Clifford-like parallelisms

Hans Havlicek Stefano Pasotti Silvia Pianta

Dedicated to Helmut Karzel on the occasion of his 90th birthday

Abstract

Given two parallelisms of a projective space we describe a construction, called blending, that yields a (possibly new) parallelism of this space. For a projective double space $(\mathbb{P}, \|_\ell, \|_r)$ over a quaternion skew field we characterise the "Clifford-like" parallelisms, *i.e.* the blends of the Clifford parallelisms $\|_\ell$ and $\|_r$, in a geometric and an algebraic way. Finally, we establish necessary and sufficient conditions for the existence of Clifford-like parallelisms that are not Clifford.

Mathematics Subject Classification (2010): 51A15, 51J15
Key words: Blend of parallelisms, Clifford parallelism, projective double space, quaternion skew field

1 Introduction

The first definition of parallel lines in the real projective 3-space dates back to 1873 and was introduced by W.K. Clifford in the metric framework of elliptic geometry (see [5]): two distinct lines M and N in the real elliptic 3-space, are said to be *Clifford parallel*, if the four lines M, N, $M^\perp$ and $N^\perp$ are elements of the same regulus. (Here $M^\perp$ denotes the polar line of M w.r.t. the "absolute", *i.e.* the imaginary quadric that determines the elliptic metric in the real projective 3-space).

Some years later, in 1890 F. Klein revived Clifford's ideas and, using the complexification of the real projective space, defined two lines to be *parallel in the sense of Clifford* if they meet the same complex conjugate pair of generators of the absolute (see [21]).

Depending on the kind of generators under consideration, one can speak of *right* parallel, or *left* parallel lines, then each fixed conjugate pair of generators "indicates" a *left* (or *right*) *parallel class*, which in fact is a regular spread, namely an elliptic linear congruence of the projective space.

Thus we can say that a *Clifford parallelism* in the real projective 3-space consists of all regular spreads, or elliptic linear congruences, whose indicator lines are pairs of complex conjugate lines of a regulus contained in an imaginary quadric. Besides, Clifford parallelisms go in pairs, and also note that all (real) Clifford parallelisms are projectively equivalent. An interesting survey on the various definitions of Clifford parallelisms can be found in [1].

Generalising this situation, H. Karzel, H.-J. Kroll and K. Sörensen in 1973 introduced the notion of a *projective double space* $(\mathbb{P}, \|_\ell, \|_r)$, that is a projective space (of unspecified dimension, over an unspecified field) equipped with two parallelism relations fulfilling a configurational property which can be expressed by the axiom (DS) of Section 3 (see [17], [18]). The real projective 3-space with left and right Clifford parallelisms is an example and it turns out that the projective double spaces $(\mathbb{P}, \|_\ell, \|_r)$ with $\|_\ell \neq \|_r$ are necessarily of dimension 3 and precisely the ones that can be obtained from a quaternion skew field H over a field F as in Section 4 (see [17], [18], [15], [22]).

In this way one obtains what in 2010 A. Blunck, S. Pasotti and S. Pianta called *generalized Clifford parallelisms* in the note [4]. If the maximal commutative subfields of H are not mutually F-isomorphic, then new "non-Clifford" regular parallelisms can be obtained by "blending" in some suitable way the left and right parallel classes (see [4, 4.13]). This method has no equivalent in the classical case, since the maximal commutative subfields of the real quaternions are mutually $\mathbb{R}$-isomorphic.

Taking up this idea, we introduce here the definition of *Clifford-like parallelism* in a projective double space $(\mathbb{P}, \|_\ell, \|_r)$, that is a parallelism $\|$ on $\mathbb{P}$ such that

$$\forall\, M, N \in \mathcal{L} :\, M \parallel N \Rightarrow (M \parallel_\ell N \;\text{ or }\; M \parallel_r N).$$

In Section 2 we start from the more general setting of equivalence relations on a set $\mathcal{L}$ and we define a *blend* of two equivalence relations π_1, π_2 as an equivalence relation π_3 such that each equivalence class of π_3 coincides with an equivalence class of π_1 or π_2. In order to obtain a characterisation of all the blends of π_1 and π_2 in Proposition 2.4, we use the equivalence relation π_{12} generated by them, which is the join of π_1 and π_2 in the lattice of equivalence relations on $\mathcal{L}$.

In Section 3 we study the blends of parallelisms of a projective space. By Theorem 3.1, we can prove that the Clifford-like parallelisms of a projective double space $(\mathbb{P}, \|_\ell, \|_r)$ are precisely the "blends" of $\|_\ell$ and $\|_r$. Therefore Clifford-like parallelisms are regular.

In Section 4 we connect a projective double space $(\mathbb{P}, \|_\ell, \|_r)$ with $\|_\ell \neq \|_r$ to a quaternion skew field H over a field F and we describe the equivalence relation

$\pi_{\ell r}$ generated by $\|_\ell$ and $\|_r$ using the maximal commutative subfields of H (see Theorem 4.2). In Theorem 4.10 we obtain a characterisation of all Clifford-like parallelisms of $(\mathbb{P}(H), \|_\ell, \|_r)$ showing that they are precisely those introduced in [4, 4.13]. Finally, in Theorems 4.12 and 4.15 we discuss the existence and some properties of Clifford-like parallelisms that are not Clifford.

To conclude, we observe that it might be interesting to investigate the blends of the left and right parallelisms of an arbitrary kinematic space in the same spirit as in [24].

2 Blends of equivalence relations

Throughout this section we consider an arbitrary set $\mathcal{L}$. Let $\pi \subseteq \mathcal{L} \times \mathcal{L}$ be an equivalence relation on $\mathcal{L}$. The partition of $\mathcal{L}$ associated with π is denoted by Π. The elements of Π are called *π-classes*. For any $M \in \mathcal{L}$ we denote by $\mathcal{C}(M)$ the π-class containing M. The same kind of notation will be used for other equivalence relations on $\mathcal{L}$ by writing, for example, π_1, Π_1 and $\mathcal{C}_1(M)$.

The following simple lemma will be used repeatedly.

Lemma 2.1. *Let π be an equivalence relation on $\mathcal{L}$ and $\mathcal{B} \subseteq \mathcal{L}$. Then the following are equivalent.*

(a) *$\mathcal{B}$ admits a partition by π-classes.*

(b) *$\{\mathcal{C}(X) \mid X \in \mathcal{B}\}$ is the only partition of $\mathcal{B}$ by π-classes.*

(c) *$\mathcal{B} = \bigcup_{X \in \mathcal{B}} \mathcal{C}(X)$.*

(d) *$\mathcal{L} \setminus \mathcal{B}$ admits a partition by π-classes.*

Proof. (a) $\Rightarrow$ (b). Let Σ be a partition of $\mathcal{B}$ by π-classes. Then Σ coincides with the partition given in (b). (b) $\Rightarrow$ (c). This is obvious. (c) $\Rightarrow$ (d). It suffices to observe that $\{\mathcal{C}(X) \mid X \in \mathcal{L} \setminus \mathcal{B}\}$ is a partition of $\mathcal{L} \setminus \mathcal{B}$ by π-classes. (d) $\Rightarrow$ (a). The existence of a partition, say Σ', of $\mathcal{L} \setminus \mathcal{B}$ by π-classes implies that $\Pi \setminus \Sigma'$ is a partition of $\mathcal{B}$ by π-classes. □

We now introduce our basic notion.

Definition 2.2. Let π_1 and π_2 be (not necessarily distinct) equivalence relations on $\mathcal{L}$. An equivalence relation π_3 on $\mathcal{L}$ is called a *blend* of π_1 and π_2 if

$$\Pi_3 \subseteq \Pi_1 \cup \Pi_2. \tag{2.1}$$

Equivalently, condition (2.1) can be written in the form

$$\forall\, M \in \mathcal{L}\colon\ \mathcal{C}_3(M) = \mathcal{C}_1(M) \ \text{ or } \ \mathcal{C}_3(M) = \mathcal{C}_2(M). \tag{2.2}$$

The *trivial blends* of π_1 and π_2 are the relations π_1 and π_2 themselves.

Our aim is to describe all blends of equivalence relations π_1 and π_2 on $\mathcal{L}$. We thereby use that all equivalence relations on $\mathcal{L}$ constitute a lattice; see, for example, [2, Ch. I, §8, Ex. 9] or [27, Sect. 50]. In this lattice, the *meet* of π_1 and π_2 equals $\pi_1 \cap \pi_2$ (however, the meet of equivalence relations is irrelevant for our investigation). The *join* of π_1 and π_2 is the intersection of all equivalence relations on $\mathcal{L}$ that contain $\pi_1 \cup \pi_2$ or, in other words, the equivalence relation *generated* by π_1 and π_2. This join is denoted by π_{12}. For all $M, N \in \mathcal{L}$, we have $M\ \pi_{12}\ N$ precisely when there exist an integer $n \geq 1$ and (not necessarily distinct) elements $N_1, N_2, \ldots, N_{2n+1} \in \mathcal{L}$ such that

$$M = N_1\ \pi_1\ N_2\ \pi_2\ N_3\ \pi_1\ \cdots\ \pi_2\ N_{2n+1} = N. \tag{2.3}$$

Also, we need another elementary lemma.

Lemma 2.3. *Let π_1 and π_2 be equivalence relations on $\mathcal{L}$ and denote by π_{12} the equivalence relation generated by π_1 and π_2. Furthermore, let $\mathcal{B} \subseteq \mathcal{L}$. Then the following statements are equivalent.*

(a) *$\mathcal{B}$ admits a partition by π_{12}-classes.*

(b) *$\mathcal{B}$ admits a partition by π_1-classes and a partition by π_2-classes.*

Proof. Let (a) be satisfied. For each $X \in \mathcal{B}$, we have $\mathcal{C}_1(X) \cup \mathcal{C}_2(X) \subseteq \mathcal{C}_{12}(X) \subseteq \mathcal{B}$, where the second inclusion follows by applying Lemma 2.1 to an existing partition of $\mathcal{B}$ by π_{12}-classes. This forces $\mathcal{B} \subseteq \bigcup_{Y \in \mathcal{B}} \mathcal{C}_1(Y) \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \bigcup_{Z \in \mathcal{B}} \mathcal{C}_2(Z) \subseteq \mathcal{B}$. These two formulas in combination with Lemma 2.1 establish (b).

Conversely, let us choose some $M \in \mathcal{B}$. Then, for all $N \in \mathcal{C}_{12}(M)$, there is a finite sequence as in (2.3), whence $N \in \mathcal{B}$. Thus $\mathcal{C}_{12}(M) \subseteq \mathcal{B}$. This shows $\mathcal{B} \subseteq \bigcup_{X \in \mathcal{B}} \mathcal{C}_{12}(X) \subseteq \mathcal{B}$, and Lemma 2.1 provides the existence of a partition of $\mathcal{B}$ by π_{12}-classes. □

Proposition 2.4. *Let π_1 and π_2 be equivalence relations on $\mathcal{L}$. Furthermore, denote by π_{12} the equivalence relation generated by π_1 and π_2.*

(a) *Upon choosing any subset $\mathcal{D}$ of $\mathcal{L}$ we let*

$$\mathcal{B} := \bigcup_{X \in \mathcal{D}} \mathcal{C}_{12}(X). \tag{2.4}$$

Then the set

$$\Pi_{\mathcal{B}} := \{\mathcal{C}_1(M) \mid M \in \mathcal{B}\} \cup \{\mathcal{C}_2(M) \mid M \in \mathcal{L} \setminus \mathcal{B}\} \tag{2.5}$$

is a partition of $\mathcal{L}$*, whose associated equivalence relation* $\pi_{\mathcal{B}}$ *is a blend of* π_1 *and* π_2*.*

(b) *Conversely, any blend of* π_1 *and* π_2 *arises according to* (a) *from at least one subset of* $\mathcal{L}$*.*

(c) *Let* $\mathcal{D}$ *and* $\tilde{\mathcal{D}}$ *be subsets of* $\mathcal{L}$*. Applying the construction from* (a) *to* $\mathcal{D}$ *and* $\tilde{\mathcal{D}}$ *gives* $\pi_{\mathcal{B}}$ *and* $\pi_{\tilde{\mathcal{B}}}$*, respectively. Then* $\pi_{\mathcal{B}}$ *coincides with* $\pi_{\tilde{\mathcal{B}}}$ *if, and only if,*

$$\{\mathcal{C}_{12}(X) \mid X \in \mathcal{D} \setminus \mathcal{L}_{12}\} = \{\mathcal{C}_{12}(\tilde{X}) \mid \tilde{X} \in \tilde{\mathcal{D}} \setminus \mathcal{L}_{12}\}, \tag{2.6}$$

where $\mathcal{L}_{12} := \{M \in \mathcal{L} \mid \mathcal{C}_1(M) = \mathcal{C}_2(M)\}$*.*

Proof. (a) We read off from (2.4) and Lemma 2.1 that $\mathcal{B}$ admits a partition by π_{12}-classes. Now Lemma 2.3 shows that $\mathcal{B}$ admits a partition by π_1-classes, namely $\{\mathcal{C}_1(M) \mid M \in \mathcal{B}\}$, and also a partition by π_2-classes. Applying Lemma 2.1 to the latter, gives the existence of a partition of $\mathcal{L} \setminus \mathcal{B}$ by π_2-classes, namely $\{\mathcal{C}_2(M) \mid M \in \mathcal{L} \setminus \mathcal{B}\}$. Therefore, in accordance with (2.5), $\Pi_{\mathcal{B}} \subseteq \Pi_1 \cup \Pi_2$ is a partition of $\mathcal{L}$, and so $\pi_{\mathcal{B}}$ is a blend of π_1 and π_2.

(b) Given any blend π_3 of π_1 and π_2 we start by defining

$$\mathcal{D} := \{X \in \mathcal{L} \mid \mathcal{C}_1(X) = \mathcal{C}_3(X)\} \text{ and } \mathcal{B}' := \bigcup_{X \in \mathcal{D}} \mathcal{C}_1(X). \tag{2.7}$$

According to its definition, $\mathcal{B}'$ admits a partition Σ' by π_1-classes. From (2.7), Σ' is a partition of $\mathcal{B}'$ by π_3-classes as well. Now Lemma 2.1 gives that $\mathcal{L} \setminus \mathcal{B}'$ admits a partition, say Σ'', by π_3-classes. No element of Σ'' can be in Π_1. Since π_3 is a blend of π_1 and π_2, we obtain $\Sigma'' \subseteq \Pi_2$. Next, by virtue of Lemma 2.1, $\mathcal{B}'$ admits also a partition by π_2-classes and, finally, Lemma 2.3 provides a partition Σ''' of $\mathcal{B}'$ by π_{12}-classes.

We now proceed as in part (a) of the current proof, commencing with the set $\mathcal{D}$ given in (2.7). From Lemma 2.1 and due to the existence of the partition Σ''' of $\mathcal{B}'$, we see that the set $\mathcal{B}$ from (2.4) equals the set $\mathcal{B}'$ appearing in (2.7). Under these circumstances we end up with $\Pi_{\mathcal{B}} = \Sigma' \cup \Sigma'' = \Pi_3$.

(c) This is an immediate consequence of (2.5). □

The set $\mathcal{D}$ from Proposition 2.4 (a) merely serves the purpose of defining the set $\mathcal{B}$ in (2.4). Formula (2.6) does not impose any restriction on $\mathcal{D} \cap \mathcal{L}_{12}$ and $\tilde{\mathcal{D}} \cap \mathcal{L}_{12}$. Therefore, whenever $\mathcal{L}_{12}$ is non-empty, there is a choice of $\mathcal{D}$ and $\tilde{\mathcal{D}}$

such that $\pi_{\mathcal{B}} = \pi_{\tilde{\mathcal{B}}}$ even though $\mathcal{B} \neq \tilde{\mathcal{B}}$. For example, $\mathcal{D} := \emptyset$ and $\tilde{\mathcal{D}} := \mathcal{L}_{12} \neq \emptyset$ give rise to $\emptyset = \mathcal{B} \neq \tilde{\mathcal{B}}$, whereas $\pi_{\mathcal{B}} = \pi_{\tilde{\mathcal{B}}} = \pi_2$.

The final result in this section will lead us to a characterisation of blends of parallelisms in Theorem 3.1. It is motivated by the following evident observation. Let π_1, π_2, π_3 be equivalence relations on $\mathcal{L}$. If π_3 is a blend of π_1 and π_2 then, by (2.2),

$$\forall\, M, N \in \mathcal{L}\colon\; M\; \pi_3\; N \Rightarrow (M\; \pi_1\; N \;\text{ or }\; M\; \pi_2\; N). \tag{2.8}$$

In our current setting, (2.8) is not sufficient for π_3 to be a blend of π_1 and π_2. Take, for example, as $\mathcal{L}$ any set with at least two elements, let $\pi_1 = \pi_2 = \mathcal{L} \times \mathcal{L}$, and let π_3 be the equality relation on $\mathcal{L}$. Then (2.8) is trivially true, but π_3 fails to be a blend of π_1 and π_2.

Proposition 2.5. *Let π_1, π_2, π_3 be equivalence relations on $\mathcal{L}$ such that* (2.8) *is satisfied. Then*

$$\forall\, M \in \mathcal{L}\colon\; \mathcal{C}_3(M) \subseteq \mathcal{C}_1(M) \;\text{ or }\; \mathcal{C}_3(M) \subseteq \mathcal{C}_2(M). \tag{2.9}$$

Proof. Assume, to the contrary, that (2.9) does not hold. So, there is an $M_0 \in \mathcal{L}$ such that $\mathcal{C}_3(M_0) \nsubseteq \mathcal{C}_1(M_0)$ and $\mathcal{C}_3(M_0) \nsubseteq \mathcal{C}_2(M_0)$. Now $\mathcal{C}_3(M_0) \nsubseteq \mathcal{C}_1(M_0)$ implies the existence of an $M_2 \in \mathcal{C}_3(M_0) \setminus \mathcal{C}_1(M_0)$. Applying (2.8) to $M_2\; \pi_3\; M_0$ and taking into account that $M_2 \notin \mathcal{C}_1(M_0)$, we obtain $M_2 \in \mathcal{C}_2(M_0)$. Hence

$$M_2 \in (\mathcal{C}_2(M_0) \cap \mathcal{C}_3(M_0)) \setminus \mathcal{C}_1(M_0). \tag{2.10}$$

Likewise, $\mathcal{C}_3(M_0) \nsubseteq \mathcal{C}_2(M_0)$ implies that there exists an element

$$M_1 \in (\mathcal{C}_1(M_0) \cap \mathcal{C}_3(M_0)) \setminus \mathcal{C}_2(M_0). \tag{2.11}$$

All in all, $M_1\; \pi_3\; M_0\; \pi_3\; M_2$ yields $M_1\; \pi_3\; M_2$. Thus, by (2.8), at least one of the following is satisfied.

(i) $M_1\; \pi_1\; M_2$. This gives $M_2 \in \mathcal{C}_1(M_1) = \mathcal{C}_1(M_0)$ and contradicts (2.10).

(ii) $M_1\; \pi_2\; M_2$. This gives $M_1 \in \mathcal{C}_2(M_2) = \mathcal{C}_2(M_0)$ and contradicts (2.11). □

3 Blends of parallelisms

We consider a projective space $\mathbb{P}$ with point set $\mathcal{P}$ and line set $\mathcal{L}$. An equivalence relation on $\mathcal{L}$ is called a *parallelism* on $\mathbb{P}$ if each point $q \in \mathcal{P}$ is incident with precisely one line from each equivalence class; see, for example, [12], [13], or [16, § 14]. The notation from the previous section will slightly be altered when dealing with parallelisms by writing $\parallel$ instead of π. In addition, if $\parallel \subseteq \mathcal{L} \times \mathcal{L}$ is a parallelism, then the equivalence class of a line $M \in \mathcal{L}$ will be called its *parallel* *class*, and it will be denoted by $\mathcal{S}(M)$ in order to emphasise the fact that $\mathcal{S}(M)$ is a *spread* of $\mathbb{P}$. On the other hand, the partition of $\mathcal{L}$ arising from $\parallel$ will be written as Π like before. In the presence of several parallelisms we shall distinguish between these objects by adding appropriate indices or attributes.

As anticipated, the next theorem provides a characterisation of blends of parallelisms by virtue of Proposition 2.5.

Theorem 3.1. *Let $\parallel_1$ and $\parallel_2$ be parallelisms on $\mathbb{P}$. Then the following hold.*

(a) *Any blend of $\parallel_1$ and $\parallel_2$ is a parallelism on $\mathbb{P}$.*

(b) *A parallelism $\parallel_3$ on $\mathbb{P}$ is a blend of $\parallel_1$ and $\parallel_2$ if, and only if,*

$$\forall\, M, N \in \mathcal{L} :\; M \parallel_3 N \Rightarrow (M \parallel_1 N \;\; or \;\; M \parallel_2 N). \tag{3.1}$$

Proof. (a) All parallel classes of the given parallelisms are spreads of $\mathbb{P}$. The same applies therefore to all equivalence classes of any blend of $\parallel_1$ and $\parallel_2$, that is, such a blend is a parallelism on $\mathbb{P}$.

(b) If $\parallel_3$ is a blend of $\parallel_1$ and $\parallel_2$ then (3.1) is nothing but a reformulation of (2.8). Conversely, we first make use of Proposition 2.5, which gives (2.9) up to some notational differences. Next, we notice that no proper subset of a spread of $\mathbb{P}$ is again a spread of $\mathbb{P}$. Since all parallel classes of $\parallel_1$, $\parallel_2$, and $\parallel_3$ are spreads of $\mathbb{P}$, we are therefore in a position to infer from (2.9) that, *mutatis mutandis*, (2.2) is satisfied. □

Suppose that a projective space $\mathbb{P}$ is endowed with parallelisms $\parallel_\ell$ and $\parallel_r$ that are called the *left* and *right* parallelism, respectively. We speak of left (right) parallel lines and left (right) parallel classes. According to [17], $(\mathbb{P}, \parallel_\ell, \parallel_r)$ constitutes a *projective double space* if the following axiom is satisfied.

(DS) For all triangles p_0, p_1, p_2 in $\mathbb{P}$ there exists a common point of the lines M_1 and M_2 that are defined as follows. M_1 is the line through p_2 that is left parallel to the join of p_0 and p_1, M_2 is the line through p_1 that is right parallel to the join of p_0 and p_2.

In case of a projective double space $(\mathbb{P}, \parallel_\ell, \parallel_r)$, each of $\parallel_\ell$ and $\parallel_r$ is referred to as a *Clifford parallelism* of $(\mathbb{P}, \parallel_\ell, \parallel_r)$. We now generalise this notion.

Definition 3.2. Let $(\mathbb{P}, \parallel_\ell, \parallel_r)$ be a projective double space. A *Clifford-like* parallelism $\parallel$ of $(\mathbb{P}, \parallel_\ell, \parallel_r)$ is a parallelism on $\mathbb{P}$ such that

$$\forall\, M, N \in \mathcal{L} :\; M \parallel N \Rightarrow (M \parallel_\ell N \;\; \text{or} \;\; M \parallel_r N).$$

By Theorem 3.1, the Clifford-like parallelisms of $(\mathbb{P}, \|_\ell, \|_r)$ are precisely the blends of $\|_\ell$ and $\|_r$. In particular, $\|_\ell$ and $\|_r$ themselves are the trivial examples of Clifford-like parallelisms of $(\mathbb{P}, \|_\ell, \|_r)$.

Next, we recall that there exist projective double spaces $(\mathbb{P}, \|_\ell, \|_r)$ such that $\|_\ell$ coincides with $\|_r$. See [8], [14] and [22] for further details, an algebraic characterisation, and geometric properties. Such a double space has only one Clifford-like parallelism, namely $\|_\ell = \|_r$. We therefore exclude this kind of double space from our further discussion.

The projective double spaces $(\mathbb{P}, \|_\ell, \|_r)$ with $\|_\ell \neq \|_r$ are precisely the ones that can be obtained from quaternion skew fields (see [17], [18], [15], [22]). A detailed account is the topic of the next section.

Finally, we observe that the "left and right Clifford parallelisms" introduced in [3] and defined by an *octonion division algebra* do not give rise to a projective double space. For further details, see [3] and the references therein.

Remark 3.3. In [10, Rem. 3.7 and Thm. 3.8] the authors gave examples of *piecewise Clifford parallelisms* with two pieces. Without going into details, let us point out that (in our terminology) these parallelisms arise from a three-dimensional Pappian projective space $\mathbb{P}$ that is made into a projective double space in two different ways, say $(\mathbb{P}, \|_{\ell,1}, \|_{r,1})$ and $(\mathbb{P}, \|_{\ell,2}, \|_{r,2})$. Thereby, it has to be assumed that $\|_{\ell,1}$ and $\|_{\ell,2}$ share a *single* parallel class. The piecewise Clifford parallelisms with two pieces are blends of $\|_{\ell,1}$ and $\|_{\ell,2}$, but none of these is Clifford-like with respect to any double space structure on $\mathbb{P}$. The proof of the last statement is beyond the scope of this article, since the methods utilised in [10] are totally different from ours.

4 Clifford-like parallelisms from quaternion skew fields

In this section we deal with a quaternion skew field H with centre F. We thereby stick to the terminology and notation from [4] and [9]. Also, we use the abbreviations $H^* := H \setminus \{0\}$ and $F^* := F \setminus \{0\}$. For a detailed account on quaternions we refer, among others, to [28, pp. 46–48] and [29, Ch. I].

The F-vector space H is equipped with a quadratic form $H \to F$, called the *norm form*, sending $q \mapsto q\overline{q} = \overline{q}q$. Here $\overline{}$ denotes the *conjugation*, which is an antiautomorphism of the skew field H. The conjugation is of order two and fixes F elementwise. Polarisation of the norm form yields the symmetric bilinear form

$$\langle\cdot,\cdot\rangle\colon H \times H \to F\colon (p,q) \mapsto \langle p,q\rangle := p\overline{q} + q\overline{p} = \overline{p}q + \overline{q}p. \qquad (4.1)$$

For any subset $X \subseteq H$ we denote by $X^\perp$ the set of those quaternions that are orthogonal to all elements of X with respect to $\langle\cdot,\cdot\rangle$.

The *projective space* $\mathbb{P}(H)$ is understood to be the set of all subspaces of the F-vector space H and *incidence* is symmetrised inclusion. We adopt the usual geometric language: *points*, *lines* and *planes* are the subspaces of vector dimension one, two, and three, respectively. The set of lines of $\mathbb{P}(H)$ will be written as $\mathcal{L}(H)$. Furthermore, we shall regard $\perp$ as a *polarity* of $\mathbb{P}(H)$ sending, for example, any line M to its polar line $M^\perp$. For one kind of line this will now be made more explicit.

Lemma 4.1. *For any line $L = F1 \oplus Fg$, where $1 \in H$ and $g \in H \setminus F$, the line $L^\perp$ is the set of all $u \in H$ subject to the condition*

$$u\overline{g} = gu. \tag{4.2}$$

Proof. From $L^\perp = 1^\perp \cap g^\perp$ and (4.1), a quaternion $u \in H$ belongs to $L^\perp$ precisely when the following system of equations is satisfied:

$$u + \overline{u} = 0, \quad g\overline{u} + u\overline{g} = 0. \tag{4.3}$$

It is immediate from (4.3) that any $u \in L^\perp$ satisfies (4.2). Conversely, if (4.2) holds for some $u \in H$ then $g(\overline{u} + u) = g\overline{u} + gu = g\overline{u} + u\overline{g} \in F$. Together with $g \notin F$ and $\overline{u} + u \in F$ this forces $\overline{u} + u = 0$, whence the system (4.3) is satisfied. □

Let $M, N \in \mathcal{L}(H)$. Then the line M is *left parallel* to the line N, in symbols $M \parallel_\ell N$, if there is a $c \in H^*$ with $cM = N$. Similarly, M is *right parallel* to N, in symbols $M \parallel_r N$, if there is a $d \in H^*$ with $Md = N$. The relations $\parallel_\ell$ and $\parallel_r$ make $\mathbb{P}(H)$ into a projective double space $(\mathbb{P}(H), \parallel_\ell, \parallel_r)$, that is, $\parallel_\ell$ and $\parallel_r$ are its *Clifford parallelisms* (see [15]). In accordance with the terminology and notation from Section 3, each line $M \in \mathcal{L}(H)$ determines its *left parallel class* $\mathcal{S}_\ell(M)$ and its *right parallel class* $\mathcal{S}_r(M)$. All left (right) parallel classes are regular spreads of $\mathbb{P}$ (see [4, 4.8 Cor.] or [9, Prop. 4.3]), that is, $\parallel_\ell$ and $\parallel_r$ are *regular* parallelisms [13, Ch. 26].

For any choice of $c, d \in H^*$ we can define the F-linear bijection $\mu_{c,d}\colon H \to H\colon p \mapsto cpd$, which acts as a projective collineation on $\mathbb{P}(H)$ preserving both the left and the right Clifford parallelism as a straightforward computation shows. Also, $\mu_{c,d}$ preserves the norm form of H up to the factor $c\overline{c}d\overline{d} \in F^*$ so that orthogonality of subspaces of H is preserved too. Two particular cases deserve special mention. For $d \in F^*$, in particular for $d = 1$, the mapping $\mu_{c,d}$ is a *left translation*. A *right translation* arises in a similar way for $c \in F^*$.

Let $\mathcal{A}(H)$ be the *star of lines* with centre $F1$ (with $1 \in H$), that is, the set of all lines of $\mathcal{L}(H)$ passing through the point $F1$. From an algebraic point of view, each left (right) parallel class has a distinguished representative, namely its only line belonging to $\mathcal{A}(H)$. The star $\mathcal{A}(H)$ is precisely the set of all two-dimensional

F-subalgebras of H or, in other words, the set of all maximal subfields of H. Given $L_1, L_2 \in \mathcal{A}(H)$ we remind that an *F-isomorphism* of L_1 onto L_2 is a ring isomorphism $L_1 \to L_2$ fixing F elementwise. If such an isomorphism exists then L_1 and L_2 are called *F-isomorphic*, in symbols $L_1 \cong L_2$.

Let $\pi_{\ell r}$ denote the equivalence relation on $\mathcal{L}(H)$ that is generated by the left and right Clifford parallelism on $\mathbb{P}(H)$. If $M \; \pi_{\ell r} \; N$ applies, then we say that M is *left-right equivalent* to N.

We now present several characterisations of left-right equivalent lines.

Theorem 4.2. *Let $M_1, M_2 \in \mathcal{L}(H)$ and let L_1 and L_2 be the uniquely determined lines through the point $F1$ such that $L_1 \parallel_\ell M_1$ and $L_2 \parallel_r M_2$. Then the following are equivalent.*

(a) $M_1 \; \pi_{\ell r} \; M_2$.

(b) *There exist $e_1, e_2 \in H^*$ with $e_1 M_1 = M_2 e_2$.*

(c) $\mathcal{S}_\ell(M_1) \cap \mathcal{S}_r(M_2) \neq \emptyset$.

(d) *There exists a line $M \in \mathcal{L}(H)$ such that $\mathcal{S}_\ell(M_1) \cap \mathcal{S}_r(M_2) = \{M, M^\perp\}$.*

(e) $L_1 \cong L_2$.

(f) *There exists an $e \in H^*$ with $e^{-1} L_1 e = L_2$.*

Proof. (a) $\Rightarrow$ (b). By the definition of $\parallel_\ell$ and $\parallel_r$ and by virtue of (2.3), we obtain that $M_1 \; \pi_{\ell r} \; M_2$ implies the existence of an integer $n \geq 1$ and elements $g_1, g_2, \ldots, g_{2n}$ such that

$$M_1 \parallel_\ell g_1 M_1 \parallel_r g_1 M_1 g_2 \parallel_\ell \cdots \parallel_r g_{2n-1} g_{2n-3} \cdots g_1 M_1 g_2 g_4 \cdots g_{2n} = M_2$$

With $e_1 := g_{2n-1} g_{2n-3} \cdots g_1$ and $e_2 := (g_2 g_4 \cdots g_{2n})^{-1}$ the assertion follows.

(b) $\Rightarrow$ (c). Clearly, $e_1 M_1 = M_2 e_2 \in \mathcal{S}_\ell(M_1) \cap \mathcal{S}_r(M_2)$.

(c) $\Rightarrow$ (d). By our assumption, there exists a line M, say, belonging to $\mathcal{S}_\ell(M_1) \cap \mathcal{S}_r(M_2)$. Also, there is a left translation $\mu_{c_1,1}$ taking M to L_1, *i.e.*, $c_1 M = L_1$. Since $\mu_{c_1,1}$ preserves not only the left and right Clifford parallelism but also the orthogonality of lines in both directions, it suffices to verify that

$$\mathcal{S}_\ell(L_1) \cap \mathcal{S}_r(L_1) = \{L_1, L_1^\perp\}.$$

To this end we pick a quaternion $g \in L_1 \setminus F$, which is maintained throughout this part of the proof.

First, we take any line $N \in \mathcal{S}_\ell(L_1) \cap \mathcal{S}_r(L_1)$. For all $u \in N^*$ we obtain from $1 \in L_1$ that $N = u L_1 = L_1 u$. Thus the inner automorphism $\mu_{u^{-1},u}$ of H restricts to an automorphism of L_1. There are two possibilities.

Case (i). $\mu_{u^{-1},u}$ fixes L_1 elementwise. Consequently, u commutes with all elements of L_1 or, equivalently, $u \in L_1$. Therefore $N = L_1$.

Case (ii). $\mu_{u^{-1},u}$ fixes F elementwise, but not L_1. Due to $[L_1 : F] = 2$, the identity on L_1 and the restriction of $\mu_{u^{-1},u}$ to L_1 are all the elements of the Galois group $\mathrm{Gal}(L_1/F)$. The restriction of the conjugation $\overline{}$ to L_1 belongs also to $\mathrm{Gal}(L_1/F)$. We proceed by showing that $g \neq \overline{g}$. If $\mathrm{Char}\, F \neq 2$ then this immediate from $g \in L \setminus F$. If $\mathrm{Char}\, F = 2$ then $g \in L \setminus F$ implies $g \neq u^{-1}gu$. Since g and $u^{-1}gu$ are distinct zeros in L_1 of the minimal polynomial of g over F, which reads $X^2 + (g + \overline{g})X + g\overline{g}$, the coefficient $g + \overline{g}$ in this polynomial does not vanish. This implies $g = -g \neq \overline{g}$. Irrespective of $\mathrm{Char}\, F$ we therefore have that $\mu_{u^{-1},u}$ and $\overline{}$ restrict to the same automorphism of L_1. In particular, $u^{-1}gu = \overline{g}$, that is, $u \in L_1^{\perp}$ by (4.2). Therefore $N = L_1^{\perp}$.

Finally, it remains to establish that $L_1^{\perp} \in \mathcal{S}_\ell(L_1) \cap \mathcal{S}_r(L_1)$. There exists a non-zero $h \in L_1^{\perp}$, whence $h\overline{g} = gh$ holds according to (4.2). Due to $g \in L_1$ this yields, for all $v \in L_1$, on the one hand $(hv)\overline{g} = g(hv)$ and, on the other hand, $(vh)\overline{g} = g(vh)$. As $L_1^{\perp}$ is characterised by (4.2), we obtain $hL_1 \subseteq L_1^{\perp}$ and $L_1h \subseteq L_1^{\perp}$. Thus $L_1^{\perp} = hL_1 = L_1h$, as required.

(d) $\Rightarrow$ (e). From $M = c_1^{-1}L_1 \in \mathcal{S}_\ell(M_1) \cap \mathcal{S}_r(M_2) = \mathcal{S}_\ell(L_1) \cap \mathcal{S}_r(L_2)$ there is a $c_2 \in H^*$ such that $M = L_2c_2^{-1}$. Therefore $L_1 = c_1L_2c_2^{-1}$, and $1 \in L_2$ gives $c_1c_2^{-1} \in L_1^*$. We read off from $1 \in L_1$ that L_1 is a subalgebra of H, and so $L_1 = L_1c_1c_2^{-1}$. Summing up, we have

$$L_2 = c_1^{-1}L_1c_2 = c_1^{-1}(L_1c_1c_2^{-1})c_2 = c_1^{-1}L_1c_1.$$

This allows us to define a mapping $L_1 \to L_2 \colon x \mapsto c_1^{-1}xc_1$, which is an F-isomorphism.

(e) $\Rightarrow$ (f). By the Skolem-Noether theorem (see [11, Thm. 4.9]), any F-isomorphism $L_1 \to L_2$ can be extended to an inner automorphism of H. So there is an $e \in H^*$ with $L_2 = e^{-1}L_1e$.

(f) $\Rightarrow$ (a). By our assumptions, there exist $d_1, d_2, e \in H^*$ with $d_1M_1 = L_1$, $M_2d_2 = L_2$, and $e^{-1}L_1e = L_2$. This implies $e^{-1}d_1M_1ed_2^{-1} = M_2$. Thus

$$M_1 \parallel_\ell e^{-1}d_1M_1 \parallel_r e^{-1}d_1M_1ed_2^{-1} = M_2,$$

and (2.3) gives $M_1 \,\pi_{\ell r}\, M_2$. □

Corollary 4.3. *For all lines $M_1, M_2 \in \mathcal{L}(H)$ the left parallel class $\mathcal{S}_\ell(M_1)$ is different from the right parallel class $\mathcal{S}_r(M_2)$.*

Proof. As H is infinite, so are $\mathcal{S}_\ell(M_1)$ and $\mathcal{S}_r(M_2)$. By Theorem 4.2, $\mathcal{S}_\ell(M_1)$ and $\mathcal{S}_r(M_2)$ have at most two lines in common, whence they cannot coincide. □

Corollary 4.4. *Let $N \in \mathcal{L}(H)$. Then $\mathcal{S}_\ell(N) \cap \mathcal{S}_r(N) = \{N, N^{\perp}\}$.*

Proof. We consider Theorem 4.2 for $M_1 = M_2 = N$. Then (c) holds and, by $N \in \mathcal{S}_\ell(N) \cap \mathcal{S}_r(N)$, the assertion follows from (d). □

Note that the result from the previous corollary is established also in [15, (2.6)] but using methods different from ours.

Corollary 4.5. *Let L be a maximal subfield of H, that is, L is a line through the point $F1$. The field extension L/F is separable if, and only if, the parallel classes $\mathcal{S}_\ell(L)$ and $\mathcal{S}_r(L)$ have two distinct lines in common.*

Proof. From Corollary 4.4, $\mathcal{S}_\ell(L) \cap \mathcal{S}_r(L) = \{L, L^\perp\}$. We consider Theorem 4.2 for $M_1 = M_2 := L$, and so $L_1 = L$. Then (c) holds, and we can repeat the proof of (c) $\Rightarrow$ (d) with $M := L$ and $g \in L \setminus F$. If L/F is separable then $g \neq \overline{g}$ and $L \neq L^\perp$. Otherwise $g = \overline{g}$ and $L = L^\perp$. □

Corollary 4.6. *Let L be a maximal subfield of H and let $u \in H^*$. Then $u^{-1}Lu = L$ is equivalent to $u \in (L \cup L^\perp) \setminus \{0\}$.*

Proof. If $u \in L \setminus \{0\}$ then $u^{-1}Lu = L$ is obviously true. If $u \in L^\perp \setminus \{0\}$ then (4.2) implies $u^{-1}Lu = L$. The converse follows from cases (i) and (ii) in the proof of Theorem 4.2, (c) $\Rightarrow$ (d). □

By the theorem of Cartan-Brauer-Hua [23, (13.17)], for each maximal subfield L of H there exists a $c \in H^*$ with $c^{-1}Lc \neq L$. Corollary 4.6 shows how all such elements c can be found.

Corollary 4.7. *Let L_1 and L_2 be maximal subfields of H. Then $L_1 \ \pi_{\ell r} \ L_2$ is equivalent to $L_1 \cong L_2$.*

All maps $\mu_{c,d}$, with c, d varying in H^*, constitute a subgroup of the general linear group $\mathrm{GL}(H)$. This subgroup acts on the line set $\mathcal{L}(H)$ in a natural way, thereby splitting $\mathcal{L}(H)$ into line orbits. From Theorem 4.2, these line orbits are precisely the classes of left-right equivalent lines. The next result gives another interpretation in terms of *flags*, that is, incident point-line pairs.

Proposition 4.8. *Let (Fp_1, M_1) and (Fp_2, M_2) be flags of the 3-dimensional projective space $\mathbb{P}(H)$. There exists a map $\mu_{c,d}$, with $c, d \in H^*$, taking (Fp_1, M_1) to (Fp_2, M_2) if, and only if, $M_1 \ \pi_{\ell r} \ M_2$.*

Proof. First, note that there always exists the left translation $\mu_{p_1^{-1},1}$ taking Fp_1 to $F1$ and the right translation $\mu_{1,p_2^{-1}}$ taking Fp_2 to $F1$. So, $L_1 := p_1^{-1}M_1$ and $L_2 := M_2 p_2^{-1}$ are the uniquely determined lines appearing in Theorem 4.2.

From Theorem 4.2, $M_1 \ \pi_{\ell r} \ M_2$ implies $e^{-1}L_1 e = L_2$ for some $e \in H^*$. Letting $c := e^{-1}p_1^{-1}$ and $d := ep_2$ gives $cp_1 d = p_2$ and $cM_1 d = M_2$, that is, the map $\mu_{c,d}$ has the required properties.

Conversely, if $\mu_{c,d}$ takes (Fp_1, M_1) to (Fp_2, M_2) then $cM_1 = M_2 d^{-1}$ forces $M_1 \ \pi_{\ell r} \ M_2$ according to Theorem 4.2. □

Remark 4.9. The group Γ of all collineations of $\mathbb{P}(H)$ that preserve both the left and the right parallelism was described in [25, Thm. 1] in terms of the factor group H^*/F^*, which thereby serves as a model for the point set $\mathbb{P}(H)$ by identifying F^*c with Fc for all $c \in H^*$. By [26, Prop. 4.1 and Prop. 4.2], a collineation γ of $\mathbb{P}(H)$ belongs to Γ if, and only if, γ can be induced by an F-semilinear transformation of H that is the product of an automorphism of the skew field H and a map $\mu_{c,d}$ for some $c, d \in H^*$ (see also [3, Thm. 4.3]).

In particular, the maps $\mu_{c,d}$ induce exactly the F-linear part of the group Γ. If $\operatorname{Char} F \neq 2$ then we know by [19, (4.16)] that they induce precisely the proper motions of the elliptic 3-space $\mathbb{P}(H)$, so the classes of left-right equivalent lines turn out to be the line orbits under the action of the elliptic proper motion group.

The following result describes *all* Clifford-like parallelisms of $(\mathbb{P}(H), \parallel_\ell, \parallel_r)$.

Theorem 4.10. *Let $\mathcal{A}(H)$ be the subset of all lines of $\mathbb{P}(H)$ through the point $F1$.*

(a) *Upon choosing any subset $\mathcal{D}$ of $\mathcal{A}(H)$ we let*

$$\mathcal{F} := \bigcup_{X \in \mathcal{D},\, c \in H^*} c^{-1} X c. \tag{4.4}$$

Then

$$\Pi_{\mathcal{F}} := \{\mathcal{S}_\ell(L) \mid L \in \mathcal{F}\} \cup \{\mathcal{S}_r(L) \mid L \in \mathcal{A}(H) \setminus \mathcal{F}\} \tag{4.5}$$

is the set of parallel classes of a Clifford-like parallelism $\parallel_{\mathcal{F}}$, say, of the projective double space $(\mathbb{P}(H), \parallel_\ell, \parallel_r)$.

(b) *Conversely, any Clifford-like parallelism $\parallel$ of $(\mathbb{P}(H), \parallel_\ell, \parallel_r)$ arises according to* (a) *from at least one subset of $\mathcal{A}(H)$.*

(c) *Let $\mathcal{D}$ and $\tilde{\mathcal{D}}$ be subsets of $\mathcal{A}(H)$. Applying the construction from* (a) *to $\mathcal{D}$ and $\tilde{\mathcal{D}}$ gives parallelisms $\parallel_{\mathcal{F}}$ and $\parallel_{\tilde{\mathcal{F}}}$, respectively. Then $\parallel_{\mathcal{F}}$ coincides with $\parallel_{\tilde{\mathcal{F}}}$ if, and only if, $\mathcal{F} = \tilde{\mathcal{F}}$.*

Proof. (a) We apply the construction from Proposition 2.4 (a) to $\mathcal{D}$; thereby we replace π_1 and π_2 with $\parallel_\ell$ and $\parallel_r$, respectively. So, starting with $\mathcal{B} := \bigcup_{X \in \mathcal{D}} \mathcal{C}_{\ell r}(X)$, we finally arrive at the partition $\Pi_{\mathcal{B}}$ from (2.5), whose associated equivalence relation on $\mathcal{L}(H)$ is a blend of $\parallel_\ell$ and $\parallel_r$. By Theorem 3.1, this $\Pi_{\mathcal{B}}$ is the set of parallel classes of a Clifford-like parallelism of $(\mathbb{P}(H), \parallel_\ell, \parallel_r)$. Each of its parallel classes has a unique line in common with $\mathcal{A}(H)$. Therefore

$$\Pi_{\mathcal{B}} = \{\mathcal{S}_\ell(L) \mid L \in \mathcal{B} \cap \mathcal{A}(H)\} \cup \{\mathcal{S}_r(L) \mid L \in \mathcal{A}(H) \setminus \mathcal{B}\}. \tag{4.6}$$

Theorem 4.2 and Corollary 4.7 show that

$$\forall\, X \in \mathcal{A}(H)\colon \{c^{-1}Xc \mid c \in H^*\} = \{Y \in \mathcal{A}(H) \mid X \cong Y\} = \mathcal{C}_{\ell r}(X) \cap \mathcal{A}(H).$$

So, substituting in (4.4) gives

$$\mathcal{F} = \bigcup_{X\in\mathcal{D}} \Big(\mathcal{C}_{\ell r}(X) \cap \mathcal{A}(H)\Big) = \mathcal{B} \cap \mathcal{A}(H). \tag{4.7}$$

Now, by comparing (4.5) with (4.6), we obtain $\Pi_{\mathcal{F}} = \Pi_{\mathcal{B}}$.

(b) The given parallelism $\parallel$ is a blend of $\parallel_\ell$ and $\parallel_r$ by Theorem 3.1. Thus $\parallel$ allows a construction as described in Proposition 2.4 (a) using $\parallel_\ell$, $\parallel_r$, and some subset, say $\tilde{\mathcal{D}}$, of $\mathcal{L}(H)$. Replacing $\tilde{\mathcal{D}}$ with the set

$$\mathcal{D} := \Big(\bigcup_{X\in\tilde{\mathcal{D}}} \mathcal{C}_{\ell r}(X)\Big) \cap \mathcal{A}(H) \tag{4.8}$$

does not alter this result, as has been pointed out in Proposition 2.4 (c). The first part of the current proof shows that we also get the parallelism $\parallel$ by applying the construction from (a) to the set $\mathcal{D}$ from (4.8).

(c) By the first part of the proof, we obtain $\parallel_{\mathcal{F}}$ and $\parallel_{\tilde{\mathcal{F}}}$ from $\mathcal{D}$ and $\tilde{\mathcal{D}}$, respectively, also via the construction in Proposition 2.4 (a). In our current setting the condition (2.6) simplifies to

$$\{\mathcal{C}_{\ell r}(X) \mid X \in \mathcal{D}\} = \{\mathcal{C}_{\ell r}(\tilde{X}) \mid \tilde{X} \in \tilde{\mathcal{D}}\}, \tag{4.9}$$

since $\mathcal{L}(H)_{\ell r} = \emptyset$ by Corollary 4.3. From (4.7), equation (4.9) is equivalent to $\mathcal{F} = \tilde{\mathcal{F}}$. It therefore suffices to make use of Proposition 2.4 (c), with (2.6) to be replaced by $\mathcal{F} = \tilde{\mathcal{F}}$, in order to complete the proof. □

Remark 4.11. Theorem 4.10 (a) was sketched without a strict proof in [4, 4.13]. However, there are some formal differences to our approach, as we avoid the indicator lines of regular spreads that have been used there. Our set of lines $\mathcal{A}(H)$ is, from an algebraic point of view, the family of all quadratic extensions L of F with $F \subseteq L \subseteq H$ from [4, 4.13]. In this way, our $\mathcal{F}$ turns into a family of subfields of H. Equation (4.4) guarantees that no subfield in $\mathcal{F}$ is F-isomorphic to a subfield in $\mathcal{A}(H) \setminus \mathcal{F}$. The latter condition is mentioned in the sketch of proof from [4, 4.13], but is missing there at that point, where the family $\mathcal{F}$ is fixed for the first time. (F-isomorphic subfields of H are termed as being "conjugate" in [4].)

Below we shall make use of the *ordinary quaternion algebra* over a *formally real* field F, *i.e.* -1 is not a square in F. This kind of algebra will be denoted as $(K/F, -1)$. According to [28, pp. 46–48] it arises (up to F-isomorphism) in the following way. The field F is extended to $K := F(i)$, where i is a square root of

$-1 \in F$. One defines $(K/F, -1)$ as the subring of the ring of 2×2 matrices over K consisting of all matrices

$$\begin{pmatrix} x_1 + ix_2 & y_1 + iy_2 \\ -y_1 + iy_2 & x_1 - ix_2 \end{pmatrix} \text{ with } x_1, x_2, y_1, y_2 \in F$$

and identifies any $x \in F$ with the matrix $\operatorname{diag}(x, x) \in (K/F, -1)$. The F-algebra $(K/F, -1)$ is a skew field if, and only if, -1 is not a sum of two squares in F. If the latter condition applies then $(K/F, -1)$ is called the *ordinary quaternion skew field* over F. For example, an ordinary quaternion skew field exists over any formally real Pythagorean field. We recall that a field is *Pythagorean* when the sum of any two squares is a square as well (see *e.g.* [20, p. 204]).

Theorem 4.12. *Let H be a quaternion skew field with centre F. Then the following are equivalent.*

(a) *F is a formally real Pythagorean field, and H is the ordinary quaternion skew field over F.*

(b) *All maximal subfields of H are mutually F-isomorphic.*

(c) *The Clifford parallelisms $\|_\ell$ and $\|_r$ are the only Clifford-like parallelisms of the projective double space $(\mathbb{P}(H), \|_\ell, \|_r)$.*

Proof. (a) $\Leftrightarrow$ (b). This was established in [7, Thm. 1 and Lemma 1] (but note that the definition of Pythagorean field used there is slightly different from ours). See also [3, Thm. 9.1] for a proof in a more general situation.

(b) $\Leftrightarrow$ (c). From Theorem 4.2, all maximal subfields of H are F-isomorphic if, and only if, for all $L \in \mathcal{A}(H)$ we have $\mathcal{A}(H) = \bigcup_{c \in H^*} c^{-1}Lc$. The last equation holds precisely when there are only two possibilities for the set $\mathcal{F}$ appearing in (4.4), namely either $\mathcal{F} = \mathcal{A}(H)$ or $\mathcal{F} = \emptyset$. This in turn is equivalent, by Theorem 4.10, to saying that $\|_\ell$ and $\|_r$ are the only Clifford-like parallelisms of $(\mathbb{P}(H), \|_\ell, \|_r)$. □

We continue by giving some explicit examples of Clifford-like parallelisms using the construction from Theorem 4.10 (a).

Example 4.13. Let $\operatorname{Char} H = 2$. We define

$$\mathcal{D} := \{L \in \mathcal{A}(H) \mid L \text{ is a separable extension of } F\}.$$

The set $\mathcal{A}(H) \setminus \mathcal{F}$ comprises precisely the inseparable quadratic extensions of F that are contained in H. We get $\mathcal{F} = \mathcal{D}$, since the group of inner automorphisms of H, in its natural action on $\mathcal{A}(H)$, leaves both $\mathcal{D}$ and $\mathcal{A}(H) \setminus \mathcal{D}$ invariant. Both $\mathcal{F}$ and $\mathcal{A}(H) \setminus \mathcal{F}$ are non-empty; see, among others, [6, pp. 103–104] or [28, pp. 46–48]. So $\mathcal{D}$ gives rise to a Clifford-like parallelism of $(\mathbb{P}(H), \|_\ell, \|_r)$ other than $\|_\ell$ and $\|_r$.

Example 4.14 (see [4, 4.12]). Let H be the ordinary quaternion skew field over the field $\mathbb{Q}$ of rational numbers. Then each quadratic field extension $\mathbb{Q}(\sqrt{-q})$ with $q \in \mathbb{Q}^*$ sum of three squares appears as a subfield of H. Any two such extensions $\mathbb{Q}(\sqrt{-q_1})$ and $\mathbb{Q}(\sqrt{-q_2})$ are $\mathbb{Q}$-isomorphic if, and only if, q_1 and q_2 are in the same square class of $\mathbb{Q}^*$, *i.e.*, there exists $c \in \mathbb{Q}^*$ such that $q_1 = c^2 q_2$. Since we have many non $\mathbb{Q}$-isomorphic quadratic extensions of $\mathbb{Q}$ contained in H, we also have many possible choices for the set $\mathcal{F}$ and consequently many different Clifford-like parallelisms.

Take notice that Clifford-like parallelisms of $(\mathbb{P}(H), \parallel_\ell, \parallel_r)$ always come in pairs. We just have to change the roles of $\mathcal{F}$ and $\mathcal{A}(H) \setminus \mathcal{F}$ in (4.5). However, with two obvious exceptions, the two parallelisms of such a pair do not make $\mathbb{P}(H)$ into a projective double space. This follows from our final theorem, which contains an even stronger result.

Theorem 4.15. *Let* $\parallel$ *be a Clifford-like parallelism of* $(\mathbb{P}(H), \parallel_\ell, \parallel_r)$ *other than* $\parallel_\ell$ *and* $\parallel_r$*. Then there is no parallelism* $\parallel'$ *on* $\mathbb{P}(H)$ *that makes* $\mathbb{P}(H)$ *into a projective double space* $(\mathbb{P}(H), \parallel, \parallel')$.

Proof. We assume, to the contrary, that there is a projective double space $(\mathbb{P}(H), \parallel, \parallel')$. Also, for all $M, N \in \mathcal{L}(H)$ let $\mathcal{R}_\ell(M, N)$ denote the set of all lines in $\mathcal{S}_\ell(M)$ that have at least one common point with N. The sets $\mathcal{R}_r(M, N)$, $\mathcal{R}(M, N)$, and $\mathcal{R}'(M, N)$ are defined in an analogous way by replacing $\parallel_\ell$ with $\parallel_r$, $\parallel$, and $\parallel'$, respectively.

We claim that there exist three distinct lines L_1, L_2, L_3 through the point $F1$ such that

$$\mathcal{S}_\ell(L_1) = \mathcal{S}(L_1), \quad \mathcal{S}_\ell(L_2) = \mathcal{S}(L_2), \quad \mathcal{S}_r(L_3) = \mathcal{S}(L_3). \tag{4.10}$$

Indeed, the existence of L_1 and L_3 is clear from $\parallel$ being different from $\parallel_\ell$ and $\parallel_r$. By Corollary 4.6, L_2 can be chosen as $L_2 := c^{-1} L_1 c$ with $c \in H^* \setminus (L_1 \cup L_1^\perp)$. We distinguish two cases.

Case (i). The parallelisms $\parallel$ and $\parallel'$ coincide. In $(\mathbb{P}(H), \parallel, \parallel)$ the double space axiom holds. This gives that each line of $\mathcal{R}(L_1, L_2)$ has a point in common with each line of $\mathcal{R}(L_2, L_1)$. Since the lines of $\mathcal{R}(L_1, L_2)$ are mutually skew, we obtain that $\mathcal{R}(L_1, L_2)$ and $\mathcal{R}(L_2, L_1)$ are mutually opposite reguli. The same kind of reasoning in $(\mathbb{P}(H), \parallel_\ell, \parallel_r)$ gives that $\mathcal{R}_\ell(L_1, L_2)$ and $\mathcal{R}_r(L_2, L_1)$ are mutually opposite reguli. From the first equation in (4.10), $\mathcal{R}_\ell(L_1, L_2) = \mathcal{R}(L_1, L_2)$ and, since a regulus has a unique opposite regulus, $\mathcal{R}_r(L_2, L_1) = \mathcal{R}(L_2, L_1)$. The second equation in (4.10) gives $\mathcal{R}_r(L_2, L_1) \subseteq \mathcal{S}_\ell(L_2)$. By Theorem 4.2, $\mathcal{S}_\ell(L_2) \cap \mathcal{S}_r(L_2)$ contains at most two lines, whereas the cardinality of the regulus $\mathcal{R}_r(L_2, L_1) \subseteq \mathcal{S}_\ell(L_2) \cap \mathcal{S}_r(L_2)$ is $|F| + 1$. So, this case is impossible.

Case (ii). The parallelisms $\parallel$ and $\parallel'$ are different. We proceed like before and obtain in a first step that $\mathcal{R}_\ell(L_1, L_3) = \mathcal{R}(L_1, L_3)$ and $\mathcal{R}_r(L_3, L_1) = \mathcal{R}'(L_3, L_1)$ are mutually opposite reguli. The third equation in (4.10) gives $\mathcal{R}'(L_3, L_1) \subseteq \mathcal{S}(L_3)$. Taking into account that $(\mathbb{P}(H), \|, \|')$ admits a description in terms of some quaternion skew field, we apply Theorem 4.2 and get $|\mathcal{S}(L_3) \cap \mathcal{S}'(L_3)| \leq 2$, whereas $\mathcal{R}'(L_3, L_1) \subseteq \mathcal{S}(L_3) \cap \mathcal{S}'(L_3)$ has $|F| + 1$ elements, a contradiction. □

As a consequence of Theorems 4.12 and 4.15, we obtain the following:

Corollary 4.16. *A projective double space $(\mathbb{P}(H), \|_\ell, \|_r)$, where H does not satisfy condition* (a) *from Theorem 4.12, admits Clifford-like parallelisms that are not Clifford w.r.t. any double space structure on $\mathbb{P}$.*

References

[1] Betten, D., Riesinger R.: *Clifford parallelism: old and new definitions, and their use*, J. Geom. **103**, no. 1, 31–73 (2012)

[2] Birkhoff, G.: Lattice Theory, third ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Providence, R.I. (1979)

[3] Blunck, A., Knarr, N., Stroppel, B., Stroppel, M.J.: *Clifford parallelisms defined by octonions*, Monatsh. Math. **187**, no. 3, 437–458 (2018)

[4] Blunck, A., Pasotti, S., Pianta, S.: *Generalized Clifford parallelisms*, Innov. Incidence Geom. **11**, 197–212 (2010)

[5] Clifford, W.K.: *Preliminary sketch of biquaternions*, Proc. Lond. Math. Soc. **4**, 381–395 (1871/73)

[6] Draxl, P.K.: Skew Fields, London Mathematical Society Lecture Note Series, vol. 81, Cambridge University Press, Cambridge (1983)

[7] Fein, B., Schacher, M.: *The ordinary quaternions over a Pythagorean field*, Proc. Amer. Math. Soc. **60** (1976), 16–18 (1977)

[8] Havlicek, H.: *A note on Clifford parallelisms in characteristic two*, Publ. Math. Debrecen **86**, no. 1-2, 119–134 (2015)

[9] Havlicek, H.: *Clifford parallelisms and external planes to the Klein quadric*, J. Geom. **107**, no. 2, 287–303 (2016)

[10] Havlicek, H., Riesinger, R.: *Pencilled regular parallelisms*, Acta Math. Hungar. **153**, no. 1, 249–264 (2017)

[11] Jacobson, N.: Basic Algebra II, Freeman, New York (1989)

[12] Johnson, N.L.: *Parallelisms of projective spaces*, J. Geom. **76**, no. 1-2, 110–182 (2003)

[13] Johnson, N.L.: Combinatorics of Spreads and Parallelisms, Pure and Applied Mathematics (Boca Raton), vol. 295, CRC Press, Boca Raton (2010)

[14] Karzel, H.: *Zweiseitige Inzidenzgruppen*, Abh. Math. Sem. Univ. Hamburg **29**, 118–136 (1965)

[15] Karzel, H., Kroll, H.-J.: *Eine inzidenzgeometrische Kennzeichnung projektiver kinematischer Räume*, Arch. Math. (Basel) **26**, 107–112 (1975)

[16] Karzel, H., Kroll, H.-J.: Geschichte der Geometrie seit Hilbert, Wissenschaftliche Buchgesellschaft, Darmstadt (1988)

[17] Karzel, H., Kroll, H.-J., Sörensen, K.: *Invariante Gruppenpartitionen und Doppelräume*, J. Reine Angew. Math. **262/263**, 153–157 (1973)

[18] Karzel, H., Kroll, H.-J., Sörensen, K.: *Projektive Doppelräume*, Arch. Math. (Basel) **25**, 206–209 (1974)

[19] Karzel, H., Pianta, S., Stanik, R.: *Generalized euclidean and elliptic geometries, their connections and automorphism groups*, J. Geom. **48**, no. 1-2, 109–143 (1993)

[20] Karzel, H., Sörensen, K., Windelberg, D.: Einführung in die Geometrie, Studia mathematica/Mathematische Lehrbücher, Taschenbuch 1, Uni-Taschenbücher, No. 184, Vandenhoeck & Ruprecht, Göttingen (1973)

[21] Klein, F.: *Zur Nicht-Euklidischen Geometrie*, Math. Ann. **37**, 544–572 (1890)

[22] Kroll, H.-J.: *Bestimmung aller projektiven Doppelräume*, Abh. Math. Sem. Univ. Hamburg **44** (1975), 139–142 (1976)

[23] Lam, T.Y.: A First Course in Noncommutative Rings, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York (2001)

[24] Pasotti, S.: *Regular parallelisms in kinematic spaces*, Discrete Math. **310**, 3120–3125 (2010)

[25] Pianta, S.: *On automorphisms for some fibered incidence groups*, J. Geom. **30**, no. 2, 164–171 (1987)

[26] Pianta S., Zizioli, E.: *Collineations of geometric structures derived from quaternion algebras*, J. Geom. **37**, no. 1-2, 142–152 (1990)

[27] Szász, G.: Introduction to Lattice Theory, Academic Press, New York London; Akadémiai Kiadó, Budapest (1963)

[28] Tits J., Weiss, R.M.: Moufang Polygons, Springer Monographs in Mathematics, Springer-Verlag, Berlin (2002)

[29] Vignéras, M.-F.: Arithmétique des algèbres de quaternions, Lecture Notes in Mathematics, vol. 800, Springer-Verlag, Berlin (1980)

Hans Havlicek
Institut für Diskrete Mathematik und Geometrie
Technische Universität
Wiedner Hauptstraße 8–10/104
A-1040 Wien
Austria
havlicek@geometrie.tuwien.ac.at

Stefano Pasotti
DICATAM-Sez. Matematica
Università degli Studi di Brescia
via Branze, 43
I-25123 Brescia
Italy
stefano.pasotti@unibs.it

Silvia Pianta
Dipartimento di Matematica e Fisica
Università Cattolica del Sacro Cuore
via Trieste, 17
I-25121 Brescia
Italy
silvia.pianta@unicatt.it