On the Geometry of Field Extensions

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Summary. We investigate the spread arising from a field extension and its chains. The major tool in this paper is the concept of transversal lines of a chain which is closely related with the Cartan-Brauer-Hua theorem. Provided that one chain has a "sufficiently large" number of such lines, both this chain as well as the given spread permit a simple geometric description by means of collineations.

0. Every field extension L over K gives rise to a spread together with a system of subsets called chains. Provided that K is in the centre of L these spreads and chains were investigated thoroughly within the wider concept of chain geometries: It is well known that through every point of a subspace belonging to a chain there goes a transversal line of this chain. So every chain is a Segre-manifold (regulus). See [5] for a survey of this topic.

In the present paper we investigate how things will alter when K is not necessarily a part of the centre of L.

1. For any vector space V over a (not necessarily commutative) field K, denote by $\mathcal{P}_K(V)$ the projective space on V. The same notation will be used for any subspace of V.

Let L be a field. The projective line over L is given by $\mathcal{P}_L(L \oplus L) =: \mathcal{P}_L$. If $K \neq L$ is a subfield of L, then the chains of $\mathcal{P}_L$ (with respect to K) are the images of the **standard chain**

$\{(k_0,k_1)L \mid (0,0) \neq (k_0,k_1) \in K \times K\}$

under the projective group $\mathrm{PGL}(\mathcal{P}_L)$. Cf. [1,320].

Regarding $L \oplus L$ as a right vector space over K yields the projective space $\mathcal{P}_K(L \oplus L) =: \mathcal{P}_K$. Every point $(l_0,l_1)L \in \mathcal{P}_L$ gives rise to the subspace $\mathcal{P}_K((l_0,l_1)L)$ of $\mathcal{P}_K$. All such subspaces form a spread **S** of $\mathcal{P}_K$. We shall write $\mathcal{U} := \mathcal{P}_K((1,0)L)$, $\mathcal{V} = \mathcal{P}_K((1,1)L)$, $\mathcal{W} := \mathcal{P}_K((0,1)L)$. Every chain of $\mathcal{P}_L$ gives rise to a subset of **S** which will be called a chain likewise.

2. Every projectivity π of $\mathcal{P}_L$ is induced by an L-linear f map of $L \oplus L$. But f is also K-linear, so that $\mathrm{PGL}(\mathcal{P}_L)$ corresponds to a group of automorphic projective collineations of **S** which operates 3-fold transitively on **S** and hence transitively on the set **C** of all chains in **S**. Therefore it is sufficient to discuss the geometrical properties of the standard chain **c**, say.

A line ℓ of $\mathcal{P}_K$ is called **transversal line** of a chain **k**, if $\mathcal{Y} \in \mathbf{k} \mapsto \ell \cap \mathcal{Y}$ defines a bijection of **k** onto ℓ.

THEOREM 1. *There exists a transversal line of the standard chain* **c** *passing through* $(a,0)K \in \mathcal{U}$ *if, and only if,* $a^{-1}Ka = K$.

Proof. Let ℓ be a line which intersects $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$. Then ℓ contains

$$(a,0)K \in \mathcal{U},\ (a',a')K \in \mathcal{V},\ (0,a'')K \in \mathcal{W},$$

say, with $a,a',a'' \in L^{\times} := L\setminus\{0\}$. By collinearity of these points $a = a' = a''$.

Now take any element $\mathcal{Y} = \mathcal{P}_K((1,y)L) \in \mathbf{c}$ with $y \in K^{\times}$. We deduce that ℓ has a point in common with $\mathcal{Y}$ if, and only if, there exist skalars $x_0,x_1 \in K^{\times}$, $b \in L^{\times}$ such that

$$b = ax_0,\ yb = ax_1.$$

As x_0,x_1 are right homogeneous coordinates, we may put $x_0 := 1$, whence $a = b$ and $a^{-1}ya \in K$. So ℓ intersects all elements of **c** if, and only if, $a^{-1}Ka \subset K$.

In the same manner as above $K \subset a^{-1}Ka$ can be shown to be necessary and sufficient that every point of ℓ lies in at least one element of **c**.■

Let $a \in L^{\times}$ and $a^{-1}Ka = K$. The restriction of the inner automorphism

$$\psi_a : L \to L,\ x \mapsto a^{-1}xa$$

to K induces an automorphism φ_a of K. If φ_a is inner, then $a^{-1}ya = u^{-1}yu$ for all $y \in K$ and some $u \in K^{\times}$, whence $a \in Z_L(K)^{\times}\cdot K^{\times}$, where $Z_L(K)$ denotes the centralizer of K in L. Conversely every $a \in Z_L(K)^{\times}\cdot K^{\times}$ gives rise to an inner automorphism of K.

Let $a \in K^{\times}$ and $a^{-1}Ka \subset K$. Then ψ_a restricted to K is an isomorphism φ_a of K onto a subfield of K. Clearly, φ_a is linear when K is regarded as a right vector space over $\mathrm{fix}(\varphi_a) := \{y \in K \mid \varphi_a(y)=y\}$. Provided that the right degree of K over $\mathrm{fix}(\varphi_a)$ is finite, φ_a turns out to be surjective or, in other words, an automorphism of K. Consequently such an $a \in K^{\times}$ yields a transversal line of **c**.

One special case is worth noting: If the centre $Z_L(L) =: Z$ of L is a subfield of K and $[K:Z]$ is finite then, by a theorem of Skolem and Noether (cf. e.g. [2,46]), every automorphism of K which fixes Z elementwise extends to an inner automorphism of L and hence gives rise to a transversal line of **c**.

3. We investigate the set of all transversal lines of the standard chain **c**. Obviously $\mathscr{k} := (1,0)K \vee (0,1)K$ is a transversal line of **c**. It will be called the **standard transversal line** of **c**. If ℓ is any transversal line of **c** then

$$\alpha : k \to \ell,\ k\cap\mathcal{Y} \mapsto \ell\cap\mathcal{Y} \text{ (with } \mathcal{Y} \in \mathbf{c})$$

is a well defined bijection of k onto ℓ. Suppose that ℓ carries the point $(a,0)K$. Then this α is given explicitly by

$$(1,y)K \mapsto (a,ya)K = \big(a,a(a^{-1}ya)\big)K,$$

whence $a \in Z_L(K)^{\times}\cdot K^{\times}$ characterizes α as being a projectivity. If α is a projectivity, then we shall say that k and ℓ are **projectively linked** transversal lines. This is an equivalence relation on the set of transversal lines of $\mathbf{c}$.

Theorem 2. *Suppose that ℓ_i $(i \in I)$ are transversal lines of the standard chain $\mathbf{c}$ and write $A_i := \ell_i\cap\mathcal{U}$.*

(a) If $\{A_i \mid i\in I\}$ is an r-frame, then all ℓ_i's are projectively linked.

(b) If ℓ_j and ℓ_k are not projectively linked whenever $j,k \in I$ are different, then $\{A_i \mid i\in I\}$ is an independent set of points and no other transversal line of $\mathbf{c}$ is incident with a point of $\operatorname{span}\{A_i \mid i\in I\}$.

Proof. (a) Let $\{A_i \mid i\in I\}$ be an r-frame with $I = \{0,\dots,r+1\}$, say. Hence for all $\mathcal{Y} \in \mathbf{c}$ the points of intersection $\ell_0\cap\mathcal{Y},\dots,\ell_{r+1}\cap\mathcal{Y}$ form a frame of an n-dimensional subspace of $\mathcal{Y}$. Thus the projection onto the line ℓ_1 with centre $\ell_2\vee\dots\vee\ell_{r+1}$ takes $\ell_0\cap\mathcal{Y}$ to $\ell_1\cap\mathcal{Y}$. The same argument holds for any two different lines ℓ_i and ℓ_j.

(b) Suppose that $\{A_i \mid i\in I\}$ is dependent. Then $\{A_i \mid i\in I'\}$ for some finite subset $I' \subset I$ is a frame of an r-dimensional subspace of $\mathcal{U}$ with $r \geq 1$. Thus two different transversal lines are projectively linked by (a), an absurdity.■

Theorem 3. *Denote by $\mathcal{T}$ the set of all points of $\mathcal{U}$ which are incident with a transversal line of $\mathbf{c}$ which is projectively linked with the standard transversal line k. With $\mathcal{T}$ being regarded as a trace space of $\mathcal{U}$, the projective space on the right vector space $Z_L(K)$ over the centre of K is isomorphic to $\mathcal{T}$. Moreover independence of points with respect to the trace space $\mathcal{T}$ is equivalent to independence with respect to $\mathcal{U}$.*

Proof. A bijection ι of the projective space on $Z_L(K)$ over $Z_K(K)$ onto $\mathcal{T}$ is given by

$$aZ_K(K) \mapsto (a,0)K \quad (a \neq 0).$$

Assume that for $a \in Z_L(K)$ there exist different elements $a_1,\dots,a_n \in Z_L(K)$ which are linearly independent over K such that

$$a = a_1x_1+\dots+a_nx_n \text{ with } x_i \in K^{\times}.$$

We read off from

$$ay = \sum_i a_i(x_iy) = ya = \sum_i ya_ix_i = \sum_i a_i(yx_i) \text{ for all } y \in K,$$

that all x_i's are in the centre of K. On the other hand any linear combination

$$\sum_i a_i x_i \quad \text{with } a_i \in Z_L(K)^{\times},\ x_i \in Z_K(K)$$

belongs to the centralizer of K in L.

Thus ι is collineation of the projective space on $Z_L(K)$ onto the trace space $\mathcal{T}$ and independence with respect to $\mathcal{T}$ and $\mathcal{U}$ is equivalent.■

Now suppose that transversal lines k and ℓ are not projectively linked. By theorem 2 the pedal point $(b,0)K$ of ℓ in $\mathcal{U}$ does not belong to span$\mathcal{T} \subset \mathcal{U}$. The standard chain **c** is elementwise invariant under the collineation

$$\mu_b : \mathcal{P}_K \to \mathcal{P}_K,\ (l_0,l_1)K \mapsto (l_0 b, l_1 b)K$$

and the transversal lines of **c** are permuted bijectively. The relation "projectively linked" is being preserved under μ_b as well as μ_b^{-1}. Since $\mu_b(k) = \ell$, all results established for k carry over to ℓ. As an immediate consequence of theorems 2 and 3 we state:

Theorem 4. *Let ℓ_i $(i \in I)$ be a family of transversal lines of the standard chain* **c**. *Suppose that ℓ_j and ℓ_k are not projectively linked whenever $j,k \in I$ are different. Denote by $\mathcal{T}_i \subset \mathcal{U}$ the set of all points which are incident with a transversal line of* **c** *that is projectively linked with ℓ_i. Then $\{\text{span}\mathcal{T}_i \mid i \in I\}$ is an independent set of isomorphic subspaces of $\mathcal{U}$.*

4. As an application of the previous results here is a simple geometric proof of the Cartan-Brauer-Hua theorem (cf. e.g. [1,323]):

Suppose that $K \neq L$ and $a^{-1}Ka = K$ for all $a \in L^{\times}$. Then every point of $\mathcal{U}$ is on a transversal line of **c**. By theorem 2 all transversal lines of **c** are projectively linked with k, whence $\varphi_a \in \text{Aut}(K)$, $y \mapsto a^{-1}ya$ is inner. We deduce from theorem 3 and $\mathcal{T} = \mathcal{U}$ that K is isomorphic to $Z_K(K)$. Therefore K is commutative and $\varphi_a = \text{id}_K$. Thus K lies in the centre of L, as required.

5. Denote by $\tilde{\mathcal{A}}$ the join of $\mathcal{W}$ with any point $(a,0)K \in \mathcal{U}$ and put $\mathcal{A} := \tilde{\mathcal{A}} \setminus \mathcal{W}$. We define a map

$$\rho : \mathbf{S} \setminus \{\mathcal{W}\} \to \mathcal{A},\ \mathcal{X} \mapsto \mathcal{A} \cap \mathcal{X}.$$

In algebraic terms we have $\mathcal{P}_K\big((l_0,l_1)L\big) \mapsto (a, l_1 l_0^{-1} a)K$, whence ρ is a bijection. Note that $\mathcal{A}$ is an affine space whose parallelism is given by $\mathcal{W}$ as hyperplane at infinity. Obviously $\mathcal{A}$ is isomorphic to the affine space on the vector space L over K. Those chains through $\mathcal{W}$ which have a transversal line in $\tilde{\mathcal{A}}$ are in one-one correspondence with the lines of $\mathcal{A}$.

Theorem 5. *Let $a_0 = 1$, $a_1 \in L^{\times}$ and write $\mathcal{A}_0, \mathcal{A}_1$ for the affine spaces given by $\big(\mathcal{W} \vee (0,a_0)K\big) \setminus \mathcal{W}$, $\big(\mathcal{W} \vee (0,a_1)K\big) \setminus \mathcal{W}$, respectively. The bijection*

$\beta_{01}: \mathcal{A}_0 \to \mathcal{A}_1,\ \mathcal{X}\cap\mathcal{A}_0 \mapsto \mathcal{X}\cap\mathcal{A}_1 \quad (\mathcal{X} \in \mathbf{S}\setminus\{\mathcal{W}\})$

is an affinity if, and only if, $a_1^{-1}Ka_1 = K$.

Proof. (a) If β_{01} is an affinity, then β_{01} extends to a collineation $\kappa_{01}: \tilde{\mathcal{A}}_0 \to \tilde{\mathcal{A}}_1$, say, which takes the standard transversal line $\mathcal{k}$ of $\mathbf{c}$ to a transversal line of $\mathbf{c}$ passing through $(a_1,0)K$. We obtain $a_1^{-1}Ka_1 = K$ by theorem 1.

(b) Suppose that $a_1^{-1}Ka_1 = K$. The bijection β_{01} maps $(1,l)K \in \mathcal{A}_0$ to $(a,la)K$. But $l\ (\in L) \mapsto la_1\ (\in L)$ is K-semilinear, so β_{01} is affine.■

Clearly the affine structure of $\mathcal{A}$ can be re-transferred to $\mathbf{S}\setminus\{\mathcal{W}\}$. This gives a **residual affine space** of $(\mathbf{S},\mathbf{C})$. We read off from theorem 5 that this affine structure on $\mathbf{S}\setminus\{\mathcal{W}\}$ is uniquely determined by $\mathcal{W}$ together with one chain $\mathbf{k}$ through $\mathcal{W}$ such that $\mathbf{k}\setminus\{\mathcal{W}\}$ is an affine line.

When K is commutative and the right degree of L over K is finite, say r+1, then a point model of $(\mathbf{S},\mathbf{C})$ may be found on a Grassmannian manifold. The map ρ can be extended to all r-dimensional subspaces of $\mathcal{P}_K$ whose intersection with $\tilde{\mathcal{A}}$ is precisely one point. Up to a projective collineation this extension of ρ equals the product of the Grassmann map γ with a suitable projection of the Grassmannian. The restriction of this projection to $\gamma(\mathbf{S})$ is "stereographic", since $\gamma(\mathcal{W})$ is the only point of $\gamma(\mathbf{S})$ without image. So the situation is similar to ordinary chain geometry; cf. [5,chapter 18.6.4].

THEOREM 6. *Suppose that there exists a basis* $\{(a_i,0)K \mid i\in I\}$ *of* $\mathcal{U}$ *with transversal lines* ℓ_i *of the standard chain* $\mathbf{c}$ *passing through these points, respectively. Let* $0 \in I$, $a_0 = 1$ *and write* $\tilde{\mathcal{A}}_i := \mathcal{W}\vee(a_i,0)K$. *Then there exist collineations* $\kappa_{0i}: \tilde{\mathcal{A}}_0 \to \tilde{\mathcal{A}}_i$ *such that*

$\mathcal{X} = \operatorname{span}\{\kappa_{0i}(\mathcal{X}\cap\mathcal{A}_0) \mid i\in I\}$ *for all* $\mathcal{X} \in \mathbf{c}$.

Proof. Define κ_{0i} as the extension of the affinity β_{0i} according to the proof of theorem 5. Then $\mathcal{X} = \operatorname{span}\{\kappa_{0i}(\mathcal{X}\cap\mathcal{A}_0) \mid i\in I\}$, since $\mathbf{S}$ is a spread.■

Generalizing a terminology introduced in [4] we may say that the spread $\mathbf{S}$ is generated by the family κ_{0i} of collineations. (Cf. [6,pp.299] for a similar, but nevertheless different result on pappian spreads.) If we restrict the collineations κ_{0i} to the standard transversal line $\mathcal{k}$, then we obtain a geometric description of the standard chain $\mathbf{c}$. By transformation under automorphic collineations of $\mathbf{S}$ this description carries over to any chain of the spread $\mathbf{S}$. Provided that K is in the centre of L, the conditions of theorem 6 are automatic and we have re-established the result that chains are Segre-manifolds.

It is easy to give examples of fields K,L such that the conditions of theorem 6 are met. Assume that L is a field of quaternions over a commutative field P with {1,i,j,k} denoting the usual basis of L over P.

1. Let $P = \mathbb{R}$ and $K = \mathbb{R}(i)$ a subfield of complex numbers. Putting $a_0 = 1$, $a_1 = j$ shows that **S** is a spread generated by a non-projective collineation. Cf. [4].
2. Let $P = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}(i)$. Put $a_0 := 1$, $a_1 := \sqrt{2}$, $a_2 := j$, $a_3 := j\sqrt{2}$. Here κ_{01} is projective while κ_{02} and κ_{03} are non-projective collineations.
3. Let $P = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}(i,j,k)$, viz. the quaternions over $\mathbb{Q}$. Put $a_0 := 1$, $a_1 := \sqrt{2}$. Thus κ_{01} is a projective collineation, every chain is a regulus in the sense of B. Segre [7,319] and **S** is an elliptic linear congruence of lines according to a definition given in [3].

In this last example we are already "very close" to the description of the spread **S** and its chains when K is a subfield of the centre of L.

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