A Dimension Formula for the Nucleus of a Veronese Variety*

Johannes Gmainer   Hans Havlicek

September 27, 1999

Abstract

The nucleus of a Veronese variety is the intersection of all its osculating hyperplanes. Various authors have given necessary and sufficient conditions for the nucleus to be empty. We present an explicit formula for the dimension of this nucleus for arbitrary characteristic of the ground field. As a corollary, we obtain a dimension formula for that subspace in the t–th symmetric power of a finite–dimensional vector space $\mathbf{V}$ which is spanned by the powers $\mathbf{a}^t$ with $\mathbf{a} \in \mathbf{V}$.

Keywords: Veronese variety, nucleus, symmetric power, multinomial coefficient.

Address of the authors:
Abteilung für Lineare Algebra und Geometrie
Technische Universität
Wiedner Hauptstraße 8–10
A–1040 Wien
Austria, Europe.
Email:
`gmainer@geometrie.tuwien.ac.at`
`havlicek@geometrie.tuwien.ac.at`
Telephone:
(+43)-1-588 01-11330
Fax:
(+43)-1-588 01-11399
All correspondence should be sent to Hans Havlicek.

*Research supported by the Austrian Science Fund (FWF), project P–12353–MAT, and by the City of Vienna (Hochschuljubiläumsstiftung der Stadt Wien), project H–39/98.

1 Introduction

It is well known that, in a projective plane over a (commutative) field F of characteristic two, the tangents of a conic have a common point called *nucleus*. In fact, conics are just specific examples of *Veronese varieties* and a tangent of a conic may be seen as an *osculating hyperplane*. So the intersection of all osculating hyperplanes of a Veronese variety will be called its *nucleus*. In case of characteristic zero such a nucleus is always empty, since all osculating hyperplanes form a Veronese variety in the dual of the ambient space. For non–zero characteristic all Veronese varieties with empty nucleus have been determined independently by H. Timmermann [11], [12], A. Herzer [6], and H. Karzel [9]. The inaugural thesis [12] contains a formula for the dimension of the nucleus of a *normal rational curve*, i.e. a Veronese image of a projective line. Another proof of that formula and further references can be found in [4]. See also J.A. Thas [10], J.W.P. Hirschfeld and J.A. Thas [7, 25.1].

In the present paper we improve the above mentioned results by giving an explicit formula for the dimension of the nucleus of a Veronese variety. We have to assume, however, that the ground field has sufficiently many elements, since otherwise a Veronese variety consists of "few" points in some "high dimensional" space.

In the second chapter we present a slightly modified version of Herzer's elegant coordinate–free approach to Veronese varieties and their osculating subspaces. See 2.5 for a motivation of our modification.

The announced dimension formula for nuclei can be found in Chapter 3, Theorem 2. Finally, we apply our results to show that three (seemingly strong) conditions are not sufficient to characterize Veronese mappings to within collineations. Cf. however [5], where quadratic Veronese mappings have been characterized in a purely geometric way.

Throughout this paper symmetric powers of vector spaces and divisibility of multinomial coefficients by primes play an essential role. In the t–th symmetric power of a finite–dimensional vector space $\mathbf{V}$ there is a distinguished subspace $\mathbf{A}$ which is generated by all powers $\mathbf{a}^t$ where $\mathbf{a}$ ranges in $\mathbf{V}$. In case of characteristic zero the subspace $\mathbf{A}$ equals the t–th symmetric power of $\mathbf{V}$.

In Corollary 2 we find a formula for the dimension of $\mathbf{A}$ for non–zero characteristic. As before, the ground field has to be sufficiently large. In fact, the codimension of $\mathbf{A}$ is, up to an additive constant, the projective dimension of the nucleus of a Veronese variety.

2 Veronese Mappings

2.1 Let $\mathbf{X}$ be an $(m+1)$–dimensional vector space over a field F with $m \in \mathbb{N} = \{0, 1, \ldots\}$. We denote by $\mathbf{X}^*$ its dual space and by $S^n\mathbf{X}^*$ the n–th symmetric power of $\mathbf{X}^*$ ($n \in \mathbb{N}$), where $S^0\mathbf{X}^* = F$. Cf., among others, [1, Chapter III, §6].

We fix one $t \in \mathbb{N} \setminus \{0\}$ and assume that $(S^t\mathbf{X}^*, \mathbf{Y})$ is a dual pair of F–vector spaces with a (non–degenerate bilinear) pairing $\langle\, ,\, \rangle : S^t\mathbf{X}^* \times \mathbf{Y} \to F$. Via $\langle\, ,\, \rangle$, the space $\mathbf{Y}$ turns into the space of symmetric t–multilinear forms on $\mathbf{X}^*$.

Each vector $\mathbf{x} \in \mathbf{X}$ defines a symmetric t–multilinear form

$$(\mathbf{X}^*)^t \to F,\ (\mathbf{a}_1^*, \mathbf{a}_2^*, \ldots, \mathbf{a}_t^*) \mapsto \mathbf{a}_1^*(\mathbf{x}) \cdot \mathbf{a}_2^*(\mathbf{x}) \cdot \ldots \cdot \mathbf{a}_t^*(\mathbf{x}).$$

By the universal property of symmetric powers, there exists a unique vector in $\mathbf{Y}$, say $g(\mathbf{x})$, with

$$\langle \mathbf{a}_1^* \cdot \mathbf{a}_2^* \cdot \ldots \cdot \mathbf{a}_t^*, g(\mathbf{x}) \rangle = \mathbf{a}_1^*(\mathbf{x}) \cdot \mathbf{a}_2^*(\mathbf{x}) \cdot \ldots \cdot \mathbf{a}_t^*(\mathbf{x}) \tag{1}$$

for all $\mathbf{a}_1^*, \mathbf{a}_2^*, \ldots, \mathbf{a}_t^* \in \mathbf{X}^*$. So we have a well defined mapping

$$g : \mathbf{X} \to \mathbf{Y},\ \mathbf{x} \mapsto g(\mathbf{x}) \tag{2}$$

which will be used to define the Veronese mapping in 2.4.

2.2 Let $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$ be a basis of $\mathbf{X}$ and put $\mathbf{b}_0^*, \mathbf{b}_1^*, \ldots, \mathbf{b}_m^* \in \mathbf{X}^*$ for the dual basis. Then the $\binom{m+t}{t}$ distinct vectors $\mathbf{b}_0^{*e_0} \cdot \mathbf{b}_1^{*e_1} \cdot \ldots \cdot \mathbf{b}_m^{*e_m}$, where $(e_0, e_1, \ldots, e_m)$ runs in the set

$$E_m^t := \{(e_0, e_1, \ldots, e_m) \in \mathbb{N}^{m+1} \mid e_0 + e_1 + \ldots + e_m = t\}, \tag{3}$$

form a basis of $S^t\mathbf{X}^*$. Denote by

$$\{\mathbf{c}_{e_0, e_1, \ldots, e_m} \mid (e_0, e_1, \ldots, e_m) \in E_m^t\} \subset \mathbf{Y} \tag{4}$$

its dual basis with respect to the pairing $\langle\, ,\, \rangle$. Hence

$$g(\sum_{i=0}^m x_i \mathbf{b}_i) = \sum_{E_m^t} x_0^{e_0} x_1^{e_1} \ldots x_m^{e_m} \mathbf{c}_{e_0, e_1, \ldots, e_m} \ \ (x_i \in F). \tag{5}$$

2.3 Given an element $\mathbf{r} \in S^t\mathbf{X}^*$ then

$$\mathbf{r}' : \mathbf{X} \to F,\ \mathbf{x} \mapsto \langle \mathbf{r}, g(\mathbf{x}) \rangle \tag{6}$$

is a homogeneous polynomial function of degree t and all such functions arise in this way according to (5). Clearly, the functions $\mathbf{r}'$ form a subspace, say $(S^t\mathbf{X}^*)'$, of the space of all functions $\mathbf{X} \to F$. It is necessary to distinguish between $S^t\mathbf{X}^*$ and $(S^t\mathbf{X}^*)'$ exactly if $g(\mathbf{X})$ does not generate $\mathbf{Y}$ or, equivalently, exactly if there is a non–zero element $\mathbf{r} \in S^t\mathbf{X}^*$ with $\mathbf{r}' = 0$.

Lemma 1 *The vector space $\mathbf{Y}$ is spanned by $g(\mathbf{X})$ if, and only if, $\#F \geq t$ or $m = \dim \mathbf{X} - 1 = 0$.*

Proof. *(a)* Assume $\#F =: q < t$ and $\dim \mathbf{X} > 1$. Choose two basis forms, say $\mathbf{b}_0^*$ and $\mathbf{b}_1^*$ (cf. 2.2), and define

$$\mathbf{r} := \mathbf{b}_0^{*q}\mathbf{b}_1^{*t-q} - \mathbf{b}_0^*\mathbf{b}_1^{*t-1} \neq 0.$$

Put $\mathbf{x} = \sum_{i=0}^m x_i \mathbf{b}_i \in \mathbf{X}$ with $x_i \in F$. From $x^q - x = 0$ for all $x \in F$ we obtain

$$\mathbf{r}'(\mathbf{x}) = x_0^q x_1^{t-q} - x_0 x_1^{t-1} = x_1^{t-q}(x_0^q - x_0 x_1^{q-1}) = 0 \text{ for all } \mathbf{x} \in \mathbf{X}.$$

(b) If $m = 0$, then $\dim \mathbf{X} = \dim S^t\mathbf{X}^* = \dim \mathbf{Y} = 1$. Hence $\mathbf{Y}$ is spanned by any $g(\mathbf{x})$ with $\mathbf{x} \in \mathbf{X} \setminus \{0\}$.

(c) Let $\#F \geq t$. If $\mathbf{r} \in S^t\mathbf{X}^*$ satisfies $\mathbf{r}' = 0$, then $\mathbf{r} = 0$ by [6, (1.2)]. From the remarks above, $\mathbf{Y}$ is spanned by $g(\mathbf{X})$. $\square$

2.4 Next we are going to interpret (2) in geometric terms. The *Veronese mapping*

$$\gamma : \mathcal{P}(\mathbf{X}) \to \mathcal{P}(\mathbf{Y}),\ F\mathbf{x} \mapsto F(g(\mathbf{x})) \tag{7}$$

assigns to each point of the projective space on $\mathbf{X}$ a point of the projective space on $\mathbf{Y}$, since (1) forces that $g(\mathbf{x}) \neq 0$ whenever $\mathbf{x} \neq 0$ and $g(w\mathbf{x}) = w^t g(\mathbf{x})$ for all $w \in F$ and all $\mathbf{x} \in \mathbf{X}$. The image set $\gamma(\mathcal{P}(\mathbf{X})) =: \mathcal{V}_m^t$ is a *Veronese variety.* According to (5) this approach coincides with the classical one [3].

2.5 Herzer's coordinate–free definition of a Veronese variety [6, (2.1)] uses a dual pair $((S^t\mathbf{X}^*)', \mathbf{Z})$ of F–vector spaces. In our approach this $\mathbf{Z}$ may be chosen as the subspace of $\mathbf{Y}$ spanned by $g(\mathbf{X})$. So, in contrast to [6, p. 144], a Veronese variety does not necessarily span $\mathcal{P}(\mathbf{Y})$. The entire space $\mathcal{P}(\mathbf{Y})$ is spanned, however, by the union of all (or sufficiently many) osculating subspaces (cf. 2.7).

For example, let $m = 1$, $t = 3$, and $\#F = 2$. Then $\mathcal{V}_1^3$ is a twisted cubic consisting of three non–collinear points. So, following [6], such a twisted cubic should be considered as a triangle in a plane $\mathcal{T}$, say. However, none of the tangents and none of the osculating planes (according to our definition) is contained in that plane $\mathcal{T}$.

2.6 The definition of a Veronese variety in the papers of Karzel [9] and Timmermann [11], [12] follows Burau [3]. It is based upon *Segre varieties* or, in algebraic language, the tensor product $\bigotimes^t \mathbf{X}$. We sketch the connection with our definition.

For each σ in the symmetric group S_t there is a unique linear automorphism f_σ of $\bigotimes^t \mathbf{X}$ such that $\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \cdots \otimes \mathbf{x}_t \mapsto \mathbf{x}_{\sigma(1)} \otimes \mathbf{x}_{\sigma(2)} \otimes \cdots \otimes \mathbf{x}_{\sigma(t)}$ for all $\mathbf{x}_i \in \mathbf{X}$. There are two distinguished subspaces in $\bigotimes^t \mathbf{X}$:

$$\mathbf{Y} := \{\mathbf{k} \in \bigotimes{}^t \mathbf{X} \mid f_\sigma(\mathbf{k}) = \mathbf{k} \text{ for all } \sigma \in S_t\} \tag{8}$$

is the space of symmetric tensors and

$$\mathbf{M} := \operatorname{span}\{\mathbf{k} - f_\sigma(\mathbf{k}) \mid \mathbf{k} \in \bigotimes{}^t \mathbf{X},\ \sigma \in S_t\} \tag{9}$$

is the kernel of the canonical mapping $\bigotimes^t \mathbf{X} \to \bigotimes^t \mathbf{X}/\mathbf{M} = S^t\mathbf{X}$.

The tensor products $\bigotimes^t \mathbf{X}^*$ and $\bigotimes^t \mathbf{X}$ form a dual pair of vector spaces with the pairing $\langle\, , \rangle_\otimes$ given as complete contraction of $(\bigotimes^t \mathbf{X}^*) \otimes (\bigotimes^t \mathbf{X})$. By virtue of this pairing, the elements of $\bigotimes^t \mathbf{X}$ are the t–multilinear forms on $\mathbf{X}^*$ and $\mathbf{Y}$ is the subspace of symmetric t–multilinear forms. Orthogonality with respect to $\langle\, , \rangle_\otimes$ will be denoted by $\perp_\otimes$.

As $\mathbf{Y}^{\perp_\otimes}$ is kernel of the canonical mapping $\bigotimes^t \mathbf{X}^* \to S^t\mathbf{X}^*$, the vector spaces $S^t\mathbf{X}^* = \bigotimes^t \mathbf{X}^*/\mathbf{Y}^{\perp_\otimes}$ and $\mathbf{Y}$ form a dual pair with the pairing $\langle\, , \rangle$, say, induced by $\langle\, , \rangle_\otimes$. This is in accordance with our approach in 2.1. In the present context the mapping (2) takes the form

$$g(\mathbf{x}) = \underbrace{\mathbf{x} \otimes \mathbf{x} \otimes \cdots \otimes \mathbf{x}}_{t} \text{ for all } \mathbf{x} \in \mathbf{X}. \tag{10}$$

Also, the basis (4) may be understood in terms of $\bigotimes^t \mathbf{X}$: If $(e_0, e_1, \ldots, e_m) \in E_m^t$, then let $\mathbf{b}_{e_0,e_1,\ldots,e_m} := \mathbf{b}_{i_1} \otimes \mathbf{b}_{i_2} \otimes \ldots \otimes \mathbf{b}_{i_t}$ with $0 \le i_1 \le \ldots \le i_t \le m$ and each basis vector $\mathbf{b}_i$ appearing exactly e_i times. As σ runs in S_t we obtain exactly

$$\binom{t}{e_0, e_1, \ldots, e_m} = \frac{t!}{e_0! e_1! \ldots e_m!}$$

distinct vectors $f_\sigma(\mathbf{b}_{e_0,e_1,\ldots,e_m})$ and their sum is easily seen to be $\mathbf{c}_{e_0,e_1,\ldots,e_m}$. The canonical mapping $\bigotimes^t \mathbf{X} \to S^t\mathbf{X}$ maps $\mathbf{c}_{e_0,e_1,\ldots,e_m} \in \mathbf{Y}$ to

$$\mathbf{c}_{e_0,e_1,\ldots,e_m} + \mathbf{M} = \binom{t}{e_0, e_1, \ldots, e_m} \mathbf{b}_0^{e_0} \cdot \mathbf{b}_1^{e_1} \cdot \ldots \cdot \mathbf{b}_m^{e_m}. \tag{11}$$

2.7 Returning to the settings of 2.1, let $\mathbf{U} \subset \mathbf{X}$ be an $(r+1)$–dimensional subspace $(0 \le r < m)$. First we show that the restriction of γ to the projective subspace $\mathcal{P}(\mathbf{U})$ is a Veronese mapping.

We write $\mathbf{U}^*$ for the dual space of $\mathbf{U}$ and $\mathbf{U}^\circ$ for the annihilator (orthogonal subspace) of $\mathbf{U}$ in $\mathbf{X}^*$. It is easily seen that there are canonical isomorphisms

$$\begin{aligned} \mathbf{U}^* &\cong \mathbf{X}^*/\mathbf{U}^\circ, \\ S^t\mathbf{U}^* &\cong S^t(\mathbf{X}^*/\mathbf{U}^\circ) \cong S^t\mathbf{X}^*/(\mathbf{U}^\circ \cdot S^{t-1}\mathbf{X}^*), \end{aligned}$$

where $\mathbf{U}^\circ \cdot S^{t-1}\mathbf{X}^*$ is a shorthand for the subspace of $S^t\mathbf{X}^*$ spanned by all products of the form

$$\mathbf{a}_1^* \cdot \mathbf{a}_2^* \cdot \ldots \cdot \mathbf{a}_t^* \text{ with } \mathbf{a}_1^* \in \mathbf{U}^\circ,\ \mathbf{a}_2^*, \ldots, \mathbf{a}_t^* \in \mathbf{X}^*.$$

So $S^t\mathbf{U}^*$ and the subspace $(\mathbf{U}^\circ \cdot S^{t-1}\mathbf{X}^*)^\perp$ of $\mathbf{Y}$ form a dual pair of vector spaces with the pairing induced by $\langle\, , \rangle$. Hence

$$g_\mathbf{U} : \mathbf{U} \to (\mathbf{U}^\circ \cdot S^{t-1}\mathbf{X}^*)^\perp,\ \mathbf{u} \mapsto g(\mathbf{u})$$

yields a Veronese mapping of $\mathcal{P}(\mathbf{U})$ and $\gamma(\mathcal{P}(\mathbf{U}))$ is a Veronese variety $\mathcal{V}_r^t$ contained in $\mathcal{V}_m^t$.

Following Herzer [6, (4.1)], we associate with $\mathbf{U}$ the subspaces

$$(S^{k+1}\mathbf{U}^\circ \cdot S^{t-k-1}\mathbf{X}^*)^\perp \text{ with } k \in \{-1, 0, \dots, t-1\}. \tag{12}$$

In projective terms they yield the *k–osculating subspaces* of $\mathcal{V}_m^t$ along the subvariety $\mathcal{V}_r^t$ arising from $\mathcal{P}(\mathbf{U})$. See [3] or [7] for geometrical interpretations of those subspaces.

We are interested in the special case that $\mathbf{U} = \ker \mathbf{a}^*$ ($\mathbf{a}^* \in \mathbf{X}^*$) is a hyperplane of $\mathbf{X}$ and that $k = t-1$ is maximal. This gives

$$(S^t\mathbf{U}^\circ \cdot S^0\mathbf{X}^*)^\perp = (F\mathbf{a}^{*t})^\perp, \tag{13}$$

i.e., a hyperplane of $\mathbf{Y}$. The corresponding projective hyperplane is the *osculating* (or *contact*) *hyperplane* of $\mathcal{V}_m^t$ along the subvariety $\mathcal{V}_{m-1}^t = \gamma(\mathcal{P}(\mathbf{U}))$. Such an osculating hyperplane meets the Veronese variety $\mathcal{V}_m^t$ exactly in a subvariety $\mathcal{V}_{m-1}^t$. Thus, finally, we have established the *dual Veronese mapping*

$$\gamma^* : \mathcal{P}(\mathbf{X}^*) \to \mathcal{P}(S^t\mathbf{X}^*),\ F\mathbf{a}^* \mapsto F\mathbf{a}^{*t}. \tag{14}$$

With the notations of 2.2 we obtain

$$(\sum_{i=0}^m a_i\mathbf{b}_i^*)^t = \sum_{E_m^t} \binom{t}{e_0, e_1, \dots, e_m} a_0^{e_0}a_1^{e_1}\dots a_m^{e_m}\mathbf{b}_0^{*e_0}\cdot\mathbf{b}_1^{*e_1}\cdot\dots\cdot\mathbf{b}_m^{*e_m} \quad (a_i \in F). \tag{15}$$

See also [3, pp. 160–163].

2.8 It is an essential property of the Veronese mapping γ that for each collineation κ of $\mathcal{P}(\mathbf{X})$ there is a collineation $\tilde{\kappa}$ of $\mathcal{P}(\mathbf{Y})$ with $\tilde{\kappa} \circ \gamma = \gamma \circ \kappa$. In our approach this is easily derived from the universal property of $S^t\mathbf{X}^*$: Let $f : \mathbf{X} \to \mathbf{X}$ be a semilinear bijection inducing κ with accompanying automorphism $\iota \in \operatorname{Aut} K$. Put $f^\top : \mathbf{X}^* \to \mathbf{X}^*$ for its transpose mapping. For each $\mathbf{y} \in \mathbf{Y}$ there is a unique vector in $\mathbf{Y}$, say $\tilde{f}(\mathbf{y})$, with

$$\langle \mathbf{a}_1^* \cdot \mathbf{a}_2^* \cdot \dots \cdot \mathbf{a}_t^*, \tilde{f}(\mathbf{y})\rangle = \iota(\langle f^\top(\mathbf{a}_1^*) \cdot f^\top(\mathbf{a}_2^*) \cdot \dots \cdot f^\top(\mathbf{a}_t^*), \mathbf{y}\rangle)$$

for all $\mathbf{a}_1^*, \mathbf{a}_2^*, \dots, \mathbf{a}_t^* \in \mathbf{X}^*$. Then $\tilde{f} : \mathbf{Y} \to \mathbf{Y}$ is a ι–semilinear bijection inducing $\tilde{\kappa}$. It is straightforward to show that $\tilde{\kappa}$ preserves the Veronese variety $\gamma(\mathcal{P}(\mathbf{X}))$ and its osculating subspaces.

Observe that we did not assert the uniqueness of $\tilde{\kappa}$. Also, there may be collineations of $\mathcal{P}(\mathbf{Y})$ fixing $\mathcal{V}_m^t$ as a set of points without preserving its osculating subspaces. Clearly, such collineations cannot arise from collineations of $\mathcal{P}(\mathbf{X})$. The existence of such "exceptional collineations" is immediate whenever $\mathcal{V}_m^t$ does not span $\mathcal{P}(\mathbf{Y})$. Another example is given in 3.6.

3 Nuclei

3.1 In this section we investigate a fixed Veronese variety $\mathcal{V}_m^t$. In order to avoid trivialities we assume $m \geq 1$ and $t \geq 2$.

Definition 1 The *nucleus* of a Veronese variety is defined as the intersection of all its osculating hyperplanes.

As we aim at a formula for the dimension of the nucleus of a $\mathcal{V}_m^t$ we shall use coordinates. However, all results do not depend on the specific choice of a basis $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$ of $\mathbf{X}$.

Theorem 1 *The nucleus $\mathcal{N}$ of a Veronese variety $\mathcal{V}_m^t$ contains exactly those base points $F\mathbf{c}_{e_0,e_1,\ldots,e_m}$ of $\mathcal{P}(\mathbf{Y})$ satisfying*

$$\binom{t}{e_0, e_1, \ldots, e_m} \equiv 0 \pmod{\operatorname{char} F}. \tag{16}$$

If $\#F \geq t$, then the nucleus is spanned by those base points.

Proof. *(a)* A fixed base point $F\mathbf{c}_{e_0,e_1,\ldots,e_m}$ belongs to $\mathcal{N}$ exactly if

$$\binom{t}{e_0, e_1, \ldots, e_m} a_0^{e_0} a_1^{e_1} \ldots a_m^{e_m} = 0 \text{ for all } a_0, a_1, \ldots a_m \in F$$

by (15). This in turn is equivalent to (16).

(b) Each vector $\mathbf{y} \in \mathbf{Y}$ defines a function

$$\mathbf{y}'' : \mathbf{X}^* \to F, \ \mathbf{a}^* \mapsto \langle \mathbf{a}^{*t}, \mathbf{y} \rangle.$$

Letting $\mathbf{y} = \sum_{E_m^t} y_{e_0,e_1,\ldots,e_m} \mathbf{c}_{e_0,e_1,\ldots,e_m}$ and $\mathbf{a}^* = \sum_{i=0}^m a_i \mathbf{b}_i^*$ gives

$$\mathbf{y}''(\mathbf{a}^*) = \sum_{E_m^t} y_{e_0,e_1,\ldots,e_m} \binom{t}{e_0, e_1, \ldots, e_m} a_0^{e_0} a_1^{e_1} \ldots a_m^{e_m}.$$

So $\mathbf{y}''$ is a homogeneous polynomial function of degree t.

Now let $F\mathbf{y}$ be a point in the nucleus, whence $\mathbf{y}'' = 0$. By $\#F \geq t$ and [6, (1.2)] we obtain

$$y_{e_0,e_1,\ldots,e_m} \binom{t}{e_0, e_1, \ldots, e_m} = 0 \text{ for all } (e_0, e_1, \ldots, e_m) \in E_m^t.$$

Therefore $\binom{t}{e_0,e_1,\ldots,e_m} \not\equiv 0 \pmod{\operatorname{char} F}$ implies $y_{e_0,e_1,\ldots,e_m} = 0$, as required. □

3.2 When comparing (5) with (15) it is tempting to define a linear mapping

$$h : \mathbf{Y} \to S^t\mathbf{X}^* \text{ by } \mathbf{c}_{e_0,e_1,\ldots,e_m} \mapsto \binom{t}{e_0, e_1, \ldots, e_m} \mathbf{b}_0^{*e_0} \cdot \mathbf{b}_1^{*e_1} \cdot \ldots \cdot \mathbf{b}_m^{*e_m}. \tag{17}$$

Clearly, such an h depends on the basis $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_m$ of $\mathbf{X}$, but the induced (possibly singular) duality will take the point set of $\mathcal{V}_m^t$ onto the set of osculating hyperplanes. However, the kernel of h has an invariant meaning. Regard $\mathbf{Y}$ as subspace of $\bigotimes^t \mathbf{X}$ (cf. 2.6). From (11), $\ker h$ is spanned by all vectors $\mathbf{c}_{e_0,e_1,\dots,e_m}$ satisfying (16). Thus $\ker h = \mathbf{Y} \cap \mathbf{M}$. This is a description "from outside $\mathbf{Y}$". For $\#F \geq t$ we may also describe that kernel "from within $\mathbf{Y}$" as subspace orthogonal to all $\mathbf{a}^{*t} \in S^t\mathbf{X}^*$ or, in projective terms, as nucleus of $\mathcal{V}_m^t$.

3.3 If $\operatorname{char} F = 0$, then the nucleus of $\mathcal{V}_m^t$ is empty by Theorem 1. In the subsequent part of this paper we assume the characteristic of F to be a prime p.

The representation of a non–negative integer $n \in \mathbb{N}$ in base p has the form $n = \sum_{\lambda \in \mathbb{N}} n_\lambda p^\lambda$ with only finitely many digits $n_\lambda \in \{0, 1, \dots, p-1\}$ different from 0. Such representations play a crucial role in the following discussion.

Theorem 2 *Let*

$$\sum_{\lambda \in \mathbb{N}} t_\lambda p^\lambda \tag{18}$$

be the representation of t *in base* $p = \operatorname{char} F > 0$. *If* $\#F \geq t$, *then the nucleus of a Veronese variety* $\mathcal{V}_m^t$ *has (projective) dimension*

$$\binom{m+t}{t} - \prod_{\lambda \in \mathbb{N}} \binom{m+t_\lambda}{t_\lambda} - 1. \tag{19}$$

Proof. There are $\binom{m+t}{t}$ base points $F\mathbf{c}_{e_0,e_1,\dots,e_m}$ of $\mathcal{P}(\mathbf{Y})$. The number of $(m+1)$–tuples $(e_0, e_1, \dots, e_m) \in E_m^t$ such that the multinomial coefficient $\binom{t}{e_0,e_1,\dots,e_m}$ is not divisible by the prime p equals

$$\prod_{\lambda \in \mathbb{N}} \binom{m+t_\lambda}{t_\lambda} = \prod_{\lambda \in \mathbb{N}} \binom{m+t_\lambda}{m}; \tag{20}$$

see [8, Theorem 3.1] or [13, Theorem 2]. This completes the proof. □

Now the following is immediate.

Corollary 1 *Let* $\mathbf{V}$ *be an* $(m+1)$*–dimensional vector space over a field* F *with characteristic* $p > 0$ *and let (18) be the representation of* $t \in \mathbb{N}$ *in base* p. *If* $\#F \geq t$, *then (20) is equal to the dimension of the subspace of the* t*–th symmetric power of* $\mathbf{V}$ *which is spanned by* $\{\mathbf{a}^t \mid \mathbf{a} \in \mathbf{V}\}$.

Theorem 2 has been established by H. Timmermann [12, 4.15] for *normal rational curves* $\mathcal{V}_1^t$. See also [4].

From (5), (15), and Lemma 1 the symmetric powers $\mathbf{a}^{*t}$ with $\mathbf{a}^* \in \mathbf{X}^*$ cannot generate $S^t\mathbf{X}^*$ when $\#F < t$. So here the nucleus of a Veronese variety $\mathcal{V}_m^t$ is non–empty. By Theorem 1, (19) gives a lower bound for the dimension of the nucleus.

3.4 We add a few remarks on multinomial coefficients. Given $d_0, d_1, \ldots, d_m \in \mathbb{N}$ with $d_0 + d_1 + \ldots + d_m \neq t$ one usually puts $\binom{t}{d_0, d_1, \ldots, d_m} := 0$.

Returning to the settings from above, choose $(e_0, e_1, \ldots, e_m) \in E_m^t$ with representations $e_i = \sum_\lambda e_{i,\lambda} p^\lambda$ in base p. Then

$$\binom{t}{e_0, e_1, \ldots, e_m} \equiv \prod_{\lambda \in \mathbb{N}} \binom{t_\lambda}{e_{0,\lambda}, e_{1,\lambda}, \ldots, e_{m,\lambda}} \pmod p \tag{21}$$

[2, 364]. For binomial coefficients this result is due to Lucas. Thus a necessary and sufficient condition for

$$\binom{t}{e_0, e_1, \ldots, e_m} \not\equiv 0 \pmod p \tag{22}$$

is that

$$t_\lambda = e_{0,\lambda} + e_{1,\lambda} + \ldots + e_{m,\lambda} \text{ for all } \lambda \in \mathbb{N}. \tag{23}$$

This means that no "carries" are made if $e_0 + e_1 + \ldots + e_m$ is calculated with digits in base p. Cf. [8, Lemma 2.1] or [13, Theorem 1].

3.5 From (23) it is easy to determine all $m, t \in \mathbb{N}$ such that (22) holds true for all $(e_0, e_1, \ldots, e_m) \in E_m^t$.

1. $m \leq 0$ or $t \leq 1$: Here (22) is always true. Recall, however, that these trivial cases have been ruled out from our discussion.

2. $2 \leq t < p$: Then (23) holds true, since $t_0 = t$ and $t_\lambda = 0$ for all $\lambda > 0$.

3. $t \geq p$ and $m = 1$: Let $J > 0$ be the highest position of a non–zero digit t_λ in (18). A binomial coefficient $\binom{t}{e_0, e_1} = \binom{t}{e_0}$ with $(e_0, e_1) \in E_1^t$ vanishes modulo p exactly if $e_{0,\lambda} > t_{0,\lambda}$ for at least one $\lambda < J$, since $e_0 \leq t$ implies $e_{0,J} \leq t_J$. So here (22) holds true for all values in E_1^t if, and only if, $t_0 = t_1 = \ldots = t_{J-1} = p - 1$, i.e., $t = t_J p^J - 1$.

4. $t \geq p$ and $m \geq 2$: Put $e_0 := 1$, $e_1 := p - 1$, $e_2 := t - p$, $e_3 = \ldots = e_m = 0$. Then (22) is not satisfied.

From this observation and the remarks at the end of 3.3, all Veronese varieties with empty nucleus are immediate.

3.6 Let $\mathbf{U}$ be an $(r + 1)$–dimensional subspace of $\mathbf{X}$. Put $\mathcal{V}_r^t$ for the Veronese image of $\mathcal{P}(\mathbf{U})$ and $\mathcal{N}_r$ for its nucleus. We may assume w.l.o.g. that $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_r$ form a basis of $\mathbf{U}$.

If $\#F \geq t$, then the nucleus $\mathcal{N}_r$ is spanned by all base points $F\mathbf{c}_{e_0,e_1,\ldots,e_m}$ subject to (16) and $e_r = e_{r+1} = \ldots = e_m = 0$. So, from Lemma 1, we obtain

$$\mathcal{N}_r = \mathcal{N} \cap \operatorname{span}(\mathcal{V}_r^t). \tag{24}$$

It should be noted here that in (24) the set $\mathcal{V}_r^t$ has to be the γ–image of a subspace and not merely a Veronese variety contained (as a point set) in $\mathcal{V}_m^t$.

Take, for example, $F = \mathrm{GF}(2)$, $m = 2$, and $r = 1$. Then $\gamma(\mathcal{P}(\mathbf{X})) = \mathcal{V}_2^2$ is a frame of the 5–dimensional projective space $\mathcal{P}(\mathbf{Y})$, i.e. a set of 7 points in general position. The nucleus of this *Veronese surface* is the plane spanned by $F\mathbf{c}_{0,1}$, $F\mathbf{c}_{0,2}$, and $F\mathbf{c}_{1,2}$. The Veronese image of a line $\mathcal{P}(\mathbf{U}) \subset \mathcal{P}(\mathbf{X})$ is a conic contained in $\mathcal{V}_2^2$. The nuclei of the seven conics that arise as γ–images of the seven lines of $\mathcal{P}(\mathbf{X})$ are given by (24). But any three points of $\mathcal{V}_2^2$ form a triangle or, in other words, a conic. Now choose a triangle Δ in $\mathcal{V}_2^2$ which is no γ–image of a line. We may suppose that Δ is the γ–image of $F\mathbf{b}_0$, $F\mathbf{b}_1$, and $F\mathbf{b}_2$. Hence $\Delta = \{F\mathbf{c}_{0,0}, F\mathbf{c}_{1,1}, F\mathbf{c}_{2,2}\}$ spans a plane skew to $\mathcal{N}$. So if Δ is regarded as a conic, then its nucleus does not arise according to (24).

This specific Veronese surface has another striking property: Each of the 7! permutations of $\mathcal{V}_2^2$ extends to a unique collineation of $\mathcal{P}(\mathbf{Y})$, since any two ordered frames determine a unique collineation. Thus, although $\mathcal{V}_2^2$ spans the entire space $\mathcal{P}(\mathbf{Y})$, there are "exceptional" automorphic collineations that do not stem from the $7 \cdot 6 \cdot 4$ collineations of $\mathcal{P}(\mathbf{X})$.

3.7 The Veronese mapping $\gamma : \mathcal{P}(\mathbf{X}) \to \mathcal{P}(\mathbf{Y})$ has the following well known properties:

(V1) γ is injective.

(V2) Each line is mapped onto a normal rational curve $\mathcal{V}_1^t$.

(V3) For each collineation κ of $\mathcal{P}(\mathbf{X})$ there is a collineation $\tilde{\kappa}$ of $\mathcal{P}(\mathbf{Y})$ with $(\tilde{\kappa} \circ \gamma)(P) = (\gamma \circ \kappa)(P)$ for all points $P \in \mathcal{P}(\mathbf{X})$.

The following example shows that (V1), (V2), and (V3) are in general not sufficient to characterize Veronese mappings to within collineations.

Let $m = 2$ and let F be an infinite field of characteristic $p = 2$. So $\mathcal{V}_2^3$ is spanning the 9–dimensional projective space $\mathcal{P}(\mathbf{Y})$. By (19), the nucleus $\mathcal{N}$ of $\mathcal{V}_2^3$ is a single point, namely $F\mathbf{c}_{1,1,1}$. Under γ the line of $\mathcal{P}(\mathbf{X})$ joining $F\mathbf{b}_0$ and $F\mathbf{b}_1$ goes over to a twisted cubic in the 3–space spanned by the four base points $F\mathbf{c}_{e_0,e_1,0}$ with $e_0 + e_1 = 3$; so that 3–space is skew to the nucleus. By (V3), these properties are shared by the γ–images of all lines. Denote by π the projection of $\mathcal{P}(\mathbf{Y})$ with centre $\mathcal{N}$ onto a complementary hyperplane. Then $\pi \circ \gamma$ satisfies (V1), (V2), and (V3). From Lemma 1, no Veronese variety $\mathcal{V}_2^s$ over F is spanning an 8–dimensional projective space.

Due to the results in 3.5, similar examples are easily found over infinite fields of any non–zero characteristic p. It is enough to let $m \geq 2$ and $t = t_J p^J - 1$ with $1 \leq t_J < p$ and $J \geq 2$. Then the Veronese mapping takes the lines of $\mathcal{P}(\mathbf{X})$ onto normal rational curves with empty nuclei, whereas the entire projective space $\mathcal{P}(\mathbf{X})$ is mapped onto a Veronese variety with a non–empty nucleus.

References

[1] Bourbaki, N., *Elements of mathematics, Algebra I*, Springer, Berlin Heidelberg New York, 1989.

[2] Brouwer, A.E., and Wilbrink, H.A., *Block designs*, in Handbook of incidence geometry, Buekenhout, F., ed., Elsevier, Amsterdam, 1995, ch. 8, pp. 349–382.

[3] Burau, W., *Mehrdimensionale projektive und höhere Geometrie*, Dt. Verlag d. Wissenschaften, Berlin, 1961.

[4] Gmainer, J., and Havlicek, H., *Nuclei of normal rational curves.* J. Geom., in print.

[5] Havlicek, H., and Zanella, C., *Quadratic embeddings*, Beitr. Algebra Geom., 38 (1997), pp. 289–298.

[6] Herzer, A., *Die Schmieghyperebenen an die Veronese–Mannigfaltigkeit bei beliebiger Charakteristik*, J. Geom., 18 (1982), pp. 140–154.

[7] Hirschfeld, J.W.P., and Thas, J.A., *General Galois geometries*, Oxford University Press, Oxford, 1991.

[8] Howard, F.T., *The number of multinomial coefficients divisible by a fixed power of a prime*, Pac. J. Math., 50 (1974), pp. 99–108.

[9] Karzel, H., *Über einen Fundamentalsatz der synthetischen algebraischen Geometrie von W. Burau und H. Timmermann*, J. Geom., 28 (1987), pp. 86–101.

[10] Thas, J.A., *Normal rational curves and $(q+2)$–arcs in a Galois space $S_{q-2,q}$* $(q = 2^h)$, Atti Accad. Naz. Lincei Rend., 47 (1969), pp. 249–252.

[11] Timmermann, H., *Descrizioni geometriche sintetiche di geometrie proiettive con caratteristica* $p > 0$, Ann. Mat. Pura Appl., IV. Ser. 114, (1977), pp. 121–139.

[12] Timmermann, H., *Zur Geometrie der Veronesemannigfaltigkeit bei endlicher Charakteristik*, Habilitationsschrift, Univ. Hamburg, 1978.

[13] Volodin, N.A., *Distribution of polynomial coefficients congruent modulo* p^N, Math. Notes, 45 (1989), pp. 195–199.