Jordan homomorphisms and harmonic mappings

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Abstract

We show that each Jordan homomorphism $R \to R'$ of rings gives rise to a harmonic mapping of one connected component of the projective line over R into the projective line over R'. If there is more than one connected component then this mapping can be extended in various ways to a harmonic mapping which is defined on the entire projective line over R.

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1 Introduction

1.1 The problem to determine all *harmonic mappings* (see 4.7) for *projective lines over rings* (see 4.1) goes back to K. von Staudt (1798–1867), who treated harmonic bijections between real projective lines; cf. [18, p. 57–58] for a survey and historical remarks. The results which have been obtained so far show that among the relevant algebraic mappings are – apart from projective transformations – not only *homomorphisms* but also *Jordan homomorphisms* of rings (see 3.1). If a ring R contains a subfield K then the point set of the associated *chain geometry* (see 5.1) is the projective line over R. The investigation of *homomorphism of chain geometries* (see 5.3) has also lead to Jordan homomorphisms of rings. There is a widespread literature on the interplay of Jordan homomorphisms, harmonic mappings, and homomorphisms of chain geometries. The interested reader should consult [1], [2], [3], [4], [5], [6], [11], [13], [14], [15], [16], [20], and [21] for further references and related results.

1.2 Suppose that we are given a Jordan homomorphism $\alpha : R \to R'$ of rings. The ring R can be embedded in the projective line $\mathbb{P}(R)$ over R via $t \mapsto R(t,1)$ and there is a similar embedding $R' \to \mathbb{P}(R')$. By virtue of these embeddings, α determines a mapping

$$R(t,1) \mapsto R'(t^\alpha, 1') \text{ with } t \in R. \qquad (1)$$

There arises the question if (1) can be extended to a mapping $\mathbb{P}(R) \to \mathbb{P}(R')$ in some "natural way". It is fairly obvious that such an extension should take each point $R(1,t)$ to $R'(1', t^\alpha)$. However, the projective line may also contain points of the form $R(a,b)$, where neither a nor b are invertible, whence they cannot be written as $R(1,t)$ or $R(t,1)$. For each of those points an "appropriate" definition of the image point is not immediate, since $R(a,b) \mapsto R(a^\alpha, b^\alpha)$ gives in general no well defined mapping.

An affirmative answer to the question above has been given by C. Bartolone [1] under the additional assumption that R is a ring of stable rank 2. Among the rings with this property are, e.g., local rings, matrix rings over fields, and finite-dimensional algebras over commutative fields. In case of stable rank 2 each point of $\mathbb{P}(R)$ can be written in the form $R(t_1t_2 - 1, t_1)$ with parameters $t_1, t_2 \in R$, and

$$R(t_1t_2 - 1, t_1) \mapsto R'(t_1^\alpha t_2^\alpha - 1', t_1^\alpha) \tag{2}$$

is a well defined extension of (1).

In the present article there will be no restriction on the rings R and R'. We shall show that the mapping (1) can be extended to a harmonic mapping $\overline{\alpha} : C \to \mathbb{P}(R')$, where $C \subseteq \mathbb{P}(R)$ denotes the *connected component* (in the graph-theoretic sense; see 4.1) of the point $R(1, 0)$. If $C \neq \mathbb{P}(R)$ then $\overline{\alpha}$ can be extended in various ways to a harmonic mapping $\mathbb{P}(R) \to \mathbb{P}(R')$. The definition of $\overline{\alpha}$ is rather involved: Each point of the connected component C can be described by some finite sequence $(t_1, t_2, \ldots, t_n)$ of parameters in R, where $n \geq 0$ is variable. Then the Jordan homomorphism acts on these parameters, i.e., the sequence $(t_1^\alpha, t_2^\alpha, \ldots, t_n^\alpha)$ determines the image point; cf. formula (3) below which is a generalization of (2). The number of parameters which is needed in order to describe all points of C may be unbounded. Furthermore, a point of C may admit many representations in terms of parameters. So the problem is to show that we have a well defined mapping. In [1] the situation is less complicated: When R is a ring of stable rank 2, the projective line $\mathbb{P}(R)$ coincides with the connected component C and, as has been mentioned above, each point of $\mathbb{P}(R)$ can be described with just two parameters.

1.3 The paper is organized as follows: In Section 2 we discuss the *elementary subgroup* $\mathrm{E}_2(R)$ of the general linear group $\mathrm{GL}_2(R)$ over a ring R. Following P.M. Cohn [12] we consider a family of matrices $E(t)$, $t \in R$, with the property that each matrix in $\mathrm{E}_2(R)$ can be written as a product $E(t_1)E(t_2)\cdots E(t_n)$ with $t_1, t_2, \ldots, t_n \in R$ and $n \geq 0$. The entries of a matrix in $\mathrm{E}_2(R)$ can be expressed with the help of an infinite family of polynomials in non-commuting indeterminates. Next, in Section 3, we introduce the concept of a *polynomial with Jordan property* and present two infinite families of such polynomials (Propositions 3.3 and 3.4). These polynomials are used in order to compare the matrices $E(t_1)E(t_2)\cdots E(t_n) \in \mathrm{E}_2(R)$ and $E(t_1^\alpha)E(t_2^\alpha)\cdots E(t_n^\alpha) \in \mathrm{E}_2(R')$. For example, if the $(1, 1)$-entry of the first matrix is a unit, then so is the $(1, 1)$-entry of the second matrix (Theorem 3.5).

Unfortunately, in general there is no well defined mapping sending $E(t_1)E(t_2)\cdots E(t_n)$ to $E(t_1^\alpha)E(t_2^\alpha)\cdots E(t_n^\alpha)$. But we can pass from $\mathrm{E}_2(R')$ to an appropriate quotient group $\mathrm{E}_2(R'')/N_\alpha$; here R'' denotes the subring of R' which is generated by the image of the Jordan homomorphism α and N_α is a normal subgroup of $\mathrm{E}_2(R'')$ which depends on α. In this way a well defined homomorphism of groups $\mathrm{E}_2(R) \to \mathrm{E}_2(R'')/N_\alpha$ can be obtained (Theorem 3.6). If N_α contains only the identity matrix then we have a well defined mapping $\mathrm{E}_2(R) \to \mathrm{E}_2(R'')$ (Corollary 3.7). This is the case when α belongs to a certain class of Jordan homomorphisms, including homomorphisms and antihomomorphisms. However, we shall see that N_α can also be non-trivial (Examples 3.8). We show that N_α is in the centre of $\mathrm{E}_2(R'')$, whence the Jordan homomorphism α gives rise to a homomorphism

$$\alpha_{\mathrm{PE}} : \mathrm{PE}_2(R) \to \mathrm{PE}_2(R'')$$

of *projective elementary groups* (see 4.1), which act on the connected component $C \subseteq \mathbb{P}(R)$ and a connected component of the subline $\mathbb{P}(R'') \subseteq \mathbb{P}(R')$, respectively (Theorem 4.4). The mapping α_{PE} is then the key to showing that a well defined mapping $\overline{\alpha}$ of the points of C is given by

$$R(1,0) \cdot E(t_1)E(t_2)\cdots E(t_n) \mapsto R'(1',0') \cdot E(t_1^\alpha)E(t_2^\alpha)\cdots E(t_n^\alpha) \qquad (3)$$

with $t_1, t_2, \dots, t_n \in R$ and $n \geq 0$. This $\overline{\alpha}$ extends (1) and it turns α_{PE} into a homomorphism of transformation groups. We show some geometric properties of the mapping $\overline{\alpha}$ and present several examples to illustrate our results.

In Section 5 we examine homomorphisms of chain geometries. In particular, it will be established that the isomorphisms of affine chain geometries discussed by A. Herzer in [15, 9.1] can be extended to homomorphisms of chain geometries without any additional assumption on the underlying rings (Theorem 5.2). Thus our results yield new examples of homomorphisms of chain geometries.

2 The elementary subgroup $\mathrm{E}_2(R)$

2.1 Throughout this paper we shall only consider associative rings with a unit element, which is preserved by homomorphisms, inherited by subrings, and acts unitally on modules. The group of *invertible elements* and the *centre* of a ring R will be denoted by R^* and $Z(R)$, respectively. Also, we shall write $\mathcal{S}(R) := R^0 \cup R^1 \cup R^2 \cup \dots$ for the set of all *finite sequences* in R, including the empty sequence.

2.2 We recall that the *elementary subgroup* $\mathrm{E}_2(R)$ of the general linear group $\mathrm{GL}_2(R)$ is generated by the set of all matrices

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \text{ with } t \in R. \qquad (4)$$

Furthermore,

$$E(t)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} = E(0) \cdot E(-t) \cdot E(0), \qquad (5)$$

whence each element of $\mathrm{E}_2(R)$ can be written in the form

$$E(t_1) \cdot E(t_2) \cdots E(t_n) =: E(t_1, t_2, \dots, t_n) =: E(T) \qquad (6)$$

where $T := (t_1, t_2, \dots, t_n) \in \mathcal{S}(R)$ denotes a sequence of $n \geq 0$ elements; cf. [12, p. 368].

It is easily seen that a (2×2)-matrix over R commutes with all matrices $E(t)$, $t \in R$, if and only if it has the form $\mathrm{diag}(a,a)$ with $a \in Z(R)$. Hence the centre of $\mathrm{E}_2(R)$ is the subgroup

$$H := \mathrm{E}_2(R) \cap \{\mathrm{diag}(a,a) \mid a \in Z(R)^*\}. \qquad (7)$$

2.3 In order to describe the entries of a matrix (6) we consider an infinite sequence $X = (x_1, x_2, \dots)$ of indeterminates over $\mathbb{Z}$ and the free $\mathbb{Z}$-algebra $\mathbb{Z}\langle X\rangle$. Its elements are polynomials in the non-commuting indeterminates $x_1, x_2, \dots$ with coefficients in $\mathbb{Z}$.

We shall frequently use the following *universal property* of $\mathbb{Z}\langle X\rangle$ [19, p. 6]: If R is an arbitrary ring and $(t_1, t_2, \dots)$ is an infinite sequence of elements in R, then there is a unique homomorphism $\mathbb{Z}\langle X\rangle \to R$ such that $x_i \mapsto t_i$ for each $i \in \{1, 2, \dots\}$. The image of $f \in \mathbb{Z}\langle X\rangle$ under this homomorphism is written as $f(t_1, t_2, \dots)$. Also, we have a homomorphism $\mathrm{E}_2(\mathbb{Z}\langle X\rangle) \to \mathrm{E}_2(R)$ by the action of $f \mapsto f(t_1, t_2, \dots)$ on the entries of a matrix. In addition, let $T = (t_1, t_2, \dots, t_n) \in \mathcal{S}(R)$ be a finite sequence which may be empty $(n < 1)$. Then we put

$$f(T) = f(t_1, t_2, \dots, t_n) := f(t_1, t_2, \dots, t_n, 0, 0, \dots). \tag{8}$$

In order to avoid misinterpretations let us point out the following particular case of (8): Assume that $f = 2x_2 + x_3 \in \mathbb{Z}\langle X\rangle$, $T = (x_2, x_3) \in \mathcal{S}(\mathbb{Z}\langle X\rangle)$, and $V = (v_1, v_2, v_3) \in \mathcal{S}(R)$. Then $f(T) = f(x_2, x_3)$ denotes that polynomial which arises from $f \in \mathbb{Z}\langle X\rangle$ if X is substituted by $(x_2, x_3, 0, 0, \dots)$. As $f(x_2, x_3) = 2x_3 \neq f$, we must not write "$f = f(x_2, x_3)$" in order to stress that f belongs to the $\mathbb{Z}$-subalgebra of $\mathbb{Z}\langle X\rangle$ generated by $\{x_2, x_3\}$. Furthermore, $f(V) = 2v_2 + v_3$, but $(f(T))(V) = (f(x_2, x_3))(V) = 2x_3(V) = 2v_3 = f(v_2, v_3)$. On the other hand, for each $g \in \mathbb{Z}\langle X\rangle$ there is a sufficiently large integer n such that $g = g(x_1, x_2, \dots, x_n)$.

2.4 Following [12, p. 376], we define a sequence of elements in $\mathbb{Z}\langle X\rangle$ recursively by

$$\left.\begin{array}{c} e^{(-2)} := -1, \quad e^{(-1)} := 0, \quad e^{(0)} := 1, \\ e^{(n)} := e^{(n-1)}\, x_n - e^{(n-2)}, \end{array}\right\} \tag{9}$$

where $n \in \{1, 2, \dots\}$. It will turn out useful to have a short notation for polynomials that arise from the ones given in (9) by a substitution (8) as follows: Given $i, j \in \mathbb{Z}$ with $i \geq 1$ and $j \geq i - 3$ we define

$$e_i^j := e^{(j-i+1)}(x_i, x_{i+1}, \dots, x_j), \quad \tilde{e}_i^j := e^{(j-i+1)}(x_j, x_{j-1}, \dots, x_i). \tag{10}$$

So j is an upper index rather than an exponent. In particular, we have

$$e_1^n = e^{(n)}(x_1, x_2, \dots, x_n) = e^{(n)} \text{ for all } n \in \{-2, -1, \dots\}. \tag{11}$$

Furthermore, each polynomial e_i^j can be written as a $\mathbb{Z}$-linear combination of monomials $x_{h_1} x_{h_2} \cdots x_{h_m}$ with $i \leq h_1 < h_2 < \cdots < h_m \leq j$ and m ranging from 0 to $j - i + 1$. Likewise $\tilde{e}_i^j$ is a $\mathbb{Z}$-linear combination of monomials $x_{h_1} x_{h_2} \cdots x_{h_m}$ with $j \geq h_1 > h_2 > \cdots > h_m \geq i$ and m ranging from 0 to $j - i + 1$. For example, $e_2^2 = e^{(1)}(x_2) = x_2$ and $e_5^6 = e^{(2)}(x_5, x_6) = x_5 x_6 - 1$. Many of the following calculations are based upon the identities

$$e_i^j = e_i^{j-1} x_j - e_i^{j-2}, \tag{12}$$
$$\tilde{e}_i^j = \tilde{e}_{i+1}^j x_i - \tilde{e}_{i+2}^j, \tag{13}$$

which follow from (9) and (10) whenever $1 \geq i \geq j$.
Next we describe certain elements of the group $\mathrm{E}_2(\mathbb{Z}\langle X\rangle)$:

Lemma 2.5 *If* $(x_1, x_2, \ldots, x_n) \in \mathcal{S}(\mathbb{Z}\langle X\rangle)$ *then*

$$E(x_1, x_2, \ldots, x_n) = \begin{pmatrix} e_1^n & e_1^{n-1} \\ -e_2^n & -e_2^{n-1} \end{pmatrix}, \tag{14}$$

$$E(x_1, x_2, \ldots, x_n)^{-1} = \begin{pmatrix} -\tilde{e}_2^{n-1} & -\tilde{e}_1^{n-1} \\ \tilde{e}_2^n & \tilde{e}_1^n \end{pmatrix}. \tag{15}$$

Proof: Clearly, for $n = 0$ we have $E() = I$, the identity in $\mathrm{E}_2(\mathbb{Z}\langle X\rangle)$. Now (14) follows easily by induction ([12, p. 376]), since for $n \geq 1$ we infer from the induction hypothesis and (12) that

$$E(x_1, x_2, \ldots, x_n) = \begin{pmatrix} e_1^{n-1} & e_1^{n-2} \\ -e_2^{n-1} & -e_2^{n-2} \end{pmatrix} \cdot \begin{pmatrix} x_n & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} e_1^n & e_1^{n-1} \\ -e_2^n & -e_2^{n-1} \end{pmatrix}.$$

The proof of (15) runs in a similar manner taking into account the first part of equation (5), (13), and $E(x_1, x_2, \ldots, x_n)^{-1} = E(x_2, x_3, \ldots, x_n)^{-1} \cdot E(x_1)^{-1}$. □

2.6 We obtain the recursion $e_1^n = x_1 e_2^n - e_3^n$ for $n \in \{1, 2, \ldots\}$ from the $(1,1)$-entry of the matrix equation $E(x_1, x_2, \ldots, x_n) = E(x_1) \cdot E(x_2, x_3, \ldots, x_n)$ together with (14). This yields

$$e_i^j = x_i e_{i+1}^j - e_{i+2}^j, \tag{16}$$

$$\tilde{e}_i^j = x_j \tilde{e}_i^{j-1} - \tilde{e}_i^{j-2}, \tag{17}$$

for $1 \geq i \geq j$ as counterparts of (12) and (13); cf. [12, p. 376].

2.7 We now return to an arbitrary ring R. For each $T = (t_1, t_2, \ldots, t_n) \in \mathcal{S}(R)$ there is the homomorphism $\mathbb{Z}\langle X\rangle \to R : f \to f(T)$; see (8). So we can transfer our calculations from $\mathbb{Z}\langle X\rangle$ to R and from $\mathrm{E}_2(\mathbb{Z}\langle X\rangle)$ to $\mathrm{E}_2(R)$. For example, (12) yields

$$e_1^n(T) = e_1^{n-1}(T)t_n - e_1^{n-2}(T) = e_1^{n-1}(t_1, t_2, \ldots, t_{n-1})t_n - e_1^{n-2}(t_1, t_2, \ldots, t_{n-2}).$$

whereas (16) gives

$$e_1^n(T) = t_1 e_2^n(T) - e_3^n(T) = t_1 e_1^{n-1}(t_2, t_3, \ldots, t_n) - e_1^{n-2}(t_3, t_4, \ldots, t_n).$$

Observe that there are numerous ways to rewrite such identities, since we may also add irrelevant ring elements. E.g., $e_1^n(T) = e_2^{n+1}(s, T)$ and $e_1^n(T) = e_1^n(T, v)$ for all $s, v \in R$.
It has been pointed out in [1, Lemma 1.5] that $e_1^3(t_1, t_2, t_3) = t_1 t_2 t_3 - t_3 - t_1 \in R^*$ implies $\tilde{e}_1^3(t_1, t_2, t_3) = t_3 t_2 t_1 - t_1 - t_3 \in R^*$ for all $(t_1, t_2, t_3) \in R^3$. The following result is a generalization of this, but we give a completely different proof using the group $\mathrm{E}_2(R)$:

Proposition 2.8 *Let* $T \in R^n$, $n \geq 0$. *Then* $e_1^n(T) \in R^*$ *implies* $\tilde{e}_1^n(T) \in R^*$.

Proof: This is obvious for $n = 0$. So let $n \geq 1$. Since $e_1^n(T) \in R^*$, there are elements s, $v \in R$ such that

$$\begin{aligned} 0 &= s e_1^n(T) - e_2^n(T) = s e_2^{n+1}(s, T) - e_3^{n+1}(s, T) = e_1^{n+1}(s, T) = e_1^{n+1}(s, T, v), \\ 0 &= e_1^n(T)v - e_1^{n-1}(T) = e_1^n(T, v)v - e_1^{n-1}(T, v) = e_1^{n+1}(T, v) = e_2^{n+2}(s, T, v), \end{aligned}$$

where we used (16) and (12). We read off from (14) and $e_2^{n+1}(s, T, v) = e_1^n(T)$ that

$$E(s, T, v) = \mathrm{diag}\big(e_1^{n+2}(s, T, v), -e_1^n(T)\big).$$

The inverse of this matrix is diagonal, too, and by (15) its $(1,1)$-entry equals $-\tilde{e}_2^{n+1}(s, T, v) = -\tilde{e}_1^n(T)$, which is therefore a unit. □

3 Jordan homomorphisms

3.1 Let R and R' be rings. We recall that a mapping $\alpha : R \to R'$ is called *Jordan homomorphism* if

$$(a+b)^\alpha = a^\alpha + b^\alpha, \quad 1^\alpha = 1', \quad (aba)^\alpha = a^\alpha b^\alpha a^\alpha \quad \text{for all } a, b \in R. \tag{18}$$

See, among others, [17, p. 2] or [15, p. 832]. For such an α and an $a \in R^*$ the equation $1' = (aa^{-2}a)^\alpha = a^\alpha (a^{-2})^\alpha a^\alpha$ shows that $a^\alpha \in R'^*$. Also, $a^\alpha = (aa^{-1}a)^\alpha = a^\alpha (a^{-1})^\alpha a^\alpha$ implies $(a^{-1})^\alpha = (a^\alpha)^{-1}$ for all $a \in R^*$. We say that α is *proper* if it is neither a homomorphism nor an antihomomorphism.
In general a Jordan homomorphism is not multiplicative, but on certain expressions, like aba in (18), it acts "like a ring homomorphism". In order to generalize this we pass to the free $\mathbb{Z}$-algebra $\mathbb{Z}\langle X\rangle$. We say that $f \in \mathbb{Z}\langle X\rangle$ is a *polynomial with Jordan property*, or a *J-polynomial* for short, if $f(T)^\alpha = f(T^\alpha)$ holds for every Jordan homomorphism $\alpha : R \to R'$ between arbitrary rings R and R' and every (finite or infinite) sequence T in R.

Examples 3.2 Clearly, 1, x_i, and $x_i x_j x_i$ are J-polynomials for all $i, j \in \{1, 2, \dots\}$. The set of all J-polynomials forms a $\mathbb{Z}$-submodule of $\mathbb{Z}\langle X\rangle$. Furthermore, suppose that G is a (finite or infinite) sequence of J-polynomials and that f is a J-polynomial. Then it is easily seen that also $f(G)$ has the Jordan property. In particular we obtain the J-polynomials

$$\begin{aligned} x_1^2 &= x_1 1 x_1, \\ x_1x_2 + x_2x_1 &= (x_1+x_2)^2 - x_1^2 - x_2^2, \\ x_1x_2x_3 + x_3x_2x_1 &= (x_1+x_3)x_2(x_1+x_3) - x_1x_2x_1 - x_3x_2x_3. \end{aligned}$$

Next we give two infinite families of J-polynomials.

Proposition 3.3 *Let $n \in \{-1, 0, \dots\}$. Then $e_1^n \tilde{e}_1^{n-1}$ is a polynomial with Jordan property.*

Proof: We proceed by induction observing that $e_1^{-1}\tilde{e}_1^{-2} = 0$, $e_1^0 \tilde{e}_1^{-1} = 0$, and $e_1^1 \tilde{e}_1^0 = x_1$ are J-polynomials. Letting $n \geq 2$ we infer from the induction hypothesis and an appropriate substitution that

$$e_i^n \tilde{e}_i^{n-1} = e_1^{n-i+1}(x_i, x_{i+1}, \dots, x_n)\tilde{e}_1^{n-i}(x_i, x_{i+1}, \dots, x_{n-1}) \tag{19}$$

is a J-polynomial for $i \in \{2, 3\}$. The keys for the following calculations are formula (16) and formula (13). So we get

$$\begin{aligned} e_1^n \tilde{e}_1^{n-1} &= (x_1 e_2^n - e_3^n)(\tilde{e}_2^{n-1} x_1 - \tilde{e}_3^{n-1}) \\ &= \underbrace{x_1(e_2^n \tilde{e}_2^{n-1})x_1 + e_3^n \tilde{e}_3^{n-1}}_{=:f_1} - (x_1 e_2^n \tilde{e}_3^{n-1} + e_3^n \tilde{e}_2^{n-1} x_1) \end{aligned}$$

where, by (19) and the examples given in 3.2, f_1 is a J-polynomial. Similarly, if $n \geq 3$ then

$$x_1 e_2^n \tilde{e}_3^{n-1} + e_3^n \tilde{e}_2^{n-1} x_1 = \underbrace{x_1 x_2 (e_3^n \tilde{e}_3^{n-1}) + (e_3^n \tilde{e}_3^{n-1}) x_2 x_1}_{=:f_2} - (x_1 e_4^n \tilde{e}_3^{n-1} + e_3^n \tilde{e}_4^{n-1} x_1)$$

with a J-polynomial f_2. Also, for $n \geq 4$ we get

$$x_1 e_4^n \tilde{e}_3^{n-1} + e_3^n \tilde{e}_4^{n-1} x_1 = \underbrace{x_1(e_4^n \tilde{e}_4^{n-1}) x_3 + x_3(e_4^n \tilde{e}_4^{n-1}) x_1}_{=: f_3} - (x_1 e_4^n \tilde{e}_5^{n-1} + e_5^n \tilde{e}_4^{n-1} x_1).$$

with a J-polynomial f_3. Proceeding in this way we arrive either at

$$f_{n-1} - (x_1 e_{n+1}^n \tilde{e}_n^{n-1} + e_n^n \tilde{e}_{n+1}^{n-1} x_1) = f_{n-1} - (x_1 \cdot 1 \cdot 1 + x_n \cdot 0 \cdot x_1),$$

when n is odd, or at

$$f_{n-1} - (x_1 e_n^n \tilde{e}_{n+1}^{n-1} + e_{n+1}^n \tilde{e}_n^{n-1} x_1) = f_{n-1} - (x_1 \cdot x_n \cdot 0 + 1 \cdot 1 \cdot x_1),$$

when n is even. But x_1 is a J-polynomial, whence the assertion follows. □

Proposition 3.4 *Let $n \in \{-2, -1, \dots\}$. Then $e_1^n \tilde{e}_1^n$ is a polynomial with Jordan property.*

Proof: This is clear when $n < 0$. If $n \geq 0$ then we infer from the $(1,2)$-entry of $E(x_1, x_2, \dots, x_n) \cdot E(x_1, x_2, \dots, x_n)^{-1} = I$ and Lemma 2.5 that

$$-e_1^n \tilde{e}_1^{n-1} + e_1^{n-1} \tilde{e}_1^n = 0.$$

Hence $e_1^{n+1} \tilde{e}_1^n = e_1^n x_{n+1} \tilde{e}_1^n - e_1^{n-1} \tilde{e}_1^n = e_1^n x_{n+1} \tilde{e}_1^n - e_1^n \tilde{e}_1^{n-1}$, where we used (12) and the equation above. So

$$e_1^n x_{n+1} \tilde{e}_1^n = e_1^{n+1} \tilde{e}_1^n + e_1^n \tilde{e}_1^{n-1},$$

and Proposition 3.3 yields that $e_1^n x_{n+1} \tilde{e}_1^n$ is a J-polynomial. This property remains unaltered if we substitute x_{n+1} by 1. □

Our next result generalizes [21, Lemma 1.2] and is based upon the previous propositions. It will be the backbone of many considerations. The theorem says that if certain entries of a matrix $E(T)$ are of a particular form, then so are the corresponding entries in $E(T^\alpha)$.

Theorem 3.5 *Let $\alpha : R \to R'$ be a Jordan homomorphism and let $T \in R^n$, $n \geq 0$. Then the following holds:*

(a) *$e_1^n(T) \in R^*$ implies $e_1^n(T^\alpha) \in R'^*$.*

(b) *If $e_1^n(T) \in R^*$ and $e_1^{n-1}(T) = 0$ then $e_1^{n-1}(T^\alpha) = 0'$.*

Proof: (a) We deduce from Proposition 2.8 that $\tilde{e}_1^n(T) \in R^*$. Proposition 3.4 and $R^{*\alpha} \subseteq R'^*$ establish that

$$e_1^n(T^\alpha) \tilde{e}_1^n(T^\alpha) = (e_1^n \tilde{e}_1^n)(T^\alpha) = \big((e_1^n \tilde{e}_1^n)(T)\big)^\alpha = \big(e_1^n(T) \tilde{e}_1^n(T)\big)^\alpha \in R'^*.$$

So $e_1^n(T^\alpha)$ is right invertible. Let $\widetilde{T}$ be the finite sequence T written in reverse order. Then

$$\tilde{e}_1^n(T^\alpha) e_1^n(T^\alpha) = e_1^n(\widetilde{T}^\alpha) \tilde{e}_1^n(\widetilde{T}^\alpha) = \big(e_1^n(\widetilde{T}) \tilde{e}_1^n(\widetilde{T})\big)^\alpha = \big(\tilde{e}_1^n(T) e_1^n(T)\big)^\alpha \in R'^*,$$

where we used the equation from above in the second step. Hence $e_1^n(T^\alpha)$ is also left invertible.
(b) We read off from (12) the identity

$$e_1^{n+1}(x_1, x_2, \dots, x_n, 1) = e_1^n \cdot 1 - e_1^{n-1}. \tag{20}$$

So $e_1^{n-1}(T) = 0$ implies $e_1^{n+1}(T,1) = e_1^n(T)$, whereas $\tilde{e}_1^n(T,1) = \tilde{e}_1^n(T)$ holds trivially. Since $e_1^{n+1}\tilde{e}_1^n$ and $e_1^n\tilde{e}_1^n$ are J-polynomials by Proposition 3.3 and Proposition 3.4, we obtain that

$$\begin{aligned} e_1^{n+1}(T^\alpha, 1')\tilde{e}_1^n(T^\alpha, 1') &= (e_1^{n+1}\tilde{e}_1^n)(T^\alpha, 1') = \big((e_1^{n+1}\tilde{e}_1^n)(T,1)\big)^\alpha \\ &= \big(e_1^n(T)\tilde{e}_1^n(T)\big)^\alpha = e_1^n(T^\alpha)\tilde{e}_1^n(T^\alpha). \end{aligned}$$

Now Proposition 2.8 and (a) allow to cancel the unit $\tilde{e}_1^n(T^\alpha, 1') = \tilde{e}_1^n(T^\alpha)$. Hence (20) forces that $e_1^{n-1}(T^\alpha) = 0'$. □

We are now in a position to show our first main result.

Theorem 3.6 *Let $\alpha : R \to R'$ be a Jordan homomorphism and denote by R'' the subring of R' generated by R^α. Then the following statements are true:*

(a) *If $T \in \mathcal{S}(R)$ then $E(T) \in H$ implies $E(T^\alpha) \in H''$, where H and H'' denote the centres of $\mathrm{E}_2(R)$ and $\mathrm{E}_2(R'')$, respectively.*

(b) *The set*

$$N_\alpha := \{E(T^\alpha) \mid T \in \mathcal{S}(R) \text{ and } E(T) = I\} \tag{21}$$

is contained in H''.

(c) *N_α is a normal subgroup of $\mathrm{E}_2(R'')$. Furthermore, the mapping*

$$\alpha_{\mathrm{E}} : \mathrm{E}_2(R) \to \mathrm{E}_2(R'')/N_\alpha : E(T) \mapsto N_\alpha \cdot E(T^\alpha), \tag{22}$$

where $T \in \mathcal{S}(R)$, is a well defined homomorphism of groups.

Proof: (a) Let $T \in R^n$ be a sequence such that $E(T) \in H$. So, by (7), there is an $a \in Z(R)^*$ with $E(T) = \mathrm{diag}(a,a)$. Put $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) := E(T^\alpha)$. By virtue of (14), all entries of this matrix are in R''. The first row of $E(T)$ is $\big(e_1^n(T), e_1^{n-1}(T)\big) = (a, 0)$. We infer from Theorem 3.5 that a' is a unit and that $b' = 0'$. Next choose any $s \in R$ and let $S := (s, T, 0, -s, 0)$. By (5),

$$\begin{aligned} E(S) &= E(s)E(T)E(s)^{-1} = E(T) = \mathrm{diag}(a,a), && (23) \\ E(S^\alpha) &= E(s^\alpha)E(T^\alpha)E(s^\alpha)^{-1} = \begin{pmatrix} d' & d's^\alpha - s^\alpha a' - c' \\ 0' & a' \end{pmatrix}. && (24) \end{aligned}$$

As before, Theorem 3.5 shows that d' is a unit and that $d's^\alpha - s^\alpha a' - c'$ vanishes for all $s \in R$. Letting $s := 0, 1$ yields $c' = 0'$ and $a' = d'$. Hence $a's^\alpha = s^\alpha a'$ for all $s \in R$ and therefore $a' \in Z(R'')$. We infer that $E(T^\alpha) \in H''$.
(b) This is immediate from (a).

(c) First, let us introduce the following notation. Given $T \in R^n$, $n \geq 0$, we define

$$\widehat{T} := (0, -t_n, 0, 0, -t_{n-1}, 0, \dots, 0, -t_1, 0) \in R^{3n}. \tag{25}$$

We observe that (5) implies $E(\widehat{T}) = E(T)^{-1}$. Also, when (25) is applied accordingly to sequences in R', then $\widehat{T}^\alpha = \widehat{T^\alpha}$.
Clearly, $I' \in N_\alpha$. Next assume that $E(T) = E(V)$ for $T, V \in \mathcal{S}(R)$. Then $E(T, \widehat{V}) = I$, and $E(T^\alpha)E(V^\alpha)^{-1} = E(T^\alpha, \widehat{V^\alpha}) = E(T^\alpha, \widehat{V}^\alpha) \in N_\alpha$. Hence $E(T^\alpha) \in N_\alpha \cdot E(V^\alpha)$.
Any two matrices in N_α can be written in the form $E(T^\alpha)$, $E(V^\alpha)$ with $E(T) = E(V) = I$. By the above, $E(T^\alpha)E(V^\alpha)^{-1} \in N_\alpha$. So N_α is a subgroup of $\mathrm{E}_2(R'')$ which is normal by (a). Summing up, the mapping α_{E} is well defined and obviously it is a homomorphism. □

The reason for introducing the subring R'' in the theorem above is that N_α need not be normal in $\mathrm{E}_2(R')$; cf. Example 3.8 (f) below.

Corollary 3.7 *If, under the assumptions of Theorem 3.6, the group N_α is trivial, then*

$$\alpha_{\mathrm{E}} : \mathrm{E}_2(R) \to \mathrm{E}_2(R'') : E(T) \mapsto E(T^\alpha),$$

where $T \in \mathcal{S}(R)$, is a well defined mapping.

Examples 3.8 (a) Let $\alpha : R \to R'$ be a homomorphism. Then there is the homomorphism $\alpha_* : \mathrm{GL}_2(R) \to \mathrm{GL}_2(R') : M \mapsto M^\alpha$, i.e., α is applied to each entry of M. Obviously, $E(t)^{\alpha_*} = E(t^\alpha)$ for all $t \in R$, whence $N_\alpha = \{I'\}$ and α_* restricts to $\alpha_{\mathrm{E}} : \mathrm{E}_2(R) \to \mathrm{E}_2(R'')$.

(b) Let $\alpha : R \to R'$ be an antihomomorphism. The mapping $\alpha_{**} : \mathrm{GL}_2(R) \to \mathrm{GL}_2(R') : M \mapsto E(0')^{-1}((M^{\mathrm{T}})^{\alpha_*})^{-1}E(0')$, where M^{T} denotes the transpose of M, is a homomorphism. It maps each $E(t) \in \mathrm{E}_2(R)$ to $E(t^\alpha)$. Hence $N_\alpha = \{I'\}$ and α_{**} restricts to α_{E}. If R' is commutative then $M^{\alpha_{**}} = \det(M^{\alpha_*})^{-1}M^{\alpha_*}$ for all $M \in \mathrm{GL}_2(R)$.

(c) Suppose that $R = \prod_{\lambda\in\Lambda} R_\lambda$ is the direct product of rings R_λ. Then, up to isomorphism, $\mathrm{GL}_2(R) = \prod_{\lambda\in\Lambda} \mathrm{GL}_2(R_\lambda)$ and $\mathrm{E}_2(R) = \prod_{\lambda\in\Lambda} \mathrm{E}_2(R_\lambda)$. Similarly, let $R' = \prod_{\lambda\in\Lambda} R'_\lambda$ and let $\alpha_\lambda : R_\lambda \to R'_\lambda$ be a family of mappings, where each α_λ is a homomorphism or antihomomorphism. Then $\alpha := \prod_{\lambda\in\Lambda} \alpha_\lambda$ is in general a proper Jordan homomorphism $R \to R'$. Now, by (a) or (b), we can choose at least one homomorphism $\beta_\lambda : \mathrm{GL}_2(R_\lambda) \to \mathrm{GL}_2(R'_\lambda)$ which restricts to $\alpha_{\lambda,\mathrm{E}}$. So $\beta := \prod_{\lambda\in\Lambda} \beta_\lambda : \mathrm{GL}_2(R) \to \mathrm{GL}_2(R')$ is a homomorphism with $E(t)^\beta = E(t^\alpha)$ for all $t \in R$. Finally, as above, $N_\alpha = \{I'\}$ and β restricts to α_{E}.

(d) The following class of examples is essentially due to A. Herzer [14, 4.2]. Although our assumptions are weaker, Herzer's proofs immediately carry over to our setting.

Suppose that D is a commutative ring and let B be a D-algebra with a D-linear homomorphism $\chi : B \to D$ of rings. Denote by M a left module over D which is endowed with a D-bilinear, alternating, and associative product $M \times M \to M$. Then $R := B \oplus M$ becomes a D-algebra, if a product is defined by

$$(b_1 + m_1)(b_2 + m_2) := b_1b_2 + b_1{}^\chi m_2 + b_2{}^\chi m_1 + m_1m_2$$

for all $b_1, b_2 \in B$ and $m_1, m_2 \in M$. (The commutativity of D guarantees that R is associative.) Assume that B', χ', and M' are given as above and that $\alpha_1 : B \to B'$ is a homomorphism or antihomomorphism of D-algebras satisfying $\chi = \alpha_1\chi'$. Also, let $\alpha_2 : M \to M'$ be an arbitrary D-linear mapping. Then

$$\alpha : R \to R' : b + m \mapsto b^{\alpha_1} + m^{\alpha_2} \ \ (b \in B,\ m \in M)$$

is a D-linear Jordan homomorphism.

(e) With the notation introduced in Example (d) let $B = B'$, $\chi = \chi'$, and $M = M' = D^3$. Write $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ for the canonical basis of D^3. A product on D^3 with the properties mentioned above is given by $\varepsilon_i^2 = 0$, $\varepsilon_i\varepsilon_j = -\varepsilon_j\varepsilon_i$, $\varepsilon_1\varepsilon_2 = \varepsilon_3$, $\varepsilon_i\varepsilon_3 = 0$ for all $i, j \in \{1, 2, 3\}$. We define a D-linear Jordan automorphism α of R by $\alpha_1 = \mathrm{id}_B$, whereas α_2 fixes ε_1 and interchanges ε_2 with ε_3. Now $N_\alpha \neq \{I\}$ follows from

$$\begin{aligned} E(\varepsilon_1)E(\varepsilon_3)E(-\varepsilon_1)E(-\varepsilon_3) &= I, \\ E(\varepsilon_1)E(\varepsilon_2)E(-\varepsilon_1)E(-\varepsilon_2) &= \mathrm{diag}(1-\varepsilon_3, 1-\varepsilon_3). \end{aligned}$$

So α is a proper Jordan automorphism. Also there is no mapping $\mathrm{E}_2(R) \to \mathrm{E}_2(R)$ with $E(T) \mapsto E(T^\alpha)$ for all $T \in \mathcal{S}(R)$.

(f) Let R be given as in Example (e) with $B = D$ and $\chi = \mathrm{id}_D$. (Then $R = D^4$ is isomorphic to the exterior algebra $\bigwedge D^2$.) Furthermore $\alpha : R \to R$ is defined as in (e), but we reserve the letter R' for the D-algebra of (4×4)-matrices over D. The right regular representation ρ of R maps each $a \in R$ to that (4×4)-matrix which describes the linear mapping $R \to R : x \mapsto xa$ in terms of the basis $(1, \varepsilon_1, \varepsilon_2, \varepsilon_3)$. The product $\alpha\rho$ is a Jordan monomorphism, say $\beta : R \to R'$. We have $R'' = R^\beta = R^\rho$.

The first row of the (4×4)-matrix $u' := (1-\varepsilon_3)^\rho \in R'$ reads $(1, 0, 0, -1)$, since $1(1-\varepsilon_3) = 1-\varepsilon_3$. So it is not in the centre of R' and there is a (4×4)-matrix $r' \in R'$ that does not commute with u'. A straightforward calculation shows that $E(r')^{-1}\mathrm{diag}(u', u')E(r')$ is not diagonal. On the other hand, all matrices in N_β are diagonal, since they are in the centre H'' of $\mathrm{E}_2(R'')$. Example (e) shows that $\mathrm{diag}(u', u') \in N_\beta$, whence N_β is not normal in $\mathrm{E}_2(R')$.

(g) With R as in Example (e) let $\alpha_1 = \mathrm{id}_B$, whereas α_2 is given by $\varepsilon_1 \mapsto \varepsilon_1$, $\varepsilon_2 \mapsto \varepsilon_2$, and $\varepsilon_3 \mapsto 0$. Then R^α is not a subring of R, since $\varepsilon_3 = \varepsilon_1^\alpha\varepsilon_2^\alpha$ is not in the image of α.

4 The projective line over a ring

4.1 Consider the free left R-module R^2. The *projective line over* R is the orbit of the free cyclic submodule $R(1, 0)$ under the natural right action of $\mathrm{GL}_2(R)$. In other words, $\mathbb{P}(R)$ is the set of all $p \leq R^2$ such that $p = R(a, b)$, where (a, b) is the first row of an invertible matrix; compare [15, p. 785]. If also (c, d) is the first row of an invertible matrix, then $R(a, b) = R(c, d)$ if and only if there is a unit $u \in R^*$ with $(c, d) = u(a, b)$ [8, Proposition 2.1].

Let $\{(a,b),(c,d)\}$ be a basis of R^2. Then the points $p=R(a,b)$ and $q=R(c,d)$ are called *distant*. In this case we write $p \vartriangle q$. The vertices of the *distant graph* on $\mathbb{P}(R)$ are the points of $\mathbb{P}(R)$, the edges of this graph are the unordered pairs of distant points. The set $\mathbb{P}(R)$ can be decomposed into *connected components* (maximal connected subsets of the distant graph); cf. [9, p. 108].
The orbit of $R(1,0)\in\mathbb{P}(R)$ under the group $\mathrm{E}_2(R)$ is exactly the connected component of $R(1,0)$ [9, Theorem 3.2]. It will be denoted by C. If a matrix $E(T)\in\mathrm{E}_2(R)$ fixes $R(1,0)$ and all points $R(t,1)$ with $t\in R$, then it is easily seen that $E(T)=\mathrm{diag}(a,a)$ with $a\in Z(R)^*$. On the other hand, each such $E(T)$ fixes all points of $\mathbb{P}(R)$. So the kernel of the group action of $\mathrm{E}_2(R)$ on C (or $\mathbb{P}(R)$) is the centre H of $\mathrm{E}_2(R)$; cf. (7). As usual, we write $\mathrm{PE}_2(R):=\mathrm{E}_2(R)/H$ for the *projective elementary group.*

Lemma 4.2 *The group* $\mathrm{E}_2(R)$ *acts* 2-$\vartriangle$-transitively *on the connected component* $C\subseteq\mathbb{P}(R)$*, i.e. transitively on the set of ordered pairs of distant points of* C*.*

Proof: Let (p,q) be a pair of distant points in C. Since all points of C are in one orbit of $\mathrm{E}_2(R)$, there exists a matrix in $\mathrm{E}_2(R)$ sending p to $R(1,0)$ and q to $q_1 \vartriangle R(1,0)$. So $q_1=R(t,1)$ for some $t\in R$. Now $E(0)E(-t)$ fixes $R(1,0)$ and takes q_1 to $R(0,1)$. □

4.3 Suppose that R'' is a subring of a ring R'. As $\mathrm{GL}_2(R'')$ is a subgroup of $\mathrm{GL}_2(R')$, there is a mapping

$$\mathbb{P}(R'')\to\mathbb{P}(R') : R''(a'',b'')\mapsto R'(a'',b'')$$

which is easily seen to be injective. It will be used to identify $\mathbb{P}(R'')$ with a subset of $\mathbb{P}(R')$. We have to distinguish between the connected components C'' and C' of $R'(1',0')$ in $\mathbb{P}(R'')$ and $\mathbb{P}(R')$, respectively. Since $\mathrm{E}_2(R'')$ is a subgroup of $\mathrm{E}_2(R')$, we have $C''\subseteq C'$.
However, unless the centre H'' of $\mathrm{E}_2(R'')$ lies in the centre H' of $\mathrm{E}_2(R')$, the group $\mathrm{PE}_2(R'')$ cannot be considered as a subgroup of $\mathrm{PE}_2(R')$: In general there are two matrices in $\mathrm{E}_2(R'')$ that induce the same transformation on C'', but distinct transformations on C'. So we cannot always consider $\mathrm{PE}_2(R'')$ as a group which acts on C'.
The reader should keep these remarks in mind with regard to the following theorem, which is our second main result:

Theorem 4.4 *Let* $\alpha : R\to R'$ *be a Jordan homomorphism and denote by* R'' *the subring of* R' *generated by* R^α*. Then the following statements are true.*

(a) *The mapping*

$$\alpha_{\mathrm{PE}} : \mathrm{PE}_2(R)\to\mathrm{PE}_2(R'') : H\cdot E(T)\mapsto H''\cdot E(T^\alpha), \tag{26}$$

where $T\in\mathcal{S}(R)$*, is a well defined homomorphism of groups.*

(b) *Consider the connected component* $C\subseteq\mathbb{P}(R)$ *and the connected component* $C''\subseteq\mathbb{P}(R'')\subseteq\mathbb{P}(R')$*. Then the mapping*

$$\overline{\alpha} : C\to C'' : R(1,0)\cdot E(T)\mapsto R'(1',0')\cdot E(T^\alpha), \tag{27}$$

where $T\in\mathcal{S}(R)$*, is well defined.*

(c) *The pair* $(\alpha_{\mathrm{PE}}, \overline{\alpha})$ *is a homomorphism of transformation groups.*

Proof: (a) By Theorem 3.6 (b), $N_\alpha \subseteq H''$. So there exists the canonical epimorphism

$$\eta : \mathrm{E}_2(R'')/N_\alpha \to \big(\mathrm{E}_2(R'')/N_\alpha\big)/(H''/N_\alpha) \cong \mathrm{E}_2(R'')/H''.$$

We identify its image with $\mathrm{PE}_2(R'')$. So, by (22), the composition $\alpha_{\mathrm{E}}\eta : \mathrm{E}_2(R) \to \mathrm{PE}_2(R'')$ is a homomorphism. It follows from Theorem 3.6 (a) that $H^{\alpha_{\mathrm{E}}} \subseteq H''/N_\alpha = \ker \eta$. Hence $H \subseteq \ker \alpha_E \eta$ and α_{PE} is a well defined homomorphism of groups.
(b) We regard $\mathrm{PE}_2(R)$ and $\mathrm{PE}_2(R'')$ as transformation groups on the connected components C and C'', respectively.
Suppose that a matrix $E(S)$, $S \in \mathcal{S}(R)$, fixes $R(1,0)$. Thus its first row has the form $(u,0)$ with $u \in R^*$. We infer from Theorem 3.5 that the first row of $E(S^\alpha)$ reads $(u', 0')$ with a unit $u' \in R''$. So $E(S^\alpha)$ leaves $R'(1',0')$ invariant. This means that under α_{PE} the stabilizer of $R(1,0)$ is mapped into the stabilizer of $R'(1',0')$.
For each point $p \in C$ there is a sequence $T \in \mathcal{S}(R)$ such that $p = R(1,0)\cdot E(T)$. Also let $p = R(1,0)\cdot E(V)$ with $V \in \mathcal{S}(R)$. So the transformation $H \cdot E(T)E(V)^{-1} \in \mathrm{PE}_2(R)$ fixes $R(1,0)$, whence the transformation $H''\cdot E(T^\alpha)E(V^\alpha)^{-1} \in \mathrm{PE}_2(R'')$ fixes $R'(1',0')$. Therefore $\overline{\alpha}$ is well defined.
(c) By (a) and (b), the diagram

$$\begin{array}{ccc} C & \xrightarrow{\;E(T)\;} & C \\ \big\downarrow \overline{\alpha} & & \big\downarrow \overline{\alpha} \\ C'' & \xrightarrow{\;E(T^\alpha)\;} & C'' \end{array} \tag{28}$$

commutes for each $T \in \mathcal{S}(R)$, whence the assertion follows. □

4.5 For each point $p \in C$ there is a smallest integer $n \geq 0$ such that $p = R(1,0)\cdot E(T)$ with $T \in R^n$. In fact, n is just the distance of p and $R(1,0)$ in the distant graph [9, formula (10)]. The supremum of all distances between points of C is a non-negative integer or ∞. It is called the *diameter* of C. Furthermore, we have $(1,0)\cdot E(0)^2 = (-1,0)$ and $(1,0)\cdot E(t) = (1,0)\cdot E(1,t+1)$ for all $t \in R$. So if the diameter of C is finite, say d, then it is enough to consider sequences $T \in R^m$ with fixed length $m := \max\{2,d\}$ in order to reach all points of C. By (14), in this case $\overline{\alpha}$ can be described by the single formula

$$R\big(e_1^m(T), e_1^{m-1}(T)\big)^{\overline{\alpha}} = R'\big(e_1^m(T^\alpha), e_1^{m-1}(T^\alpha)\big) \text{ with } T \in R^m. \tag{29}$$

This generalizes [1, Theorem 2.4], where $m = 2$ and R is a ring of stable rank 2. See also [10, Remark 5.4] for the special case of an antiisomorphism of rings. We shall see in Example 4.11 that there are rings where one needs sequences $T \in R^n$ for infinitely many $n \geq 0$ in order to describe $\overline{\alpha}$.

4.6 Let us state some immediate consequences of Theorem 4.4. If (a,b) and (a',b') are the first rows of the matrices $E(V) \in \mathrm{E}_2(R)$ and $E(V^\alpha)$, respectively, then (28) implies that

$$\big(R(a,b)\cdot E(T)\big)^{\overline{\alpha}} = R'(a',b')\cdot E(T^\alpha) \tag{30}$$

for all $T \in \mathcal{S}(R)$. Letting $E(V)=E(0)$ we get $(a,b)=(0,1)$ and $(a',b')=(0',1')$. Therefore also the second rows of the matrices $E(T)$ and $E(T^\alpha)$, with $T \in \mathcal{S}(R)$, represent points corresponding under $\overline{\alpha}$. We deduce from (27), (30), and (5) that

$$R(t,1)^{\overline{\alpha}} = \big(R(1,0)\cdot E(t)\big)^{\overline{\alpha}} = R'(1',0')\cdot E(t^\alpha) = R'(t^\alpha,1'), \tag{31}$$

$$R(1,t)^{\overline{\alpha}} = \big(R(0,1)\cdot E(0,-t,0)\big)^{\overline{\alpha}} = R'(0',1')\cdot E(0',-t^\alpha,0') = R'(1',t^\alpha), \tag{32}$$

for all $t \in R$. In particular, $\overline{\alpha}$ is indeed an extension of the mapping described in (1). Furthermore, $\overline{\alpha}$ is a *fundamental mapping*; this means that $R(1,0)^{\overline{\alpha}} = R'(1',0')$, $R(0,1)^{\overline{\alpha}} = R'(0',1')$, and $R(1,1)^{\overline{\alpha}} = R'(1',1')$.
If $R^\alpha = R''$ then each point of C'' can be written as $R'(1',0')\cdot E(T^\alpha)$, whence $C^{\overline{\alpha}} = C''$. Similarly, if $R^\alpha = R'$ then $C^{\overline{\alpha}} = C' = C''$. If $\overline{\alpha}$ injective then (31) implies the injectivity of α. Also, if α is bijective then $\alpha^{-1}: R' \to R$ is a Jordan isomorphism and $\overline{\alpha^{-1}}$ is easily seen to be the inverse of $\overline{\alpha}$. What remains open here is whether or not $C^{\overline{\alpha}} = C''$ implies that $R^\alpha = R''$ and whether or not $\overline{\alpha}$ is injective if α is injective.

4.7 A mapping $\mathbb{P}(R) \to \mathbb{P}(R')$ is said to be *harmonic* if it preserves harmonic quadruples or, in other words, if it preserves cross ratio -1 [15, 1.3.5]. If (p_0,p_1,p_2,p_3) is a harmonic quadruple in $\mathbb{P}(R)$ then p_0, p_1, and p_i are mutually distant for $i \in \{2,3\}$. Furthermore, all four points are mutually distant if and only if $-1 \neq 1$ in R. Otherwise $p_2 = p_3$, whence in this case harmonic mappings do not deserve our interest.
In order to state the next result we have to allow the domain and the codomain of a harmonic mapping to be a subset of a projective line.

Proposition 4.8 *Let $\alpha: R \to R'$ and $\overline{\alpha}: C \to C''$ be given as in Theorem 4.4. Then the following statements are true:*

(a) *$\overline{\alpha}$ takes pairs of distant points to pairs of distant points.*

(b) *$\overline{\alpha}$ is a harmonic mapping.*

Proof: (a) Let (p,q) be a pair of distant points in C. By Lemma 4.2, there exists a matrix in $\mathrm{E}_2(R)$ sending p to $R(1,0)$ and q to $R(0,1)$. Clearly, the $\overline{\alpha}$-images of $R(1,0)$ and $R(0,1)$ are distant and, by (28), the points $p^{\overline{\alpha}}$ and $q^{\overline{\alpha}}$ are distant, too.
(b) Suppose that p_1, p_2 are points of $\mathbb{P}(R)$. Then $\big(R(1,0),R(0,1),p_1,p_2\big)$ is a harmonic quadruple if and only if there is a $u \in R^*$ with $p_1 = R(u,1)$ and $p_2 = R(-u,1)$.
If we are given a harmonic quadruple in C then, by Lemma 4.2 and (28), we may assume without loss of generality that the first two points are $R(1,0)$ and $R(0,1)$. So the remaining two points can be described as above. We deduce from (31) and $u^\alpha \in R'^*$ that under $\overline{\alpha}$ a harmonic quadruple is obtained. □

The first part of Proposition 4.8 implies that

$$\operatorname{dist}(p^{\overline{\alpha}}, q^{\overline{\alpha}}) \leq \operatorname{dist}(p,q) \text{ for all } p,q \in C. \tag{33}$$

Here $\operatorname{dist}(p,q)$ denotes the distance of two points in the distant graph.

4.9 We turn to following question: If $C \neq \mathbb{P}(R)$, how should one extend $\overline{\alpha}$ to a mapping $\gamma : \mathbb{P}(R) \to \mathbb{P}(R')$? In view of Proposition 4.8 such an extension should at least preserve distant pairs and harmonic quadruples. Here are solutions to this problem.

Examples 4.10 (a) For each connected component C_μ of $\mathbb{P}(R)$ other than C choose a matrix $A_\mu \in \mathrm{GL}_2(R)$ such that the first row of A_μ represents a point of C_μ. Also select a matrix $A'_\mu \in \mathrm{GL}_2(R')$. Then A_μ^{-1} maps C_μ onto C, $\overline{\alpha}$ takes C into C', and A'_μ maps C' into some connected component of $\mathbb{P}(R')$. In this way we obtain a solution γ by pasting together all these mappings.

(b) In Example (a) the matrices A'_μ can be chosen at random. Suppose now that there is a homomorphism $\sigma : \mathrm{GL}_2(R) \to \mathrm{GL}_2(R')$ such that σ restricts to α_{E} and such that σ takes the stabilizer of $R(1,0)$ into the stabilizer of $R'(1',0')$. Then

$$\overline{\sigma} : \mathbb{P}(R) \to \mathbb{P}(R') : R(1,0)\cdot M \mapsto R'(1',0')\cdot M^\sigma \ \text{ with } M \in \mathrm{GL}_2(R) \tag{34}$$

is a well defined extension of $\overline{\alpha}$ and $(\sigma, \overline{\sigma})$ is a homomorphism of group actions. The mapping $\overline{\sigma}$ fits into Example (a) by choosing $A'_\mu = A_\mu^\sigma$.

The homomorphisms α_*, α_{**}, and β which have been introduced in 3.8 (a), (b), and (c), respectively, satisfy the conditions above: A matrix $M \in \mathrm{GL}_2(R)$ stabilizes $R(1,0)$ if and only if there are elements $a, d \in R^*$ and $c \in R$ with $M = \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$. If α is a homomorphism of rings then the assertion is immediate. Furthermore, in this case we get the well known mapping

$$\overline{\alpha_*} : \mathbb{P}(R) \to \mathbb{P}(R') : R(a,b) \mapsto R'(a^\alpha, b^\alpha). \tag{35}$$

If α is an antihomomorphism then a straightforward calculation shows

$$M^{\alpha_{**}} = \begin{pmatrix} (d^\alpha)^{-1} & 0' \\ (a^\alpha)^{-1} c^\alpha (d^\alpha)^{-1} & (a^\alpha)^{-1} \end{pmatrix},$$

whence the assertion follows also in the remaining cases, cf. also [10, Remark 5.4].

We end this section with an example where $\mathbb{P}(R)$ has more than one connected component, the connected components of $\mathbb{P}(R)$ have infinite diameter, and α is a proper Jordan endomorphism.

Example 4.11 We specify and slightly modify the data of Example 3.8 (e) as follows: Let $D = K$ be a commutative field, let $B = K[x,y]$ be the algebra of polynomials in commuting indeterminates x, y over K, and let $\chi : K[x,y] \to K : f \mapsto f(0,0)$. The module $M = K^3$ and its multiplication remain unchanged. Hence $R = K[x,y] \oplus K^3$. Again $\alpha_1 = \mathrm{id}_{K[x,y]}$, but now α_2 is chosen to be any K-linear mapping with ε_3 not being an eigenvector. Then α is a

proper K-linear Jordan endomorphism, since $(\varepsilon_1\varepsilon_2)^\alpha = \varepsilon_3^\alpha \notin K\varepsilon_3$, whereas $\varepsilon_1^\alpha\varepsilon_2^\alpha \in K\varepsilon_3$ and $\varepsilon_2^\alpha\varepsilon_1^\alpha \in K\varepsilon_3$.
The projection $\pi : R \to K[x,y]$ is an epimorphism of K-algebras with kernel $\{0\} \oplus K^3$. By (35), it gives rise to a mapping $\overline{\pi_*} : \mathbb{P}(R) \to \mathbb{P}(K[x,y])$ which is surjective [8, Proposition 3.5 (3)]. Under $\overline{\pi_*}$ the connected component of $R(1,0)$ is mapped onto the connected component of $K[x,y](1,0)$, since π_{PE} is a surjection of $\mathrm{PE}_2(R)$ onto $\mathrm{PE}_2(K[x,y])$. The projective line $\mathbb{P}(K[x,y])$ has more than one connected component and all its connected components have infinite diameter [9, Example 5.7 (c)]. So (33) implies that also $\mathbb{P}(R)$ has more than one connected component and that $C \subset \mathbb{P}(R)$ has infinite diameter. By [9, Theorem 3.2 (a)], then all connected components of $\mathbb{P}(R)$ have infinite diameter.

5 On homomorphisms of chain geometries

5.1 If $K \subseteq R$ is a (not necessarily commutative) subfield, then the projective line $\mathbb{P}(K)$ can be identified with a subset of $\mathbb{P}(R)$; cf. 4.3. The orbit of $\mathbb{P}(K)$ under the group $\mathrm{GL}_2(R)$ is the set of *K-chains.* It turns $\mathbb{P}(R)$ into a *chain geometry* $\Sigma(K,R)$. The following basic properties of chain geometries can be found in [7]: Any three mutually distant points are on at least one K-chain. Two distinct points are distant if and only if they are on a common K-chain. Therefore each K-chain is contained in a unique connected component. In contrast to [15] it is not assumed that K is in the centre of R, whence in [7] we used the term *generalized chain geometry* for what is here called a chain geometry.
We now consider two chain geometries $\Sigma(K,R)$, $\Sigma(K',R')$ and the mapping (27). The following result is a generalization of [1, Theorem 2.4] and [15, 9.1]:

Theorem 5.2 *Let $\alpha : R \to R'$ be a Jordan homomorphism. The mapping $\overline{\alpha} : C \to C''$ maps K-chains into K'-chains if and only if for each $c \in R^*$ there is a $u'_c \in R'^*$ such that*

$$(Kc)^\alpha \subseteq ({u'_c}^{-1}K'u'_c)c^\alpha. \tag{36}$$

Proof: For each $c \in R^*$ the point set

$$\mathcal{D}_c := \{R(kc,1) \mid k \in K\} \cup \{R(1,0)\} \tag{37}$$

is a K-chain through $R(1,0)$, $R(0,1)$, and $R(c,1)$. The K'-chains passing through $R'(1',0')$, $R'(0',1')$, and $R'(c^\alpha,1')$ are exactly the sets

$$\{R\big(({u'}^{-1}k'u')c^\alpha, 1'\big) \mid k' \in K'\} \cup \{R'(1',0')\} \tag{38}$$

where u' ranges in R'^*.
Suppose that $\overline{\alpha}$ maps K-chains into K'-chains. So for each $c \in R^*$ the point set $\mathcal{D}_c^{\overline{\alpha}}$ is a subset of a K'-chain. Now (31) implies that this chain contains the points $R'(1',0')$, $R'(0',1')$, and $R'(c^\alpha,1')$, whence it can be written in the form (38) for some $u'_c \in R'^*$ depending on c. Applying (31) to each point of (37) shows that condition (36) is satisfied.

Conversely, (36) forces that each K-chain $\mathcal{D}_c$ given by (37) is mapped into one of the K'-chains given by (38). By Lemma 4.2, every K-chain $\mathcal{D} \subseteq C$ is $\mathrm{E}_2(R)$-equivalent to some K-chain through $R(1,0)$ and $R(0,1)$. Such a chain has the form

$$\{R(ka,b) \mid k \in K\} \cup \{R(1,0)\} \text{ with } a,b \in R^*.$$

Since $\mathrm{diag}(b,b^{-1}) = E(-b)E(-b^{-1})E(-b) \in \mathrm{E}_2(R)$, the chains $\mathcal{D}$ and $\mathcal{D}_c$, where $c := ab$, are in one orbit of $\mathrm{E}_2(R)$. Now (28) shows that also $\mathcal{D}^{\overline{\alpha}}$ is a subset of a K'-chain. □

Condition (36) reduces to $(Kc)^\alpha \subseteq K'c^\alpha$ provided that K' is invariant under all inner automorphisms of R'. This is the case whenever K' is in the centre of R', but there are also other possibilities [7, Examples 2.5].

5.3 We close with some remarks on a mapping $\overline{\alpha}$ where α satisfies the conditions of Theorem 5.2. If $\mathbb{P}(R) = C$ then $\overline{\alpha}$ is a *homomorphism of chain geometries*, i.e., K-chains are mapped into K'-chains. If $\mathbb{P}(R) \neq C$ then $\overline{\alpha}$ can be extended to a mapping $\gamma : \mathbb{P}(R) \to \mathbb{P}(R')$ according to Example 4.10 (a). As the general linear group preserves chains, any such γ is a homomorphism of chain geometries. Explicit examples for this latter case arise from Example 4.11, because all Jordan endomorphisms described there are K-linear and thus fulfil condition (36).

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