Characterising Clifford parallelisms among Clifford-like parallelisms

Hans Havlicek Stefano Pasotti[*] Silvia Pianta[*†]

June 24, 2020

Dedicated to Mario Marchi on the occasion of his 80th birthday, in friendship

Abstract

We recall the notions of *Clifford* and *Clifford-like* parallelisms in a 3-dimensional projective double space. In a previous paper the authors proved that the linear part of the full automorphism group of a Clifford parallelism is the same for all Clifford-like parallelisms which can be associated to it. In this paper, instead, we study the action of such group on parallel classes thus achieving our main results on characterisation of the Clifford parallelisms among Clifford-like ones.

Mathematics Subject Classification (2010): 51A15, 51J15
Key words: Clifford parallelism, Clifford-like parallelism, projective double space, kinematic algebra, automorphism

1 Introduction

It is a widely used strategy in mathematics to define a new structure by modifying a given one. The definition of a Clifford-like parallelism from [7] and [17], which is recalled in Section 2, follows these lines. The starting point is a *projective double space* $(\mathbb{P}, \|_\ell, \|_r)$, that is, a projective space $\mathbb{P}$ together with a *left parallelism* $\|_\ell$ and a *right parallelism* $\|_r$ on its line set such that the so-called *double space axiom* (DS) is satisfied. The given parallelisms $\|_\ell$ and $\|_r$ are called the *Clifford parallelisms* of $(\mathbb{P}, \|_\ell, \|_r)$ in analogy to the classical example arising from

---

[*] This work was partially supported by GNSAGA of INdAM (Italy)

[†] This work was partially supported by Università Cattolica del Sacro Cuore (Milano, Italy) in the framework of Call D.1 2019

the three-dimensional elliptic space over the real numbers. The parallel classes of $\|_\ell$ and $\|_r$ are then used to define parallelisms that are *Clifford-like* w.r.t. $(\mathbb{P}, \|_\ell, \|_r)$. Among them are the initially given parallelisms $\|_\ell$ and $\|_r$. We restrict ourselves most of the time to the case when $\mathbb{P}$ is three-dimensional, and we make use of an algebraic description of such a double space in terms of an appropriate four-dimensional algebra H over a commutative field F. Thereby we adopt the notation $(\mathbb{P}(H_F), \|_\ell, \|_r)$ and we have to distinguish two cases, (A) and (B). In case (A), H is a quaternion skew field with centre F, the left and right parallelisms do not coincide and, in general, there are Clifford-like parallelisms of $(\mathbb{P}(H_F), \|_\ell, \|_r)$ different from $\|_\ell$ and $\|_r$. In case (B), H is a commutative extension field of F satisfying some extra property, and $\|_\ell = \|_r$ is the only Clifford-like parallelism of $(\mathbb{P}(H_F), \|_\ell, \|_r)$. We include case (B) for the sake of completeness and in order to obtain a unified exposition that covers both cases, even though several of our results are trivial in case (B).

In Section 3 we study automorphisms of a Clifford-like parallelism of a projective double space $(\mathbb{P}(H_F), \|_\ell, \|_r)$ being motivated by the following result: if a projective collineation of $\mathbb{P}(H_F)$ preserves *at least one* Clifford-like parallelism of $(\mathbb{P}(H_F), \|_\ell, \|_r)$, then *all* its Clifford-like parallelisms are preserved.[1] This follows from [18, Thm. 3.5] in case (A) and holds trivially in case (B). In our algebraic setting these projective collineations are induced by F-linear transformations of H which are described in Subsection 3.1, where we determine all F-semilinear automorphisms of the right parallelism. In preparation for Section 4, we exhibit for a quaternion skew field H the orbits of certain points and lines of $\mathbb{P}(H_F)$ under the group of inner automorphisms of H and we determine all $\|_r$-classes that are fixed under a left translation of H.

The main results are stated in Section 4. In Theorem 4.1, we consider a three-dimensional projective space $\mathbb{P}$ that is made into a double space in two ways. If there exists a parallelism $\|$ on $\mathbb{P}$ that is Clifford-like w.r.t. both double space structures then the given double spaces coincide up to a change of the attributes “left” and “right” in one of them. This finding improves [17, Thm. 4.15] (see Corollary 4.2) and it simplifies matters considerably. Indeed, when dealing with a Clifford-like parallelism, there is only one corresponding double space structure in the background. In Theorems 4.3, 4.5 and 4.6 we characterise the Clifford parallelisms among the Clifford-like parallelism of $(\mathbb{P}(H_F), \|_\ell, \|_r)$ via the existence of automorphisms with specific properties. For example, Theorem 4.3 establishes that a Clifford-like parallelism of $(\mathbb{P}(H_F), \|_\ell, \|_r)$ is Clifford precisely when it admits an automorphism that fixes all its parallel classes and acts non-trivially on the point set of the projective space $\mathbb{P}(H_F)$.

---

[1] The situation gets intricate when dealing with a non-projective collineation that preserves at least one Clifford-like parallelism of $(\mathbb{P}(H_F), \|_\ell, \|_r)$. See the examples in [18, Sect. 4].

Next, let us emphasise that some of our investigations are in continuity with classical results on *dilatations* in *kinematic spaces*. For example, in our proof of Theorem 4.3 we could use the fact that the existence of a *proper* non-trivial dilatation (namely a non-identical collineation with a fixed point and the property that all parallel classes remain invariant) is possible only in the commutative case, *i.e.* in our case (B) (see [35, Teorema 2] or [27, (II.10)]). We decided instead to include a short direct proof in order to keep the paper self-contained. There are also neat connections to the theory of *Sperner spaces* and *(generalised) translation structures*; we refer the interested reader to [1], [38], [39] and the many references given there.

Finally, another remark seems appropriate. Any Clifford-like parallelism on the three-dimensional real projective space is Clifford (see Remark 3.6). The Clifford parallelisms on this space are the only topological parallelisms that admit an automorphism group of dimension at least 4; see [33] and the intimately related articles [3], [5], [31], [32]. In contrast to our considerations, in this beautiful result only the "size" of an automorphism group is taken into account and not its action on the parallel classes.

2 Preliminaries on Clifford and Clifford-like parallelisms

A *parallelism* $\parallel$ on a projective space $\mathbb{P}$ is an equivalence relation on the set $\mathcal{L}$ of lines such that each point of $\mathbb{P}$ is incident with precisely one line from each equivalence class. (If $\mathbb{P}$ is a finite projective space then a parallelism is also called a *packing* or a *resolution*.) For each line $M \in \mathcal{L}$ we write $\mathcal{S}(M)$ for the *parallel class* of M, that is, the equivalence class containing M. This notation arises quite naturally, since any parallel class is in fact a *spread* (of lines) of $\mathbb{P}$. When considering several parallelisms, we distinguish among the above notions and symbols by adding appropriate attributes, subscripts or superscripts. We refer to [2], [20, Ch. 17], [22], [23] and [24, § 14] for a wealth of results about parallelisms and further references.

Let $\mathbb{P}$ and $\mathbb{P}'$ be projective spaces with parallelisms $\parallel$ and $\parallel'$, respectively and let κ be a collineation of $\mathbb{P}$ to $\mathbb{P}'$ such that, for all lines $M, N \in \mathcal{L}$, $M \parallel N$ implies $\kappa(M) \parallel' \kappa(N)$. Then κ takes any $\parallel$-class to a $\parallel'$-class by [18, Lemma 2.1]. Such a κ is frequently called an *isomorphism*[2] of $(\mathbb{P}, \parallel)$ to $(\mathbb{P}', \parallel')$.

Suppose that a projective space $\mathbb{P}$ is endowed with two (not necessarily distinct) parallelisms, a *left* parallelism $\parallel_\ell$ and a *right* parallelism $\parallel_r$. Following [25],

---

[2] A slightly different terminology will be used when dealing with projective spaces over vector spaces; see the first paragraph of Section 3.1.

$(\mathbb{P}, \|_\ell, \|_r)$ constitutes a *projective double space* if the following axiom is satisfied.

(DS) For all triangles p_0, p_1, p_2 in $\mathbb{P}$ there exists a common point of the lines M_1 and M_2 that are defined as follows. M_1 is the line through p_2 that is left parallel to the join of p_0 and p_1, M_2 is the line through p_1 that is right parallel to the join of p_0 and p_2.

Given a projective double space $(\mathbb{P}, \|_\ell, \|_r)$ each of $\|_\ell$ and $\|_r$ is referred to as a *Clifford parallelism*[3] of $(\mathbb{P}, \|_\ell, \|_r)$. More generally, a *Clifford-like parallelism* of $(\mathbb{P}, \|_\ell, \|_r)$ is defined as a parallelism $\|$ on $\mathbb{P}$ such that, for all $M, N \in \mathcal{L}$, $M \parallel N$ implies $M \|_\ell N$ or $M \|_r N$ (see [17, Def. 3.2]). Each parallel class of a Clifford-like parallelism $\|$ of $(\mathbb{P}, \|_\ell, \|_r)$ is a left or a right parallel class: see [17, Thm. 3.1], where this topic appears in the wider context of "blends" of parallelisms. A Clifford-like parallelism of $(\mathbb{P}, \|_\ell, \|_r)$ is said to be *proper* if it does not coincide with one of $\|_\ell$ and $\|_r$. In what follows, whenever we say that a parallelism $\|$ on a projective space $\mathbb{P}$ is Clifford (respectively Clifford-like) it is intended that $\mathbb{P}$ can be made into a double space $(\mathbb{P}, \|_\ell, \|_r)$ such that $\|$ is one of its Clifford (respectively Clifford-like) parallelisms.

An algebraic description—up to isomorphism—of *all* projective double spaces $(\mathbb{P}, \|_\ell, \|_r)$ that contain at least two distinct lines and satisfy the so-called "prism axiom" was given in [25]. It is based on quaternion skew fields and purely inseparable commutative field extensions of characteristic two. According to [26, Satz 1] and [28, Satz 2], the prism axiom appearing in [25] is redundant; see also the surveys in [24, § 14] and [22, pp. 112–115]. This is why we omit to consider this axiom here. From now on we exhibit exclusively three-dimensional projective double spaces.[4] We therefore recall only their algebraic description in the next few paragraphs.

We adopt the following settings throughout this article: F denotes a commutative field and H is an F-algebra with unit 1_H satisfying one of the following conditions.

(A) H is a quaternion skew field with centre $F1_H$.

(B) H is a commutative field with degree $[H : F1_H] = 4$ and such that $h^2 \in F1_H$ for all $h \in H$.

In what follows, we identify any $f \in F$ with $f1_H \in H$, whence F turns into a subfield of H. If E is a subfield of H, then H is a left vector space and a

---

[3] This definition does not include Clifford parallelisms that arise from octonions (see [6], [41], [42], [43], [44]). The (generalised) Clifford parallelisms appearing in [11, Kap. 12] and [40] are not fully covered.

[4] In any other dimension (DS) implies $\|_\ell = \|_r$, whence proper Clifford-like parallelisms of $(\mathbb{P}, \|_\ell, \|_r)$ do not exist.

right vector space over E. We denote these spaces as ${}_E H$ and H_E, respectively. Whenever E is contained in the centre of H, we do not distinguish between ${}_E H$ and H_E. In each of the cases (A) and (B), H_F is an infinite *kinematic* (or, in a different terminology: *quadratic*) F-algebra, *i.e.*,

$$h^2 \in F + Fh \text{ for all } h \in H. \tag{1}$$

If (B) applies then the characteristic $\operatorname{Char} F$ equals two and H is a purely inseparable extension of F.

All F-linear endomorphisms of H_F constitute the F-algebra $\operatorname{End}(H_F)$. The *left regular representation* $\lambda\colon H \to \operatorname{End}(H_F)$ sends each $h \in H$ to the mapping $\lambda(h) =: \lambda_h$ given as $\lambda_h(x) := hx$ for all $x \in H$. The image $\lambda(H)$ is an isomorphic copy of the field H within $\operatorname{End}(H_F)$. The elements of the multiplicative group[5] $\lambda(H^*) = \operatorname{GL}(H_H)$ are the *left translations*. Similarly, the *right regular representation* $\rho\colon H \to \operatorname{End}(H_F)$ sends each $h \in H$ to $\rho(h) =: \rho_h$ given as $\rho_h(x) := xh$ for all $x \in H$. In this way we obtain $\rho(H)$ as an antiisomorphic copy of H within $\operatorname{End}(H_F)$ and the group of *right translations*[6] $\rho(H^*) = \operatorname{GL}({}_H H)$. For all $g, h \in H$, the mappings λ_g and ρ_h commute. The multiplicative group H^* admits the representation $\tilde{(\ )}\colon H^* \to \operatorname{GL}(H_F)$ sending each $h \in H^*$ to $\tilde{h} := \lambda_h^{-1} \circ \rho_h$, which is an inner automorphism of the field H. Clearly, in case (B) the group $\widetilde{H^*}$ comprises only the identity id_H.

The *projective space* on the vector space H_F, in symbols $\mathbb{P}(H_F)$, is understood to be the set of all subspaces of H_F with *incidence* being symmetrised inclusion. We adopt the usual geometric terms: *Points*, *lines*, and *planes* of $\mathbb{P}(H_F)$ are the subspaces of H_F with vector dimension one, two, and three, respectively; the set of all lines is written as $\mathcal{L}(H_F)$. The following notions rely on H_F being an F-algebra. In $\mathbb{P}(H_F)$, lines M and N are defined to be *left parallel*, $M \parallel_\ell N$, if $\lambda_c(M) = N$ for some $c \in H^*$. Similarly, M and N are said to be *right parallel*, $M \parallel_r N$, if $\rho_c(M) = N$ for some $c \in H^*$. Then $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ is a *projective double space*. The parallelisms $\parallel_\ell$ and $\parallel_r$ are distinct in case (A) and identical in case (B).

Remark 2.1. The left and right parallelism w.r.t. $(H, +, \cdot)$ are the same as the right and left parallelism defined by the opposite field of H. So, from a geometric point of view, the choice of the attributes “left” and “right” is immaterial.

The multiplication on the field $(H, +, \cdot)$ may be altered without changing the associated projective double space $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$. Let us choose any $e \in H^*$. Then we can define a multiplication $\cdot^e$ on H via $x \cdot^e y := x \cdot e^{-1} \cdot y$ for all $x, y \in H$. This makes $(H, +, \cdot^e)$ into an F-algebra, which will briefly be written as H^e. The left translation λ_e (w.r.t. H) is an F-linear isomorphism of H to H^e, whence the arbitrarily chosen element $e \in H^*$ turns out to be the unit element of H^e. The

---

[5] We abbreviate $H \setminus \{0\}$ as H^* and use the same kind of notation for any field.

[6] Observe that the zero endomorphism $\lambda_0 = \rho_0$ is *not* among the left and right translations.

projective double spaces arising from the F-algebras H and H^e are the same, since $\lambda_h = \lambda^e_{h\cdot e}$ and $\rho_h = \rho^e_{e\cdot h}$ for all $h \in H^*$.

Let us briefly sketch a more conceptual verification of our second observation. The point Fe and the parallelisms $\parallel_\ell$ and $\parallel_r$ can be used to make the point set $\mathbb{P}(H_F)$ into a two-sided incidence group with unit element Fe [25, §3]. (The prism axiom appearing in [25] can be avoided [26, Satz 1], [28, Satz 2].) Then, using the group structure on $\mathbb{P}(H_F)$, the F-vector space H can be endowed with a multiplication making it into a field with unit element e (see [9, Satz 1] and [45, Hauptsatz]). This field, which coincides with our H^e, therefore provides an alternative description of the projective double space $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$.

Remark 2.2. There are various other ways to define a *Clifford parallelism* on a three-dimensional (necessarily pappian) projective space. We refer to [4], [7], [13, p. 46], [14, Sect. 2], [15], [16] and the references given there. On that account, it is our aim to make use only of the above algebraic approach.

Let $\mathcal{A}(H_F) \subset \mathcal{L}(H_F)$ denote the star of lines with centre $F1$. By (1), each line $L \in \mathcal{A}(H_F)$ is readily seen to be a maximal commutative subfield of H and hence an F-subalgebra. Next, we recall an explicit construction that gives *all* Clifford-like parallelisms of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$. Upon choosing any $\widetilde{H^*}$-invariant subset $\mathcal{F} \subseteq \mathcal{A}(H_F)$, one obtains a partition of $\mathcal{L}(H_F)$ by taking the left parallel classes of all lines in $\mathcal{F}$ and the right parallel classes of all lines in $\mathcal{A}(H_F) \setminus \mathcal{F}$. This partition determines an equivalence relation, which turns out to be a Clifford-like parallelism $\parallel$ of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$. See [17, Thm. 4.10] for a proof in the case when (A) applies; in case (B) the result is trivial due to $\parallel = \parallel_\ell = \parallel_r$.

Remark 2.3. Let $\parallel$ be any parallelism on $\mathbb{P}(H_F)$ and let $\mathcal{S}(M)$, $M \in \mathcal{L}(H_F)$, be one of its parallel classes. We recall that the *kernel* of the spread $\mathcal{S}(M)$ consists of all endomorphisms φ of the abelian group $(H, +)$ such that $\varphi(N) \subseteq N$ for all $N \in \mathcal{S}(M)$. This kernel, which will be denoted by $\mathrm{K}(H, \mathcal{S}(M))$, is a field; see, for example, [34, Thm. 1.6]. Consequently, if $\varphi \in \mathrm{K}(H, \mathcal{S}(M))$ and $\varphi \neq 0$, then $\varphi(N) = N$ for all $N \in \mathcal{S}(M)$. The following simple reasoning will repeatedly be used. If $\varphi_1, \varphi_2 \in \mathrm{K}(H, \mathcal{S}(M))$ satisfy $\varphi_1(g) = \varphi_2(g)$ for some $g \in H^*$, then $(\varphi_1 - \varphi_2)(g) = 0$ forces that $\varphi_1 - \varphi_2$ is not injective. Therefore $\varphi_1 - \varphi_2$ is the zero endomorphism or, in other words, $\varphi_1 = \varphi_2$.

Proposition 2.4. *If Clifford parallelisms* $\parallel$ *and* $\parallel'$ *on a three-dimensional projective space have two distinct parallel classes in common, then these parallelisms coincide.*

Proof. By virtue of the algebraic description of all projective double spaces and by Remark 2.1, we may assume the following. The parallelism $\parallel$ is the right parallelism $\parallel_r$ coming from an F-algebra $(H, +, \cdot)$ subject to (A) or (B). There is a multiplication $\cdot' \colon H \times H \to H$ making the F-vector space H_F into an F-algebra

$(H, +, \cdot')$ subject to (A) or (B) such that $\|'$ coincides with the right parallelism $\|'_r$ arising from $(H, +, \cdot')$. These algebras share a common unit element $1 \in H^*$, say.

By our assumption, there are distinct lines $L_1, L_2 \in \mathcal{A}(H_F)$ such that $\mathcal{S}_r(L_1) = \mathcal{S}'_r(L_1)$ and $\mathcal{S}_r(L_2) = \mathcal{S}'_r(L_2)$. Choose any $z \in L_n$ where $n \in \{1, 2\}$. Then λ_z and λ'_z are both in $\mathrm{K}(H, \mathcal{S}_r(L_n))$. According to Remark 2.3, $\lambda_z(1) = z = \lambda'_z(1)$ implies $\lambda_z = \lambda'_z$. Hence

$$z \cdot x = \lambda_z(x) = \lambda'_z(x) = z \cdot' x \ \text{ for all } \ x \in H \ \text{ and all } \ z \in L_1 \cup L_2. \tag{2}$$

More generally, the equality in (2) is fulfilled for all $x \in H$ and all z from the subfield of $(H, +, \cdot)$ that is generated by $L_1 \cup L_2$. This subfield coincides with $(H, +, \cdot)$, since L_1 is a maximal subfield of $(H, +, \cdot)$. All things considered, we obtain $(H, +, \cdot) = (H, +, \cdot')$ and therefore $\| = \|_r = \|'_r = \|'$. □

Remark 2.5. Note that the above theorem may alternatively be established by using the one-to-one correspondence between Clifford parallelisms and external planes to the Klein quadric (see [16, Cor. 4.5]).

3 Automorphisms, their orbits and actions

This section is devoted to deepen the study of the automorphisms of the Clifford parallelisms of a three-dimensional projective double space $(\mathbb{P}(H_F), \|_\ell, \|_r)$ as described in Section 2. In particular we obtain a description of the orbits of certain points and lines under the action of the group $\widetilde{H^*}$, and we characterise the right parallel classes fixed (as a set) by a given left translation. In order to avoid trivialities, we shall repeatedly confine ourselves to case (A). These findings will lead us in Section 4 to the proof of our main results.

3.1 Automorphisms

In this subsection H always denotes an F-algebra subject to (A) or (B). Given any parallelism $\|$ on $\mathbb{P}(H_F)$, we are going to use from now on the phrase *automorphism of* $\|$ for any β in the general semilinear group $\Gamma\mathrm{L}(H_F)$ that acts as a $\|$-preserving collineation on $\mathbb{P}(H_F)$. The symbol $\Gamma_\|$ denotes the *automorphism group* of $\|$. This terminology is in accordance with the one in [18].

The Clifford parallelisms of the projective double space $(\mathbb{P}(H_F), \|_\ell, \|_r)$ give rise to automorphism groups $\Gamma_{\|_\ell} =: \Gamma_\ell$ and $\Gamma_{\|_r} =: \Gamma_r$. These groups coincide, that is,

$$\Gamma_\ell = \Gamma_r. \tag{3}$$

In case (A), a proof can be derived from [36, p. 166]; see [18, Sect. 2] for further details. In case (B), equation (3) is trivial. The group $\lambda(H^*)$ of left translations, the group $\rho(H^*)$ of right translations and the group $\widetilde{H^*}$ of inner automorphisms are subgroups of $\Gamma_\ell = \Gamma_r$.

Lemma 3.1. *Let $\mathcal{S}_r(M)$ be the right parallel class of a line $M \in \mathcal{L}(H_F)$. The elements of the kernel $\mathrm{K}(H, \mathcal{S}_r(M))$ are precisely the mappings λ_g with g ranging in the line that contains the point $F1$ and is right parallel to M. Consequently,*

$$\lambda(H) = \bigcup_{L \in \mathcal{A}(H_F)} \mathrm{K}(H, \mathcal{S}_r(L)) = \bigcup_{M \in \mathcal{L}(H_F)} \mathrm{K}(H, \mathcal{S}_r(M)). \tag{4}$$

A similar result holds with the role of "left" and "right" interchanged.

Proof. There is a $d \in H^*$ such that $F1 \subseteq \rho_d(M) = Md$. Choose any $g \in Md$. Then, for all $h \in H^*$, $\lambda_g(Mdh) = g(Mdh) = (gMd)h \subseteq Mdh$, whence $\lambda_g \in \mathrm{K}(H, \mathcal{S}_r(M))$. Conversely, let $\varphi \in \mathrm{K}(H, \mathcal{S}_r(M))$. Then $\varphi(1) \in Md$ gives $\lambda_{\varphi(1)} \in \mathrm{K}(H, \mathcal{S}_r(M))$, and $\varphi(1) = \lambda_{\varphi(1)}(1)$ implies $\varphi = \lambda_{\varphi(1)}$ according to Remark 2.3.

Equation (4) is now immediate, since each element of H is contained in at least one line of the star $\mathcal{A}(H_F)$ and each right parallel class contains a line passing through $F1$. □

Any line $L \in \mathcal{A}(H_F)$ is a commutative quadratic extension field of F contained in H. The above Lemma illustrates the rather obvious result that the restriction to L of the representation λ (respectively ρ) provides an isomorphism of the field L onto the kernel of the right (respectively left) parallel class of the line L. This proves anew that all left and right parallel classes are regular spreads (see [7, 4.8 Cor.], [15, Prop. 3.5] or [16, Prop. 4.3]). Maybe less obvious is the following conclusion. Any semilinear transformation $\varphi \in \Gamma\mathrm{L}(H_F)$ that fixes all lines of one right (respectively left) parallel class is a left (respectively right) translation and therefore in the automorphism group $\Gamma_\ell = \Gamma_r$.

In the next proposition we describe the automorphism group $\Gamma_\ell = \Gamma_r$. Alternative proofs, which cover only the case when H is a quaternion skew field, can be retrieved from [6, Sect. 4], [36, Thm. 1] and [37, Prop. 4.1 and 4.2]. Below, we follow the exposition in [18, Sect. 2].

Proposition 3.2. *Let $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ be a projective double space, where H is an F-algebra subject to* (A) *or* (B). *The automorphism group of the right parallelism satisfies*

$$\Gamma_r = \lambda(H^*) \rtimes \mathrm{Aut}(H/F), \tag{5}$$

where $\mathrm{Aut}(H/F)$ *denotes the group of all automorphisms of the field H that fix F as a set.*

Proof. A direct verification shows that the group $\mathrm{Aut}(H/F)$ is a subgroup of Γ_r. As we noted at the beginning of this subsection, the same applies for the group $\lambda(H^*)$. For all $\gamma \in \Gamma_r$ and all lines $M \in \mathcal{L}(H_F)$, we have

$$\mathrm{K}(H, \mathcal{S}_r(\gamma(M))) = \gamma \circ \mathrm{K}(H, \mathcal{S}_r(M)) \circ \gamma^{-1}. \tag{6}$$

Using (4), this implies that $\lambda(H^*)$ is a normal subgroup of Γ_r.

Let us choose any $\beta \in \Gamma_r$. We define $\varphi := \lambda_{\beta(1)}^{-1} \circ \beta$, whence $\varphi \in \Gamma_r$ fixes $1 \in H$. In order to verify

$$\varphi \circ \lambda_z \circ \varphi^{-1} = \lambda_{\varphi(z)} \text{ for all } z \in H, \tag{7}$$

we proceed as follows. There is a line L with $1, z \in L$. Applying (6) to $\gamma := \varphi$ and $M := L$ gives that $\varphi \circ \lambda_z \circ \varphi^{-1}$ as well as $\lambda_{\varphi(z)}$ belongs to $\mathrm{K}(H, \mathcal{S}_r(\varphi(L)))$. Now $(\varphi \circ \lambda_z \circ \varphi^{-1})(1) = \lambda_{\varphi(z)}(1)$ together with Remark 2.3 establishes (7). For all $x, y \in H$, we have

$$\begin{aligned}
\varphi(xy) &= (\varphi \circ \lambda_x \circ \lambda_y)(1) \\
&= ((\varphi \circ \lambda_x \circ \varphi^{-1}) \circ (\varphi \circ \lambda_y \circ \varphi^{-1}))(1) \\
&= (\lambda_{\varphi(x)} \circ \lambda_{\varphi(y)})(1) \\
&= \varphi(x)\varphi(y)
\end{aligned}$$

so that φ is an automorphism of the field H. Furthermore, $\varphi(1) = 1$ together with φ being F-semilinear implies $\varphi(F) = F$. □

Take notice that F is the centre of the quaternion skew field H in case (A) and so under these circumstances $\mathrm{Aut}(H/F) = \mathrm{Aut}(H)$.

Suppose that $\varphi \in \mathrm{Aut}(H/F)$ is F-linear or, equivalently, that φ fixes F elementwise. Then $\varphi \in \widetilde{H^*}$ is an inner automorphism of the field H. In case (A), this follows from the theorem of Skolem-Noether [21, Thm. 4.9]. In case (B), any inner automorphism of H is trivial and $\varphi = \mathrm{id}_H$, since any $h \in H^* \setminus F^*$ is a double zero of the polynomial $h^2 + t^2 \in F[t]$, which is the minimal polynomial of h over F. So, by (5), the group of all *F-linear automorphisms* of $\|_r$ can be written in the form

$$\Gamma_r \cap \mathrm{GL}(H_F) = \lambda(H^*) \rtimes \widetilde{H^*}. \tag{8}$$

Let $\|$ be any Clifford-like parallelism of $(\mathbb{P}(H_F), \|_\ell, \|_r)$. The group appearing in (8) coincides with the group $\Gamma_\| \cap \mathrm{GL}(H_F)$ comprising all *F-linear automorphisms* of $\|$ (see [18, Thm. 3.5]). The problem to determine the full automorphism group $\Gamma_\|$ without extra assumptions on H, F or $\|$ seems to be open. Partial solutions can be found [18, Sect. 3]. The examples in [18, Sect. 4] show the existence of proper Clifford-like parallelisms $\|$ satisfying $\Gamma_\| = \Gamma_\ell = \Gamma_r$ and also of proper Clifford-like parallelisms $\|$ satisfying $\Gamma_\| \subset \Gamma_\ell = \Gamma_r$.

3.2 Orbits under the group of inner automorphisms

In this subsection H denotes an F-algebra subject to (A), that is, a quaternion skew field with centre F. The following outcomes fail in case (B), since there the group of inner automorphisms is trivial.

Recall that, given any $h \in H$, the *trace* and the *norm* of h are the elements of F defined, respectively, by $\mathrm{tr}(h) = h + \overline{h}$ and $N(h) = h\overline{h} = \overline{h}h$, where $\overline{h}$ denotes the *conjugate* of h. The conjugation is an antiautomorphism of H of order 2 that fixes F elementwise. The identity $h^2 - \mathrm{tr}(h)h + N(h) = 0$ holds for any $h \in H$. The norm N is a multiplicative quadratic form and its associated symmetric bilinear form is

$$\langle \cdot , \cdot \rangle \colon H \times H \to F \colon (x, y) \mapsto \langle x, y\rangle = \mathrm{tr}(x\overline{y}) = x\overline{y} + y\overline{x}. \tag{9}$$

The form $\langle \cdot , \cdot \rangle$ is non-degenerate and so the mapping sending each subspace X of H_F to its orthogonal subspace $X^\perp$ is a polarity of $\mathbb{P}(H_F)$.

The next result is briefly mentioned in [7, Rem. 4.5] and [30, p. 76, Ex. 10] ($\mathrm{Char}\, F \neq 2$ only). For the sake of completeness, a proof will be presented below.

Lemma 3.3. *Given quaternions $q_1, q_2 \in H$ there exists an inner automorphism of H taking q_1 to q_2 if, and only if,* $\mathrm{tr}(q_1) = \mathrm{tr}(q_2)$ *and* $N(q_1) = N(q_2)$.

Proof. From $\mathrm{tr}(q_1) = \mathrm{tr}(q_2)$ and $N(q_1) = N(q_2)$, the quaternions q_1, q_2 are zeros of the polynomial $m(t) = t^2 - \mathrm{tr}(q_1)t + N(q_1) \in F[t]$. If $m(t)$ is reducible over F, then $m(t)$ has no zeros in H outside F. Thus $q_1 \in F$ and $m(t) = (t - q_1)^2$. Now $m(q_2) = 0$ yields $q_2 = q_1$, whence the identity id_H is a solution. On the other hand, if $m(t)$ is irreducible over F, then id_F can be extended in a unique way to an isomorphism γ of the commutative field $F1 \oplus Fq_1 \subset H$ onto the commutative field $F1 \oplus Fq_2 \subset H$ such that $\gamma(q_1) = q_2$; see, for example, [8, Prop. 7.2.2]. By the theorem of Skolem-Noether [21, Thm. 4.9], this γ extends to an inner automorphism of H.

The proof of the converse is straightforward. □

The above result describes the orbits under the action of the inner automorphism group $\widetilde{H^*}$ on quaternions.[7] By considering the vector space H_F as an affine space, the orbit of any $q \in H$ is the intersection of the affine quadric $\{x \in H \mid N(x) = N(q)\}$ with the hyperplane $\{x \in H \mid \mathrm{tr}(x) = \mathrm{tr}(q)\}$. Here, however, we aim at providing a description of the orbits of the points of $\mathbb{P}(H_F)$ under the action of $\widetilde{H^*}$. Since the behaviour of the points of the plane $(F1)^\perp = \{x \in H \mid \mathrm{tr}(x) = 0\}$ is different from that of any other point, these points will be excluded in the next proposition.

---

[7] After extending H to a *projective line over H* by adding an extra point ∞, these $\widetilde{H^*}$-orbits turn into orbits of the group of projectivities that fix the points 0, 1 and ∞. This approach results in an alternative description, as can be seen from [12].

Proposition 3.4. *Let H be a quaternion skew field with centre F and let Fq, $q \in H^*$, be a point of $\mathbb{P}(H_F)$ such that* $\operatorname{tr}(q) \neq 0$. *Then the following hold.*

(a) *The orbit of Fq under the action of the group $\widetilde{H^*}$ of inner automorphisms of H is a quadric of $\mathbb{P}(H_F)$, say $\mathcal{O}_q$, which is given by the quadratic form*

$$\omega_q \colon H \to F \colon x \mapsto \operatorname{tr}(q)^2\, N(x) - N(q)\, \operatorname{tr}(x)^2.$$

(b) *If $q \in F^*$, then $\mathcal{O}_q$ consists of a single point.*

(c) *If $q \in H^* \setminus F^*$, then $\mathcal{O}_q$ is an elliptic quadric, no line through $F1$ is tangent to $\mathcal{O}_q$, and the polar form of ω_q is non-degenerate.*

Proof. (a) If Fp is in the $\widetilde{H^*}$-orbit of Fq, then there are $c \in F^*$ and $h \in H^*$ such that $p = ch^{-1}qh$. Consequently, $N(p) = c^2 N(q)$ and $\operatorname{tr}(p) = c\operatorname{tr}(q)$. This entails $\omega_q(p) = c^2\omega_q(q) = 0$.

Conversely, let a point Fp', $p' \in H^*$, be given with $\omega_q(p') = 0$. Then $p' \neq 0$ implies $N(p') \neq 0$ and so $\operatorname{tr}(p') \neq 0$ follows from $\omega_q(p') = 0$. We define

$$p := \operatorname{tr}(q)\operatorname{tr}(p')^{-1} p'.$$

Then $\operatorname{tr}(p) = \operatorname{tr}(q) \neq 0$, and $\omega_q(p) = 0$ establishes $N(p) = N(q)$. Now Lemma 3.3 implies the existence of an $h \in H^*$ such that $p = h^{-1}qh$.

(b) The quadric $\mathcal{O}_q$, $q \in F^*$, is the $\widetilde{H^*}$-orbit of $F1$, whence it consists of this single point only.

(c) The point $F1$ is not in the $\widetilde{H^*}$-orbit of Fq and so $F1$ is off the quadric $\mathcal{O}_q$. From $q + \overline{q} = \operatorname{tr}(q) \in F^*$ and $\omega_q(\overline{q}) = 0$, the line joining Fq and $F1$ meets $\mathcal{O}_q$ residually at $F\overline{q} \neq Fq$ and so it is not tangent to $\mathcal{O}_q$. Also, the point Fq is a regular point of $\mathcal{O}_q$. By the transitive action of the group $\widetilde{H^*}$ on the points of $\mathcal{O}_q$, the same applies to all other points of $\mathcal{O}_q$. The quadric $\mathcal{O}_q$ cannot be ruled, because it does not contain any point of the plane $\{x \in H \mid \operatorname{tr}(x) = 0\}$.

The polar form of ω_q is

$$(x, y) \mapsto \operatorname{tr}(q)^2\langle x, y\rangle - 2N(q)\operatorname{tr}(x)\operatorname{tr}(y) = \operatorname{tr}(q)^2\operatorname{tr}(x\overline{y}) - 2N(q)\operatorname{tr}(x)\operatorname{tr}(y).$$

If $\operatorname{Char} F \neq 2$ then the polar form of ω_q is non-degenerate, since otherwise $\mathcal{O}_q$ would contain a singular point. In the case of $\operatorname{Char} F = 2$ the form ω_q is non-degenerate, because it merely is a non-zero scalar multiple of the non-degenerate alternating bilinear form $\langle\,\cdot\,,\cdot\,\rangle$ from (9). □

Proposition 3.5. *Let H be a quaternion skew field with centre F and, in $\mathbb{P}(H_F)$, let L be a line that passes through the point $F1$ and is not contained in the plane $(F1)^\perp$. Every plane through an arbitrary line in the $\widetilde{H^*}$-orbit of L contains infinitely many lines of this orbit.*

Proof. By virtue of the action of $\widetilde{H^*}$ on $\widetilde{H^*}(L)$, it is enough to show the assertion for an arbitrary plane E passing through L.

On the line L, we can pick one point, say Fq, other than $F1$ such that $\operatorname{tr}(q) \neq 0$. By Proposition 3.4, the orbit of Fq is an elliptic quadric $\mathcal{O}_q$. Furthermore, the line L is a bisecant of this quadric that meets $\mathcal{O}_q$ at Fq and $F\overline{q} \neq Fq$. The plane E contains the bisecant L of $\mathcal{O}_q$ and so E cannot be a tangent plane of $\mathcal{O}_q$. This implies that E intersects $\mathcal{O}_q$ along a regular conic. As F is infinite, so is this conic. By joining each of the points of the conic with $F1$ we get infinitely many lines through $F1$ in the plane E. All of them are in $\widetilde{H^*}(L)$. □

Remark 3.6. The orbit of any line $L \in \mathcal{A}(H_F)$ under the group $\widetilde{H^*}$ is infinite [10, Thm. 3]. This result was improved in [46] by showing that any such orbit has cardinality $|F|$. Limited to the case of quaternion skew fields and lines of $\mathcal{A}(H_F)$ that are not in $(F1)^\perp$, the last proposition enriches this result with a geometric insight.

From [17, Thm. 4.12], the group $\widetilde{H^*}$ acts transitively on $\mathcal{A}(H_F)$ if, and only if, F is a formally real pythagorean field and H is an "ordinary" quaternion skew field with centre F. Precisely under these circumstances, $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ admits no proper Clifford-like parallelisms.

3.3 Parallel classes fixed by automorphisms

First, let $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ be a projective double space as specified in Section 2. Suppose that a left translation λ_g, $g \in H^*$, acts as a non-identical collineation on $\mathbb{P}(H_F)$. Hence $g \in H^* \setminus F^*$. Any line $M \in \mathcal{L}(H_F)$ is left parallel to its image $\lambda_g(M)$ and so λ_g fixes all left parallel classes. As we saw in Lemma 3.1, $\mathcal{S}_r(F1 \oplus Fg)$ is the only right parallel class that is fixed linewise under λ_g. If λ_g fixes also all lines of a left parallel class, then Lemma 3.1 forces λ_g to be a right translation as well, that is, g has to be in the centre of H. In case (A) this gives a contradiction. In case (B), H is a commutative field and so this condition imposes no restriction on g; due to $\parallel_\ell = \parallel_r$, the given λ_g fixes precisely one left parallel class linewise, namely $\mathcal{S}_\ell(F1 \oplus Fg)$.

For the rest of this subsection we confine ourselves to the case (A).

Proposition 3.7. *Let H be a quaternion skew field with centre F and let $g \in H^* \setminus F^*$. In $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$, a right parallel class is invariant under the left translation λ_g precisely when it is of the form $\mathcal{S}_r(M)$, where M is a line satisfying at least one of the following conditions:*

$$M = F1 \oplus Fg; \tag{10}$$

$$F1 \subseteq M \subseteq g^{-1}(F1)^\perp. \tag{11}$$

Proof. (a) Suppose that (10) holds. From Lemma 3.1, all lines of the right parallel class $\mathcal{S}_r(M)$ are fixed under λ_g.

(b) Suppose that a line M satisfies (11). The line $M^\perp$ is left parallel and right parallel to M (see [17, Cor. 4.4]) and it is contained in $(F1)^\perp$. The line gM is also left parallel to M. As $M^\perp$ and gM are incident with the plane $(F1)^\perp$, they share a common point and so they must coincide. Taking into account that $\lambda_g \in \Gamma_r$ and $M^\perp \parallel_r M$ we obtain $\lambda_g(\mathcal{S}_r(M)) = \mathcal{S}_r(gM) = \mathcal{S}_r(M^\perp) = \mathcal{S}_r(M)$, as required.

(c) Conversely, any λ_g-invariant right parallel class can be written as $\mathcal{S}_r(M)$ with $F1 \subseteq M$. Then $\lambda_g(M) \parallel_\ell M \parallel_r \lambda_g(M)$. Again from [17, Cor. 4.4], there are only two possibilities. First, $\lambda_g(M) = gM = M$, which implies $Fg \subseteq M$ and establishes (10). Second, $\lambda_g(M) = gM = M^\perp$. From $F1 \subseteq M$ we obtain $\lambda_g(M) = M^\perp \subseteq (F1)^\perp$. Applying λ_g^{-1} results in $M \subseteq g^{-1}(F1)^\perp$, whence (11) holds. □

Remark 3.8. Figures 1 and 2 depict the possible cases in Proposition 3.7 under the assumption $\operatorname{Char} F \neq 2$ and $\operatorname{Char} F = 2$, respectively. In all cases, there are distinct points $F1$ and Fg as well as distinct planes $(F1)^\perp$ and $g^{-1}(F1)^\perp$. Furthermore, $(F1)^\perp \cap (g^{-1}(F1)^\perp) = (F1 \oplus Fg)^\perp$.

The pictures on the left-hand side show the situation when $F1 \not\subseteq g^{-1}(F1)^\perp$ or, in other words, when $Fg \not\subseteq (F1)^\perp$, which in turn is equivalent to $\operatorname{tr}(g) \neq 0$. Here there are no lines M subject to (11). The pictures on the right-hand side show the opposite situation. Here the set of all lines M that satisfy (11) comprises a pencil of lines. In detail, the circumstances are as follows.

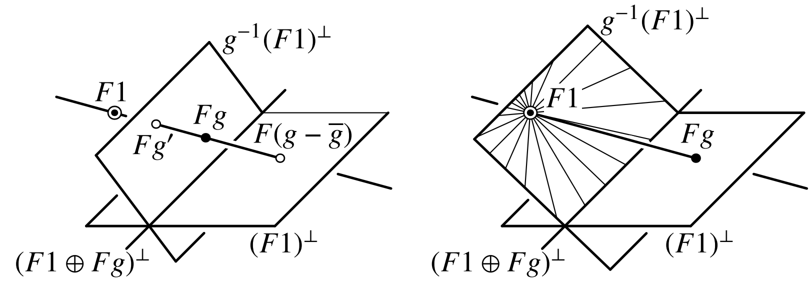

Figure 1: $\operatorname{Char} F \neq 2$

Figure 1, left: the line $F1 \oplus Fg$ intersects the plane $(F1)^\perp$ at $F(g-\overline{g})$ and the plane $g^{-1}(F1)^\perp$ at Fg', $g' := g^{-1}(g-\overline{g})$; the points $F1$, Fg, $F(g-\overline{g})$ and Fg' are mutually distinct; the lines $F1 \oplus Fg$ and $(F1 \oplus Fg)^\perp$ are skew.

Figure 1, right: $(F1 \oplus Fg) \cap (F1)^\perp = Fg$, $(F1 \oplus Fg) \cap g^{-1}(F1)^\perp = F1$; the lines $F1 \oplus Fg$ and $(F1 \oplus Fg)^\perp$ are skew.

Figure 2, left: $(F1 \oplus Fg) \cap (F1)^\perp = F1$, $(F1 \oplus Fg) \cap g^{-1}(F1)^\perp = Fg^{-1}$; the points $F1$, Fg and Fg^{-1} are mutually distinct; the lines $F1 \oplus Fg$ and $(F1 \oplus Fg)^\perp$ are skew.

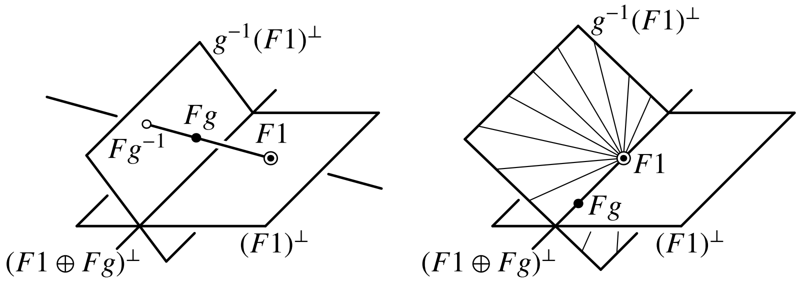

Figure 2: Char $F = 2$

Figure 2, right: the line $F1 \oplus Fg$ coincides with $(F1 \oplus Fg)^\perp$.

Finally, note that the situations depicted on the right-hand side, namely $Fg \subseteq (F1)^\perp$, comprises precisely the cases when the left translation λ_g acts as an involution on the projective space.

4 Main results

The definition of a Clifford-like parallelism in [17, Def. 3.2] is essentially based on a given projective double space $(\mathbb{P}, \|_\ell, \|_r)$. We are thus led to the problem of whether or not distinct projective double spaces can share a Clifford-like parallelism.

Theorem 4.1. *Let $(\mathbb{P}(H_F), \|_\ell, \|_r)$ be a projective double space, where H is an F-algebra subject to* (A) *or* (B)*. Furthermore, let $\|'_\ell$ and $\|'_r$ be parallelisms such that $(\mathbb{P}(H_F), \|'_\ell, \|'_r)$ is also a projective double space. Suppose that a parallelism $\|$ of $\mathbb{P}(H_F)$ is Clifford-like with respect to both double space structures. Then, possibly up to a change of the attributes "left" and "right" in one of these double spaces, $\|_\ell = \|'_\ell$ and $\|_r = \|'_r$.*

Proof. First, we consider case (A). We take any line of the star $\mathcal{A}(H_F)$. We noted in Remark 3.6 that the orbit of this line under the group $\widetilde{H^*}$ of all inner automorphisms of H is infinite. Thus there are three mutually distinct lines, say L_1, L_2 and L_3, in this orbit. From [17, Thm. 4.10], the $\|$-classes of these lines are of the same kind w.r.t. $(\mathbb{P}(H_F), \|_\ell, \|_r)$, *i.e.*, we have either $\mathcal{S}(L_n) = \mathcal{S}_\ell(L_n)$ for all $n \in \{1, 2, 3\}$ or $\mathcal{S}(L_n) = \mathcal{S}_r(L_n)$ for all $n \in \{1, 2, 3\}$.

Next, we turn to case (B). There exist three mutually distinct lines $L_1, L_2, L_3 \in \mathcal{A}(H_F)$. Their $\|$-classes are of the same kind w.r.t. $(\mathbb{P}(H_F), \|_\ell, \|_r)$ due to $\|_\ell = \|_r = \|$.

In both cases, the parallel classes $\mathcal{S}(L_n)$, $n \in \{1, 2, 3\}$, are mutually distinct. Consequently, among them there are at least two distinct classes of the same kind w.r.t. the double space $(\mathbb{P}(H_F), \|'_\ell, \|'_r)$. Up to a change of notation, we may assume $\mathcal{S}(L_n) = \mathcal{S}_r(L_n) = \mathcal{S}'_r(L_n)$ for $n \in \{1, 2\}$. Now Proposition 2.4 shows that the

Clifford parallelisms $\|_r$ and $\|_r'$ coincide. This in turn forces $\|_\ell = \|_\ell'$, since the left parallelism is uniquely determined by the right one (see [24, pp. 75–76] or [19, §6]). □

Corollary 4.2. *Any Clifford-like parallelism* $\|$ *of* $(\mathbb{P}(H_F), \|_\ell, \|_r)$ *other than* $\|_\ell$ *and* $\|_r$ *is not Clifford.*

Proof. Assume to the contrary that $\| =: \|_\ell'$ is Clifford. Then there is a parallelism, say $\|_r'$, such that $(\mathbb{P}(H_F), \|_\ell', \|_r')$ is a projective double space. Applying Theorem 4.1 gives therefore $\| = \|_\ell$ or $\| = \|_r$, a contradiction. □

The above corollary, when restricted to case (A), is just a reformulation of [17, Thm. 4.15]. Therefore, the rather technical proof in [17], which relies on H being a quaternion skew field, can now be avoided.

Our final results provide the announced characterisations of Clifford parallelisms among Clifford-like parallelisms.

Theorem 4.3. *Let* $\|$ *be a Clifford-like parallelism of* $(\mathbb{P}(H_F), \|_\ell, \|_r)$*, where* H *is an* F*-algebra subject to* (A) *or* (B)*. Then the following assertions are equivalent.*

- (a) *The parallelism* $\|$ *is Clifford.*
- (b) *The parallelism* $\|$ *admits an automorphism* $\beta \in \Gamma_{\|}$ *that stabilises all its parallel classes and acts as a non-identical collineation on the projective space* $\mathbb{P}(H_F)$*.*

Proof. (a) ⇒ (b). There exists a $g \in H^* \setminus F^*$. Corollary 4.2 shows that $\| = \|_\ell$ or $\| = \|_r$. In the first case the left translation λ_g has the required properties, in the second case the same applies to the right translation ρ_g.

(b) ⇒ (a) In case (B), $\|_\ell = \|_r$ implies that $\| = \|_\ell$ is Clifford.

From now on we deal with case (A) only. We select one line N_1 through $F1$ that is not in $(F1)^\perp$. We assume w.l.o.g. that the parallel class $\mathcal{S}(N_1)$ is a *left* parallel class. (Otherwise, we have to interchange the attributes "left" and "right" in what follows.) Let $g := \beta(1)$. We consider the left translation λ_g and the product

$$\alpha := \lambda_g^{-1} \circ \beta. \tag{12}$$

We choose one $N \in \widetilde{H^*}(N_1)$. Then the parallel class $\mathcal{S}(N)$ is a left parallel class. Thus

$$N \|_\ell \beta(N) \|_\ell g^{-1}\beta(N) = \alpha(N). \tag{13}$$

Formula (13) and $\alpha(1) = 1 \in N$ together force $\alpha(N) = N$. By Proposition 3.5, every plane through N contains at least two lines from the orbit $\widetilde{H^*}(N_1)$, and so any such plane is fixed under α. The lines and planes through $F1$ are the "points" and “lines” of a projective plane; “incidence” is given by symmetrised inclusion. Our α acts on this projective plane as a collineation. By the above, all “lines” through the “point” N are fixed under α, and so N serves as a “centre” of this collineation. But N may vary in the orbit $\widetilde{H^*}(N_1)$, which comprises more than one line by the theorem of Cartan-Brauer-Hua [29, (13.17)]. Consequently, this collineation has more than one “centre”, that is, α fixes all lines of the star $\mathcal{A}(H_F)$.

We now consider the action of α on the projective space $\mathbb{P}(H_F)$. Since all lines of $\mathcal{A}(H_F)$ are fixed, α acts as a perspective collineation with centre $F1$. This implies that α is F-linear. Since α and λ_g^{-1} are F-linear, so is β. From $\beta \in \Gamma_{\|} \cap \mathrm{GL}(H_F) = \Gamma_\ell \cap \mathrm{GL}(H_F)$ (see [18, Thm. 3.5]) and $\lambda_g^{-1} \in \Gamma_\ell \cap \mathrm{GL}(H_F)$ follows $\alpha \in \Gamma_\ell \cap \mathrm{GL}(H_F)$. Now pick any line $L \in \mathcal{L}(H_F)$. The left parallel line to L through $F1$ is fixed under $\alpha \in \Gamma_\ell$, whence we have $L \parallel_\ell \alpha(L)$. On the other hand, L is incident with at least one plane through $F1$. This plane is α-invariant. Therefore the left parallel lines L and $\alpha(L)$ are coplanar, which in turn implies $L = \alpha(L)$. So we arrive at $\alpha = c\,\mathrm{id}_H$ for some $c \in F^*$. Now, using $\alpha(1) = 1$, we end up with $\alpha = \mathrm{id}_H$.

Next, we give an explicit description of β. By virtue of (12), our assumption that β does not fix all lines of $\mathbb{P}(H_F)$, and $\alpha = \mathrm{id}_H$, we have

$$\beta = \lambda_g \text{ and } g \in H^* \setminus F^*.$$

Finally, we claim that $\parallel = \parallel_\ell$. Assume to the contrary that $\parallel \neq \parallel_\ell$. So there is a line M_1 with $\mathcal{S}(M_1) = \mathcal{S}_r(M_1)$ and $F1 \subseteq M_1$. Then $\mathcal{S}(M) = \mathcal{S}_r(M)$ for all lines $M \in \widetilde{H^*}(M_1)$, which forces

$$\beta(\mathcal{S}_r(M)) = \mathcal{S}_r(M) \text{ for all } M \in \widetilde{H^*}(M_1). \tag{14}$$

We now distinguish three cases.

Case (i). Let $F1 \not\subseteq g^{-1}(F1)^\perp$. From Proposition 3.7 and (14), any line $M \in \widetilde{H^*}(M_1)$ has to satisfy (10). This implies $\widetilde{H^*}(M_1) = \{F1 \oplus Fg\}$ and contradicts the theorem of Cartan-Brauer-Hua [29, (13.17)], which says $|\widetilde{H^*}(M_1)| > 1$.

Case (ii). Let $F1 \subseteq g^{-1}(F1)^\perp$ and $M_1 \not\subseteq (F1)^\perp$. We choose any plane E other than $g^{-1}(F1)^\perp$ through the line M_1. Let $\mathcal{M}_E$ denote the set of all lines that are incident with E and belong to $\widetilde{H^*}(M_1)$. By Proposition 3.5, the set $\mathcal{M}_E$ is infinite. From Proposition 3.7 and (14), any line $M \in \mathcal{M}_E$ has to satisfy (10) or (11), that is $M = F1 \oplus Fg$ or $M = g^{-1}(F1)^\perp \cap E$. This implies $|\mathcal{M}_E| \leq 2$, an absurdity.

Case (iii). Let $F1 \subseteq g^{-1}(F1)^\perp$ and $M_1 \subseteq (F1)^\perp$. From Remark 3.8, this applies precisely when

$$M_1 = F1 \oplus Fg = (F1)^\perp \cap (g^{-1}(F1)^\perp); \tag{15}$$

see the right-hand side of Figure 2. The plane $(F1)^\perp$ is $\widetilde{H^*}$-invariant, whence it contains all lines of $\widetilde{H^*}(M_1)$. From Proposition 3.7 and (14), any line $M \in \widetilde{H^*}(M_1)$

has to satisfy (10) or (11). By virtue of the second equation in (15), this implies $\widetilde{H^*}(M_1) = \{F1 \oplus Fg\}$ and, as in Case (i), contradicts the theorem of Cartan-Brauer-Hua. □

Remark 4.4. Note that, as a consequence of the previous theorem, the group of automorphisms that preserve all parallel classes with respect to a given Clifford-like parallelism $\parallel$ of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$ is contained in $\mathrm{GL}(H_F)$. Moreover this group is the group of left translations (or right translations respectively) precisely when $\parallel = \parallel_r$ (respectively $\parallel = \parallel_\ell$). If, on the other hand, $\parallel$ is a proper Clifford-like parallelism, then this group is the group of all λ_g with $g \in F^*$, thus, from the projective point of view, it comprises only the identity map.

Theorem 4.5. *Let $\parallel$ be a Clifford-like parallelism of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$, where H is an F-algebra subject to* (A) *or* (B). *Then the following assertions are equivalent.*

(a) *The parallelism $\parallel$ is Clifford and $\parallel_\ell \neq \parallel_r$.*

(b) *The parallelism $\parallel$ admits an automorphism $\beta \in \Gamma_\parallel$ that stabilises a single parallel class of $\parallel$ and, furthermore, fixes all lines of this particular parallel class.*

Proof. (a) ⇒ (b). Corollary 4.2 shows that $\parallel = \parallel_\ell$ or $\parallel = \parallel_r$. Let, for example, $\parallel = \parallel_r$. We infer from $\parallel_\ell \neq \parallel_r$ that H is a quaternion skew field. There exists a $g \in H \setminus (F1 \cup (F1)^\perp)$; cf. the left-hand sides of Fig. 1 and Fig. 2 for illustrations. Then no line $M \in \mathcal{L}(H_F)$ satisfies (11). By Proposition 3.7, $\beta := \lambda_g$ stabilises a single right parallel class, namely $\mathcal{S}_r(F1 \oplus Fg)$, and, furthermore, β fixes all lines of $\mathcal{S}_r(F1 \oplus Fg)$.

(b) ⇒ (a). The only β-invariant parallel class can be written in the form $\mathcal{S}(L)$ with $L \in \mathcal{A}(H_F)$. Let us assume that $\mathcal{S}(L)$ is a right parallel class. Since all lines of $\mathcal{S}_r(L)$ are fixed under β, we obtain $\beta \in \mathrm{K}(H, \mathcal{S}_r(L))^* = \lambda(L^*)$ from Lemma 3.1. Consequently, all left parallel classes are stabilised under β, whence none of them is a parallel class of $\parallel$. This shows $\parallel_\ell \neq \parallel = \parallel_r$. □

Theorem 4.6. *Let $\parallel$ be a Clifford-like parallelism of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$, where H is an F-algebra subject to* (A) *or* (B). *Then the following assertions are equivalent.*

(a) *The parallelism $\parallel$ is Clifford and $\parallel_\ell = \parallel_r$.*

(b) *If an automorphism $\beta \in \Gamma_\parallel$ fixes all lines of at least one parallel class of $\parallel$, then all parallel classes of $\parallel$ are stabilised under β.*

(c) *The parallelism $\parallel$ admits an automorphism $\beta \in \Gamma_\parallel$ that stabilises all its parallel classes, fixes at least one of its parallel classes linewise, and acts as a non-identical collineation on the projective space $\mathbb{P}(H_F)$.*

Proof. (a) ⇒ (b). We have $\parallel = \parallel_\ell = \parallel_r$. Let $\beta \in \Gamma_\parallel$ fix all lines of a right[8] parallel class, which will be written as $\mathcal{S}_r(L)$ with $L \in \mathcal{A}(H_F)$. From Lemma 3.1, $\beta \in \mathrm{K}(H, \mathcal{S}_r(L))^* = \lambda(L^*)$ and so β stabilises all left parallel classes or, said differently, all $\parallel$-classes.

(b) ⇒ (c). We may assume w.l.o.g. that there exists a line $L \in \mathcal{A}(H_F)$ with the property $\mathcal{S}(L) = \mathcal{S}_r(L)$. There is a $g \in L^* \setminus F^*$. The left translation $\lambda_g =: \beta$ fixes all lines of $\mathcal{S}(L) = \mathcal{S}_r(L)$ and acts as a non-identical collineation on $\mathbb{P}(H_F)$. So, by our assumption, β stabilises all $\parallel$-classes. Thus β meets all the requirements appearing in (c).

(c) ⇒ (a). We may assume w.l.o.g. that β fixes all lines of a right parallel class, $\mathcal{S}_r(L) = \mathcal{S}(L)$ with $L \in \mathcal{A}(H_F)$. There are two possibilities.

Case (i). $\parallel_\ell \neq \parallel_r$. Theorem 4.3 gives that $\parallel$ is a Clifford parallelism of $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$. From [17, Cor. 4.3], the parallelisms $\parallel_\ell$ and $\parallel_r$ have no parallel classes in common. Consequently, $\mathcal{S}_r(L)$ being one of the $\parallel$-classes yields $\parallel = \parallel_r$. From Lemma 3.1, the given automorphism β is a left translation λ_g for some $g \in H^*$. Since β acts non-identical on $\mathbb{P}(H_F)$, we have $g \in H^* \setminus F^*$. Hence, by Proposition 3.7, at least one right parallel class is not stabilised under β, a contradiction.

Case (ii). $\parallel_\ell = \parallel_r$. Now $\parallel_\ell = \parallel_r$ is the only Clifford-like parallelism of the projective double space $(\mathbb{P}(H_F), \parallel_\ell, \parallel_r)$, whence $\parallel$ turns out to be Clifford. □

References

[1] L. Bader and G. Lunardon, *Desarguesian spreads*, Ric. Mat. **60** (2011), no. 1, 15–37.

[2] A. Betten, S. Topalova, and S. Zhelezova, *Parallelisms of* PG(3, 4) *invariant under cyclic groups of order* 4, Algebraic informatics (M. Ćirić, M. Droste, and J.-É. Pin, eds.), Lecture Notes in Comput. Sci., vol. 11545, Springer, Cham, 2019, 8th International Conference, CAI 2019, Niš, Serbia, June 30–July 4, 2019, pp. 88–99.

[3] D. Betten and R. Löwen, *Compactness of the automorphism group of a topological parallelism on real projective* 3*-space*, Results Math. **72** (2017), no. 1-2, 1021–1030.

[4] D. Betten and R. Riesinger, *Clifford parallelism: old and new definitions, and their use*, J. Geom. **103** (2012), no. 1, 31–73.

---

[8] We use the attributes “left” and “right” in accordance with the general situation, as described elsewhere. Of course, the distinction between “left” and “right” is immaterial here.

[5] ———, *Collineation groups of topological parallelisms*, Adv. Geom. **14** (2014), no. 1, 175–189.

[6] A. Blunck, N. Knarr, B. Stroppel, and M. J. Stroppel, *Clifford parallelisms defined by octonions*, Monatsh. Math. **187** (2018), no. 3, 437–458.

[7] A. Blunck, S. Pasotti, and S. Pianta, *Generalized Clifford parallelisms*, Innov. Incidence Geom. **11** (2010), 197–212.

[8] P. M. Cohn, *Basic algebra*, Springer-Verlag, London, 2003, Groups, Rings and Fields.

[9] E. Ellers and H. Karzel, *Kennzeichnung elliptischer Gruppenräume*, Abh. Math. Sem. Univ. Hamburg **26** (1963), 55–77.

[10] C. C. Faith, *On conjugates in division rings*, Canadian J. Math. **10** (1958), 374–380.

[11] O. Giering, *Vorlesungen über höhere Geometrie*, Vieweg, Braunschweig, 1982.

[12] H. Havlicek, *Durch Kollineationsgruppen bestimmte projektive Räume*, Beiträge Algebra Geom. **27** (1988), 175–184.

[13] H. Havlicek, *On Plücker transformations of generalized elliptic spaces*, Rend. Mat. Appl. (7) **15** (1995), no. 1, 39–56.

[14] ———, *A characteristic property of elliptic Plücker transformations*, J. Geom. **58** (1997), no. 1-2, 106–116.

[15] H. Havlicek, *A note on Clifford parallelisms in characteristic two*, Publ. Math. Debrecen **86** (2015), no. 1-2, 119–134.

[16] ———, *Clifford parallelisms and external planes to the Klein quadric*, J. Geom. **107** (2016), no. 2, 287–303.

[17] H. Havlicek, S. Pasotti, and S. Pianta, *Clifford-like parallelisms*, J. Geom. **110** (2019), no. 1, Art. 2, 18 pp.

[18] ———, *Automorphisms of a Clifford-like parallelism*, Adv. Geom. (to appear).

[19] A. Herzer, *Halbprojektive Translationsgeometrien*, Mitt. Math. Sem. Giessen **127** (1977), ii+136 pp.

[20] J. W. P. Hirschfeld, *Finite projective spaces of three dimensions*, Oxford University Press, Oxford, 1985.

[21] N. Jacobson, *Basic algebra II*, Freeman, New York, 1989.

[22] N. L. Johnson, *Parallelisms of projective spaces*, J. Geom. **76** (2003), no. 1-2, 110–182, Combinatorics, 2002 (Maratea).

[23] ______, *Combinatorics of spreads and parallelisms*, Pure and Applied Mathematics (Boca Raton), vol. 295, CRC Press, Boca Raton, 2010.

[24] H. Karzel and H.-J. Kroll, *Geschichte der Geometrie seit Hilbert*, Wissenschaftliche Buchgesellschaft, Darmstadt, 1988.

[25] H. Karzel, H.-J. Kroll, and K. Sörensen, *Invariante Gruppenpartitionen und Doppelräume*, J. Reine Angew. Math. **262/263** (1973), 153–157.

[26] ______, *Projektive Doppelräume*, Arch. Math. (Basel) **25** (1974), 206–209.

[27] H. Karzel and C. J. Maxson, *Kinematic spaces with dilatations*, J. Geom. **22** (1984), no. 2, 196–201.

[28] H.-J. Kroll, *Bestimmung aller projektiven Doppelräume*, Abh. Math. Sem. Univ. Hamburg **44** (1975), 139–142 (1976).

[29] T. Y. Lam, *A first course in noncommutative rings*, second ed., Graduate Texts in Mathematics, vol. 131, Springer, New York, 2001.

[30] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005.

[31] R. Löwen, *Compactness of the automorphism group of a topological parallelism on real projective* 3*-space: The disconnected case*, Bull. Belg. Math. Soc. Simon Stevin **25** (2018), no. 4, 629–640.

[32] ______, *A characterization of Clifford parallelism by automorphisms*, Innov. Incidence Geom. **17** (2019), no. 1, 43–46.

[33] ______, *Parallelisms of* $PG(3,\mathbb{R})$ *admitting a* 3*-dimensional group*, Beiträge Algebra Geom. **60** (2019), no. 2, 333–337.

[34] H. Lüneburg, *Translation planes*, Springer, Berlin, 1980.

[35] M. Marchi and C. Perelli Cippo, *Su una particolare classe di* S*-spazi*, Rend. Semin. Mat. Brescia **4** (1980), 3–42.

[36] S. Pianta, *On automorphisms for some fibered incidence groups*, J. Geom. **30** (1987), no. 2, 164–171.

[37] S. Pianta and E. Zizioli, *Collineations of geometric structures derived from quaternion algebras*, J. Geom. **37** (1990), no. 1-2, 142–152.

[38] W. Seier, *Kollineationen von Translationsstrukturen*, J. Geom. **1** (1971), no. 2, 183–195.

[39] ———, *Isomorphismen verallgemeinerter Parallelstrukturen*, J. Geom. **3** (1973), no. 2, 165–178.

[40] J. A. Tyrrell and J. G. Semple, *Generalized Clifford parallelism*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 61, Cambridge University Press, London New York, 1971.

[41] J. van Buggenhaut, *Algèbres d'octaves et parallélisme dans l'espace elliptique à* 7 *dimensions*, Acad. Roy. Belg. Bull. Cl. Sci. (5) **54** (1968), 662–670.

[42] ———, *Deux généralisations du parallélisme de Clifford*, Bull. Soc. Math. Belg. **20** (1968), 406–412.

[43] ———, *Principe de trialité et parallélisme dans l'espace elliptique à* 7 *dimensions*, Acad. Roy. Belg. Bull. Cl. Sci. (5) **54** (1968), 577–584.

[44] F. Vaney, *Le parallélisme absolu dans les espaces elliptiques réels à* 3 *et* 7 *dimensions et le principe de trialité dans l'espace elliptique à* 7 *dimensions*, Thèse, Université de Paris, 1929, Gauthier-Villars.

[45] H. Wähling, *Darstellung zweiseitiger Inzidenzgruppen durch Divisionsalgebren*, Abh. Math. Sem. Univ. Hamburg **30** (1967), 220–240.

[46] ———, *Konjugierte Teilkörper eines Körpers*, Arch. Math. (Basel) **37** (1981), no. 1, 52–58.

Hans Havlicek
Institut für Diskrete Mathematik und Geometrie
Technische Universität
Wiedner Hauptstraße 8–10/104
A-1040 Wien
Austria
`havlicek@geometrie.tuwien.ac.at`

Stefano Pasotti
DICATAM-Sez. Matematica
Università degli Studi di Brescia
via Branze, 43
I-25123 Brescia
Italy
`stefano.pasotti@unibs.it`

Silvia Pianta
Dipartimento di Matematica e Fisica
Università Cattolica del Sacro Cuore
via Trieste, 17
I-25121 Brescia
Italy
`silvia.pianta@unicatt.it`