B$ at a point V, say, whence $A^\varphi = V^\varphi = X^\varphi = Y^\varphi = C^\varphi$, a contradiction.

(III) If $\{A,B,C\}_\varphi$ is a line of $\mathcal{P}'$ then $\varphi|\mathcal{E}$ may be injective.

(IV) If $\{A,B,C\}_\varphi$ is a plane then the restriction of φ to every side of the triangle $\{A,B,C\}$ is an injection, whence $\varphi|\mathcal{E}$ is injective and preserves non-collinearity of points in $\mathcal{E}$. Thus $\mathcal{E}^\varphi$ is a subplane of $\mathcal{E}_\varphi$.

We shall speak of planes of type (I) - (IV) (with respect to φ).

3.5. If g,h are lines of a plane $\mathcal{E}$ and $C \in \mathcal{E}\setminus(g\cup h)$ then $g \xrightarrow{C} h$ is to denote the perspectivity of g onto h with centre C. We shall need a

Lemma. *Let μ be a projectivity of a line ℓ in a Desarguesian projective space $\mathcal{P}$, $\dim\mathcal{P} \geq 2$. Assume that μ can be factorized in the form*

$$\ell \xrightarrow{C} f \xrightarrow{D} \ell \text{ with } C \neq D \text{ and } \ell \neq f.$$

Given distinct points $E,F \in \mathcal{P}\setminus\ell$ satisfying $(C\vee D)\cap\ell = (E\vee F)\cap\ell$ then there exists a line g which meets ℓ at $f\cap\ell$ such that μ equals the projectivity

$$\ell \xrightarrow{E} g \xrightarrow{F} \ell.$$

Proof. There exists a perspective collineation $\kappa : \mathcal{P} \to \mathcal{P}$ that fixes ℓ pointwise such that the lines $\ell, E\vee F, (C\vee D)^\kappa$ are coplanar and distinct. Set $Z := (E\vee C^\kappa)\cap(F\vee D^\kappa)$ and let $\rho : \mathcal{P} \to \mathcal{P}$ be a perspective collineation with centre Z, $C^\kappa \mapsto E$ and an axis $\mathcal{H}$ subject to $\mathcal{H}\cap(\ell\vee C^\kappa) = \ell$. Then $D^{\kappa\rho} = F$ and $g := f^{\kappa\rho}$ has the required properties. ∎

3.6. The following Theorem 1 is a collection of four propositions:

Theorem 1. *Let $\mathcal{R}$ and $\mathcal{P}'$ be projective spaces, and let $\mathcal{P}'$ be Desarguesian. Suppose that the mapping $\varphi : \mathcal{R} \to \mathcal{P}'$ satisfies conditions* (WL1) *and* (WL2)*.*

1. *$\mathcal{R}$ is a Desarguesian projective space, $\dim\mathcal{R} \geq 2$, and an underlying field of $\mathcal{R}$ is isomorphic to a subfield of an underlying field of $\mathcal{P}'$.*

Proof. There exists a triangle $\{A^\varphi,B^\varphi,C^\varphi\} \subset \mathcal{R}^\varphi$, say, whence $\{A,B,C\} \subset \mathcal{R}$ is a triangle too. Thus $\dim\mathcal{R} \geq 2$, $\mathcal{E} := A\vee B\vee C$ is a plane of type (IV) and $\mathcal{E}^\varphi$ is a subplane of $\mathcal{E}_\varphi$. Now the other assertions are obviously true. ∎

2. *Let* $\sigma : \ell \xrightarrow{A} m$ *be a perspectivity of distinct lines* $\ell, m \subset \mathcal{R}$ *and let* $\ell_\varphi = m_\varphi$ *be a line of* $\mathcal{P}'$. *Then the mapping*

$$\ell^\varphi \to m^\varphi,\ X^\varphi \mapsto X^{\sigma\varphi}\ (X \in \ell)$$

extends to a projectivity of ℓ_φ *which permits a factorization*

$$\ell_\varphi \xrightarrow{C'} \mathfrak{f}' \xrightarrow{D'} \ell_\varphi$$

for collinear points A^φ, C', D' *and a line* $\mathfrak{f}' \subset \mathcal{P}'$ *with* $\ell_\varphi \cap \mathfrak{f}' = (\ell \cap m)^\varphi$.

Proof. Both $\varphi|\ell$ and $\varphi|m$ are injective, whence the plane $\ell \vee m$ is of type (II) or (III). By $\#\ell \geq 3$, $X^\varphi \neq X^{\sigma\varphi}$ for at least one $X \in \ell$. It follows from (WL2) that there exists a point $C \in \mathcal{R}$ such that C^φ is off ℓ_φ. Then $A^\varphi \neq C^\varphi$ and there is a point $D \in (A \vee C)\setminus\{A, C\}$. Because of $\dim(\ell \vee m \vee C) = 3$, $\mathfrak{f} := (C \vee \ell) \cap (D \vee m)$ is a line. We infer from the collinearity of A, C, D that σ coincides with $\ell \xrightarrow{C} \mathfrak{f} \xrightarrow{D} m$; cf. figure 1. The planes $C \vee \ell$ and $D \vee m$ are of type (IV), whence putting

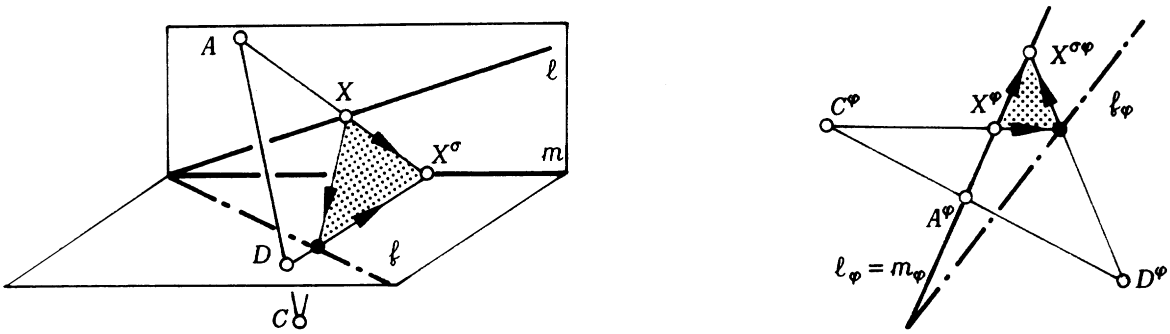

Fig. 1

$\mathfrak{f}_\varphi =: \mathfrak{f}'$, $C^\varphi =: C'$ and $D^\varphi =: D'$ completes the proof. ∎

3. *If there exists a hyperplane* $\mathcal{J}$ *of* $\mathcal{R}$ *such that* $\mathcal{J}_\varphi \neq \mathcal{R}_\varphi$ *then* $\mathcal{J}_\varphi$ *is a hyperplane of* $\mathcal{R}_\varphi$. *The restriction* $\varphi|(\mathcal{R}\setminus\mathcal{J})$ *is injective and* $(\mathcal{R}\setminus\mathcal{J})^\varphi \cap \mathcal{J}_\varphi = \emptyset$.

Proof. There exists a point $A \in \mathcal{R}$ such that $A^\varphi \notin \mathcal{J}_\varphi$, whence $A \notin \mathcal{J}$. Writing $\mathcal{R} = A \vee \mathcal{J}$ yields $\mathcal{R}_\varphi = A^\varphi \vee \mathcal{J}_\varphi$. Thus $\mathcal{J}_\varphi$ is a hyperplane of $\mathcal{R}_\varphi$. If $X, Y \in \mathcal{R}\setminus\mathcal{J}$ are distinct then $\ell := X \vee Y$ either is of type (I) and $X^\varphi = Y^\varphi = (\ell \cap \mathcal{J})^\varphi = \ell^\varphi \subset \mathcal{J}_\varphi$, or ℓ is of type (II) and $\varphi|\ell$ is injective. Putting $Y := A$ shows $X^\varphi \notin \mathcal{J}_\varphi$ for all $X \in \mathcal{R}\setminus\mathcal{J}$. Hence only the second possibility can occur. Thus $\varphi|(\mathcal{R}\setminus\mathcal{J})$ is injective and $(\mathcal{R}\setminus\mathcal{J})^\varphi \cap \mathcal{J}_\varphi = \emptyset$. ∎

4. *Suppose that* $\mathcal{P}' \neq \mathcal{R}_\varphi = \mathcal{H}_\varphi$ *for all hyperplanes* $\mathcal{H}$ *of* $\mathcal{R}$. *Choose a hyperplane* $\mathcal{J}'$ *of* $\mathcal{P}'$ *which contains* $\mathcal{R}_\varphi$ *and a point* $Z' \in \mathcal{P}'\setminus\mathcal{J}'$. *Denote by*

$$\omega : \mathcal{P}' \rightarrowtail \mathcal{J}'$$

the projection with centre Z' *onto* $\mathcal{J}'$*. Then there exists a mapping* $\lambda : \mathcal{R} \to \mathcal{P}'$ *which satisfies* (WL1) *and* (WL2) *such that*

$$X^{\varphi} = X^{\lambda\omega} \text{ for all } X \in \mathcal{R} \tag{3}$$

and there is a hyperplane of $\mathcal{R}$*, say* $\mathcal{J}$*, such that* $\mathcal{J}_{\lambda} = \mathcal{J}' \neq \mathcal{R}_{\lambda}$.

Proof. (a) Fix any hyperplane $\mathcal{J} \subset \mathcal{R}$ and any point $A \in \mathcal{R}\backslash\mathcal{J}$. Write $\mathcal{A}$ for the pre-image of A^{φ} and denote by

$$\psi : \mathcal{R} \rightarrowtail \mathcal{J}$$

the projection with centre A onto the hyperplane $\mathcal{J}$. In $\mathcal{P}'$ fix a point

$$A' \in (A^{\varphi} \vee Z')\backslash\{A^{\varphi}, Z'\}.$$

In a first step (cf. figure 2) we define a mapping

$$\tau : \{A\} \cup \mathcal{J} \cup (\mathcal{R}\backslash\mathcal{A}) \to \mathcal{P}', \; X \mapsto \begin{cases} A' & \text{if } X = A, \\ X^{\varphi} & \text{if } X \in \mathcal{J}, \\ (Z' \vee X^{\varphi}) \cap (A' \vee X^{\psi\varphi}) & \text{if } X \in \mathcal{R}\backslash(\mathcal{A} \cup \mathcal{J})). \end{cases} \tag{4}$$

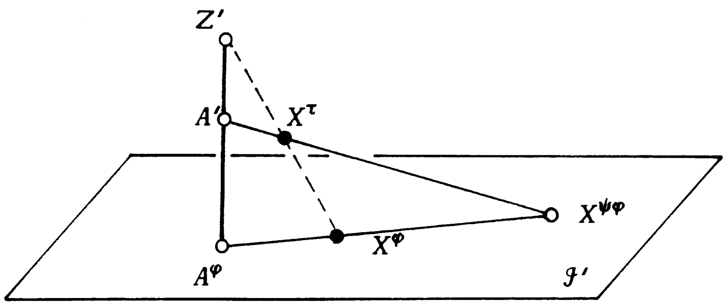

Fig. 2

This mapping τ will be extended in (c) to give the required mapping λ.

(b) We discuss the image set ℓ^{τ} of a line $\ell \subset \mathcal{R}$ that is neither a part of the hyperplane $\mathcal{J}$ nor contained in the subspace $\mathcal{A}$.

● Suppose that $\ell \cap \mathcal{A} = \varnothing$. We infer from $A \notin \ell$ that ℓ^{ψ} is a line of $\mathcal{J}$. The plane $\mathcal{E} := A \vee \ell$ cannot of type (I).

If $\mathcal{E}$ is of type (II) and ℓ^{φ} is a single point then $\ell^{\psi}{}_{\varphi} = \mathcal{E}_{\varphi}$ is a line through A^{φ}. Hence $\ell^{\tau} \subset Z' \vee \ell^{\varphi}$.

If $\mathcal{E}$ is of type (II) and $\ell^{\psi\varphi}$ is a single point then $\ell_{\varphi} = \mathcal{E}_{\varphi}$ is a line through A^{φ}. Hence $\ell^{\tau} \subset A' \vee \ell^{\psi\varphi}$.

If $\mathcal{E}$ is of type (II) and both ℓ_{φ} and $\ell^{\psi}{}_{\varphi}$ are lines of $\mathcal{P}'$, or if $\mathcal{E}$ is of type

(III), then we apply Theorem 1.2 to the mapping

$$\ell^{\varphi} \to \ell^{\psi\varphi},\ X^{\varphi} \mapsto X^{\psi\varphi}\ (X \in \ell)$$

and the Lemma to the extending projectivity and points A',Z'. This shows that ℓ^{τ} is part of a line that runs neither through A' nor through Z'.

If $\mathcal{E}$ is of type (IV) then $A^{\varphi} \notin \ell_{\varphi}$, $A^{\varphi} \notin \ell^{\psi}{}_{\varphi}$, $\ell_{\varphi} \neq \ell^{\psi}{}_{\varphi}$ and ℓ^{τ} is a subset of the line $(A'\vee\ell^{\psi}{}_{\varphi})\cap(Z'\vee\ell_{\varphi})$.

● Suppose that $\ell\cap\mathcal{A} =: A_1$ is a single point. Hence ℓ_{φ} is a line of type (IV).

If $A_1 \neq A$ then $\mathcal{E} := A\vee\ell$ is a plane of type (II) and $\ell^{\psi}{}_{\varphi}$ is a line. As before, by Theorem 1.2 and the Lemma, ℓ^{τ} is part of a line that meets $A'\vee Z'$ at a unique point A_1', say. This A_1' is distinct from A^{φ},A' and Z'.

If $A_1 = A$ then $\ell^{\psi\varphi} \neq A^{\varphi}$ is a single point and $\ell^{\tau} \subset A'\vee\ell^{\psi\varphi}$ is part of a line.

Irrespective of the various cases the restriction $\tau|(\ell\cap\operatorname{dom}\tau)$ is injective: If ℓ_{φ} is a line then this implied by the injectivity of $\varphi|\ell$ and the fact that Z',X^{φ},X^{τ} are collinear for all $X \in \ell\cap\operatorname{dom}\tau$. If ℓ_{φ} is not a line then $\varphi|\ell^{\psi}$ is injective and $A',X^{\psi\varphi},X^{\tau}$ are collinear for all $X \in \ell\cap\operatorname{dom}\tau$. Thus ℓ_{τ} is always a line.

(c) Given $A_1 \in \mathcal{A}\backslash\operatorname{dom}\tau$ there exists a point $B \in \mathcal{I}\backslash\mathcal{A}$, whence $B^{\varphi} \neq A_1{}^{\varphi} = A^{\varphi}$. As follows from (b), $(A_1\vee B)_{\tau}$ is a line and there exists a point

$$A_1' := (A'\vee Z')\cap(A_1\vee B)_{\tau}.$$

We show that A_1' does not depend on the choice of $B \in \mathcal{I}\backslash\mathcal{A}$.

If B is replaced by $C \in \mathcal{I}\backslash\mathcal{A}$ such that $\{A,B,C\}^{\varphi}$ is a triangle then $\mathcal{E} := A_1\vee B\vee C$ is a plane of type (IV). Hence $\mathcal{E}\cap\mathcal{A} = A_1$. Choose a point $B_1 \in (A_1\vee B)\backslash\{A_1,B\}$ as well as a point $C_1 \in (A_1\vee C)\backslash\{A_1,C\}$. Lines $B_1\vee C_1$ and $B\vee C$ meet at some point $D \neq B,C$. Now, going over to the τ-images, the lines $(B\vee C)_{\tau} \subset \mathcal{I}'$ and $(B_1\vee C_1)_{\tau} \not\subset \mathcal{I}'$ are distinct and they are spanning a plane $\mathcal{E}'$ that contains $(A_1\vee B)_{\tau}$ and $(A_1\vee C)_{\tau}$. But $\mathcal{E}'\cap(A'\vee Z')$ is a single point, whence $A_1' \in (A_1\vee C)_{\tau}$.

If B is replaced by $C \in \mathcal{I}\backslash\mathcal{A}$ such that $\{A,B,C\}^{\varphi}$ is not a triangle then, by $\mathcal{I}_{\varphi} = \mathcal{R}_{\varphi}$, there exists a point $E \in \mathcal{I}\backslash\mathcal{A}$ for that $\{A,B,E\}^{\varphi}$ is a triangle, whence $\{A,C,E\}^{\varphi}$ is a triangle too. Repeated application of the arguments used above shows $A_1' \in (A_1\vee C)_{\tau}$.

Hence we may unambigously define

$$\lambda : \mathcal{R} \to \mathcal{P}', \ X \ \mapsto \begin{cases} X^{\tau} & \text{if } X \in \operatorname{dom}\tau \\ (A'\vee Z')\cap(X\vee B)_{\tau} & \text{if } X \notin \operatorname{dom}\tau \quad (B \in \mathcal{J}\setminus\mathcal{A} \text{ arbitrary}) \end{cases} \tag{5}$$

Thus $(\mathcal{A}\cap\mathcal{J})^{\lambda} = A^{\varphi}$, $A^{\lambda} = A'$ and $\big(\mathcal{A}\setminus(\mathcal{J}\cup\{A\})\big)^{\lambda} \subset (A'\vee Z')\setminus\{A^{\varphi},Z',A'\}$. Conditions (3), $\mathcal{J}_{\lambda} = \mathcal{J}' \neq \mathcal{R}_{\lambda}$ and (WL2) obviously are true for λ.

(d) Finally, we have to check if λ satisfies (WL1). By the definition of λ it is sufficient to show that (WL1) holds for distinct points X,Y in $\mathcal{A}$, but not both in $\mathcal{J}$. Put $\ell := X\vee Y$. We are finished if we can show that $\lambda|\ell$ is injective.

There exists a point $B \in \mathcal{J}$ such that $A^{\varphi} \neq B^{\varphi}$. Hence $\mathcal{E} := B\vee\ell$ is a plane and $\mathcal{E}\cap\mathcal{A} = \ell$. The restriction of λ to $\mathcal{E}$ preserves collinearity of points, since $\ell^{\lambda} \subset A'\vee Z'$. Choose a line $k \subset \mathcal{E}$ through $\ell\cap\mathcal{J}$, but not through B and distinct from the line ℓ. Hence k_{λ} lies in the plane $(A'\vee Z'\vee B^{\varphi})$, runs through $A^{\varphi} = (\ell\cap\mathcal{J})^{\varphi}$, but is not incident with A' or B^{φ}. The restriction of λ to k is injective and, by perspectivity $k_{\lambda} \to (A'\vee Z')$ with centre B^{φ}, also $\lambda|\ell$ is injective. ■

3.7. The mapping λ, as has been defined before, will be called a *lifting* of the mapping φ. Provided that $\mathcal{P}' = \mathcal{R}_{\varphi} = \mathcal{H}_{\varphi}$ for all hyperplanes $\mathcal{H}$ of $\mathcal{R}$, the Desarguesian projective space $\mathcal{P}'$ may be embedded as hyperplane in a projective space $\mathcal{P}''$. Thus the condition $\mathcal{P}' \neq \mathcal{R}_{\varphi}$ is not essential. We remark that the technique for defining λ generalizes a construction used in [31]. An algebraic approach to certain liftings was given in [21].

3.8. Recall the notations introduced in 2.2. Here is our main result:

Theorem 2. *Let $\mathcal{P}(\mathbf{V})$ and $\mathcal{P}(\mathbf{W})$ be projective spaces on vector spaces $\mathbf{V}$ over K and $\mathbf{W}$ over L, respectively. Assume that $\varphi : \mathcal{P}(\mathbf{V}) \rightarrowtail \mathcal{P}(\mathbf{W})$ satisfies* (WL1) *and* (WL2). *Then φ is a weak linear mapping.*

Proof. (a) At first we make the additional assumptions that $\operatorname{dom}\varphi = \mathcal{P}(\mathbf{V})$ and that there exist a hyperplane $\mathcal{J} \subset \mathcal{P}(\mathbf{V})$ with $\mathcal{J}_{\varphi} \neq \mathcal{P}(\mathbf{V})_{\varphi}$. Hence we may put $\mathbf{V} = \mathbf{v}K\oplus\mathbf{V}_1$, where $\mathbf{V}_1$ is a hyperplane of $\mathbf{V}$ with $\mathcal{P}(\mathbf{V}_1) = \mathcal{J}$ and $\mathbf{v} \in \mathbf{V}\setminus\mathbf{V}_1$. Similarily $\mathbf{W} = \mathbf{w}K\oplus\mathbf{W}_1$ for some hyperplane $\mathbf{W}_1$ with $\mathcal{P}(\mathbf{W}_1) =: \mathcal{J}'$, $\mathcal{J}_{\varphi} = \mathcal{P}(\mathbf{V})_{\varphi}\cap\mathcal{J}'$ and $\mathbf{w} \in (\mathbf{v}K)^{\varphi}\setminus\{\mathbf{o}\}$.

We shall regard $\mathcal{J}$ as hyperplane at infinity and $\mathcal{P}(\mathbf{V})\setminus\mathcal{J}$ as an affine space. Also $\mathbf{V}_1$ may be viewed as an affine space and, mutatis mutandis, this carries over to $\mathbf{W}$. It is well known that

$$\alpha : \mathbf{V}_1 \to \mathcal{P}(\mathbf{V})\setminus\mathcal{J},\ \mathbf{x} \mapsto (\mathbf{v}+\mathbf{x})K \text{ and } \beta : \mathbf{W}_1 \to \mathcal{P}(\mathbf{W})\setminus\mathcal{J}',\ \mathbf{y} \mapsto (\mathbf{w}+\mathbf{y})K$$

are affinities (i.e. collineations preserving parallelism). We use α and β to define

$$\alpha\varphi\beta^{-1} =: g : \mathbf{V}_1 \to \mathbf{W}_1. \tag{6}$$

This definition does make sense, since $\big(\mathcal{P}(\mathbf{V})\setminus\mathcal{J}\big)^{\varphi} \subset \mathcal{P}(\mathbf{W})\setminus\mathcal{J}'$. By (6) we have

$$\big((\mathbf{v}+\mathbf{x})K\big)^{\varphi} = (\mathbf{w}+\mathbf{x}^{g})L \text{ for all } \mathbf{x} \in \mathbf{V}_1.$$

Thus g is injective, because $\varphi\,|\,\big(\mathcal{P}(\mathbf{V})\setminus\mathcal{J}\big)$ is an injection by Theorem 1.3.

By (WL2) and Theorem 1.3 the subspace $\mathcal{J}_{\varphi}$ contains a line. Hence there are at least two linearly independent vectors in ${\mathbf{V}_1}^{g}$. We claim that g is additive: $\mathbf{o}^{g} = \mathbf{o}$ follows from $\mathbf{w} \in (\mathbf{v}K)^{\varphi}\setminus\{\mathbf{o}\}$. So we have to show additivity for $\mathbf{x},\mathbf{y} \in \mathbf{V}_1\setminus\{\mathbf{o}\}$ only:

If $\mathbf{x}^{g},\mathbf{y}^{g}$ are linearly independent then $\{\mathbf{o},\mathbf{x},\mathbf{y},\mathbf{x}+\mathbf{y}\}$ is a parallelogram and this property is shared by the image under $\alpha\varphi$, since the action of φ on the hyperplane at infinity shows that φ takes parallel lines in $\mathcal{P}(\mathbf{V})\setminus\mathcal{J}$ to parallellines in $\mathcal{P}(\mathbf{W})\setminus\mathcal{J}'$. The affinity β^{-1} preserves parallelograms, whence $(\mathbf{x}+\mathbf{y})^{g} = \mathbf{x}^{g}+\mathbf{y}^{g}$.

If $\mathbf{y} = -\mathbf{x}$ then there exists $\mathbf{z} \in \mathbf{V}_1$ such that $\mathbf{z}^{g} \notin \mathbf{x}^{g}L$. Thus $\{\mathbf{o},-\mathbf{x},\mathbf{x}+\mathbf{z},\mathbf{z}\}$ is a parallelogram, whence $(-\mathbf{x})^{g} = -(\mathbf{x}^{g})$ and $\big(\mathbf{x}+(-\mathbf{x})\big)^{g} = \mathbf{x}^{g}+(-\mathbf{x})^{g}$.

If $\mathbf{x}^{g},\mathbf{y}^{g} \neq -\mathbf{x}^{g}$ are linearly dependent then there exists a vector $\mathbf{z} \in \mathbf{V}_1$ such that $\mathbf{z}^{g} \notin \mathbf{x}^{g}L$. By the injectivity of g we obtain $\mathbf{x}+\mathbf{y} \neq \mathbf{o}$ and

$$(\mathbf{x}+\mathbf{y})^{g} = (\mathbf{x}+\mathbf{y}-\mathbf{z}+\mathbf{z})^{g} = (\mathbf{x}+\mathbf{y}-\mathbf{z})^{g}+\mathbf{z}^{g} = \mathbf{x}^{g}+(\mathbf{y}-\mathbf{z})^{g}+\mathbf{z}^{g} = \mathbf{x}^{g}+\mathbf{y}^{g}-\mathbf{z}^{g}+\mathbf{z}^{g} = \mathbf{x}^{g}+\mathbf{y}^{g}.$$

Given $\mathbf{x} \in \mathbf{V}_1\setminus\{\mathbf{o}\}$ then $\mathbf{x}K \subset \mathbf{V}_1$ is a line of the affine space on $\mathbf{V}_1$ and its g-image is part of a line of the affine space on $\mathbf{W}_1$ that is passing through $\mathbf{o} \in \mathbf{W}$. We deduce $(\mathbf{x}K)^{g} \subset (\mathbf{x}^{g})L$ from the injectivity of g. By ZICK's result, stated in 2.2, g is a weak semilinear mapping[8] with respect to a monomorphism $\zeta : K \to L$.

Finally, define

$$f : \mathbf{V} \to \mathbf{W},\ \mathbf{v}x+\mathbf{x}_1 \mapsto \mathbf{w}x^{\zeta}+{\mathbf{x}_1}^{g} \quad (x \in K,\ \mathbf{x}_1 \in \mathbf{V}_1).$$

This f is a weak semilinear mapping and $(\mathbf{x}K)^{\varphi} = (\mathbf{x}^{f})L$ for all $\mathbf{x} \in \mathbf{V}\setminus\{\mathbf{o}\}$.

(b) Now the general case is studied: We use a factorization $\varphi = \pi(\varphi\,|\,\mathcal{R})$ as

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[8]Alternatively, this may be shown by modifying the proof in [29,104f].

described in formula (2); when $\operatorname{dom}\varphi = \mathcal{P}(\mathbf{V})$ then $\mathcal{R} = \mathcal{P}(\mathbf{V})$ and π is the identity in $\mathcal{P}(\mathbf{V})$. If $\mathcal{R}_\varphi = \mathcal{H}_\varphi$ for all hyperplanes $\mathcal{H}$ of $\mathcal{R}$ then write $\varphi|\mathcal{R}$ as product of a lifting, say $\lambda : \mathcal{R} \to \mathcal{P}(L\oplus\mathbf{W})$, and a projection $\omega : \mathcal{P}(L\oplus\mathbf{W}) \rightarrowtail \mathcal{P}(\mathbf{W})$. Otherwise let $\lambda := \varphi|\mathcal{R}$ and let ω be the identity of $\mathcal{P}(\mathbf{W})$. So under all circumstances we have

$$\varphi = \pi\lambda\omega.$$

Projections are induced by idempotent linear mappings of vector spaces and, according to (a), λ is induced by a weak semilinear mapping. ■

4. Remarks

4.1. We use the notations of Theorem 2. If $\varphi : \mathcal{P}(\mathbf{V}) \rightarrowtail \mathcal{P}(\mathbf{W})$ is a weak linear mapping and if one line of $\mathcal{P}(\mathbf{V})$ is mapped onto a line of $\mathcal{P}(\mathbf{W})$ then $\zeta : K \to L$ is surjective, whence φ is linear. Conversely, if ζ is surjective then φ is linear. Thus, especially for real projective spaces, Theorem 2 characterizes linear mappings with non-collinear image set, since every monomorphism $\mathbb{R} \to \mathbb{R}$ is an identity mapping. Cf., e.g., [1,88f]. See also [9] and [10] for other examples of fields that admit only surjective monomorphisms.

On the other hand let L be a proper extension field of a field K. Assume that $n \geq 2$, $\mathbf{V} = L^n$ (viewed as vector space over K), $\mathbf{W} = L^n$ (viewed as vector space over L) and that f is the identity mapping of $\mathbf{V} = \mathbf{W}$. Then f is a weak semilinear bijection with respect to the inclusion mapping of K into L. The induced mapping $\varphi : \mathcal{P}(\mathbf{V}) \to \mathcal{P}(\mathbf{W})$ is surjective although a line of $\mathcal{P}(\mathbf{V})$ never is mapped onto a line of $\mathcal{P}(\mathbf{W})$.

4.2. Obviously (WL1) is not sufficient to characterize weak linear mappings whose image set is part of a line. To illustrate this just take an injective mapping of a real projective line that is not a projectivity.

Moreover we want to point out that (WL1) does not even characterize those mappings that are products of a weak linear mapping into a line and an injection of that line into itself.

To this end let L be an extension of K with right degree 2, $\dim_K \mathbf{V} = 4$, $\dim_L \mathbf{W} = 2$ and let $\varphi : \mathcal{P}(\mathbf{V}) \to \mathcal{P}(\mathbf{W})$ be a weak linear mapping with more than one image point. Hence, with the notations of 2.2, there exists a projective space

$\mathcal{P}(\mathbf{Y})$ over F that is an isomorphic copy of $\mathcal{P}(\mathbf{V})$, a semilinear bijection $i:\mathbf{V}\to\mathbf{Y}$ and a linear mapping $h:\mathbf{Y}\otimes_F L\to\mathbf{W}$ such that $(\mathbf{x}K)^{\varphi}=((\mathbf{x}^i\otimes 1)^h)L$. The rank of h equals 2, since a basis $\{\mathbf{b}_0,\ldots,\mathbf{b}_3\}$ of $\mathbf{Y}$ gives rise to a basis $\{\mathbf{b}_0\otimes 1,\ldots,\mathbf{b}_3\otimes 1\}$ of $\mathbf{Y}\otimes_F L$. Thus $\mathcal{P}(\ker h)$ is a line of $\mathcal{P}(\mathbf{Y}\otimes_F L)$. It follows from $\operatorname{dom}\varphi=\mathcal{P}(\mathbf{V})$ and [23] that $\mathcal{P}(\ker h)$ is an indicator set of a Desarguesian spread of $\mathcal{P}(\mathbf{Y})$, whence the set of fibres of φ is a Desarguesian spread too.

Next we specialize as follows: Let K, L be commutative and let $K\neq\mathrm{GF}(2)$. Hence the fibres of φ form an elliptic linear congruence of lines (regular spread) $\mathcal{S}$; cf., e.g., [2], [23]. If we replace one regulus $\mathcal{M}$ of $\mathcal{S}$ by its opposite regulus $\overline{\mathcal{M}}$ then we obtain a subregular spread $\overline{\mathcal{S}}$. There exists a bijection $\delta:\overline{\mathcal{M}}\to\mathcal{M}$; use, e.g., a non-identical automorphic perspective collineation of the doubly ruled quadric Q carrying $\mathcal{M}$ and $\overline{\mathcal{M}}$. Now define

$$\overline{\varphi}:\mathcal{P}(\mathbf{V})\to\mathcal{P}(\mathbf{W})\begin{cases} X \mapsto X^{\varphi} & \text{if } X\notin Q,\\ X \mapsto (\overline{\ell}\cap\overline{\ell}^{\delta})^{\varphi} & \text{if } X\in\overline{\ell}\in\overline{\mathcal{M}}.\end{cases}$$

The fibres of $\overline{\varphi}$ are exactly the lines of the spread $\overline{\mathcal{S}}$, whence $\overline{\varphi}$ satisfies (WL1). However, by $K\neq\mathrm{GF}(2)$, the spread $\overline{\mathcal{S}}$ is not Desarguesian; cf., e.g., [26,§17]. Thus the mapping $\overline{\varphi}$ cannot be the product of a weak linear mapping and an injection of $\mathcal{P}(\mathbf{W})$.

On the other hand Brauner's Theorem [4] includes a characterization of those mappings $\mathcal{P}(\mathbf{V})\rightarrowtail\mathcal{P}(\mathbf{W})$ that are the product of a linear mapping onto a line and a bijection of that line onto itself.

4.3. The description of φ in 4.2 looks rather complicated, for it involves not only a field L and its subfield K, but also a subfield F of L that is isomorphic to K.

If there exists an automorphism of L that takes K to F then the semilinear bijection $i:\mathbf{V}\to\mathbf{Y}$ and the linear mapping $h:\mathbf{Y}\otimes_F L\to\mathbf{W}$ can be replaced by a semilinear mapping $\hat{h}:\mathbf{V}\otimes_K L\to\mathbf{W}$ to obtain an algebraic description of φ. However, such an automorphism need not exist. Take, e.g., an element t, transcendental over $\mathrm{GF}(2)$, and set $L:=\mathrm{GF}(2)(t)$, $K:=\mathrm{GF}(2)(t^2)$, $F:=\mathrm{GF}(2)(t^4)$. If f is weak semilinear with respect to $\zeta:K\to L$, $x\mapsto x^2$ then $F=K^{\zeta}$, but there is no automorphism of L that takes K to F, since $|L:K|=2$ and $|L:F|=4$.

Other counterexamples (with $|L:K|\neq 2$) are given for $L=\mathbb{R}$ and choosing K

and F as two subfields of $\mathbb{R}$ that are distinct simple transcendental extensions of $\mathbb{Q}$.

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