

Extending the Concept of Chain Geometry

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Dedicated to H. Reiner Salzmann on the occasion of his 70th birthday.

Abstract

We introduce the chain geometry $\Sigma(K, R)$ over a ring R with a distinguished subfield K , thus extending the usual concept where R has to be an algebra over K . A chain is uniquely determined by three of its points, if, and only if, the multiplicative group of K is normal in the group of units of R . This condition is not equivalent to R being a K -algebra. The chains through a fixed point fall into *compatibility classes* which allow to describe the residue at a point in terms of a family of affine spaces with a common set of points.

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1 Introduction

Chain geometries $\Sigma(K, R)$, where R is an associative algebra over some field K , have been investigated by many authors. They were first introduced for the case of arbitrary commutative algebras by W. Benz; compare his monograph [1]. Later, A. Herzer and others considered also algebras that are not commutative. For a survey, see [8].

All these chain geometries are so-called *chain spaces* (compare [8]). A chain space is an incidence structure $\Sigma = (\mathbb{P}, \mathfrak{C})$, consisting of a point set \mathbb{P} and a set \mathfrak{C} of certain subsets of \mathbb{P} , called *chains*, satisfying the following axioms:

CS1 Each point lies on a chain, and each chain contains at least three points.

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CS2 Any three pairwise distant points lie together on exactly one chain.

Here two points $p, q \in \mathbb{P}$ are called *distant* (denoted by $p \triangle q$), if they are different and joined by at least one chain.

CS3 For each point $p \in \mathbb{P}$, the *residue* $\Sigma_p := (\mathbb{P}_p, \mathfrak{C}_p)$, where $\mathbb{P}_p := \{q \in \mathbb{P} \mid q \triangle p\}$ and $\mathfrak{C}_p := \{\mathcal{C} \setminus \{p\} \mid p \in \mathcal{C} \in \mathfrak{C}\}$, is a *partial affine space*, i.e., an incidence structure resulting from an affine space by removing certain parallel classes of lines.

In this paper we want to discuss how things alter if the subfield K of the ring R is not necessarily contained in the center of R , and hence R is not necessarily a K -algebra. The chain geometries arising then will turn out to fulfil certain weakened versions of the axioms above: In general there is more than one chain joining three pairwise distant points. The block set of a residue Σ_p is partitioned into equivalence classes such that for each class the whole point set of Σ_p together with the blocks of this class form a partial affine space.

In the literature, so far only special cases have been discussed. See [1], IV§2, [4], [5], [6], and [10], where K and R are skew fields, and [2], where R is the ring of endomorphisms of a left vector space over the skew field K .

2 Definition and basic results

Let K be a (not necessarily commutative) field, and let R be a ring with 1 such that $K \subseteq R$ and $1_K = 1_R$. By R^* we denote the set of invertible elements (the *units*) of R . The ring R is a K -algebra exactly if the field K belongs to the *center* $Z(R) := \{z \in R \mid \forall r \in R : rz = zr\}$ of R .

We define the *chain geometry* $\Sigma(K, R)$ over (K, R) as this is done for algebras in [1] and [8].

The most important ingredient of the definition of $\Sigma(K, R)$ is the group

$$\Gamma := \mathrm{GL}_2(R)$$

of invertible 2×2 -matrices with entries in R . It acts in a natural way (from the right) on the free left R -module R^2 .

The *point set* of $\Sigma(K, R)$ is the *projective line* over the ring R . This is the orbit $\mathbb{P}(R) := R(1, 0)^\Gamma$ of the free cyclic submodule $R(1, 0) \leq R^2$ under the action of Γ . So

$$\mathbb{P}(R) = \{R(a, b) \mid \exists c, d \in R : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\}.$$

In other words, the elements of $\mathbb{P}(R)$ are exactly those free cyclic submodules of R^2 that possess a free cyclic complement. Compare [8] for basic properties of $\mathbb{P}(R)$.

Note that in the special case that the ring R is a field, this definition coincides with the usual one, namely, $\mathbb{P}(R)$ then is the set of all 1-dimensional subspaces of the vector space R^2 .

Since we assume that K is a subfield of R , the projective line $\mathbb{P}(K)$ over K can be embedded into $\mathbb{P}(R)$ via $K(k, l) \mapsto R(k, l)$. We call $\mathbb{P}(K)$ (considered as a subset of $\mathbb{P}(R)$) the *standard chain* of $\Sigma(K, R)$, and denote it by \mathcal{C} .

The *chain set* of $\Sigma(K, R)$ is the orbit

$$\mathfrak{C}(K, R) := \mathcal{C}^\Gamma.$$

Altogether, the *chain geometry* over (K, R) is the incidence structure

$$\Sigma(K, R) = (\mathbb{P}(R), \mathfrak{C}(K, R)).$$

By construction, $\Sigma(K, R)$ satisfies axiom **CS1** of a chain space.

The kernel of the action of $\Gamma = \mathrm{GL}_2(R)$ on $\mathbb{P}(R)$ is the center $Z(\Gamma)$ of Γ , which coincides with $\left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \mid z \in Z(R)^* \right\}$. So the group $\mathrm{PGL}_2(R)$ of permutations of $\mathbb{P}(R)$ induced by Γ is isomorphic with $\Gamma/Z(\Gamma)$. Since $\mathfrak{C}(K, R)$ is an orbit under Γ , the group $\mathrm{PGL}_2(R)$ consists of *automorphisms* of $\Sigma(K, R)$.

Before investigating the incidence structure $\Sigma(K, R)$, we first introduce the relation ‘distant’ on $\mathbb{P}(R)$. It can be defined on the projective line over any ring (cf. [8]). Below we will see that in our case it coincides with the relation ‘distant’ defined in **CS2**. Note that some authors consider the relation ‘not distant’ instead and call it ‘parallel’ (see, e.g., [1]).

Points $p = R(a, b)$ and $q = R(c, d)$ of $\mathbb{P}(R)$ are called *distant* ($p\Delta q$), if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\Gamma = \mathrm{GL}_2(R)$, i.e., if $(a, b), (c, d)$ is a basis of R^2 . Note that by this the relation Δ is well defined.

Just as $\mathbb{P}(R)$ and $\mathfrak{C}(K, R)$, also Δ can be considered as an orbit under Γ , namely,

$$\Delta = (R(1, 0), R(0, 1))^\Gamma. \tag{1}$$

Obviously, the relation Δ is anti-reflexive and symmetric. Moreover, if R is a field, then Δ equals the relation \neq .

This leads us to a characterization of Δ in terms of the chain geometry $\Sigma(K, R)$ (see [8], 2.4.2, for the case of algebras):

Lemma 2.1 *Let $p, q \in \mathbb{P}(R)$ be different points of $\Sigma(K, R)$. Then $p\Delta q$ holds exactly if there is a chain $\mathcal{D} \in \mathfrak{C}(K, R)$ joining p and q .*

Proof: By (1) we know that $p\Delta q$ implies $p = R(1,0)^\gamma$, $q = R(0,1)^\gamma$ for some $\gamma \in \Gamma$. Hence in this case $p, q \in \mathcal{C}^\gamma \in \mathfrak{C}(K, R)$.

Conversely, if $p, q \in \mathcal{C}^\gamma \in \mathfrak{C}(K, R)$ (with $\gamma \in \Gamma$), then $p^{\gamma^{-1}}$ and $q^{\gamma^{-1}}$ are different points of $\mathcal{C} = \mathbb{P}(K)$. On \mathcal{C} one has two ‘distance’ relations: The ordinary one on $\mathbb{P}(K)$ (which is the relation \neq), and the one inherited from $\mathbb{P}(R)$. However, one can easily check that the two relations coincide, because $\text{GL}_2(K) = \text{M}(2 \times 2, K) \cap \text{GL}_2(R)$, where $\text{M}(2 \times 2, K)$ denotes the ring of 2×2 -matrices over K . So we have $p^{\gamma^{-1}}\Delta q^{\gamma^{-1}}$. Since γ preserves Δ , this proves the assertion. \square

We now want to determine the chains through three given pairwise distant points. Note that, by Lemma 2.1, any two points on a chain must be distant.

Proposition 2.2 *Let $p, q, r \in \mathbb{P}(R)$ be pairwise distant. Then there is at least one chain $\mathcal{D} \in \mathfrak{C}(K, R)$ containing p, q , and r .*

Proof: The group Γ acts 3- Δ -transitively on $\mathbb{P}(R)$, i.e., transitively on the set of triples of pairwise distant points of $\mathbb{P}(R)$ (see [8], 1.3.3). So there exists a $\gamma \in \Gamma$ with $p = R(1,0)^\gamma$, $q = R(0,1)^\gamma$, $r = R(1,1)^\gamma$, and $\mathcal{D} := \mathcal{C}^\gamma$ is a chain through p, q , and r . \square

This means that $\Sigma(K, R)$ fulfils the existence part of axiom **CS2**. The uniqueness statement of **CS2**, however, will not hold in general.

The essential result on the group action of Γ on $\Sigma(K, R)$ is as follows:

Theorem 2.3 *Let $\mathcal{D}, \mathcal{D}' \in \mathfrak{C}(K, R)$, and let $p, q, r \in \mathcal{D}$ and $p', q', r' \in \mathcal{D}'$ be three pairwise distant points, respectively. Then there exists a $\gamma \in \Gamma$ such that $p^\gamma = p'$, $q^\gamma = q'$, $r^\gamma = r'$, and $\mathcal{D}^\gamma = \mathcal{D}'$.*

Proof: There exists a $\gamma_1 \in \Gamma$ mapping \mathcal{D} to the standard chain \mathcal{C} . Put $p_1 := p^{\gamma_1}$, $q_1 := q^{\gamma_1}$, $r_1 := r^{\gamma_1}$. The group $\text{GL}_2(K) \leq \Gamma$ leaves \mathcal{C} invariant and acts triply transitively on \mathcal{C} . Hence there is a $\gamma_2 \in \text{GL}_2(K)$ with $p_1^{\gamma_2} = R(1,0)$, $q_1^{\gamma_2} = R(0,1)$, $r_1^{\gamma_2} = R(1,1)$ (and $\mathcal{C}^{\gamma_2} = \mathcal{C}$). Define γ'_1 and γ'_2 accordingly. Then $\gamma = \gamma_1\gamma_2\gamma_2'^{-1}\gamma_1'^{-1}$ has the required properties. \square

Theorem 2.4 *Let $\Sigma = \Sigma(K, R)$ and let*

$$N := N_{R^*}(K^*) = \{n \in R^* \mid n^{-1}Kn = K\} \quad (2)$$

be the normalizer of K^ in R^* .*

Then the set of chains through a triple of pairwise distant points of Σ is in 1-1-correspondence with the set

$$R^*/N := \{Nr \mid r \in R^*\}$$

of right cosets of N .

In particular, in Σ there exists exactly one chain through each triple of pairwise distant points if, and only if, K^* is normal in R^* .

Proof: The subgroup

$$\Omega := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in R^* \right\} \cong R^* \quad (3)$$

of Γ is the stabilizer of the triple $(R(1, 0), R(0, 1), R(1, 1))$ of *standard points*. So, by Theorem 2.3 the chains through $R(1, 0)$, $R(0, 1)$, $R(1, 1)$ are exactly the images \mathcal{C}^ω , $\omega \in \Omega$. Since the stabilizer of \mathcal{C} in Ω is

$$\Omega_{\mathcal{C}} = \left\{ \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \mid n \in N \right\} \cong N, \quad (4)$$

the assertion follows for the standard points and thus, by Theorem 2.3, for any three pairwise distant points. \square

So axiom **CS2** holds in $\Sigma(K, R)$ exactly if $R^* = N$. Of course, the condition $R^* = N$ is satisfied if K belongs to the center of R , i.e., if R is a K -algebra. Hence Theorem 2.4 reconfirms that **CS2** is valid for chain geometries over algebras (compare Section 1).

However, $R^* = N$ does not necessarily mean that K is central in R . We give two examples:

Examples 2.5 (a) Let K be a non-commutative field, and let $R = K[X]$ be the polynomial ring over K in the central indeterminate X . Then $R^* = K^*$, and hence $N = R^*$. However, $K \not\subseteq Z(R) = Z(K)[X]$.

By 2.4, in $\Sigma(K, R)$ there is exactly one chain through any three given pairwise distant points. But here the situation is even more special because of the lack of units outside K^* : Any *two* distant points are joined by *exactly one* chain since the stabilizer of the pair $(R(1, 0), R(0, 1))$ belongs to $\text{GL}_2(K)$.

(b) Let $K = \text{GF}(4)$, the field with 4 elements. Then K can be represented by certain 2×2 -matrices over $F = \text{GF}(2)$, i.e., we may assume $K \subseteq R = \text{M}(2 \times 2, F)$ (with $1_K = 1_R$). Of course, $Z(R) = F \not\supseteq K$. Since $|R^*| = 6$, the group K^* has index 2 in R^* and hence is normal. This means that $N = R^*$.

Also here the situation is rather special: The chains are exactly the maximal sets of pairwise distant points.

Remark 2.6 In the following special cases the validity of $R^* = N$ implies centrality of K :

- (a) Let K and R be (not necessarily commutative) fields such that $R^* = N$ and $K \neq R$. Then $K \subseteq Z(R)$ (Cartan-Brauer-Hua, see [1], p. 323).
- (b) Let U be a left vector space over K , and let $R = \text{End}_K(U)$ be the ring of endomorphisms of U . We embed K into R with respect to a fixed basis $(b_i)_{i \in I}$ of U via $k \mapsto \lambda_k : (b_i \mapsto kb_i)$.

If now $R^* = N$ holds, and $R \neq K$ (i.e., $\dim U > 1$), then $K \subseteq Z(R)$. This can be shown by calculating $\varphi^{-1}\lambda_k\varphi$ for certain elementary transvections $\varphi \in R^*$ (see [2], proof of 4.1). In particular, in this case K is commutative and so $K = Z(R)$.

Since the kernel of the canonical epimorphism $\text{GL}_2(R) \rightarrow \text{PGL}_2(R)$ consists exactly of the elements of Ω (see (3)) where the scalar a belongs to $Z(R)^* = Z(R) \cap R^*$, we have the following:

Remark 2.7 The action of $\text{PGL}_2(R)$ on $\mathbb{P}(R)$ is *sharply* 3- Δ -transitive, exactly if the multiplicative group R^* is contained in the center $Z(R)$.

Note that the condition $R^* \subseteq Z(R)$ does not imply that R is commutative. Consider, e.g., the polynomial ring $K[X, Y]$ over a commutative field K in the *non-commuting* indeterminates X and Y .

We turn back to the example given in 2.5(b). It can be generalized to arbitrary quadratic extensions of not necessarily commutative fields (cf. [3], Section 3.6). We restrict ourselves to the finite case. Then we can count the chains through three pairwise distant points:

Example 2.8 Let q be any prime power $\neq 1$, and let $F = \text{GF}(q)$ be the field with q elements. Moreover, let $K = \text{GF}(q^2)$ and $R = \text{M}(2 \times 2, F)$. Then $K = F + Fi$ ($i \in K \setminus F$) with $i^2 = s + ti$ (for suitable $s, t \in F$). The right regular representation of K yields an embedding of K into R , namely, $a + bi \mapsto \begin{pmatrix} a & b \\ bs & a + bt \end{pmatrix}$.

We now consider the elements of R also as endomorphisms of the vector space $K^2 \supseteq F^2$. A matrix in R describes an element of K , i.e., it has the form $\begin{pmatrix} a & b \\ bs & a + bt \end{pmatrix}$, exactly if $(-i, 1)$ is one of its eigenvectors (see [5], Theorem 1). The second eigenvector then must be $(-\bar{i}, 1)$, where \bar{i} is the *conjugate* of i w.r.t. the Galois group of K/F . So N is the subgroup of R^* that leaves the set $\{K(-i, 1), K(-\bar{i}, 1)\}$ invariant.

Now the group R^* is the product of K^* with the subgroup $\left\{ \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \mid x \in F^*, y \in F \right\}$, which acts sharply transitively on the set of vectors $(z, 1)$, $z \in K \setminus F$. We conclude that $N = K^* \cdot \langle \kappa \rangle$, where κ is the unique matrix of

type $\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}$ mapping $(-i, 1)$ to $(-\bar{i}, 1)$. Obviously, κ is an involution. Hence $|N| = 2|K^*| = 2(q^2 - 1)$. Since $|R^*| = (q^2 - 1)(q^2 - q)$, this means that in $\Sigma(K, R)$ there are exactly

$$\frac{1}{2}(q^2 - q)$$

chains through three pairwise distant points. In case $q = 2$ this number equals 1, as asserted in 2.5(b).

We now determine the intersection of all chains through three pairwise distant points of a chain geometry $\Sigma(K, R)$.

Proposition 2.9 *Let $p, q, r \in \mathbb{P}(R)$ be pairwise distant. Then the intersection of all chains through p, q, r is an F -chain, i.e., the image of the projective line $\mathbb{P}(F)$ over the subfield*

$$F := \bigcap_{u \in R^*} u^{-1}Ku \quad (5)$$

of K under a suitable $\gamma \in \Gamma$.

Proof: We consider w.l.o.g. the standard points $R(1, 0)$, $R(0, 1)$, $R(1, 1)$. The chains joining them are exactly the images \mathcal{C}^ω , $\omega \in \Omega$ (compare (3)). We compute

$$\bigcap_{\omega \in \Omega} \mathcal{C}^\omega = \{R(1, 0)\} \cup \bigcap_{u \in R^*} \{R(u^{-1}ku, 1) \mid k \in K\},$$

which equals $\mathbb{P}(F)$, considered as a subset of $\mathbb{P}(R)$. \square

Of course, in (5) it suffices to let u run over a system of representatives for R^*/N . In particular, if $R^* = N$, then $F = K$. This is clear also because $R^* = N$ means that the chain through p, q, r is unique. In case $F \neq K$ the theorem of Cartan-Brauer-Hua (see 2.6(a)), applied to K , implies $F \subseteq Z(K)$. Moreover, F always contains the subfield $K \cap Z(R)$ of K . In certain cases, F and $K \cap Z(R)$ coincide:

Examples 2.10 (a) Let R be a skew field and $R \neq K$. Then $F \subseteq Z(R)$ by the theorem of Cartan-Brauer-Hua and hence $F = K \cap Z(R)$.

(b) Let K and R be as in 2.8. Then for any $u \in R^* \setminus N$ the field $K \cap u^{-1}Ku$ is a proper subfield of $K = \text{GF}(q^2)$ and hence equals $\text{GF}(q) = Z(R)$.

(c) Let $R = \text{End}_K U$ for some left vector space U over K with $\dim U > 1$. Then $F = Z(K) = Z(R)$ (compare [2]).

3 Compatibility of chains

Now we want to investigate the set of all chains through a fixed point. For $p \in \mathbb{P}(R)$ let

$$\mathbb{P}_p := \{q \in \mathbb{P}(R) \mid q \triangle p\} \quad (6)$$

and

$$\mathfrak{C}(p) := \{\mathcal{D} \in \mathfrak{C}(K, R) \mid p \in \mathcal{D}\}. \quad (7)$$

We are going to introduce an equivalence relation called *compatibility* on $\mathfrak{C}(p)$. Since the group $\mathrm{PGL}_2(R)$ acts transitively on $\mathbb{P}(R)$ and consists of automorphisms of $\Sigma(K, R)$, we may restrict ourselves to the case $p = R(1, 0)$. We shall denote this point also by the symbol ∞ .

With 2.3 we obtain that the set $\mathfrak{C}(\infty)$ consists exactly of the images of the standard chain \mathcal{C} under the group

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mid a, d \in R^*, c \in R \right\}, \quad (8)$$

which is the stabilizer of ∞ in $\Gamma = \mathrm{GL}_2(R)$.

The chains $\mathcal{B}, \mathcal{D} \in \mathfrak{C}(\infty)$ are called *compatible at ∞* (denoted by $\mathcal{B} \sim_\infty \mathcal{D}$), if they belong to the same orbit under the action of the group

$$\Delta := \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \mid a \in R^*, c \in R \right\} \trianglelefteq \Gamma_\infty \quad (9)$$

on $\mathfrak{C}(\infty)$.

By definition, compatibility is an equivalence relation on $\mathfrak{C}(\infty)$. Since Δ is normal in Γ_∞ , compatibility is invariant under the action of Γ_∞ .

The equivalence classes w.r.t. \sim_∞ are called *compatibility classes*, the compatibility class of $\mathcal{B} \in \mathfrak{C}(\infty)$ is denoted by $[\mathcal{B}]_\infty$.

For an element $\delta = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \in \Delta$ and a point $p = R(x, 1) \in \mathbb{P}_\infty$ we compute $p^\delta = R(xa + c, 1)$. This shows that the action of Δ on \mathbb{P}_∞ is *sharply 2- Δ -transitive*.

The following theorem is essential:

Theorem 3.1 *Let $\mathcal{D}, \mathcal{D}' \in \mathfrak{C}(\infty)$, and let $p, q \in \mathcal{D} \setminus \{\infty\}$ and $p', q' \in \mathcal{D}' \setminus \{\infty\}$ be different points, respectively. Moreover, let δ be the unique element of Δ with $p^\delta = p'$ and $q^\delta = q'$. Then $\mathcal{D} \sim_\infty \mathcal{D}'$ holds exactly if $\mathcal{D}' = \mathcal{D}^\delta$.*

Proof: Let $\mathcal{D} \sim_\infty \mathcal{D}'$. Then $\mathcal{D}' = \mathcal{D}^{\delta'}$ for some $\delta' \in \Delta$. Since the group $\Delta(K) := \Delta \cap \mathrm{GL}_2(K)$ acts 2-transitively on $\mathcal{C} \setminus \{\infty\}$, there is a subgroup of Δ (conjugate to $\Delta(K)$) acting 2-transitively on $\mathcal{D} \setminus \{\infty\}$. So we may w.l.o.g. assume $p^{\delta'} = p', q^{\delta'} = q'$. Uniqueness of δ implies $\delta' = \delta$ and hence $\mathcal{D}' = \mathcal{D}^\delta$. The proof of the converse is obvious. \square

Theorem 3.2 *Let $\mathcal{D} \in \mathfrak{C}(\infty)$, and let p', q' be arbitrary distant points of \mathbb{P}_∞ . Then there is a unique chain $\mathcal{D}' \in \mathfrak{C}(\infty)$ with $\mathcal{D}' \sim_\infty \mathcal{D}$ and $p', q' \in \mathcal{D}'$.*

In particular, each compatibility class in $\mathfrak{C}(\infty)$ has a unique representative through the standard points.

Proof: Choose different points $p, q \in \mathcal{D} \setminus \{\infty\}$. There is a unique $\delta \in \Delta$ such that $p^\delta = p', q^\delta = q'$. By Theorem 3.1, $\mathcal{D}' := \mathcal{D}^\delta$ is the only chain with the required properties. \square

So the set $\{\mathcal{C}^\omega \mid \omega \in \Omega\}$ (cf. (3)) of all chains through the standard points is a complete set of representatives for the compatibility classes in $\mathfrak{C}(\infty)$. By 2.4, this means that $\mathfrak{C}(\infty)/\sim_\infty = \{[\mathcal{B}]_\infty \mid \mathcal{B} \in \mathfrak{C}(\infty)\}$ is in 1-1-correspondence with R^*/N .

We shall need the following explicit description of the chains compatible at ∞ with the standard chain \mathcal{C} :

Lemma 3.3 *Let $\mathcal{D} = \mathcal{C}^\gamma \in \mathfrak{C}(\infty)$, where $\gamma = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \Gamma_\infty$. Then $\mathcal{D} \sim_\infty \mathcal{C}$ is equivalent to $d \in N$.*

Proof: The unique $\delta \in \Delta$ with $R(0, 1)^\delta = R(0, 1)^\gamma$ and $R(1, 1)^\delta = R(1, 1)^\gamma$ equals $\begin{pmatrix} d^{-1}a & 0 \\ d^{-1}c & 1 \end{pmatrix}$. By Theorem 3.1, we have $\mathcal{C} \sim_\infty \mathcal{C}^\gamma$ if, and only if, $\mathcal{C}^\gamma = \mathcal{C}^\delta$, or, in other words, if $\gamma = \omega\delta$ with $\omega = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \in \Omega_{\mathcal{C}}$. This in turn is equivalent to $d \in N$ (cf. (4)). \square

The relation of compatibility can be carried over to the set $\mathfrak{C}(p)$ of chains through an arbitrary point $p \in \mathbb{P}(R)$ in a natural way: We say that $\mathcal{B}, \mathcal{D} \in \mathfrak{C}(p)$ are *compatible at p* exactly if $\mathcal{B}^\gamma \sim_\infty \mathcal{D}^\gamma$ holds for $\gamma \in \Gamma$ with $p^\gamma = \infty$. This is independent of the choice of γ because Δ is normal in Γ_∞ .

Theorem 3.2 implies that any two different chains with at least three common points are *non-compatible* at each point of intersection.

For chains meeting only in two points the situation is different. We study the chains containing ∞ and $0 := R(0, 1)$.

Proposition 3.4 *Let \mathcal{D} be a chain through ∞ and 0 . Then $\mathcal{D} \sim_\infty \mathcal{C}$ and $\mathcal{D} \sim_0 \mathcal{C}$ holds exactly if $\mathcal{D} = \mathcal{C}^\delta$, where $\delta = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ for some $a \in N$.*

Proof: Let \mathcal{D} be a chain through ∞ and 0 compatible at ∞ with \mathcal{C} . By Theorem 3.1, we may assume $\mathcal{D} = \mathcal{C}^\delta$ for $\delta = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ (with $a \in R^*$). The chains \mathcal{C} and \mathcal{D} are compatible at 0 exactly if $\mathcal{C}^\gamma \sim_\infty \mathcal{D}^\gamma$ holds for some $\gamma \in \Gamma$ mapping 0 to ∞ . We choose $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma^{-1}$, and hence obtain $\mathcal{C}^\gamma = \mathcal{C}$ and $\mathcal{D}^\gamma = \mathcal{C}^{\gamma\delta\gamma}$. Since $\gamma\delta\gamma = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, we conclude from 3.3 that

$\mathcal{C} \sim_0 \mathcal{D}$ holds exactly if $a \in N$. \square

So, whenever there is more than one compatibility class at ∞ (i.e., in case $R^* \neq N$), one can find chains through ∞ and 0 compatible at ∞ and not compatible at 0. Of course the same holds for every other pair of distant points. In particular, compatibility cannot be considered as a global equivalence relation on the whole chain set.

4 The residue at a point

We define the residue at a point of the chain geometry $\Sigma(K, R)$ exactly as (for arbitrary incidence structures) in Section 1.

For a point $p \in \mathbb{P}(R)$ we consider the point set \mathbb{P}_p as defined in (6) and the *block set*

$$\mathfrak{C}_p := \{\mathcal{D} \setminus \{p\} \mid \mathcal{D} \in \mathfrak{C}(p)\}.$$

The incidence structure

$$\Sigma_p := (\mathbb{P}_p, \mathfrak{C}_p)$$

is the *residue of $\Sigma := \Sigma(K, R)$ at p* .

Again we may restrict ourselves to the case $p = \infty$. Each residue of Σ is isomorphic to Σ_∞ .

We compute $\mathbb{P}_\infty = \{R(x, 1) \mid x \in R\}$. We often identify the point $R(x, 1)$ with the element $x \in R$, and thus the set \mathbb{P}_∞ with R .

Next we investigate the block set \mathfrak{C}_∞ . We introduce the *standard block* of Σ_∞ , this is $C := \mathcal{C} \setminus \{\infty\}$.

The relation of compatibility is carried over to \mathfrak{C}_∞ from the set $\mathfrak{C}(\infty)$: Two blocks $B, D \in \mathfrak{C}_\infty$ are called *compatible* (at ∞), if the chains $B \cup \{\infty\}$ and $D \cup \{\infty\}$ are compatible at ∞ .

The compatibility class of the block $B \in \mathfrak{C}_\infty$ is written as $[B]_\infty$. We are going to study the incidence structure $(\mathbb{P}_\infty, [B]_\infty)$. Because of 3.2 we may always assume $B = C^\omega$ with $\omega \in \Omega$.

Theorem 4.1 *Let $u \in R^*$, $\omega = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$, $B = C^\omega$, and $\alpha : x \mapsto u^{-1}xu$. Then the following statements hold:*

- (a) *The fields K and $u^{-1}Ku$ are isomorphic subfields of R . The associated affine spaces $\mathbb{A}(K, R)$ and $\mathbb{A}(u^{-1}Ku, R)$ are isomorphic via the semilinear bijection α .*
- (b) *The incidence structure $(\mathbb{P}_\infty, [B]_\infty)$ is a partial affine space in the affine space $\mathbb{A}(uKu^{-1}, R)$.*

(c) The isomorphism $\alpha : \mathbb{A}(K, R) \rightarrow \mathbb{A}(uKu^{-1}, R)$ of affine spaces induces the isomorphism $\omega|_{\mathbb{P}_\infty} : (\mathbb{P}_\infty, [C]_\infty) \rightarrow (\mathbb{P}_\infty, [B]_\infty)$ of partial affine spaces.

Proof: (a): This is a straightforward calculation.

(b): The compatibility class $[B]_\infty$ consists of all sets

$$\{R(kua + uc, u) \mid k \in K\} = (u^{-1}Ku)a + c,$$

where $a \in R^*, c \in R$. Hence the blocks of \mathfrak{C}_∞ compatible with B are certain lines of the affine space $\mathbb{A}(u^{-1}Ku, R)$. More exactly, a line $u^{-1}Kux + y$ ($x, y \in R, x \neq 0$) of $\mathbb{A}(u^{-1}Ku, R)$ is a block of $[B]_\infty$ if, and only if, x is a unit in R .

(c) follows from (a) and (b). \square

Because of 3.3, the compatibility class $[C]_\infty$ equals the entire block set \mathfrak{C}_∞ of the residue Σ_∞ exactly if $R^* = N$. So in this case the residue Σ_∞ (and thus also every other residue of Σ) is a partial affine space, i.e., axiom **CS3** holds in Σ .

Together with 2.4 we have

Theorem 4.2 *The chain geometry $\Sigma(K, R)$ is a chain space if, and only if, the multiplicative group K^* is normal in R^* .*

The examples of 2.5 yield chain spaces $\Sigma(K, R)$ where R is not a K -algebra. For the chain space of 2.5(b) one can even show even more:

Proposition 4.3 *Let $K = \text{GF}(4)$ and $R = \text{M}(2 \times 2, \text{GF}(2))$. Then the chain space $\Sigma = \Sigma(K, R)$ is not isomorphic to any chain geometry $\Sigma(L, S)$ over some L -algebra S . Moreover, Σ cannot be embedded into any chain geometry $\Sigma(L, S)$ over a strong L -algebra S as a subspace.*

(For the definition of a subspace cf. [7] and [9], for the definition of a strong algebra, cf. [7].)

Proof: Assume first that $\Sigma \cong \Sigma(L, S)$ for some L -algebra S . Then $|\mathcal{C}| = 5$ implies $L \cong K = \text{GF}(4)$, and $|\mathbb{P}_\infty| = |R| = 16$ implies $\dim_L S = 2$. There are three types of 2-dimensional algebras over L , and the residues of the associated chain geometries are obtained by removing at most two parallel classes of lines from the affine plane over L (see [1] or [8]). However, in Σ_∞ there are only two blocks through 0, each containing 3 of the 6 elements of R^* . This means that *three* parallel classes of lines of the affine plane are missing, a contradiction.

Now assume that Σ is isomorphic to a subspace Σ' of some $\Sigma(L, S)$, where S is a strong L -algebra. Again we have $L \cong K = \text{GF}(4)$. By [7], Theorem 2, the subspace Σ' can be described by a 2-dimensional vector subspace J of S (considered as a vector space over L) which contains 1 and is closed w.r.t. squaring. This already implies that J itself is an L -algebra, and $\Sigma \cong \Sigma(L, J)$ contradicts the first assertion. \square

In the second part of this proposition we had to restrict ourselves to the case of strong algebras because otherwise the proof of the coordinatization theorem for subspaces ([7], Theorem 2) does not work. Since there seems to be no algebraic description of the subspaces of the chain geometry over an arbitrary L -algebra S , it remains open whether the chain space $\Sigma = \Sigma(K, R)$ can be embedded into some $\Sigma(L, S)$ or not.

By Theorem 4.1, the set $\mathbb{P}_\infty = R$ is the common point set of the family $\mathbb{A}(u^{-1}Ku, R)$ of affine spaces, where $u \in R^*$.

If we fix one such affine space, say $\mathbb{A}(K, R)$, then some blocks of \mathfrak{C}_∞ are lines of that space. In the following Proposition, we describe all blocks, using the fact that each block is a line in some affine space $\mathbb{A}(u^{-1}Ku, R)$.

Proposition 4.4 *Let $u \in R^*$ and let B be a block appearing as a line of $\mathbb{A}(u^{-1}Ku, R)$. Then the trace space induced on B by $\mathbb{A}(K, R)$ is isomorphic to the affine space $\mathbb{A}(F_u, K)$, where F_u is the subfield $F_u := K \cap uKu^{-1}$ of K .*

Proof: The translation groups of $\mathbb{A}(K, R)$ and $\mathbb{A}(u^{-1}Ku, R)$ are the same. Hence we may assume that B contains 0. So $B = u^{-1}Kua$ for some $a \in R^*$. One easily checks that B is a left vector space over $F_{u^{-1}} = K \cap u^{-1}Ku$. Moreover, K is a left vector space over F_u , and $\alpha : x \mapsto u^{-1}xu$ is an isomorphism $F_u \rightarrow F_{u^{-1}}$. The mapping $\iota : k \mapsto u^{-1}kua = k^\alpha a$ is a semilinear bijection $K \rightarrow B$ with accompanying isomorphism α .

We have to show that ι maps the line set of $\mathbb{A}(F_u, K)$ onto the set of all those intersections of B with lines of $\mathbb{A}(K, R)$ that contain at least two points.

Obviously, ι preserves collinearity. Now consider three (different) points $k_0^\alpha a, k_1^\alpha a, k_2^\alpha a$ of B that are collinear in $\mathbb{A}(K, R)$. Then $k_0^\alpha a = xk_1^\alpha a + (1-x)k_2^\alpha a$ holds for some $x \in K$. We compute $(k_0 - k_2)^\alpha a = x(k_1 - k_2)^\alpha a$. Since $a \in R^*$, this means $x = ((k_0 - k_2)(k_1 - k_2)^{-1})^\alpha \in K \cap K^\alpha = F_{u^{-1}}$. Hence k_0, k_1, k_2 are collinear in $\mathbb{A}(F_u, K)$. \square

A special case is the following:

Example 4.5 Let R be a quaternion skew field and let K be one of its maximal commutative subfields. Then $\dim_K(R) = 2$, i.e., $\mathbb{A}(K, R)$ is an

affine plane. All lines of $\mathbb{A}(K, R)$ appear as blocks compatible at ∞ with the standard block (since $R^* = R \setminus \{0\}$). By [6], Theorem 2, the other elements of \mathfrak{C}_∞ are *affine Baer subplanes* of $\mathbb{A}(K, R)$, i.e., isomorphic to the affine plane $\mathbb{A}(Z, K)$ over the center Z of R , which in this case coincides with the field F_u for each $u \in R \setminus Z$.

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