DEGENERATE CONICS REVISITED

Dedicated to Armin Herzer on his 60th birthday.

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In a desarguesian projective plane we discuss those degenerate conics whose parameters have degree two over the centre of the underlying skew field. Such degenerate conics are closely related with Baer subplanes. This relationship enables us to improve and reformulate in a geometric language some theorems which previously have been established in a purely algebraic way. A geometric description of the family of fundamental chains of these degenerate conics is given.

1. Throughout this article K denotes a *non-commutative field*. $Z(L)$ stands for the *centralizer* in K of $L \subset K$. If $L \subset K$ is a (skew or commutative) subfield, then a *right vector space* $\mathfrak{B}$ over L is also written as $\mathfrak{B}_L$. We shall frequently regard K as a right vector space K_L over L. The *affine* and *projective space* on $\mathfrak{B}_L$ is denoted by $\mathcal{A}(\mathfrak{B}_L)$ and $\mathcal{P}(\mathfrak{B}_L)$, respectively. Let $(\mathcal{S},\mathcal{L})$ be a *linear space* with point set $\mathcal{S}$ and line set $\mathcal{L}$. Given $\mathcal{T} \subset \mathcal{S}$ then put $\mathcal{L}_{\mathcal{T}} := \{l \cap \mathcal{T} \mid l \in \mathcal{L} \wedge |l \cap \mathcal{T}| \geq 2\}$. Hence $(\mathcal{T}, \mathcal{L}_{\mathcal{T}})$ is the *trace space* of $(\mathcal{S},\mathcal{L})$ determined by $\mathcal{T}$.

The reader is referred to [10,62-73], [16] and [18,325-333] for definitions and results on *degenerate conics*. Our exposition follows [10].

Suppose that $\mathcal{P}(\mathfrak{B}_K)$ is a projective plane on a vector space $\mathfrak{B}_K$ and write $\mathcal{L}$ for the set of lines of $\mathcal{P}(\mathfrak{B}_K)$. Let Γ be a degenerate conic of $\mathcal{P}(\mathfrak{B}_K)$. Then there exists an ordered basis $(\mathfrak{p},\mathfrak{q},\mathfrak{a})$ of $\mathfrak{B}_K$ and an element $a \in K \backslash Z(K)$ such that the *proper* and *improper part* of

(1) $\Gamma = \Gamma^{\circ} \cup \Gamma^{\times}$

are given by

(2) $\Gamma^{\circ} = \{(\mathfrak{p}t + \mathfrak{q}ta + \mathfrak{a})K \mid t \in K\}$

and

(3) $\Gamma^{\times} = PQ$ with $P = \mathfrak{p}K$, $Q = \mathfrak{q}K$,

respectively [10,65]. In the terminology of [18] Γ° is a *C-configuration*. The element a is called a *parameter* of Γ. Neither $(\mathfrak{p},\mathfrak{q},\mathfrak{a})$ nor a is uniquely determined. We denote by A the subfield

of K which is generated by $Z(K) \cup \{a\}$. Any two parameters of Γ are in A and have the same degree[1] over $Z(K)$ [10,67]. The set

(4) $\quad n = \{(\mathfrak{p}t+\mathfrak{q}ta)K \,|\, t \in K\}$

is formed by all points on PQ which are collinear with at least two different points of Γ° [10,65].

THEOREM 1. *Let Γ° be the proper part of a degenerate conic Γ with parameter $a \in K \backslash Z(K)$. Then the trace space $(\Gamma^{\circ}, \mathscr{L}_{\Gamma}\circ)$ is isomorphic to the affine space $\mathcal{A}(K_{Z(A)})$ and $(\Gamma^{\circ}, \mathscr{L}_{\Gamma}\circ)$ is an affine plane if, and only if, the degree $|a{:}Z(K)| = 2$.*

    Proof. The subset $\mathfrak{B} := \{\mathfrak{p}t+\mathfrak{q}ta \,|\, t \in K\}$ of $\mathfrak{B}$ is closed under linear combinations with coefficients in $Z(A)$, i.e. $\mathfrak{B}$ is a right vector space over $Z(A)$. The map

    $\iota{:}K \to \mathfrak{B}$, $\quad t \mapsto \mathfrak{p}t+\mathfrak{q}ta$,

is an isomorphism of the vector spaces $K_{Z(A)}$ and $\mathfrak{B}_{Z(A)}$. By (2), there exists a bijection

    $\varepsilon{:}\mathfrak{B} \to \Gamma^{\circ}$, $\quad \mathfrak{p}t+\mathfrak{q}ta \mapsto (\mathfrak{p}t+\mathfrak{q}ta+\mathfrak{a})K$.

If points t_0, t_1, t_2 are collinear in $\mathcal{A}(K_{Z(A)})$, then their images under ι and $\iota\varepsilon$ are collinear in $\mathcal{A}(\mathfrak{B}_{Z(A)})$ and $\mathcal{P}(\mathfrak{B}_K)$, respectively. On the other hand, given three pairwise different collinear points $(\mathfrak{p}t_i+\mathfrak{q}t_i a+\mathfrak{a})K$ $(i = 0,1,2)$ in $\mathcal{P}(\mathfrak{B}_K)$ there are uniquely determined elements r, $s \in K$ with $r+s = 1$ and $t_2 = t_0 r+t_1 s$. By collinearity, $t_2 a = t_0 ar+t_1 as$, whence r, $s \in Z(A)$. This in turn shows that $\mathcal{A}(K_{Z(A)})$ and $(\Gamma^{\circ}, \mathscr{L}_{\Gamma}\circ)$ are ismorphic linear spaces.

    The dimension of $(\Gamma^{\circ}, \mathscr{L}_{\Gamma}\circ)$ equals the right degree $|K{:}Z(A)|$. By the centralizer theorem (cf. e.g. [9,42]), $|K{:}Z(A)| = 2$ if, and only if, $|A{:}Z(K)| = |a{:}Z(K)| = 2$ as required.$\square$

    Since $|Z(A)| \geq 3$, there is a unique parallelism in $(\Gamma^{\circ}, \mathscr{L}_{\Gamma}\circ)$ which coincides with the image under $\iota\varepsilon$ of the parallelism relation on $\mathcal{A}(K_{Z(A)})$. Two lines of $(\Gamma^{\circ}, \mathscr{L}_{\Gamma}\circ)$ are parallel if, and only if, their hulls in $\mathcal{P}(\mathfrak{B}_K)$ have a common point on n. Thus the points of $n \subset PQ$ may be identified with the hyperplane at infinity[2] of the affine space $(\Gamma^{\circ}, \mathscr{L}_{\Gamma}\circ)$. If $|K{:}Z(A)| = 2$, then $(\Gamma^{\circ}, \mathscr{L}_{\Gamma}\circ)$ is an affine plane and n is a $Z(A)$-chain of PQ. As an immediate consequence of Theorem 1 and $|K{:}Z(A)| = |a{:}Z(K)| = 2$ we infer the following

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[1]All degrees written down in this articles are *right* degrees. Cf. however Theorem 3.1.2 in [8,31].

[2] This is one of those projective spaces discussed in [11].

COROLLARY. *If* $|a:Z(K)| = 2$, *then* $(\Gamma^\circ \cup n, \mathcal{L}_{\Gamma^\circ \cup n})$ *is a Baer subplane of* $\mathcal{P}(\mathfrak{B}_K)$ *with* $Z(A)$ *as underlying field.*

2. The following sections of this article are subject to the assumption

$$|a:Z(K)| = 2,$$

whence there exist m_0, $m_1 \in Z(K)$ such that

(5) $a^2 = m_0 + m_1 a.$

We shall frequently make use of the identity $a^{-1} m_0 = (a - m_1)$ without further notice.

The group **G** of automorphic collineations of Γ has been determined in [10,71]. Now we have an alternative description of **G**:

If $\gamma \in$ **G**, then $\gamma | \Gamma^\circ$ is an affinity of $(\Gamma^\circ, \mathcal{L}_{\Gamma^\circ})$. Conversely, let β be an affinity of $(\Gamma^\circ, \mathcal{L}_{\Gamma^\circ})$. Hence $\varepsilon \beta \varepsilon^{-1}$ is product of a translation and a semi-linear isomorphism of $\mathfrak{B}_{Z(A)}$ with respect to $\dot{b} \in \mathrm{Aut}(Z(A))$, say. This β is extendable to a collineation γ of $\mathcal{P}(\mathfrak{B}_K)$ if, and only if, $\dot{b}$ is restriction of an automorphism of K. If γ exists[3], then $\gamma \in$ **G**.

3. It is our goal to discuss the partition of $PQ \backslash n$ which is given by the *fundamental chains* of Γ [10,69]. This will be done in terms of affine geometries which may be regarded as affine derivations of the geometry of L-chains on PQ [5,320], where L denotes a proper subfield of K. But first we give an alternative description of fundamental chains:

THEOREM 2. *Let* Ψ *be the group of projective collineations of* $\mathcal{P}(\mathfrak{B}_K)$ *leaving* $\Gamma^\circ \cup n$ *pointwise invariant*[4]. *The orbit* X^Ψ *of any* $X \in PQ \backslash n$ *equals the only fundamental chain* c_X *of* Γ *passing through* X.

Proof. Suppose $\psi \in \Psi$. Clearly $X^\psi \in c_X$ for $X^\psi = X$. Restricting ψ to the pencil of lines with vertex X yields a projectivity α, say. If $X \neq X^\psi$, then Γ is a subset of the point set generated by α [10,43], since $(PQ)^\alpha = PQ$ and $Y = (XY) \cap (XY)^\alpha$ for all $Y \in \Gamma^\circ$. But every line through X other than PQ intersects Γ°, as follows from the fact that X is element of a fundamental pair of Γ [10,69].

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[3] The collineation γ need not exist. Take the field $\mathbb{H}$ of real quaternions. The centralizer of any $a \in \mathbb{H} \backslash \mathbb{R}$ where $\mathbb{R} = Z(\mathbb{H})$ is a field of complex numbers $\mathbb{C}$, say. Then there exists $\dot{b} \in \mathrm{Aut}(\mathbb{C})$ such that $\mathbb{R}$ is not invariant, whence has $\dot{b}$ has no extension to $\mathbb{H}$. Cf. e.g. [14], [17].

[4] Any $\psi \in \Psi$ is a quasiperspective collineation.

Hence α is a generating map of Γ and $X^\Psi \subset c_X$.

On the other hand, assume $X = (\mathfrak{p}v + \mathfrak{q}w)K \in PQ\backslash n$. We read off from formula (3.21) in [10,69]

$$c_X = \{(\mathfrak{p}(w - m_1 v + \mathfrak{q}m_0 v)z_0 + (\mathfrak{p}v + \mathfrak{q}w)z_1)K \mid (0,0) \neq (z_0, z_1) \in Z(K)^2\}.$$

Those maps of $GL(\mathfrak{B}_K)$ having matrices

$$\begin{pmatrix} z_1 - m_1 z_0 & z_0 & 0 \\ m_0 z_0 & z_1 & 0 \\ 0 & 0 & (a - m_1)z_0 + z_1 \end{pmatrix}, \quad (0,0) \neq (z_0, z_1) \in Z(K)^2,$$

with respect to the ordered basis $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$ induce a subset Φ of Ψ, since every vector $\mathfrak{p}t + \mathfrak{q}ta + \mathfrak{a}$ $(t \in K)$ is eigenvector with

$$(a - m_1)z_0 + z_1 = m_0 z_0 a^{-1} + z_1 \neq 0$$

being its (right) eigenvalue. (In fact $\Phi = \Psi$.) By construction, $X^\Psi \supset X^\Phi = c_X$ which completes the proof. $\square$

We remark that a non-projective collineation ν which leaves $\Gamma^\circ \cup n$ elementwise invariant takes $X \in PQ\backslash n$ to $X^\nu \notin c_X$: Firstly, $X \neq X^\nu$ as there is no subline of PQ which is properly between n and PQ due to $|K{:}Z(A)| = 2$. Secondly, assume $X \neq X^\nu \in c_X$. The restriction of ν to the pencil of lines with vertex X is a bijection μ, say, such that $\Gamma^\circ = \{XY \cap (XY)^\mu \mid Y \in \Gamma^\circ\}$. But (X, X^ν) is a fundamental pair of Γ. Hence μ is a projectivity which generates Γ, a contradiction.

Let $\{b_0, b_1\}$ be a basis of $K_{Z(A)}$ and write $\mathfrak{b}_i := \mathfrak{p}b_i + \mathfrak{q}b_i a$. Then $n = \{(\mathfrak{b}_0 t_0 + \mathfrak{b}_1 t_1)K \mid (0,0) \neq (t_0, t_1) \in Z(A)^2\}$. We use $\mathfrak{b}_0$, $\mathfrak{b}_1$ to define an affine coordinatization

$$\sigma{:}PQ\backslash\{\mathfrak{b}_1 K\} \to K, \quad (\mathfrak{b}_0 + \mathfrak{b}_1 t)K \mapsto t.$$

We shall describe the partition of $K\backslash Z(A) = (PQ\backslash n)^\sigma$ given by the orbits of $\Psi^\sigma := \sigma^{-1}\Psi\sigma$. The re-transfer under σ^{-1} is immediate.

The group Ψ^σ is formed by all inner automorphisms of K such that $x \mapsto cxc^{-1}$ with $c \in A\backslash\{0\}$, since $Z(Z(A)) = A$ by the centralizer theorem [9,42]. If $c = az_0 - z_1$ $(z_0, z_1 \in Z(K))$, then

$$c^{-1} = \left((a - m_1)z_0 + z_1 \right)\left(m_0 z_0^2 + m_1 z_0 z_1 - z_1^2 \right)^{-1}$$

and

(6) $\quad cxc^{-1} = \left(axa^{-1}m_0 z_0^2 + (ax - xa + m_1 x)z_0 z_1 - xz_1^2 \right)\left(m_0 z_0^2 + m_1 z_0 z_1 - z_1^2 \right)^{-1}.$

Ψ^σ is a subgroup of the Galois group of $K|Z(A)$ [8,40], but we are only concerned with inner automorphisms. (Cf. [1,283], [2], [3], [4], [5,176] for geometric interpretations of Galois groups.) However if $|K{:}Z(K)| < \infty$, then $id_{Z(A)}$ is only extendable to inner automorphisms of K (cf. e.g. the Corollary in [8,46]) and consequently every collineation which leaves $\Gamma^\circ \cup n$ elementwise

invariant belongs to Ψ.

See [1,280] or Theorem 23 in [15,126] for results similar to the following

LEMMA. *In the affine plane $\mathcal{A}(K_{Z(A)})$ the group Ψ^σ is a group of perspective affinities with common axis $Z(A)$. The pencil of lines parallel to $(x\text{-}axa^{-1})Z(A)$ ($x \in K\backslash Z(A)$ arbitrary) is elementwise invariant under all transformations of Ψ^σ. The element $a \in K$ is inseparable over the centre of K if, and only if, Ψ^σ is a group of shears.*

Proof. Clearly Ψ^σ is a group of perspective affinities with common axis $Z(A)$. We deduce from

(7) $ax\text{-}xa+m_1x = x(m_1\text{-}a)+(axa^{-1})a$

and (6) that the orbit of $x \in K\backslash Z(A)$ under Ψ^σ is a subset of the line joining x and axa^{-1}. Thus all lines parallel to this one are invariant under all affinities of Ψ^σ and every orbit of Ψ^σ is subset of such an invariant line. The group Ψ^σ is a group of shears if, and only if,

$(x\text{-}axa^{-1}) \in Z(A)$ for all $x \in K$

or, in other words,

$a(x\text{-}axa^{-1})a^{-1} = (x\text{-}axa^{-1})(m_1a^{-1}\text{-}1) = x\text{-}axa^{-1}$,

where the first sign of equality follows from a streightforward calculation. But $m_1a^{-1}\text{-}1 = 1$ is equivalent to $\text{Char}K = 2$ and $m_1 = 0$ which in turn characterizes a as inseparable over $Z(K)$.□

Suppose that Ψ^σ is no group of shears. Then every Ψ^σ-invariant line $x + (x\text{-}axa^{-1})Z(A)$ where $x \in K\backslash Z(A)$ contains exactly one point $x^\pi \in Z(A)$ which is given by

(8) $x^\pi = \left(axa^{-1}(2m_0+m_1a)+x(2m_0\text{-}m_1a+m_1^2) \right)\left(4m_0+m_1^2 \right)^{-1}$.

Hence $\pi: K \to Z(A)$, $x \mapsto x^\pi$, is a projection in $K_{Z(A)}$. Now let Ψ^σ be a group of shears. Here we put

(9) $x^\pi := ax\text{-}xa = ax+xa$.

This map $\pi: K \to Z(A)$, $x \mapsto x^\pi$, is $Z(A)$-linear with kernel $Z(A)$ and image $Z(A)$.

In the affine space[5] $\mathcal{A}(K_A)$ the orbit c_x of $x \in K\backslash Z(A)$ under Ψ^σ is a subset of a *line* M_x, because (6) and (7) are linear combinations with coefficients in A. If a is separable over $Z(K)$,

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[5]$A = Z(A)$ if, and only if, A is a maximal commutative subfield of K, whence $|K:Z(K)| = 4$ by (5), formula (10) in [9,40] and Theorem 4 in [9,45].

then M_x is given by $x + (x-x^\pi)A$. All such lines M_y within $x + (x-x^\pi)Z(A)$ are incident with x^π. If a is inseparable over $Z(K)$, then the line M_x equals $x + x^\pi A$ and all such lines M_y within the subspace $x + x^\pi Z(A)$ are parallel to $x^\pi A$.

Now, finally, take the affine space $\mathcal{A}(K_{Z(K)})$. Our previous discussions show that c_x lies in the Ψ^σ-invariant *plane* M_x which in turn is contained in the Ψ^σ-invariant subspace $x + (x^\pi - x)Z(A)$ or $x + x^\pi Z(A)$ in the separable or inseparable case, respectively. The configurational properties of these planes and subspaces of $\mathcal{A}(K_{Z(K)})$ follow immediately from the corresponding results in $\mathcal{A}(K_A)$ and $\mathcal{A}(K_{Z(A)})$. Thus we can restrict our investigations to a fixed plane M_x.

It will be conveniant to use the projective closure $\mathcal{P}\left((Z(K) \oplus K)_{Z(K)}\right)$ of $\mathcal{A}(K_{Z(K)})$ with $u \mapsto (1,u)Z(K)$ being the canonical embedding. In this projective space the orbit c_x, as is given by (6), may be re-written as the set of all points

$$\left((m_0, axa^{-1}m_0)z_0^2 + (m_1, ax-xa+m_1 x)z_0 z_1 + (-1,-x)z_1^2\right)Z(K)$$

with $(0,0) \neq (z_0, z_1) \in Z(K)^2$. Thus c_x is a conic without points at infinity or, in other words, an ellipse. Cf. e.g. [7,31]. This is in accordance with [5,225]. Recall that the tangents of c_x in $(m_0, axa^{-1}m_0)Z(K)$ and $(-1,-x)Z(K)$ intersect each other in

(10) $(m_1, ax-xa+m_1 x)Z(K)$.

Let us return to affine terms: The plane $M_x \subset K$ carries the *affine Möbius-geometry* $\mathcal{A}(Z(K),A)$ in the sense of [5,101] (cf. also [12,221] and [13]) when identifying

(11) $t \in A$ with $y_0 + (y_1 - y_0)t \in M_x$

for any two different $y_0, y_1 \in M_x$. This identification does not essentially depend on y_0, y_1, because it is unique to within circle preserving affinities of $\mathcal{A}(Z(K),A)$. Hence we may speak unambigously of *circles* within the plane M_x. (Letting $t \in K$ in (11) yields a bijection $K \to K$ which is a circle preserving affinity of the affine Möbius-geometry $\mathcal{A}(Z(K),K)$.)

THEOREM 3. *The orbit c_x of $x \in K \backslash Z(A)$ under Ψ^σ is a circle of the plane M_x.*

Case 1 – a is separable over $Z(K)$: The mid-point (Char$K \neq 2$) or nucleus (Char$K = 2$) of the circle $c_x \subset M_x$ is the point $x^\pi \in Z(A)$ as given by (8). A circle of M_x is an orbit of Ψ^σ if, and only if, its mid-point respectively nucleus equals x^π.

Case 2 – a is inseparable over $Z(K)$: All tangents of c_x are

parallel to the line $x^\pi Z(K)$ with x^π given by (9). A circle of the plane M_x is an orbit of Ψ^σ if, and only if, there exists a translation mapping it into c_x.

Proof. The orbit c_x is a circle of M_x, since its image under σ^{-1} is a $Z(K)$-chain of PQ.

Case 1: For $\operatorname{Char} K = 2$ we deduce from $m_1 \neq 0$ and (10) that the nucleus of c_x is not at infinity. The mid-point respectively nucleus of c_x is invariant under all transformations of Ψ^σ, whence it has to be x^π. But $y^\pi = x^\pi$ for all $y \in M_x$, so that all orbits c_y ($y \in M_x \backslash Z(A)$) have common mid-point or nucleus, respectively.

Conversely, let c be a circle of M_x with mid-point respectively nucleus x^π. Choose $y \in c$. As a circle is uniquely determined by one of its points and its mid-point respectively nucleus, we get $c = c_y$.

Case 2: We read off from (10) together with $\operatorname{Char} K = 2$ and $m_1 = 0$ that all tangents of c_x are parallel to $x^\pi Z(K)$.

If $y \in M_x$, then $y = x + x^\pi s$ with $s \in A$ and $cyc^{-1} = cxc^{-1} + x^\pi s$. Thus the translation $\tau : M_x \to M_x$, $u \mapsto u + x^\pi s$, maps c_x into c_y. The proof is completed by reversing the above arguments.□

We remark that in case 2 there are circles of M_x whose tangents are parallel to $x^\pi Z(K)$ without being orbits of Ψ^σ. An example is given by a circle which touches an orbit of Ψ^σ in exactly one point.

Finally, take as K the real quaternions $\mathbb{H}$. Then M_x is a plane of complex numbers and Ψ^σ may be interpreted as a full group of rotations of the euclidian space $\mathcal{A}(\mathbb{R}, \mathbb{H})$ with a pointwise invariant plane. Inner automorphisms of $\mathbb{H}$ (cf. formula (6)) are a standard tool for the representation by quaternions of motions of the euclidian 2-sphere. Cf. e.g. [6].

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