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Isometries and Collineations of the Cayley Surface

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Abstract

Let F be Cayley's ruled cubic surface in a projective three-space over any commutative field K. We determine all collineations fixing F, as a set, and all cubic forms defining F. For both problems the cases $|K| = 2, 3$ turn out to be exceptional. On the other hand, if $|K| \geq 4$ then the set of simple points of F can be endowed with a non-symmetric distance function. We describe the corresponding circles, and we establish that each isometry extends to a unique projective collineation of the ambient space.

Keywords: Cayley surface, non-symmetric distance, isometry

MSC 2000: 51N25, 51N35, 51B15

1 Introduction

We investigate Cayley's ruled cubic surface F in a three-dimensional projective space over an arbitrary commutative ground field K. It is fairly obvious that "most" of the results that are known from the classical case ($K = \mathbb{R}, \mathbb{C}$) will remain valid. However, a closer look shows that the situation is sometimes rather intricate.

In Section 3 we determine all collineations of the Cayley surface. If $|K| \leq 3$ then there are "more" such collineations than in the general case. From the proof of this result it is immediate that for $|K| \leq 3$ there are non-proportional cubic forms defining F. However, that proof does not answer the question of finding all such cubic forms to within a non-zero factor. We pay attention to this question, since it governs the interplay between incidence geometry and algebraic geometry. In Section 4 we show that the number of solutions to this problem equals 64 if $|K| = 2$, two if $|K| = 3$, and one otherwise. Our first attempt was to solve this problem by "brute force" with the help of a computer algebra system. However, due to the presence of polynomial identities of high degree, we could not succeed without assuming $|K|$ being rather large. Therefore, in our current approach, we first use a lot of geometric reasoning before we enter the realm of algebra. In this way we obtain the result for $|K| \geq 3$. By virtue of a theorem due to G. TALLINI [12], it is easy to treat the remaining case $|K| = 2$.

The Cayley surface has an interesting "inner geometry" which can be based upon a distance function appearing (in the real case) in an article of H. BRAUNER [4]. In Section 5, using a completely different, purely geometrical approach, we generalize this distance function to the case of an arbitrary ground field K with more than three elements. Our distance function δ fits into the very general concepts developed by W. BENZ [1]. It is non-symmetric; this means that the distance from A to B is in general not the distance from B to A. It will be established that δ is a defining function for the group of automorphic projective collineations of the Cayley surface.

Occasionally, we shall also come across phenomena reflecting the characteristic of the ground field, like the presence of a line of nuclei in case $\operatorname{Char} K = 3$ (cf. formula (24)), or the absence of circles with more than one mid-point in case of $\operatorname{Char} K = 2$ (cf. Proposition 5.3).

2 Preliminaries

Throughout this article we consider the three-dimensional projective space $\mathbb{P}_3(K)$ over a commutative field K. The points of $\mathbb{P}_3(K)$ are the one-dimensional subspaces of the column space $K^{4\times 1}$, viz. they are of the form $K\boldsymbol{p}$ with $(0,0,0,0)^{\mathrm{T}} \neq \boldsymbol{p} = (p_0, p_1, p_2, p_3)^{\mathrm{T}} \in K^{4\times 1}$.

Let $K[X_0, X_1, X_2, X_3]$ be the polynomial ring which arises from K by adjoining independent indeterminates X_0, X_1, X_2, X_3. We shall use the shorthand $\boldsymbol{X} := (X_0, X_1, X_2, X_3)$. Each polynomial $g(\boldsymbol{X}) \in K[\boldsymbol{X}]$ determines a *polynomial function*

$$K^{4\times 1} \to K : (p_0, p_1, p_2, p_3)^{\mathrm{T}} \mapsto g(p_0, p_1, p_2, p_3). \tag{1}$$

Since K may be a finite field, it is necessary to distinguish between a polynomial and the associated polynomial function. We shall mainly be concerned with homogeneous polynomials (forms) in $K[\boldsymbol{X}]$. By virtue of (1), the subspace of homogeneous polynomials of degree one in $K[\boldsymbol{X}]$ is in bijective correspondence with the space of linear mappings $K^{4\times 1} \to K$ (the dual space of $K^{4\times 1}$), which in turn can be viewed as the row space $K^{1\times 4}$. This bijection allows to identify $K[\boldsymbol{X}]$ with the *symmetric algebra* on the row space $K^{1\times 4}$; cf., for example, [6, pp. 155–156].

We refer to [8, pp. 48–51] for those basic notions of algebraic geometry which will be used in this paper. However, our notation differs from [8], as we write, for example,

$$\mathcal{V}\big(g_1(\boldsymbol{X}), \ldots, g_r(\boldsymbol{X})\big) := \big\{K\boldsymbol{p} \in \mathbb{P}_3(K) \mid g_1(\boldsymbol{p}) = \cdots = g_r(\boldsymbol{p}) = 0\big\}$$

for the set of K-rational points of the variety given by homogeneous polynomials $g_1(\boldsymbol{X}), g_2(\boldsymbol{X}), \ldots, g_r(\boldsymbol{X}) \in K[\boldsymbol{X}]$.

The plane $\omega := \mathcal{V}(X_0)$ will be considered as *plane at infinity*, thus turning $\mathbb{P}_3(K)$ into a projectively closed affine space. Finally, let $Q_i := K(\delta_{0i}, \delta_{1i}, \delta_{2i}, \delta_{3i})^{\mathrm{T}}$, where δ_{ji} is the Kronecker delta and $i \in \{0, 1, 2, 3\}$, be the base points of the standard frame of reference.

Let us turn to *Cayley's ruled cubic surface* or, for short, the *Cayley surface*. It is, to within projective collineations, the point set $F := \mathcal{V}\big(f(\boldsymbol{X})\big)$, where

$$f(\boldsymbol{X}) := X_0X_1X_2 - X_1^3 - X_0^2X_3 \in K[\boldsymbol{X}]. \qquad (2)$$

We collect some of its properties (see [3], [4], and [10] for the classical case): The parametrization

$$K^2 \to \mathbb{P}_3(K) : (u_1, u_2) \mapsto K(1, u_1, u_2, u_1u_2 - u_1^3)^{\mathrm{T}} =: P(u_1, u_2) \qquad (3)$$

is injective, and its image coincides with $F \setminus \omega$ (the affine part of F). The intersection

$$F \cap \omega = \mathcal{V}(X_0, X_1) =: g_\infty \qquad (4)$$

is a line. By the above, the Cayley surface has $|K|^2 + |K| + 1$ points; cf. [9, Teorema 6]. Hence, in case of a finite ground field, it does not fit into the characterizations given by G. TALLINI [11]. The plane $\mathcal{V}(X_3)$ meets F along the line $\mathcal{V}(X_1, X_3)$ and the parabola

$$l := \mathcal{V}(X_0X_2 - X_1^2, X_3). \qquad (5)$$

The mapping

$$\beta : l \to g_\infty : K(s_0^2, s_0s_1, s_1^2, 0)^{\mathrm{T}} \mapsto K(0, 0, s_0, s_1)^{\mathrm{T}}, \qquad (6)$$

where $(0, 0) \neq (s_0, s_1) \in K^2$, is projective, and each point of l is distinct from its image point. Let $g(s_0, s_1)$ denote the line joining the two points given in (6). Thus, in particular, we obtain $g(0, 1) = g_\infty$.

It is immediate that every line $g(s_0, s_1)$ is a *generator* of F, i.e., it is contained in F. Conversely, let $h \subset F$ be a line. If $h \subset \omega$ then $h = g_\infty$, by (4). Otherwise, h has a unique point at infinity which necessarily belongs to g_∞. The plane which is spanned by h and g_∞ has the form $\pi = \mathcal{V}(a_0X_0 - X_1)$ with $a_0 \in K$. The intersection $F \cap \pi$ is given by

$$\mathcal{V}\left(a_0X_0 - X_1, X_0^2(a_0X_2 - a_0^3X_0 - X_3)\right), \qquad (7)$$

whence it consists of two distinct lines. This shows $F \cap \pi = g_\infty \cup h$ and $h = g(1, a_0)$.

According to (6), the line g_∞ is not only a generator of F, but also a *directrix*, as it has non-empty intersection with every generator. Each point of g_∞, except the point Q_3, is on precisely two generators of F; each affine point of F is incident with precisely one generator. Thus the projectivity (6) can be used to "generate" the Cayley surface in a purely geometric way. This is nicely illustrated in [10, p. 89] for the real projective three-space.

3 Automorphic collineations of F

Each matrix $M = (m_{ij})_{0\leq i,j\leq 3} \in \mathrm{GL}_4(K)$ acts on the row space $K^{1\times 4}$ by multiplication from the right hand side. By identifying each row vector

$(d_0, d_1, d_2, d_3) \in K^{1\times 4}$ with $d_0X_0 + d_1X_1 + d_2X_2 + d_3X_3 \in K[\boldsymbol{X}]$, the matrix M yields a linear bijection of the subspace of homogeneous polynomials of degree one; in particular,

$$X_i \mapsto \sum_{j=0}^{3} m_{ij}X_j \text{ for } i \in \{0,1,2,3\}. \tag{8}$$

By the universal property of symmetric algebras, this linear bijection extends to a K-algebra automorphism of $K[\boldsymbol{X}]$; cf., e.g., [6, p. 156]. Thus, altogether, $\mathrm{GL}_4(K)$ acts on $K[\boldsymbol{X}]$.

On the other hand, M acts on the column space $K^{4\times 1}$ by left multiplication, and therefore as a projective collineation on $\mathbb{P}_3(K)$. Given a form $g(\boldsymbol{X}) \in K[\boldsymbol{X}]$ and its image under M, say $h(\boldsymbol{X})$, this collineation takes $\mathcal{V}\big(h(\boldsymbol{X})\big)$ to $\mathcal{V}\big(g(\boldsymbol{X})\big)$, since $g(M\cdot\boldsymbol{p}) = h(\boldsymbol{p})$ for all $\boldsymbol{p} \in K^{4\times 1}$. If, moreover, $h(\boldsymbol{X}) \sim g(\boldsymbol{X})$, i.e., the polynomials are proportional by a non-zero scalar in K, then $\mathcal{V}\big(g(\boldsymbol{X})\big) = \mathcal{V}\big(h(\boldsymbol{X})\big)$.

The following result holds:

Lemma 3.1. *The set of all matrices*

$$M_{a,b,c} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & c & 0 & 0 \\ b & 3\,ac & c^2 & 0 \\ ab-a^3 & bc & ac^2 & c^3 \end{pmatrix} \tag{9}$$

where $a, b \in K$ and $c \in K^\times := K\setminus\{0\}$ is a group, say $G(F)$, under multiplication. Each matrix in $G(F)$ leaves invariant the cubic form $f(\boldsymbol{X}) = X_0X_1X_2 - X_1^3 - X_0^2X_3$ to within the factor c^3. Consequently, the group $G(F)$ acts on F as a group of projective collineations.

Proof. We obtain, for all $a, b, c, x, y, z \in K$ with $c, z \neq 0$, $M_{a,b,c}^{-1} = M_{a',b',c'}$, where $a' := -ac^{-1}$, $b' := (3a^2-b)c^{-2}$, $c' := c^{-1}$, and $M_{a,b,c}\cdot M_{x,y,z} = M_{a'',b'',c''}$, where $a'' := a+cx$, $b'' := b+3acx+c^2y$, $c'' := cz$. The rest is a straightforward calculation: By (8), the image of $f(\boldsymbol{X})$ under the action of $M_{a,b,c}$ equals $c^3f(\boldsymbol{X})$. □

Lemma 3.2. *For each automorphism $\zeta \in \mathrm{Aut}(K)$ the collineation $\mathbb{P}_3(K) \to \mathbb{P}_3(K) : K(p_0,p_1,p_2,p_3)^{\mathrm{T}} \mapsto K\big(\zeta(p_0),\zeta(p_1),\zeta(p_2),\zeta(p_3)\big)^{\mathrm{T}}$ leaves invariant the Cayley surface F.*

Proof. Observe that all coefficients of the polynomial $f(\boldsymbol{X})$ are in the prime field of K, whence they are fixed under ζ. Therefore $f(\boldsymbol{p}) = f\big(\zeta(\boldsymbol{p})\big)$ for all $\boldsymbol{p} \in K^{4\times 1}$. □

We now turn to the problem of finding all automorphic collineations of F. The following lemma is preliminary, a stronger result will be established in Theorem 5.4.

Lemma 3.3. *The group $G(F)$ acts transitively on $F \setminus g_\infty$.*

Proof. We fix the base point $Q_0 \in F \setminus g_\infty$. By (2), an arbitrarily chosen affine point of F has the form $P(u_1,u_2)$ with $(u_1,u_2) \in K^2$. Hence the matrix $M_{u_1,u_2,1}$ takes $Q_0 = P(0,0)$ to $P(u_1,u_2)$, and the assertion follows. □

We remark that $\{M_{a,b,1} \mid a,b \in K\}$ is a commutative subgroup of $G(F)$. By the previous proof, this group acts regularly on $F \setminus g_\infty$. Summing up our three lemmas, we obtain

Proposition 3.4. *Each collineation κ of $\mathbb{P}_3(K)$ which fixes the Cayley surface F can be written as $\kappa = \kappa_3 \circ \kappa_2 \circ \kappa_1$, where κ_1 is given as in Lemma 3.2, κ_2 is a projective collineation which stabilizes F and the base point $Q_0 = K(1,0,0,0)$, and κ_3 is induced by a matrix in $G(F)$.*

We are thus lead to our first main result:

Theorem 3.5. *Let $\mathrm{Stab}(F, Q_0)$ be the group of all projective collineations of $\mathbb{P}_3(K)$ which stabilize F and the base point Q_0. Depending on the ground field K, this stabilizer is determined by the following subgroups of $\mathrm{GL}_4(K)$.*

$$|K| = 2 \quad : \quad \{M_{0,0,1}, N\}, \tag{10}$$
$$|K| = 3 \quad : \quad \{M_{0,0,c}, N_c \mid c \in K^\times\}, \tag{11}$$
$$|K| \geq 4 \quad : \quad \{M_{0,0,c} \mid c \in K^\times\}, \tag{12}$$

where

$$N := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \text{ and } N_c := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & c & 0 & 2c \end{pmatrix}. \tag{13}$$

Proof. Let $\sigma \in \mathrm{Stab}(F, Q_0)$. We saw at the end of Section 2 that only the points of $g_\infty \setminus \{Q_3\}$ are on two distinct generators, whereas each other point of F is incident with one generator only. Thus $\sigma(g_\infty) = g_\infty$, and $\sigma(Q_3) = Q_3$. Also, ω is the only plane through g_∞ which does not contain a second generator, so that $\sigma(\omega) = \omega$. Also, since Q_0 is fixed, so is the only generator $g(1,0)$ through this point, whence $g(1,0) \cap g_\infty$, i.e. the base point Q_2, is fixed too. Consequently, σ is induced by a lower triangular matrix

$$(x_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 \\ 0 & x_{21} & x_{22} & 0 \\ 0 & x_{31} & 0 & x_{33} \end{pmatrix} \in \mathrm{GL}_4(K).$$

It remains to determine the unknown entries of this matrix, where obviously

$$\det(x_{ij}) = x_{11} x_{22} x_{33} \neq 0. \tag{14}$$

First, fix a scalar $t \in K^\times$ and consider the generator $g(1,t)$. There is an $s \in K^\times$ such that $\sigma\big(g(1,t)\big) = g(1,s)$. Thus for each $\lambda \in K$ exists an element $\mu \in K$ with $\sigma(K(1,t,t^2+\lambda,\lambda t)^{\mathrm{T}}) = K(1,s,s^2+\mu,\mu s)^{\mathrm{T}}$. So

$$x_{11} t = s, \tag{15}$$
$$x_{21} t + x_{22}\left(t^2 + \lambda\right) = s^2 + \mu, \tag{16}$$
$$x_{31} t + x_{33} \lambda t = \mu s. \tag{17}$$

We divide (17) by s, subtract it then from (16), and substitute $s = x_{11} t$ according to (15). Hence

$$x_{21} t + x_{22} t^2 - \frac{x_{31}}{x_{11}} + \left(x_{22} - \frac{x_{33}}{x_{11}}\right) \lambda = x_{11}^2 t^2 \text{ for all } \lambda \in K. \tag{18}$$

This implies

$$x_{33} = x_{11}x_{22}. \tag{19}$$

Next, we assume t to be variable, whence (18) gives

$$\left(x_{22} - x_{11}^2\right) t^2 + x_{21}t - \frac{x_{31}}{x_{11}} = 0 \text{ for all } t \in K^\times. \tag{20}$$

According to the cardinality of K there are three cases:

$|K| = 2$: By (14), $x_{11} = x_{22} = x_{33} = 1$ and (20) reads $x_{21} \cdot 1 = x_{31}$, so that $(x_{ij}) = M_{0,0,1}$ or $(x_{ij}) = N$.

$|K| = 3$: Then $x_{11}^2 = 1 = t^2$ for $t \in \{1, 2\} = K^\times$, and there are two possibilities: (i) $x_{22} = 1$, whence (20) reads $x_{21}t - x_{31}/x_{11} = 0$ for $t \in \{1, 2\}$, so that $x_{21} = x_{31} = 0$, and $(x_{ij}) = M_{0,0,c}$ with $c := x_{11}$. (ii) $x_{22} = 2$, whence (20) turns into $1 \cdot 1 + x_{21}t - x_{31}/x_{11} = 0$ for $t \in \{1, 2\}$, so that $x_{21} = 0$, $x_{31} = x_{11}$, and $(x_{ij}) = N_c$ with $c := x_{11}$.

$|K| \geq 4$: From $|K^\times| \geq 3$ and (20) follows $x_{22} = x_{11}^2$, $x_{21} = x_{31} = 0$, and $(x_{ij}) = M_{0,0,c}$ with $c := x_{11}$.

In either case it is easy to see that the given matrices form a subgroup of $\mathrm{GL}_4(K)$. □

We denote by $G_{\text{ext}}(F)$ the *extended group* of the Cayley surface, i.e. the group of *all* matrices $(x_{ij})_{0 \leq i,j, \leq 3} \in \mathrm{GL}_4(K)$, subject to the condition $x_{00} = 1$, leaving invariant the Cayley surface F. By the first paragraph of the previous proof, each automorphic projective collineation of F is induced by precisely one matrix in $G_{\text{ext}}(F)$. Furthermore, for $|K| \geq 4$, we have $G(F) = G_{\text{ext}}(F)$, whereas for $|K| \leq 3$ the groups $G(F)$ and $G_{\text{ext}}(F)$ are distinct, since none of the matrices N and N_c is in $G(F)$. We are thus lead to the following result:

Proposition 3.6. *Let*

$$f_{(|K|)}(\boldsymbol{X}) := \begin{cases} X_0X_1^2 + X_0X_1X_2 + X_1^3 + X_0^2X_1 + X_0^2X_3 \text{ when } |K| = 2, \\ 2X_0X_1X_2 + 2X_1^3 + 2X_0^2X_1 + X_0^2X_3 \text{ when } |K| = 3. \end{cases} \tag{21}$$

Then, for $|K| \leq 3$, the Cayley surface $F = \mathcal{V}\big(f(\boldsymbol{X})\big)$ coincides with $\mathcal{V}\big(f_{(|K|)}(\boldsymbol{X})\big)$.

Proof. Let $|K| = 2$. The image of $f(\boldsymbol{X})$ under the action of N gives the polynomial $f_{(2)}(\boldsymbol{X})$. Likewise, for $|K| = 3$, the polynomial $f_{(3)}(\boldsymbol{X})$, multiplied by $c \in \{1, 2\}$, arises as the image of $f(\boldsymbol{X})$ under the action of N_c. □

Observe that here “to coincide” just refers to sets of points and not to algebraic varieties in the sense of [8, p. 48]. Thus, for $|K| \leq 3$, the point set of the Cayley surface F may also be considered as the algebraic curve $\mathcal{V}\big(f(\boldsymbol{X}), f_{|K|}(\boldsymbol{X})\big)$.

4 All cubic forms defining F

In discussing the Cayley surface F we have to distinguish between properties which stem from the defining polynomial $f(\boldsymbol{X})$ and geometric properties, i.e., properties which are invariant with respect to the action of the group $G_{\text{ext}}(F)$.

First, we recall some notions which can be defined in terms of $f(\boldsymbol{X})$. Let $\partial_i := \frac{\partial}{\partial X_i}$. We start by calculating the partial derivatives

$$\begin{array}{rclrcl} \partial_0 f(\boldsymbol{X}) & = & X_1X_2 - 2X_0X_3, & \partial_1 f(\boldsymbol{X}) & = & X_0X_2 - 3X_1^2, \\ \partial_2 f(\boldsymbol{X}) & = & X_0X_1, & \partial_3 f(\boldsymbol{X}) & = & -X_0^2. \end{array} \tag{22}$$

They vanish simultaneously at $(p_0, p_1, p_2, p_3)^{\mathrm{T}} \in K^{4\times 1}$ if, and only if, at least one of the following conditions holds:

$$p_0 = p_1 = 0; \tag{23}$$

$$p_0 = p_2 = 0 \text{ and } \operatorname{Char} K = 3. \tag{24}$$

When K is a field of characteristic $\operatorname{Char} K = 3$ then, by (24), $\mathcal{V}(X_0, X_2)$ is a distinguished line in the ambient space of the Cayley surface F. Each of its points is a *nucleus* of F. See [8, p. 50] and [5, Proposition 3.17], where nuclei are defined in a slightly different way. All points subject to (23) are *singular*; they comprise the line $g_\infty \subset F$. We obtain, for all $s_2, s_3 \in K$, that

$$f\big((0,0,s_2,s_3) + T\boldsymbol{X}\big) = T^2 X_0(s_2X_1 - s_3X_0) + T^3(*) \in K[\boldsymbol{X}, T].$$

Hence all points of g_∞ are *double points*. The *tangent cone* (see [8, p. 49], where the term *tangent space* is used instead) at a point $Y = K(0,0,s_2,s_3)^{\mathrm{T}}$, $(s_2, s_3) \neq (0,0)$, is

$$\mathcal{V}\big(X_0(s_2X_1 - s_3X_0)\big), \tag{25}$$

whence we refer to the plane at infinity as the *tangent plane* at $Y = Q_3$. For $Y = K(0,0,1,s_3)^{\mathrm{T}}$ the tangent cone is the union of the plane at infinity and the plane spanned by g_∞ and the generator $g(1, s_3)$. We call each of these planes a *tangent plane* at Y. By (22), all points of $F \setminus g_\infty$ are *simple*. The *tangent plane* at $P(u_1, u_2)$ (see (3)) equals

$$\mathcal{V}\left((2u_1^3 - u_1u_2)X_0 + (-3u_1^2 + u_2)X_1 + u_1X_2 - X_3\right). \tag{26}$$

Next, we present a characterization of tangent planes:

Proposition 4.1. *Let $\tau = \mathcal{V}\left(\sum_{i=0}^{3} a_iX_i\right)$, where $a_i \in K$, be a plane. Then the following assertions are equivalent.*

(a) *τ is a tangent plane of F with respect to $f(\boldsymbol{X})$.*

(b) *The coefficients a_i satisfy the equation $a_0a_3^2 - a_1a_2a_3 + a_2^3 = 0$.*

(c) *τ contains a generator of F.*

Proof. (a) $\Rightarrow$ (b): If τ is the tangent plane at an affine point of F then the coefficients a_i are proportional to the coefficients of a polynomial as in (26), otherwise we obtain $a_2 = a_3 = 0$. In any case (b) holds.

(b) $\Rightarrow$ (c): If $a_3 = 0$ then so is a_2. Consequently $g_\infty \subset \tau$. If $a_3 \neq 0$ then we may let w.l.o.g. $a_3 = -1$, whence $g(1, a_2) \subset \tau$.

(c) $\Rightarrow$ (a): This is immediate from (25) and (26). $\square$

By (b), the set of all tangent planes with respect to $f(\boldsymbol{X})$ is a Cayley surface in the dual projective space. In view of (26), it is somewhat surprising that this holds irrespective of the characteristic of K.

Clearly, the notions from the above are not independent of the homogeneous polynomial which is used for defining F. For example, we have $F = \mathcal{V}\big(f(\boldsymbol{X})^2\big)$, but no point of F is simple with respect to $f(\boldsymbol{X})^2$. However, by restricting ourselves to cubic forms defining F, we obtain the next two theorems.

Theorem 4.2. *Let $|K| \geq 3$ and suppose that $p(\boldsymbol{X}) \in K[\boldsymbol{X}]$ is a cubic form such that $\mathcal{V}\big(p(\boldsymbol{X})\big)$ equals the Cayley surface $F = \mathcal{V}\big(f(\boldsymbol{X})\big)$. Then $p(\boldsymbol{X}) \sim f(\boldsymbol{X})$ or, only for $|K| = 3$, $p(\boldsymbol{X}) \sim f_3(\boldsymbol{X})$, where $f_3(\boldsymbol{X})$ is given by (21).*

Proof. (a) Suppose that $p(\boldsymbol{X}) = \sum_{0\leq i\leq j\leq k\leq 3} a_{ijk}X_iX_jX_k$ is a form of degree three such that $\mathcal{V}\big(p(\boldsymbol{X})\big) = F$. We aim at finding the twenty coefficients a_{ijk} to within a common non-zero factor, and we adopt the following convention: Within this proof, concepts like "simple point", "double point", "intersection multiplicity", and "tangent plane" are tacitly understood with respect to $p(\boldsymbol{X})$, unless explicitly stated otherwise.

Obviously, $Q_0, Q_2, Q_3 \in F$, whereas $Q_1 \notin F$. Hence

$$a_{111} \neq a_{000} = a_{222} = a_{333} = 0.$$

The line $g(1,0)$ is on F, whence $p\big((1,0,t,0)^{\mathrm{T}}\big) = t(a_{002} + a_{022}t) = 0$ for all $t \in K$. From $|K| \geq 3$, we obtain

$$a_{002} = a_{022} = 0.$$

Likewise, $g_\infty \subset F$ forces

$$a_{223} = a_{233} = 0.$$

(b) We proceed by establishing four auxiliary results:

(I) *Each affine point $Y \in F$ is simple.* It suffices to show that there exists a line $m \ni Y$ such that $|m \cap F| = 3$, since such a line meets F at Y with multiplicity one. First, let $Y = Q_0$. We consider the line m_0, say, joining Q_0 and $K(0,1,\alpha+1,\alpha)^{\mathrm{T}} \notin F$, where $\alpha \in K \setminus \{0,1\}$; such an α exists by $|K| \geq 3$. The intersection $m_0 \cap F$ equals the set of all points $K\big(1,\xi,\xi(\alpha+1),\xi\alpha\big)^{\mathrm{T}}$, $\xi \in K$, with

$$f\big((1,\xi,\xi(\alpha+1),\xi\alpha)^{\mathrm{T}}\big) = \xi^2(\alpha+1) - \xi^3 - \xi\alpha = -\xi(\xi-1)(\xi-\alpha) = 0.$$

Hence m_0 has the required property. By the transitive action of $G(F)$ on $F \setminus g_\infty$, the assertion follows for all $Y \in F \setminus g_\infty$.

(II) *$\mathcal{V}(X_3)$ is the tangent plane of at least one affine point, say R, on $g(1,0)$.* We know from (5) that $F \cap \mathcal{V}(X_3)$ is the union of the generator $g(1,0)$ and the parabola l. Hence $p(\boldsymbol{X}) = q(\boldsymbol{X})X_1 + X_3(*)$, where

$$q(\boldsymbol{X}) = a_{001}X_0^2 + a_{011}X_0X_1 + a_{012}X_0X_2 + a_{111}X_1^2 + a_{112}X_1X_2 + a_{122}X_2^2.$$

The planar quadric $\widetilde{l} := \mathcal{V}(q(\boldsymbol{X}), X_3)$ and the parabola l have the same points outside the line $g(1,0)$. There are at least two such points because of $|l| = |K| + 1 \geq 4$. Therefore $|\widetilde{l}| \geq 2$. It is well known that a planar quadric with at least two points is either a (non-degenerate) conic, a pair of distinct lines, or a repeated line. As $g(1,0)$ is the only line contained in $F \cap \mathcal{V}(X_3)$, we see that $\widetilde{l}$ has to be a conic. If $|K|$ is finite then $|l| = |\widetilde{l}|$ implies that $|g(1,0) \cap \widetilde{l}| = 2$; thus we can choose a point $R \in \big(g(1,0) \cap \widetilde{l}\,\big) \setminus \omega$. If $|K|$ is infinite then l and $\widetilde{l}$ have infinitely many common points outside $g(1,0)$. So we obtain $l = \widetilde{l}$, and we let $R := Q_0$. In any case, the tangent plane of F at R contains the tangent of $\widetilde{l}$ at R (which is also a tangent of F with respect to $p(\boldsymbol{X})$), and the generator $g(1,0)$. As these two lines do not coincide, the tangent plane of F at R is $\mathcal{V}(X_3)$.

(III) *The tangent plane at each affine point of F does not pass through Q_3.* The planar section $F \cap \mathcal{V}(X_1)$ consists precisely of the two lines $g(1,0)$ and g_∞. Therefore

$$p(\boldsymbol{X}) = X_0 \underbrace{(a_{003}X_0 + a_{023}X_2 + a_{033}X_3)}_{=:\, r(\boldsymbol{X})} X_3 + X_1(*),$$

where $r(\boldsymbol{X}) \sim X_0$ or $r(\boldsymbol{X}) \sim X_3$, whence

$$a_{023} = 0. \tag{27}$$

Moreover, precisely one of the coefficients a_{003} and a_{033} vanishes. We claim that

$$a_{003} \neq a_{033} = 0. \tag{28}$$

Assume to the contrary that $a_{003} = 0 \neq a_{033}$. Hence we would have $p(\boldsymbol{X}) = a_{033}X_0X_3^2 + X_1(*)$. Then the line joining Q_3 with the point $R \in g(1,0)$ from (II) would meet F at R with multiplicity two, whence the tangent plane at the simple point R would be $\mathcal{V}(X_1)$, a contradiction to (II).

Next, choose any affine point $Y \in g(1,0)$. The line Q_3Y meets F at Y with multiplicity one, due to (27) and (28). Thus it is not a tangent, and the assertion follows for all affine points of $g(1,0)$.

Finally, consider an arbitrary affine point Y of F. By Lemma 3.3, there exists a matrix $M_{a,b,c} \in G(F)$ taking Q_0 to Y. Let $\widetilde{p}(\boldsymbol{X})$ be the image of $p(\boldsymbol{X})$ under the action of $M_{a,b,c}$. So we obtain $\mathcal{V}\big(\widetilde{p}(\boldsymbol{X})\big) = F$. From the above, applied to the cubic form $\widetilde{p}(\boldsymbol{X})$, we infer that the $\widetilde{p}(\boldsymbol{X})$-tangent plane of F at Q_0 does not pass through Q_3, whence the tangent plane of F at Y does not pass through $Q_3 = \kappa(Q_3)$ either.

(IV) *All points $Z \in g_\infty$ are double points of F.* Let Y be an affine point of F and $Z \in g_\infty$. The line YZ is either a generator of F, or we have $YZ \cap F = \{Y, Z\}$; cf. formula (7). If $YZ \notin F$ then, by (III), YZ meets F at Y and Z with multiplicities one and two, respectively. As Y varies in $F \setminus g_\infty$, the lines YZ generate the whole space. Thus Z cannot be a simple point.

(c) The planar section $F \cap \mathcal{V}(X_0)$ equals the line g_∞. By (IV), all points of g_∞ are double points. Thus each line at infinity $\neq g_\infty$ meets F at a point of g_∞ with multiplicity three. So

$$X_1^3 \sim a_{111}X_1^3 + a_{112}X_1^2X_2 + a_{113}X_1^2X_3 + a_{122}X_1X_2^2 + a_{123}X_1X_2X_3 + a_{133}X_1X_3^2,$$

whence

$$a_{111} \neq a_{112} = a_{113} = a_{122} = a_{123} = a_{133} = 0.$$

Now we consider the line through Q_3 and a point $P(u_1, u_2)$, where $(u_1, u_2) \in K^2$. Since Q_3 is a double point of F, the intersection multiplicity at $P(u_1, u_2)$

equals one. This implies, for all $(u_1, u_2) \in K^2$,

$$\begin{aligned} T &\sim p\left(1, u_1, u_2, (u_1u_2 - u_1^3) + T\right)^{\mathrm{T}} \\ &= wT + a_{001}u_1 + a_{011}u_1^2 + a_{012}u_1u_2 + a_{111}u_1^3 + w(u_1u_2 - u_1^3) \in K[T], \end{aligned}$$

where we used the shorthand $w := a_{003} + a_{013}u_1$. Since w must not vanish, we obtain

$$a_{013} = 0.$$

We now substitute $u_1 = 1$ in the constant term of $p\big(1, u_1, u_2, (u_1u_2 - u_1^3) + T\big)^{\mathrm{T}}$. Hence

$$(a_{003} + a_{012})u_2 + a_{001} - a_{003} + a_{011} + a_{111} = 0 \text{ for all } u_2 \in K,$$

so that

$$a_{012} = -a_{003}.$$

Altogether, the constant term of $p\big(1, u_1, u_2, (u_1u_2 - u_1^3) + T\big)^{\mathrm{T}}$ yields the identity

$$(-a_{003} + a_{111})u_1^3 + a_{011}u_1^2 + a_{001}u_1 = 0 \text{ for all } u_1 \in K. \tag{29}$$

There are two cases:

If (29) holds trivially then $a_{111} = a_{003} \neq 0$, $a_{011} = a_{001} = 0$, and $p(\boldsymbol{X}) \sim f(\boldsymbol{X})$. This has to be the case when $|K| \geq 4$.

If (29) is a non-trivial identity in u_1 then, of course, $|K| = 3$. Up to a factor ± 1, $T^3 + 2T \in K[T]$ is the only cubic polynomial which vanishes for all elements of K. So we let $a_{001} := 2$, whence $a_{111} = 1 + a_{003}$. However, a_{111} and a_{003} must not be zero. Thus, finally, we arrive at $a_{111} = 2$ and $a_{003} = 1$, as required. □

In the proof from the above we repeatedly used the assumption $|K| \geq 3$. If it is dropped then the situation changes drastically.

Theorem 4.3. *Let $|K| = 2$ and let $p(\boldsymbol{X}) \in K[\boldsymbol{X}]$ be a cubic form. The Cayley surface $F = \mathcal{V}\big(f(\boldsymbol{X})\big)$ coincides with $\mathcal{V}\big(p(\boldsymbol{X})\big)$ if, and only if,*

$$f(\boldsymbol{X}) - p(\boldsymbol{X}) = \sum_{0 \leq i < j \leq 3} b_{ij}(X_i^2 X_j + X_i X_j^2) \text{ with } b_{ij} \in K = \{0, 1\}. \tag{30}$$

Proof. Because of $K = \{0, 1\}$, $\mathcal{V}\big(f(\boldsymbol{X})\big) = \mathcal{V}\big(p(\boldsymbol{X})\big)$ holds precisely when the cubic form $f(\boldsymbol{X}) - p(\boldsymbol{X}) \in K[\boldsymbol{X}]$ yields the zero function on $K^{4\times 1}$. By a result of G. Tallini [12, formula (1)], a cubic form in $K[\boldsymbol{X}]$ has that property if, and only if, it is given as in (30). □

By the above, we obtain 64 cubic forms $p(\boldsymbol{X})$ for $|K| = 2$, and we refrain from a further discussion.

If $|K| \leq 3$ then each of the polynomials $f(\boldsymbol{X})$ and $f_{|K|}(\boldsymbol{X})$ yields the same simple (double) points and the same *set* of tangent planes for F. This is in accordance with the characterization of tangent planes in Proposition 4.1. However, for each simple point the two polynomials yield *distinct* tangent planes.

If $|K| \geq 4$ then, by following ideas from the proof of (II), it is easy to recover the unique point of tangency of a plane τ containing a generator $g(1, s)$, $s \in K$, but not the point Q_3: We know $\tau \cap F = g(1, s) \cup k$, where k is a parabola. This k is uniquely determined by F, because $g(1, s) \cap \omega$ is its only point at infinity, and because there are at least three points of k outside $g(1, s)$. Thus k meets $g(1, s)$ residually at the unique point of F with tangent plane τ.

5 Isometries of the Cayley surface F

We shall assume $|K| \geq 4$ throughout this section.

Two (possibly identical) points of $F \setminus g_\infty$ are said to be *parallel* if they are on a common generator of F. This parallelism is an equivalence relation; it will be denoted by $\|$.

Let $A = P(u_1, u_2)$ and $B = P(v_1, v_2)$, where $u_1, u_2, v_1, v_2 \in K$, be non-parallel points of $F \setminus g_\infty$. Thus $u_1 \neq v_1$. The points of intersection of the line AB and F are in one-one correspondence with the zeros in K of the polynomial

$$\begin{aligned} & f\big((1-T)(1, u_1, u_2, u_1u_2 - u_1^3) + T(1, v_1, v_2, v_1v_2 - v_1^3)\big) \\ = \quad & T(T-1)(u_1 - v_1)\big((u_1 - v_1)^2 T - 2u_1^2 + u_2 + u_1v_1 - v_2 + v_1^2\big) \in K[T], \end{aligned}$$

taking into account multiplicities. We read off that those zeros are $0, 1$, and

$$\delta(A, B) := \frac{2u_1^2 - u_2 - u_1v_1 + v_2 - v_1^2}{(u_1 - v_1)^2}\,. \tag{31}$$

So $AB \cap F = \{A, B, C\}$ where, in terms of a cross ratio (CR), the point C is given by

$$\mathrm{CR}(C, B, A, I) = \delta(A, B) \text{ with } \{I\} := AB \cap \omega. \tag{32}$$

If $AB \cap F = \{A, B\}$, i.e. when $\delta(A, B) \in \{0, 1\}$, our definition of C is based upon the intersection multiplicity of AB at A and B. This can be avoided as follows: By the last remark of the previous section, it is possible to decide in a purely geometric way whether AB lies in the tangent plane of F at A, whence $C = A$, or at B, whence $C = B$. (For this reason we adopted the assumption $|K| \geq 4$.)

Moreover, we define $\delta(A, B) = \infty$ whenever $A \parallel B$. So we are in a position to regard δ as a *distance function*

$$\delta : (F \setminus g_\infty) \times (F \setminus g_\infty) \to K \cup \{\infty\}.$$

It turns the affine part of the Cayley surface into a *distance space* in the sense of W. Benz [1, p. 33]. We obtain

$$\delta(A, A) = \infty \text{ and } \delta(A, B) = 1 - \delta(B, A) \text{ for all } A, B \in F \setminus g_\infty, \tag{33}$$

provided that we set $1 - \infty := \infty$. This distance function can be found in a paper by H. Brauner [4, p. 115] for $K = \mathbb{R}$ in a slightly different form. In terms of our δ, Brauner's distance function can be expressed as

$$\widehat{\delta}(A, B) := \frac{3}{2}\left(\frac{1}{2} - \delta(A, B)\right)^{-1};$$

the easy verification is left to the reader. However, the approach in [4] is completely different, using differential geometry and Lie groups. A major advantage of $\widehat{\delta}$ is that instead of (33) one obtains the much more suggestive formulas $\widehat{\delta}(A, A) = 0$ and $\widehat{\delta}(A, B) = -\widehat{\delta}(B, A)$. Since we do not want to impose any restriction on the characteristic of the ground field, it is impossible for us to make use of that function $\widehat{\delta}$.

Given a point $A \in F \setminus g_\infty$ and an element $\rho \in K \cup \{\infty\}$ we define the *circle* with midpoint A and radius ρ in the obvious way as

$$\mathcal{C}(A, \rho) := \{Y \in F \setminus g_\infty \mid \delta(A, Y) = \rho\}.$$

By the *extended circle* $\mathcal{E}(A, \rho)$ we mean the circle $\mathcal{C}(A, \rho)$ together with its midpoint A.

If $\rho = \infty$ then $\mathcal{C}(A, \rho) = \mathcal{E}(A, \rho)$ is the generator of F through A, but without its point at infinity. In order to describe the remaining circles, let us introduce, for $\alpha, \beta, \gamma \in K$, the rationally parameterized curve

$$\mathcal{R}_{\alpha,\beta,\gamma} := \left\{K(1, t, \alpha + \beta t + (\gamma + 1)t^2, \alpha t + \beta t^2 + \gamma t^3)^{\mathrm{T}} \mid t \in K \cup \{\infty\}\right\}, \quad (34)$$

lying on F. It is a parabola for $\gamma = 0$, a planar cubic for $\gamma = -1$, and a twisted cubic parabola (i.e. a twisted cubic having the plane at infinity as an osculating plane) otherwise.

Lemma 5.1. *Let $P(u_{1i}, u_{2i})$, $u_{ji} \in K$ with $i \in \{1, 2, 3\}$, be three mutually non-parallel points of $F \setminus g_\infty$. Then there is a unique triad $(\alpha, \beta, \gamma) \in K^3$ such that the curve $\mathcal{R}_{\alpha,\beta,\gamma}$ contains the three given points.*

Proof. By Lagrange's interpolation formula, there is a unique triad $(\alpha, \beta, \gamma) \in K^3$ such that $u_{2i} = \alpha + \beta u_{1i} + (\gamma + 1)u_{1i}^2$ for $i \in \{1, 2, 3\}$. Hence the assertion follows. □

We add in passing that $F \setminus g_\infty$ together with the affine traces of the curves (34) is isomorphic to the affine chain geometry on the ring $K[\varepsilon]$ of dual numbers over K. An isomorphism is given by $P(u_1, u_2) \mapsto u_1 + \varepsilon u_2$. The interested reader should compare with [7, p. 796].

Next we describe circles and extended circles:

Proposition 5.2. *Suppose that a point $A = P(a_1, a_2)$, $a_1, a_2 \in K$, and $\rho \in K$ are given. Let*

$$\alpha := (\rho - 2)a_1^2 + a_2,\ \beta := (1 - 2\rho)a_1,\ \gamma := \rho. \quad (35)$$

Then (α, β, γ) is the only triad in K^3 such that the curve $\mathcal{R}_{\alpha,\beta,\gamma}$ contains the circle $\mathcal{C}(A, \rho)$. Moreover, the extended circle $\mathcal{E}(A, \rho)$ equals the set of affine points of $\mathcal{R}_{\alpha,\beta,\gamma}$.

Proof. Let $Y = P(u_1, u_2) \in F \setminus g_\infty$, where $u_1, u_2 \in K$. Using (31) for $Y \nparallel A$, and a direct verification otherwise, shows that $Y \in \mathcal{E}(A, \rho)$ if, and only if,

$$2a_1^2 - a_2 - a_1 u_1 + u_2 - u_1^2 = \rho(a_1 - u_1)^2$$

which in turn is equivalent to

$$u_2 = (\rho - 2)a_1^2 + a_2 + (1 - 2\rho)a_1 u_1 + (1 + \rho)u_1^2.$$

So $\mathcal{E}(A, \rho) = \mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega$, with α, β, γ as in (35). The uniqueness of (α, β, γ) is immediate from $|\mathcal{C}(A, \rho)| = |\mathcal{R}_{\alpha,\beta,\gamma}| - 2 = |K| - 1 \geq 3$ and Lemma 5.1. □

Proposition 5.3. *Given a curve $\mathcal{R}_{\alpha,\beta,\gamma}$, with $\alpha, \beta, \gamma \in K$, there are three possibilities.*

(a) $1-2\gamma \neq 0$: $\mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega$ *coincides with the extended circle* $\mathcal{E}(A,\rho)$, *where*

$$A := P\left(\frac{\beta}{1-2\gamma}, \alpha - \frac{(\gamma-2)\beta^2}{(1-2\gamma)^2}\right) \text{ and } \rho := \gamma. \tag{36}$$

(b) $1-2\gamma = 0 \neq \beta$: $\mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega$ *is not an extended circle.*

(c) $1-2\gamma = 0 = \beta$: $\mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega$ *is an extended circle* $\mathcal{E}(A,\frac{1}{2})$ *for all points* $A \in \mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega$.

Proof. We distinguish three cases according to the above:

(a) We infer from (35) that $\mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega = \mathcal{E}(A,\rho)$, with ρ and A as in (36).

(b) Assume to the contrary that $\mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega = \mathcal{E}(A,\rho)$. Now $1-2\gamma = 0$ yields $2\gamma \neq 0$, so that $\operatorname{Char} K \neq 2$ and $\gamma = \frac{1}{2}$. Applying Theorem 5.2 to $\mathcal{C}(A,\rho)$ yields $\rho = \gamma = \frac{1}{2}$ and, consequently, $\beta = 0$, an absurdity.

(c) We proceed as in (b) thus obtaining $\operatorname{Char} K \neq 2$, $\rho = \gamma = \frac{1}{2}$, and $a_2 = \alpha + \frac{3}{2}a_1^2$, where $a_1 \in K$ can be chosen arbitrarily. This means that every point $A = P(a_1, a_2)$ of the given curve $\mathcal{R}_{\alpha,\beta,\gamma}$ can be considered as midpoint. □

As an application of the distance function δ, we investigate various actions of the group $G(F)$ arising from its action on the projective space $\mathbb{P}_3(K)$. Given a point $P \in F \setminus g_\infty$ and a line $g \subset F$ with $P \notin g \neq g_\infty$ the pair (P,g) will be called an *antiflag* of $F \setminus g_\infty$. Following [1, p. 33] an *isometry* of $F \setminus g_\infty$ is just a mapping $\mu : F \setminus g_\infty \to F \setminus g_\infty$ such that $\delta(A,B) = \delta\big(\mu(A),\mu(B)\big)$ for all $A, B \in F \setminus g_\infty$.

Theorem 5.4. *The matrix group $G(F)$ has the following properties:*

(a) $G(F)$ *acts on* $F \setminus g_\infty$ *as a group of isometries.*

(b) $G(F)$ *acts regularly on the set of antiflags of* $F \setminus g_\infty$.

(c) *For each* $d \in K$ *the group* $G(F)$ *acts regularly on the set*

$$\Delta_d := \{(A,B) \in (F \setminus g_\infty)^2 \mid \delta(A,B) = d\}.$$

(d) *Let* $A \parallel B$ *and* $A' \parallel B'$ *be points of* $F \setminus g_\infty$. *Write* $A = P(u_1,u_2)$, $B = P(u_1,v_2)$, $A' = P(u_1',u_2')$, *and* $B' = P(u_1',v_2')$ *with* $u_1, u_2, \ldots, v_2' \in K$. *Then the number of matrices in $G(F)$ mapping (A,B) to (A',B') equals the number of distinct elements $c \in K^\times$ such that*

$$c^2(v_2 - u_2) = (v_2' - u_2'). \tag{37}$$

Proof. (a) Let $A, B \in F \setminus g_\infty$. Suppose that $M \in G(F)$ takes A to A' and B to B'. If $\delta(A,B) \neq 0, 1, \infty$ then the line AB meets the Cayley surface at three distinct points A, B, and C, say. Since M preserves cross ratios, $\delta(A',B') = \delta(A,B)$ is immediate from (32). If $\delta = 0$ then AB is a tangent of F at A. By the remark below (32), this tangent is mapped to a tangent of F at A', whence $\delta(A',B') = 0$, as required. The case $\delta(A,B) = 1$ can be treated similarly. Finally, $\delta(A,B) = \infty$ means that A, B are on a common generator, a property which is shared by their images, whence the assertion follows.

(b) Since $G(F)$ acts transitively on $F \setminus g_\infty$, it is sufficient to show that the stabilizer of Q_0 in $G(F)$, i.e. $\{M_{0,0,c} \mid c \in K^\times\}$, acts regularly on $\{g(1,c) \mid c \in$

$K^\times\}$. In fact, if we are given generators $g(1,c_1)$ and $g(1,c_2)$ with $c_1, c_2 \in K^\times$ then $M_{0,0,c_2c_1^{-1}}$ is the only solution.

(c) Let (A,B) and (A',B') be elements of Δ_d. By Lemma 3.3, we may assume w.l.o.g. that $A = A' = P(0,0)$. We infer from (31) that a point $Y = P(y_1, y_2)$, $y_1, y_2 \in K$, satisfies $\delta(A,Y) = d$ if, and only if, $y_2 = (d+1)y_1^2$ and $y_1 \in K^\times$. So there exist elements $u_1, u_1' \in K^\times$ with

$$B = P(u_1, (d+1)u_1^2),\ B' = P(u_1', (d+1)u_1'^2).$$

Letting $c := u_1' u_1^{-1}$, the matrix $M_{0,0,c}$ has the required property. The point A and the unique generator through B form an antiflag; the same holds for A' and the unique generator through B'. So the asserted regularity is a consequence of (b).

(d) The matrix $M_{-u_1,0,1}M_{0,3u_1^2-u_2,1} \in G(F)$ maps (A,B) to $\big(P(0,0), P(0, v_2-u_2)\big)$. Similarly, we can take (A',B') to $\big(P(0,0), P(0, v_2'-u_2')\big)$ by a matrix in $G(F)$. By (12), a matrix in $G(F)$ stabilizes $P(0,0) = Q_0$ if, and only if, it has the form $M_{0,0,c}$, where $c \in K^\times$. As (37) is a necessary and sufficient condition for such a matrix to take $P(0, v_2-u_2)$ to $P(0, v_2'-u_2')$, the assertion follows. □

The previous result shows that the action of $G(F)$ on pairs of parallel points depends on the square classes of $K^\times$. If $K^\times$ has a single square class (e.g. when K is quadratically closed or when K is a finite field of even order) then all pairs of distinct parallel points are in one orbit of $G(F)$. If $K^\times$ has precisely two square classes and if -1 is not a square (e.g. when $K = \mathbb{R}$ or when K is a finite field with $|K| \equiv 3 \pmod 4$) then all 2-subsets of parallel points are in one orbit of $G(F)$.

We are now in a position to describe all isometries of $F \setminus g_\infty$. Recall that we do not assume an isometry to be a bijection.

Theorem 5.5. *Each isometry $\mu : F \setminus g_\infty \to F \setminus g_\infty$ is induced by a unique matrix in $G(F)$. Consequently, μ is bijective and it can be extended in a unique way to a projective collineation of $\mathbb{P}_3(K)$ fixing the Cayley surface F.*

Proof. By Theorem 5.4 (a) and (c), it is sufficient to verify the assertion for an isometry μ fixing the points $P(0,0)$ and $P(1,0)$. Since $G(F)$ acts faithfully on $F \setminus g_\infty$, the proof will then be accomplished by showing $\mu = \mathrm{id}_{F\setminus g_\infty}$.

For all $u_2 \in K$ we obtain $\delta\big(P(1,0), P(1,u_2)\big) = \infty$, $\delta\big(P(0,0), P(1,u_2)\big) = u_2 - 1$, $\delta\big(P(0,0), P(0,u_2)\big) = \infty$ and $\delta\big(P(1,0), P(0,u_2)\big) = u_2 + 2$. So, by the isometricity of μ, all affine points of the generators through $P(1,0)$ and $P(0,0)$ remain fixed under μ.

Next choose any point $P(u_1, u_2)$, where $u_1 \in K \setminus \{0,1\}$ and $u_2 \in K$. We determine all $(v_2, w_2) \in K^2$ subject to $\delta\big(P(0,v_2), P(u_1,u_2)\big) = 0$ and $\delta\big(P(1,w_2), P(u_1,u_2)\big) = 0$. The unique solution is $(v_2, w_2) := (u_2 - u_1^2, -u_1^2 - u_1 + u_2 + 2)$. A point $P(x_1, x_2)$, $x_1, x_2 \in K$, belongs to the circle $\mathcal{C}\big(P(0,v_2), 0\big)$ if, and only if,

$$x_1 \neq 0 \text{ and } \delta\big(P(0,v_2), P(x_1,x_2)\big) = \frac{-x_1^2 + x_2 + u_1^2 - u_2}{x_1^2} = 0. \qquad (38)$$

Similarly, $P(x_1, x_2) \in \mathcal{C}\big(P(1, w_2), 0\big)$ if, and only if,

$$x_1 \neq 1 \text{ and } \delta\big(P(1, w_2), P(x_1, x_2)\big) = \frac{-x_1^2 - x_1 + x_2 + u_1^2 + u_1 - u_2}{(x_1 - 1)^2} = 0.$$

So, if $P(x_1, x_2)$ belongs to both circles it has to satisfy

$$0 = (-x_1^2 + x_2 + u_1^2 - u_2) - (-x_1^2 - x_1 + x_2 + u_1^2 + u_1 - u_2) = -x_1 + u_1 \ ,$$

whence $x_1 = u_1$ and, by (38), $x_2 = u_2$. Clearly, under μ the two circles remain fixed, so that $\mu\big(P(u_1, u_2)\big) = P(u_1, u_2)$. □

In the previous theorem we used that an isometry leaves invariant *all* distances. Thus we established that δ is a defining function (see [2, p. 23]) for the group of automorphic projective collineations of F. It would be interesting to know if this assumption could be weakened, for example, by requiring that only *some* distances are being preserved.

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