Incidence and Combinatorial Properties of Linear Complexes

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Dedicated to Helmut Karzel on the occasion of his 80th birthday

Abstract. In this paper a generalisation of the notion of polarity is exhibited which allows to completely describe, in an incidence-geometric way, the linear complexes of h-subspaces. A generalised polarity is defined to be a partial map which maps $(h-1)$-subspaces to hyperplanes, satisfying suitable linearity and reciprocity properties. Generalised polarities with the null property give rise to a linear complexes and vice versa. Given that there exists for $h > 1$ a linear complex of h-subspaces which contains no star –this seems to be an open problem over an arbitrary ground field –the combinatorial structure of a partition of the line set of the projective space into non-geometric spreads of its hyperplanes can be obtained. This line partition has an additional linearity property which turns out to be characteristic.

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1. Introduction

The notion of a linear complex has been investigated for longer than a century, and one could regard it as completely known. The classical approach is to define a linear complex of h-dimensional subspaces (shortly: h-subspaces) in $\mathrm{PG}(n, F)$, i.e., the n-dimensional projective space over a commutative field F, as the set of all h-subspaces whose Grassmann coordinates satisfy a non-trivial linear equation or, in a coordinate-free way, via a hyperplane section of the corresponding Grassmann variety. It is well known that each linear complex of *lines* ($h = 1$) is the set of all lines which are contained in their polar subspace with respect to a (possibly degenerate) null polarity. The classification of these complexes reduces to the well known classification of non-zero alternating matrices over F. Also, a lot was written by leading classical authors on the theory of linear complexes of h-subspaces for $h > 1$, but seemingly, such a theory was developed for the complex numbers only. As regards the topics which will be dealt with in this paper see, among others, [2], [3], [5], [6], [17], and [19].

To our knowledge, no incidence-geometric approach to linear complexes exists for $h > 1$. Our goal is to adopt such a point of view and coherently describe linear complexes of h-subspaces. To accomplish this task, we first collect some results about *linear mappings* and *primes* (geometric hyperplanes) of Grassmannians over arbitrary fields (commutative or skew). In a Grassmannian over a proper skew field there is only one kind of prime, whence we rule out skew fields at an early stage of our investigation. In a Grassmannian over a commutative field primes and linear complexes are the same, but this result gives no explicit information about linear complexes.

In Section 5 the notion of *polarity* is generalised. This leads us to an incidence-geometric construction of linear complexes of h-subspaces in $\mathrm{PG}(n, F)$ in terms of *generalised null polarities*. A crucial problem for linear complexes of h-subspaces in $\mathrm{PG}(n, F)$ is the existence of *singular* $(h-1)$*-subspaces* for $h > 1$. This problem is addressed in Section 6, where we also sketch a result from [2] for the case of complex numbers. However, a solution over an arbitrary field F presently seems beyond reach. If there exists a linear complex of h-subspaces, $h > 1$, without singular $(h-1)$-subspaces, then there is also a linear complex of planes without singular lines. Therefore, we focus on linear complexes of planes without singular lines, where we establish an intriguing relation with *linear line spreads*. Any linear complex of planes without singular lines gives rise to linear line spread, which turns out to be non-geometric in several cases, e. g. for a finite ground field. Finally, Section 7 is devoted to a connection between linear complexes of planes without singular lines and *linear line partitions*. This connection is used to show that certain projective spaces do not admit linear line partitions.

2. The Grassmannians of a projective space

A *semilinear space* is a pair $\Sigma = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of *points*, $\mathcal{B} \subseteq 2^{\mathcal{P}}$ is a set of *lines*, and the following axioms hold:

(*i*) $|\ell| \geq 2$ for each $\ell \in \mathcal{B}$.
(*ii*) For every $X \in \mathcal{P}$ there exists at least one $\ell \in \mathcal{B}$ such that $X \in \ell$.
(*iii*) $|\ell \cap \ell'| \leq 1$ for every $\ell, \ell' \in \mathcal{B}$, $\ell \neq \ell'$.

Note that some authors use slightly different axioms for a semilinear space; others speak of a *partial linear space* instead.

Given two points $X, Y \in \mathcal{P}$ we write $X \sim Y$, where "$\sim$" is to be read as *collinear*, if there is an $\ell \in \mathcal{B}$ such that $X, Y \in \ell$. Otherwise, X and Y are said to be *non-collinear*, $X \not\sim Y$. If $X \sim Y$ and $X \neq Y$, then the unique line $\ell \in \mathcal{B}$ such that $X, Y \in \ell$ is denoted by XY.

Let F be a (commutative or non-commutative) field. We denote by $\mathrm{PG}(n,F)$ the n-dimensional projective space coordinatised by the field F and collect some basic notions which will be used throughout this article:

If X is a subspace of $\mathrm{PG}(n,F)$ and $\dim X = d$, then X is called a *d-subspace*. Let U and W be two subspaces such that $U \subseteq W$. The *interval* $[U,W]$ is the set of all subspaces of $\mathrm{PG}(n,F)$ containing U and contained in W. The set of all d-subspaces of $\mathrm{PG}(n,F)$ belonging to $[U,W]$ is denoted by $[U,W]_d$. In particular, for an interval $[U,W]$ with $\dim U = h-1$ and $\dim W = h+1$ the set $[U,W]_h$ is the *pencil* of h-subspaces determined by U and W.

Let $[U,W]$ be an interval in $\mathrm{PG}(n,F)$, $\dim U =: k-1$. Then $[U,W]$ carries the structure of a projective space, whose d-dimensional subspaces are precisely the $(k+d)$-subspaces of $\mathrm{PG}(n,F)$ containing U and contained in W. The projective space $[U,W]$ is isomorphic to $\mathrm{PG}(\dim W - k, F)$.

The *h-th Grassmannian* of $\mathrm{PG}(n,F)$, $0 \le h \le n-1$, is the semilinear space $\Gamma(n,h,F) = (\mathcal{P},\mathcal{B})$, where $\mathcal{P}$ is the set of all h-subspaces of $\mathrm{PG}(n,F)$, and $\mathcal{B}$ is the set of all pencils of h-subspaces. When the field is clear from the context we simply write $\Gamma(n,h)$. If $\mathrm{PG}(n,F)$ admits a correlation (e. g., for a commutative field F) then $\Gamma(n,h,F)$ is isomorphic to $\Gamma(n,n-h-1,F)$. In order to avoid confusion, the elements of $\mathcal{P}$ and $\mathcal{B}$ will be called *G-points* and *G-lines*, respectively, where "G" abbreviates "Grassmann".

We consider the Grassmannian $\Gamma(n,h,F)$ for a fixed h. If a set I of G-points is a d-dimensional projective space with respect to the G-lines contained in I, then I is called a *d-G-subspace*. A d-G-subspace I is thus a set of h-subspaces of $\mathrm{PG}(n,F)$ pairwise meeting in $(h-1)$-subspaces. Hence there exists an $(h-1)$-subspace U of $\mathrm{PG}(n,F)$ such that $U \subset X$ for all $X \in I$, or there exists an $(h+1)$-subspace V of $\mathrm{PG}(n,F)$ such that $X \subset V$ for all $X \in I$; both conditions hold for $d \le 1$.

Next, we describe the maximal G-subspaces of $\Gamma(n,h,F)$. We start by recalling two notions: If U is an $(h-1)$-subspace of $\mathrm{PG}(n,F)$, the *star* with *centre* U is the set of all h-subspaces of $\mathrm{PG}(n,F)$ containing U. Consequently, a star is an $(n-h)$-G-subspace. Let $\mathcal{S}$ denote the set of stars. A *dual star* is the set of all h-subspaces of $\mathrm{PG}(n,F)$ contained in a fixed $(h+1)$-subspace. The dual stars are $(h+1)$-G-subspaces, and the set of all dual stars will be denoted by $\mathcal{T}$. For $h=1$ a dual star is a *ruled plane*. If $h = 0, n-1$, then $\mathcal{P}$ is the only maximal G-subspace. Otherwise, the set of all maximal G-subspaces partitions into the families $\mathcal{S}$ and $\mathcal{T}$.

3. Linear mappings

Let $\mathcal{P}$ and $\mathcal{P}'$ be sets. We say that χ is a *partial map* of $\mathcal{P}$ into $\mathcal{P}'$, if there is a subset $\mathbb{D}(\chi)$ of $\mathcal{P}$ such that $\chi : \mathbb{D}(\chi) \to \mathcal{P}'$ is a map in the usual sense. The set $\mathbb{D}(\chi)$ is called the *domain* of χ. Let $\mathbb{A}(\chi) = \mathcal{P} \setminus \mathbb{D}(\chi)$ be the *exceptional subset* of χ. The elements of $\mathbb{A}(\chi)$ are called *exceptional points* of χ. If $\mathbb{A}(\chi) = \emptyset$, then χ is *global*. We maintain the notation $\chi : \mathcal{P} \to \mathcal{P}'$ even when χ is not global.

In order to have a less complicated notation we extend the definition of χ to the power set of $\mathcal{P}$ by defining

$$\phi^\chi := (\phi \cap \mathbb{D}(\chi))^\chi = \{X^\chi \mid X \in \phi \cap \mathbb{D}(\chi)\} \text{ for every } \phi \in 2^{\mathcal{P}}.$$

Thus χ assigns to *every* subset of $\mathcal{P}$ a subset of $\mathcal{P}'$.

Next, let $\Sigma = (\mathcal{P}, \mathcal{B})$ and $\Sigma' = (\mathcal{P}', \mathcal{B}')$ be semilinear spaces. A partial map $\chi : \mathcal{P} \to \mathcal{P}'$ is called a *linear mapping*, if for each $\ell \in \mathcal{B}$ one of the following holds:

(*i*) $\ell^\chi \in \mathcal{B}'$, and χ maps ℓ bijectively onto ℓ^χ.
(*ii*) $\ell^\chi = \{P'\}$, where $P' \in \mathcal{P}'$, and $|\ell \cap \mathbb{A}(\chi)| = 1$.
(*iii*) $\ell \subseteq \mathbb{A}(\chi)$.

Obviously, these three conditions are mutually exclusive. This linear mapping will also be denoted by $\chi : \Sigma \to \Sigma'$.

If X, Y are distinct collinear points in $\mathbb{A}(\chi)$, then the line XY is a subset of $\mathbb{A}(\chi)$ by (iii). In particular, for a projective space Σ this implies that $\mathbb{A}(\chi)$ is a subspace of Σ. In the case that χ is global, condition (i) holds for all lines, whence distinct collinear points have distinct images. However, the images of non-collinear points may coincide.

There is one type of linear mapping deserving special mention: A *full projective embedding* is a global and injective linear map $\chi : \Sigma \to \Sigma'$ for which Σ' is a projective space.

The linear mappings between projective spaces allow the following explicit description:

Theorem 1. [4, 14] *Let Σ and Σ' be projective spaces, and let $\chi : \Sigma \to \Sigma'$ be a linear mapping. Then the partial map χ splits into a projection from $\mathbb{A}(\chi)$ onto a complementary subspace in Σ, say U, and a collineation between U and a subspace of Σ'.*

The linear mappings between Desarguesian projective spaces are –up to one particular case – the geometric counterpart of the semilinear maps between vector spaces; the exceptional points correspond to non-zero vectors in the kernel. The situation is different if the image of the linear mapping is a line. Here the collineation from the above theorem is just a bijection which, of course, need not be induced by a semilinear bijection between the underlying vector spaces.

We now shortly exhibit full projective embeddings of a Grassmannian $\Gamma(n, h, F)$. In doing so, the Grassmannians $\Gamma(n, 0, F)$ and $\Gamma(n, n-1, F)$ will be excluded, since they are a priori projective spaces. Hence we have $0 < h < n - 1$ which implies $n \geq 3$.

First, assume that F is a non-commutative field. By [14, Satz 3.6], no full projective embedding of $\Gamma(n, h, F)$ exists for $0 < h < n - 1$.

Next, we assume F to be a commutative field. Let $N = \binom{n+1}{h+1} - 1$ and

$$\wp_{n,h} : \Gamma(n, h, F) \to \mathrm{PG}(N, F)$$

be the map defined by $X^{\wp_{n,h}} = F(v_0 \wedge v_1 \wedge \ldots \wedge v_h)$, where $v_0, v_1, \ldots, v_h$ is a basis of X meant as an $(h+1)$-dimensional subspace of F^{n+1}, and $\mathrm{PG}(N, F)$ is represented as $(h+1)$-th exterior power of F^{n+1}. This definition is independent on the choice of the basis, and $\wp_{n,h}$ is a full projective embedding, called the *Plücker embedding.* Its image $\mathcal{P}^{\wp_{n,h}} = \mathcal{G}_{n,h}$ is a *Grassmann variety.* Each Grassmann variety is intersection of quadrics [18, pp. 184–188]; in particular, $\mathcal{G}_{3,1}$ is the well-known *Klein quadric*.

So, the question remains of describing all full projective embeddings of $\Gamma(n,h,F)$ when F is commutative. An answer can be given in terms of linear mappings.

Theorem 2. [14] *Let F be a commutative field and let Σ' be a projective space. If $\chi : \Gamma(n,h,F) \to \Sigma'$ is a linear mapping, then there exists a unique linear mapping $\mu : \mathrm{PG}(N,F) \to \Sigma'$ such that $\chi = \wp_{n,h}\mu$.*

By this universal property, we may obtain all full projective embeddings of $\Gamma(n,h,F)$ (to within collineations) as a product of $\wp_{n,h}$ by a (possibly trivial) projection whose centre does not meet any secant of the Grassmann variety $\mathcal{G}_{n,h}$. Therefore, each fully embedded Grassmannian is a (possibly trivial) projection of a Grassmann variety $\mathcal{G}_{n,h}$. In [22] and [23] sufficient conditions for the (non-) existence of a non-trivial projection are given.

For other questions and literature on linear mappings the reader is referred to the book [7] (different terminology), [4] (including a survey of older literature), and [15].

4. Primes and linear complexes

A *prime* of a semilinear space $(\mathcal{P},\mathcal{B})$ is a proper subset L of $\mathcal{P}$, such that for each $\phi \in \mathcal{B}$ either $\phi \subseteq L$, or $|\phi \cap L| = 1$. Note that some authors call such a subset a *geometric hyperplane.*

From now on $\Gamma(n,h,F) = (\mathcal{P},\mathcal{B})$ is a Grassmannian. It is easily seen that the set of all h-subspaces having non-empty intersection with a fixed $(n-h-1)$-subspace is a prime of $\Gamma(n,h,F)$. Somewhat surprisingly, $\Gamma(n,h,F)$ has no other primes, if F is a non-commutative field. This beautiful result can be found in [13].

Next, let F be a commutative field and $\wp_{n,h} : \Gamma(n,h,F) \to \mathrm{PG}(N,F)$ the related Plücker embedding. If H is a hyperplane of $\mathrm{PG}(N,F)$, then

$$K = (H \cap \mathcal{G}_{n,h})^{\wp_{n,h}{}^{-1}}$$

is called a *linear complex of h-subspaces* in $\mathrm{PG}(n,F)$. Cf., e. g. [5, p. 322]. If φ is a G-line, then either $\varphi \subseteq K$, or $|\varphi \cap K| = 1$. Since $\mathcal{G}_{n,h}$ generates $\mathrm{PG}(N,F)$, we have $K \neq \mathcal{P}$. Thus each linear complex is a prime of $\Gamma(n,h,F)$. Conversely, we have the following result; its short proof is taken from [14, p. 179]:

Proposition 3. *If F is commutative, then each prime of $\Gamma(n,h,F)$ is a linear complex.*

Proof. Let L be a prime of $\Gamma(n,h,F)$. Define a partial map $\chi : \Gamma(n,h,F) \to \Sigma'$, where Σ' is a point, by setting $\mathbb{A}(\chi) = L$. Obviously, this χ is linear. By Theorem 2,

$\chi = \wp_{n,h}\mu$, where $\mu : \mathrm{PG}(N,F) \to \Sigma'$ is a linear mapping. Then

$$X \in L \;\Leftrightarrow\; X \in \mathbb{A}(\chi) \;\Leftrightarrow\; X^{\wp_{n,h}} \in \mathbb{A}(\mu),$$

and the latter exceptional set is a hyperplane by Theorem 1. □

In [20] a self-contained proof of Proposition 3 is given.

5. Generalised polarities arising from a linear complex

Up to the end of the paper F denotes a *commutative* field.

Our first aim is to generalise the concept of polarity. Suppose that $\chi : \Gamma(n,k) \to \mathrm{PG}(n,F)^*$, $0 \le k \le n-1$, is linear. So, χ is a partial map of the set of k-subspaces of $\mathrm{PG}(n,F)$ into its hyperplane set. We say that such a χ is a (*generalised*) *polarity* if for all $U_1, U_2 \in \Gamma(n,k)$ with $U_1 \sim U_2$ the following holds:

$$U_1 \subseteq U_2^\chi \text{ implies } U_2 \subseteq U_1^\chi. \tag{1}$$

In (1) it is understood that for $U \in \mathbb{A}(\chi)$, $U^\chi = \{U\}^\chi$ is the whole projective space. When speaking of "polarities" below, we always mean "generalised polarities".

For $k = 0$ our definition is in accordance with the usual definition of a (possibly degenerate) polarity. Likewise, the following result is well known for $k = 0$ in the non-degenerate case. See, e. g. [1, p. 110, Cor. 1]: *"Null systems are polarities"*.

Theorem 4. *Let $\chi : \Gamma(n,k) \to \mathrm{PG}(n,F)^*$, $0 \le k \le n-1$, be a linear mapping, and assume that χ satisfies the* null property

$$U \subseteq U^\chi \textit{ for each } U \in \mathbb{D}(\chi). \tag{2}$$

Then χ is a polarity.

Proof. Assume that U_1 and U_2 are distinct k-subspaces such that $U_1 \sim U_2$ in $\Gamma(n,k)$. By the linearity of χ, one of the following holds: (i) $U_1, U_2 \in \mathbb{D}(\chi)$ and $U_1^\chi \ne U_2^\chi$; then $(U_1U_2)^\chi$ is the pencil of hyperplanes through $U_1^\chi \cap U_2^\chi$. (ii) There is a unique $U_0 \in U_1U_2$ such that $U_0 \in \mathbb{A}(\chi)$, and a hyperplane E of $\mathrm{PG}(n,F)$ exists such that $U \in U_1U_2$, $U \ne U_0$ implies $U^\chi = E$. (iii) $U_1, U_2 \in \mathbb{A}(\chi)$ so that $U_1U_2 \subseteq \mathbb{A}(\chi)$.

Let P be a point. In any case we have

$$P \in U_1^\chi \cap U_2^\chi \Rightarrow (P \in U^\chi \text{ for all } U \in U_1U_2). \tag{3}$$

In order to prove (1), assume that the above subspaces satisfy $U_1 \subseteq U_2^\chi$. Since $U_1 \subseteq U_1^\chi$, by (3) we have, for each $U \in U_1U_2$, that $U_1 \subseteq U^\chi$. On the other hand, $U \subseteq U^\chi$ implies $(U \vee U_1) \subseteq U^\chi$, where $U \vee U_1$ denotes the join in $\mathrm{PG}(n,F)$. Now take $U_1', U_2' \in U_1U_2$ with $U_1' \ne U_2' \ne U_1 \ne U_1'$. For $i = 1, 2$ we obtain

$$U_1 \vee U_2 = U_i' \vee U_1 \subseteq U_i'^\chi.$$

So, (3) yields $U_1 \vee U_2 \subseteq U_1^\chi$, and in particular $U_2 \subseteq U_1^\chi$. □

Taking into account the above theorem, a linear mapping $\Gamma(n,k) \to \mathrm{PG}(n,F)^*$ with the null property (2) will be called a (*generalised*) *null polarity.*

Now we turn back to linear complexes:

Proposition 5. *Let K be a linear complex of h-subspaces of $\Sigma = \mathrm{PG}(n,F)$, $1 \le h \le n-1$, and let $[U,W]$ be an interval of Σ with $\dim U \le h-1$ and $\dim W \ge h+1$. Then*

$$K(U,W) := K \cap [U,W]_h,$$

i.e., the set consisting of all elements of K containing U and contained in W, is a linear complex of $(h-1-\dim U)$-subspaces in the projective space $[U,W]$, unless $[U,W]_h \subseteq K$.

Proof. A pencil φ of $(h-1-\dim U)$-subspaces with respect to the projective space $[U,W]$ is a pencil of h-subspaces in Σ; from $\varphi \subseteq K$ or $|\varphi \cap K| = 1$ we have $\varphi \subseteq K(U,W)$ or $|\varphi \cap K(U,W)| = 1$, respectively. Therefore, if $[U,W]_h \nsubseteq K$, then $K(U,W)$ is a prime, and hence a linear complex, of the $(h-1-\dim U)$-th Grassmannian of $[U,W]$. □

As a particular case of the above proposition, if P is a point of Σ, then the set $K_P = K(P,\Sigma)$, consisting of all elements of K incident with P, is a linear complex of $(h-1)$-subspaces in the $(n-1)$-dimensional projective space $[P,\Sigma]$, unless $K_P = [P,\Sigma]_h$.

As a further consequence, if $S = [U,\Sigma]_h$ is the star with centre an $(h-1)$-subspace U, then either (i) $S \subseteq K$, or (ii) there is a hyperplane E of $\mathrm{PG}(n,F)$ such that for each $X \in S$, we have $X \in K$ if, and only if, $X \subseteq E$. If (i) holds, then U is called a *singular* $(h-1)$*-subspace* of K; otherwise E is the *polar hyperplane* of U. Similarly, if $T = [\emptyset, V]_h$, $\dim V = h+1$, is a dual star, then either (i) $T \subseteq K$, or (ii) there is a point P of $\mathrm{PG}(n,F)$ such that for each $X \in T$, we have $X \in K$ if and only if $P \in X$. If (i) holds, then V is called a *total* $(h+1)$*-subspace* of K; otherwise P is the *pole* of V.

We are now in a position to introduce the following crucial notion. If K is a linear complex of h-subspaces, $1 \le h \le n-1$, we will denote by $\uparrow K$ the partial map of the set of all $(h-1)$-subspaces of $\mathrm{PG}(n,F)$ into the dual projective space $\mathrm{PG}(n,F)^*$, defined as follows: given an $(h-1)$-subspace, say U, if U is singular, then $U \in \mathbb{A}(\uparrow K)$; otherwise $U^{\uparrow K}$ is the polar hyperplane of U, i.e., the union of all elements of K containing U. In view of Theorem 6 below, this mapping $\uparrow K$ will be called the *null polarity defined by* K.

Theorem 6. *Let K be a linear complex of h-subspaces in $\Sigma = \mathrm{PG}(n,F)$, $1 \le h \le n-1$. Then the following assertions hold:*

(a) *The partial map $\uparrow K : \Gamma(n,h-1) \to \mathrm{PG}(n,F)^*$ is a null polarity with non-empty domain.*

(b) *The image of $\uparrow K$ generates a subspace of $\mathrm{PG}(n,F)^*$ with dimension at least h.*

Proof. (a) We use induction on h; for $h = 1$ this is well known [5, p. 322]. So, let $h > 1$ and let φ be a pencil of $(h-1)$-subspaces. There is a point, say P, incident with every $U \in \varphi$. If $K_P = [P, \Sigma]_h$, then $\varphi \subseteq \mathbb{A}(\uparrow K)$. Otherwise, by induction assumption, $\uparrow K_P$ is a linear mapping. Thus the linearity of $\uparrow K$ follows by observing that the restrictions of $\uparrow K$ and $\uparrow K_P$ to φ coincide. Since $U \subseteq U^{\uparrow K}$ holds for all $U \in \mathbb{D}(\uparrow K)$ by definition, $\uparrow K$ is a null polarity according to Theorem 4. Finally, $K \neq \Gamma(n,h)$ implies that not all $(h-1)$-subspaces can be singular, whence the domain of $\uparrow K$ is non-empty.

(b) Assume to the contrary that the image of $\uparrow K$ generates a subspace of $\mathrm{PG}(n,F)^*$ with a smaller dimension. So the intersection of all hyperplanes in the image of $\uparrow K$ contains an $(n-h)$-subspace X, say. We show that this implies the contradiction $\mathbb{D}(\uparrow K) = \emptyset$: Given an $(h-1)$-subspace U we argue by induction on $k := \dim U \cap X$. For $k = -1$ there is no hyperplane passing through $U \vee X$, whence $U \in \mathbb{A}(\uparrow K)$. Next, assume $k > -1$. Then there is a point $P \in \Sigma$ outside $U \vee X$. Also, there exists a point $Q \in U \cap X$. Let V be a complement of Q with respect to U. Then U is an element of the pencil $[V, V \vee P \vee Q]_{h-1}$. All other elements of this pencil meet X in a subspace of dimension $k-1$, whence they belong to $\mathbb{A}(\uparrow K)$ by the induction hypothesis. The linearity of $\uparrow K$ yields $U \in \mathbb{A}(\uparrow K)$, as required. □

Corollary 7. *The set of all singular $(h-1)$-subspaces of the linear complex K is equal to $(\mathcal{G}_{n,h-1} \cap W)^{\wp^{-1}_{n,h-1}}$ for some subspace W of $\mathrm{PG}\left(\binom{n+1}{h} - 1, F\right)$ with*

$$\binom{n+1}{h} - (n+2) \leq \dim W \leq \binom{n+1}{h} - (h+2).$$

Corollary 8. *If U_1 and U_2 are singular $(h-1)$-subspaces of K, and $U_1 \sim U_2$ in $\Gamma(n,h-1)$, then each element of the pencil determined by U_1 and U_2 is a singular $(h-1)$-subspace of K.*

We have seen that $\uparrow K$ is a null polarity. Conversely, we have:

Theorem 9. *Let $\chi : \Gamma(n,h-1) \to \mathrm{PG}(n,F)^*$, $1 \leq h \leq n-1$, be a linear mapping with non-empty domain satisfying the null property* (2). *Then there is a unique linear complex of h-subspaces, say K, such that $\chi = \uparrow K$.*

Proof. We define a set K of h-subspaces by setting $X \in K$ if, and only if, there is an $(h-1)$-subspace $U \subseteq X$ such that $X \subseteq U^\chi$. Any linear complex with the required properties necessarily has to coincide with this K.

The mapping χ is a polarity by Theorem 4. From (1) we infer that if $X \in K$, then $X \subseteq W^\chi$ for every $(h-1)$-subspace W of X. If φ is the pencil of h-subspaces determined by the $(h-1)$-subspace U_φ and the $(h+1)$-subspace V_φ, $U_\varphi \subseteq V_\varphi$, then the G-points of φ belonging to K are exactly the elements $X \in \varphi$ such that $X \subseteq U_\varphi^\chi$. Therefore, either $\varphi \subseteq K$ or $|\varphi \cap K| = 1$, whence K is a prime. The proof is now accomplished by applying Proposition 3. □

The above theorem is a generalisation of the classical one, characterising a linear complex of lines as a set arising from a (possibly degenerate) null polarity $\mathrm{PG}(n,F) \to \mathrm{PG}(n,F)^*$.

By dual arguments we obtain a linear mapping $\downarrow K : \Gamma(n, h+1) \to \mathrm{PG}(n, F)$, $0 \le h \le n-2$, the *dual polarity* of K. There holds:

Theorem 10. *Let $\chi : \Gamma(n, h+1) \to \mathrm{PG}(n, F)$, $0 \le h \le n-2$, be a linear mapping with non-empty domain, and assume that for each $V \in \mathbb{D}(\chi)$, $V^\chi \subseteq V$. Then there is a unique linear complex of h-subspaces, say K, such that $\chi = \downarrow K$.*

6. Existence of singular $(h-1)$-subspaces

We start with the following technical result.

Proposition 11. *Assume that k, h, and n are integers such that $1 \le k \le h \le n-1$. Let K be a linear complex of h-subspaces of $\mathrm{PG}(n, F)$ having no singular $(h-1)$-subspace. Then $\mathrm{PG}(n+k-h, F)$ contains a linear complex of k-subspaces having no singular $(k-1)$-subspace.*

Proof. Let W be any $(h-k-1)$-subspace of $\Sigma = \mathrm{PG}(n, F)$. The intersection $K_W = K \cap [W, \Sigma]_h$ is a set of k-subspaces of $[W, \Sigma] \cong \mathrm{PG}(n+k-h, F)$. If L is a $(k-1)$-subspace of $[W, \Sigma]$, then L is an $(h-1)$-subspace of Σ, containing W. By assumption an h-subspace X of Σ exists that contains L and does not belong to K. Such X is a k-subspace of $[W, \Sigma]$, containing L and not belonging to K_W. Now Proposition 3 easily yields the assertion. □

A linear complex K of lines is the set of all self-conjugate lines of a null polarity with non-empty domain in $\mathrm{PG}(n, F)$. Since each alternating matrix has even rank, K always has a singular point for n even; on the other hand, for each odd n there are linear complexes of lines without singular points. Thus, by Proposition 11 with $k = 1$, we have that if $n - h \equiv 1 \pmod 2$, then each linear complex of h-subspaces in $\mathrm{PG}(n, F)$ has a singular $(h-1)$-subspace. In our proof of Proposition 11 the subspace W was chosen arbitrarily. Hence this result can be refined as follows:

Proposition 12. *Let K be a linear complex of h-subspaces in $\mathrm{PG}(n, F)$. If $1 \le h \le n-1$ and $n - h \equiv 1 \pmod 2$, then each $(h-2)$-subspace in $\mathrm{PG}(n, F)$ is contained in a singular $(h-1)$-subspace.*

The question concerning the existence of total $(h+1)$-subspaces is dealt with in [2] *over the field $\mathbb{C}$ of complex numbers* as follows. The *product* of a linear complex of h-subspaces, say K, and a linear complex of points, i.e., a hyperplane H, is the set

$$K \cdot H := \{X \in \Gamma(n, h+1, \mathbb{C}) \mid \exists Y \in K : Y \subseteq X \cap H\}$$

If $[\emptyset, H]_h \subseteq K$ then H is called a *total hyperplane* of K; in this case $K \cdot H$ is the set of all $(h+1)$-subspaces of $\mathrm{PG}(n, \mathbb{C})$. Otherwise, $K \cdot H$ is a linear complex of $(h+1)$-subspaces in $\mathrm{PG}(n, \mathbb{C})$. Let H_0, H_1, …, H_n be independent hyperplanes in $\mathrm{PG}(n, \mathbb{C})$. Then the set Θ of all total $(h+1)$-subspaces of K is the intersection of $K \cdot H_0, K \cdot H_1, \ldots, K \cdot H_n$; that is, Θ is represented on the Grassmann variety $\mathcal{G}_{n,h+1}$

as intersection with at most $n+1$ hyperplanes. (This is dual to our Corollary 7, and holds over an arbitrary ground field.) Since $\mathcal{G}_{n,h+1}$ is an algebraic variety with dimension $(h+2)(n-h-1)$, a sufficient condition for the existence of total $(h+1)$-subspaces is

$$(h+2)(n-h-1)-(n+1)\geq 0,$$

that is

$$(h+1)(n-h-2)\geq 1.$$

Since $h+1\geq 1$, the condition reads $n-h-2>0$ or, equivalently, $h<n-2$. Summarising, we obtain:

Theorem 13. [2] *Each linear complex of h-subspaces in* $\mathrm{PG}(n,\mathbb{C})$*, with $h<n-2$, has a total $(h+1)$-subspace.*

Dually, there holds:

Corollary 14. *Each linear complex of h-subspaces in* $\mathrm{PG}(n,\mathbb{C})$*, with $h\geq 2$, has a singular $(h-1)$-subspace.*

A *line spread* of a projective space $\mathrm{PG}(n,F)$ is a set of lines, say $\mathcal{F}$, such that each point of $\mathrm{PG}(n,F)$ belongs to exactly one line of $\mathcal{F}$. A line spread is called *geometric* (or *normal*) if for every pair of distinct lines of $\mathcal{F}$, say ℓ, m, the lines of $\mathcal{F}$ in the solid $\ell\vee m$ form a spread of $\ell\vee m$. The line spread $\mathcal{F}$ is *linear* if $\mathcal{F}^{\wp_{n,1}}$ is the intersection of the Grassmann variety $\mathcal{G}_{n,1}$ with a subspace of its ambient space $\mathrm{PG}(M,F)$ with $M:=(n^2+n-2)/2$.

Proposition 15. *Let K be a linear complex of planes in* $\mathrm{PG}(n,F)$ *having no singular line. Then n is even. For each hyperplane H in* $\mathrm{PG}(n,F)$*, let $\mathcal{F}_H$ be the set of all lines whose polar hyperplane is H. Then $\mathcal{F}_H$ is a line spread of H.*

Proof. By Proposition 12, n is even. Let A and H be a point and a hyperplane in $\mathrm{PG}(n,F)$, respectively, such that $A\in H$. The set S of all lines of $\mathrm{PG}(n,F)$ containing the point A is a subspace of $\Gamma(n,1)$ isomorphic to $\mathrm{PG}(n-1,F)$. Since $\uparrow K$ is global, its restriction to S is a collineation. As a consequence, $S^{\uparrow K}$ is the set of *all* hyperplanes through A. So, a unique line ℓ through A exists such that $\ell^{\uparrow K}=H$. This proves that $\mathcal{F}_H$ is a line spread of H. □

Proposition 16. *Under the assumptions of Proposition* 15 *the line spread $\mathcal{F}_H$ is linear. More precisely,*

$$\mathcal{F}_H^{\wp_{n,1}}=R\cap\mathcal{G}_{n,1},$$

where R is an $(M-n)$-subspace of the ambient space $\mathrm{PG}(M,F)$ *of the Grassmann variety $\mathcal{G}_{n,1}$. Furthermore, in each of the following cases the line spread $\mathcal{F}_H$ is not geometric.*

(a) *The field F is quadratically closed.*
(b) *The field F is finite.*

Proof. By Theorems 2 and 6, the null polarity $\uparrow K$ can be written as

$$\uparrow K = \wp_{n,1}\pi\kappa,$$

where $\wp_{n,1}$ is the Plücker embedding, π is a projection of $\mathrm{PG}(M,F)$ from a subspace C with dimension $(M-n-1)$ onto a complementary n-subspace D, and $\kappa : D \to \mathrm{PG}(n,F)^*$ is a collineation. The non-existence of singular lines yields

$$C \cap \mathcal{G}_{n,1} = \emptyset.$$

So, $\mathcal{F}_H^{\wp_{n,1}}$ is the intersection of $\mathcal{G}_{n,1}$ with the subspace $R := C \vee H^{\kappa^{-1}}$ of $\mathrm{PG}(M,F)$.

Now suppose that (a) or (b) holds. We assume that $\mathcal{F}_H$ is a geometric line spread. By the above, there is a hyperplane, say J, of $\mathrm{PG}(M,F)$, such that $J \cap R = C$, whence

$$J \cap \mathcal{F}_H^{\wp_{n,1}} = \emptyset.$$

As $\mathcal{F}_H$ is geometric, there is a solid U of $\mathrm{PG}(n,F)$ such that the lines of U belonging to $\mathcal{F}_H$ form a line spread of U, say $\mathcal{F}_1$. Furthermore, $\mathcal{F}_1^{\wp_{n,1}}$ is the intersection of R with a quadric Q_5^+ (the Klein quadric representing the lines of U). By the table in [16, pp. 29–31] (which remains true, mutatis mutandis, also for an infinite field), $\mathcal{F}_1^{\wp_{n,1}}$ has to be an elliptic quadric Q_3^-. This is impossible if (a) holds. On the other hand, for a finite field F we get from $J \cap Q_3^- = \emptyset$ the contradiction that Q_3^- would have an exterior plane [16, p. 17]. □

The proof from the above cannot be carried over to all infinite fields, since an elliptic quadric may have an exterior plane.

Remark 1. Proposition 11 for $k = 2$ together with the previous theorem implies the following: Any linear complex of h-subspaces, $h > 1$, having no singular $(h-1)$-subspaces, yields a non-geometric linear line spread if F satisfies one of the conditions (a) or (b). The authors do not know, whether under these circumstances non-geometric linear line spreads exist or not. In case of their non-existence, Theorem 13 of Baldassarri and Corollary 14 would also hold for projective spaces over finite or quadratically closed fields.

7. Linear line partitions

A *line partition* Ω of a projective space $\mathrm{PG}(n,F)$ is a partition of its line set into line spreads of hyperplanes such that each hyperplane contains precisely one of these spreads. Each line partition of $\mathrm{PG}(n,F)$ induces a surjective map π_Ω of the line set onto the dual space $\mathrm{PG}(n,F)^*$ as follows: It assigns to each line ℓ the unique hyperplane containing the equivalence class of ℓ. Observe that this π_Ω is globally defined on the line set.

Line partitions of finite projective spaces were investigated in [9, 10, 11, 12, 21]. Here n necessarily has to be even. In particular, we quote the following result:

Theorem 17. [10, 11] *Each finite projective space* $\mathrm{PG}(2^i-2,q)$ *with* $i \geq 2$ *admits a line partition.*

A line partition Ω will be called *linear* if π_Ω from the above is a linear mapping. The linear line partitions are closely related with certain linear complexes.

Theorem 18. *Let Ω be a line partition of $\mathrm{PG}(n,F)$. Denote by K the set of all planes ε such that a line ℓ exists which satisfies the condition*

$$\ell \subseteq \varepsilon \subseteq \ell^{\pi_\Omega}.$$

This K is a linear complex of planes if, and only if, Ω is linear. In this case, K has no singular line.

Proof. If Ω is linear, then $\chi := \pi_\Omega$ satisfies the assumptions of Theorem 9. This implies that K is a linear complex of planes without singular lines.

Conversely, assume that the given set K is a linear complex of planes. Let ℓ be a non-singular line. Then ℓ^{π_Ω} and $\ell^{\uparrow K}$ are hyperplanes which obviously are identical. Next, we show that singular lines do not exist. Assume to the contrary that ℓ_1' is a singular line. Since $\uparrow K$ has a non-empty domain, also a non-singular line ℓ_2' exists. There is a line which has a point in common with ℓ_1' and ℓ_2', respectively. This line is either singular or non-singular. Hence there exists a pencil of lines containing a singular line ℓ_1 and a non-singular line ℓ_2, say. Since $\uparrow K$ is linear, there is another non-singular line $\ell_3 \neq \ell_2$ in this pencil. We infer from the above and from the linearity of $\uparrow K$ that

$$\ell_2^{\pi_\Omega} = \ell_2^{\uparrow K} = \ell_3^{\uparrow K} = \ell_3^{\pi_\Omega}.$$

Thus the distinct incident lines ℓ_2 and ℓ_3 belong to the same line spread, a contradiction. Therefore $\pi_\Omega = \uparrow K$ which in turn implies the linearity of Ω. □

By Proposition 15 and Theorem 18, there is a bijective correspondence between linear complexes of planes without singular lines and linear line partitions. The linear complex K defined in the theorem will be called the linear complex of planes *related* to the linear line partition Ω.

Proposition 19. *No projective space $\mathrm{PG}(4,F)$ admits linear line partitions.*

Proof. A linear complex of planes, say K, in $\mathrm{PG}(4,F)$ is dual to a linear complex of lines K^* in $\mathrm{PG}(4,F)^*$. The singular lines of K are dual to ruled planes of K^*. Let S be the set of all singular points of K^* (actually, hyperplanes of $\mathrm{PG}(4,F)$). If S is a dual plane, then obviously K^* contains ruled planes. Otherwise S is a point and there is a line ℓ of K^* such that $S \notin \ell$. The plane $S \vee \ell$ is a ruled plane in K^*. This proves that each linear complex of planes in $\mathrm{PG}(4,F)$ has at least one singular line. □

Some finite four-dimensional projective spaces admit line partitions. In particular this holds in $\mathrm{PG}(4,q)$ for $q = 2,3$ [10] and for $q = 5,8,9$ [21]. By the above all these line partitions are necessarily non-linear.

Proposition 20. *No finite projective space $\mathrm{PG}(6,q)$ admits linear line partitions.*

Proof. Let K be a linear complex of planes in $\mathrm{PG}(6, q)$, without singular lines. Denote by K_H the set of planes in K which are contained in some hyperplane H. This K_H is a linear complex of planes in the five-dimensional finite projective space H. Let $\mathcal{F}_H$ be the line spread in H given according to Proposition 15. The elements of $\mathcal{F}_H$ are precisely the singular lines of K_H. In [8, Theorem 14] it is shown that (to within projective transformations) a unique linear complex of planes in $\mathrm{PG}(5, q)$ exists such that its singular lines form a line spread; moreover, this spread turns out to be geometric. This contradicts Proposition 16. □

Remark 2. It seems to be unknown whether or not linear line partitions do exist. In particular we do not know, if the line partitions of $\mathrm{PG}(2^i - 2, q)$ by Fuji-Hara and Vanstone (cf. Theorem 17) are linear. As a matter of fact, to our knowledge, the proof of their existence is only sketched in the literature.

Remark 3. The authors conducted an extensive computer-based search for linear complexes of planes without singular lines in $\mathrm{PG}(8, q)$, $q = 2, 3$, but no examples were found. Thus a proof for the (non)-existence of linear complexes of h-subspaces, $h \geq 2$, having no singular $(h-1)$-subspaces remains an enticing open problem, even in the finite case.

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