

# Projective Lines over Jordan Systems and Geometry of Hermitian Matrices

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*Dedicated to Mario Marchi on the occasion of his 70th birthday*

## Abstract

Any set of  $\sigma$ -Hermitian matrices of size  $n \times n$  over a field with involution  $\sigma$  gives rise to a *projective line* in the sense of ring geometry and a *projective space* in the sense of matrix geometry. It is shown that the two concepts are based upon the same set of points, up to some notational differences.

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## 1 Introduction

Let  $R = K^{n \times n}$  be the ring of  $n \times n$  matrices over a (not necessarily commutative) field  $K$  which admits an involution  $\sigma$ . We denote by  $H_\sigma \subset R$  the subset of  $\sigma$ -Hermitian matrices. We exhibit two well known constructions: The *projective line over the Jordan system*  $H_\sigma \subset R$  is a subset of the point set of the *projective line over the matrix ring*  $R$ . It comprises all points which can be written in the form  $R(T_2 T_1 - I, T_2)$  with  $\sigma$ -Hermitian matrices  $T_1, T_2$ ; cf. [6, 3.1.14]. The *projective space of  $\sigma$ -Hermitian  $n \times n$  matrices* is a subset of the point set of the *projective space of  $n \times n$  matrices over  $K$* . Its points are the left row spaces of those matrices  $(A, B) \in K^{n \times 2n}$  which have full left row rank and are composed of blocks  $A, B \in K^{n \times n}$  satisfying  $A(B^\sigma)^\top = B(A^\sigma)^\top$ ; cf. [16, 6.8].

We recall in Section 2 that the point set of the projective line over  $R$  is, up to a natural identification, the Grassmannian of  $n$ -dimensional subspaces of  $K^{2n}$  which in turn is nothing but the point set of the projective space of  $n \times n$  matrices over  $K$ . In Section 3 we exhibit the two subsets which arise from  $\sigma$ -Hermitian matrices according to the above mentioned constructions. The coincidence of these two

subsets is not obvious. Indeed, in the ring-geometric setting we get a set of points in terms of a *parametric representation*, whereas in the matrix-geometric setting there is a *matrix equation* which has to be satisfied. Our main result (Theorem 1) states that the two subsets coincide. The proof of one inclusion simply amounts to plugging in the parametrisation in the matrix equation. Our proof of the other inclusion is more involved. It uses the rather technical Lemma 1 and Lemma 2, which is geometric in flavour, as it deals with maximal totally isotropic subspaces of a  $\sigma$ -anti-Hermitian sesquilinear form. For a commutative field  $K$  and  $\sigma = \text{id}_K$  our Lemma 2 turns into the result [6, Satz 10.2.3].

As an application we show in Remarks 1–5 how several results from ring geometry can be translated to projective matrix spaces.

## 2 Square matrices

Let  $K$  be any (not necessarily commutative) field and  $n \geq 1$ . We shall be concerned with the ring  $R := K^{n \times n}$  of  $n \times n$  matrices with entries in  $K$ . Any  $r \times s$  matrix over  $R$  can be viewed as an  $rn \times sn$  matrix over  $K$  which is partitioned into  $rs$  blocks of size  $n \times n$  and vice versa. An  $r \times r$  matrix over  $R$  is invertible if, and only if, it is invertible as an  $rn \times rn$  matrix over  $K$ .

Consider the free left  $R$ -module  $R^2$  and the group  $\text{GL}_2(R) = \text{GL}_{2n}(K)$  of invertible  $2 \times 2$ -matrices with entries in  $R$ . A pair  $(A, B) \in R^2$  is called *admissible*, if there exists a matrix in  $\text{GL}_2(R)$  with  $(A, B)$  being its first row. Following [11, p. 785] and [6], the *projective line over  $R$*  is the orbit of the free cyclic submodule  $R(I, 0)$  under the natural right action of  $\text{GL}_2(R)$ , where  $I$  and  $0$  denote the  $n \times n$  identity and  $n \times n$  zero matrix over  $K$ , respectively. So

$$\mathbb{P}(R) := R(I, 0)^{\text{GL}_2(R)} \quad (1)$$

or, in other words,  $\mathbb{P}(R)$  is the set of all  $p \subset R^2$  such that  $p = R(A, B)$  for an admissible pair  $(A, B) \in R^2$ . Two admissible pairs represent the same point precisely when they are left-proportional by a unit in  $R$ , i. e., a matrix from  $\text{GL}_n(K)$ . Conversely, if for some pair  $(A', B') \in R^2$  and an admissible pair  $(A, B) \in R^2$  we have  $R(A', B') = R(A, B)$  then  $(A', B')$  is admissible too. This follows from [3, Proposition 2.2], because in  $R$  the notions of “right invertibility” and “invertibility” coincide.

The projective line over  $R$  allows the following description which makes use of the *left row rank* of a matrix  $X$  over  $K$  (in symbols:  $\text{rank } X$ ):

$$\mathbb{P}(R) = \{R(A, B) \mid A, B \in R, \text{rank}(A, B) = n\}. \quad (2)$$

Here  $(A, B)$  has to be interpreted as the matrix arising from  $A$  and  $B$  by means of horizontal augmentation. By (2), the point set of  $\mathbb{P}(R)$  is in bijective corre-

spondence with the Grassmannian  $\text{Gr}_{2n,n}(K)$  of  $n$ -dimensional subspaces of  $K^{2n}$  via

$$\mathbb{P}(R) \rightarrow \text{Gr}_{2n,n}(K) : R(A, B) \mapsto \text{left row space of } (A, B). \quad (3)$$

See [2], [3], and [13] for this result and its generalisations.

**Convention.** *We do not distinguish between a point of the projective line  $\mathbb{P}(R)$  and its corresponding subspace of  $K^{2n}$  via (3).*

Following [6] a ring will be called *stable* if it has stable rank 2. By [15, 2.6], our matrix ring  $R = K^{n \times n}$  is stable. This means that for each  $(A, B) \in R^2$  which is *unimodular*, i. e., there are  $X, Y \in R$  with  $AX + BY = I$ , there exists a matrix  $W \in R$  such that  $A + BW \in \text{GL}_n(K)$ . See [15, § 2]. Due to stableness two important results hold: Firstly, any unimodular pair  $(A, B) \in R^2$  generates a point [15, 2.11]. Secondly, *Bartolone's parametrisation*

$$R^2 \rightarrow \mathbb{P}(R) : (T_1, T_2) \mapsto R(T_2T_1 - I, T_2) \quad (4)$$

is well defined and surjective [1]. It allows us to write the projective line  $\mathbb{P}(R)$  in the form

$$\mathbb{P}(R) = \{R(T_2T_1 - I, T_2) \mid T_1, T_2 \in R\}. \quad (5)$$

Formula (5) was put in a more general context in [4], and we shall follow the notation from there. Altogether we have four equivalent descriptions of the projective line over a matrix ring  $R = K^{n \times n}$ .

The point set  $\mathbb{P}(R)$  is endowed with the symmetric and anti-reflexive relation *distant* ( $\Delta$ ) defined by

$$\Delta := (R(I, 0), R(0, I))^{\text{GL}_2(R)}.$$

For arbitrary points  $p = R(A, B)$  and  $q = R(C, D)$  of  $\mathbb{P}(R)$  we obtain

$$p \Delta q \Leftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_2(R) = \text{GL}_{2n}(K).$$

This in turn is equivalent to the complementarity of the  $n$ -dimensional subspaces of  $K^{2n}$  which correspond to  $p$  and  $q$  via (3). The vertices of the *distant graph* on  $\mathbb{P}(R)$  are the points of  $\mathbb{P}(R)$ , two vertices of this graph are joined by an edge if, and only if, they are distant. A crucial property of the projective line over our ring  $R$ , and more generally over any stable ring, is as follows [11, 1.4.2]: Given any two points  $p$  and  $q$  there exists some point  $r$  such that  $p \Delta r \Delta q$ . This implies that the distant graph on  $\mathbb{P}(R)$  is connected and that its diameter is  $\leq 2$ .

For example, given  $p = R(I, 0)$  and any other point  $q \in \mathbb{P}(R)$  we have  $q = R(T_2T_1 - I, T_2)$  by (5). Then  $r := R(T_1, I)$  has the required property

$$R(I, 0) \Delta R(T_1, I) \Delta R(T_2T_1 - I, T_2). \quad (6)$$

Comparing the description of the point set  $\mathbb{P}(K^{n \times n}) = \mathbb{P}(R)$  in (2) with the definition of the point set of the *projective space of  $m \times n$  matrices over  $K$*  in [16, 3.6] one sees immediately that the two definitions coincide for  $m = n \geq 2$  (due to our convention from above). So in our setting proper rectangular matrices as in [16] are not allowed and, to be compatible with [16], we assume from now on that  $n \geq 2$ , i. e., we disregard the projective line  $\mathbb{P}(K^{1 \times 1})$ . There is an immaterial difference though, as we make use of the vector space  $K^{2n}$  rather than the projective space on  $K^{2n}$  as in [16]. This is only done in order to simplify notation.

The major difference in the two approaches concerns the *additional structure* which is imposed: In the ring-theoretic setting this is the notion of *distance*, whereas in the matrix-theoretic setting the concept of *adjacency* ( $\sim$ ) is used. Recall that two  $n$ -dimensional subspaces of  $K^{2n}$  are called adjacent if, and only if, their intersection has dimension  $n - 1$ . The vertices of the *Grassmann graph* on  $\text{Gr}_{2n,n}(K)$  are the elements of  $\text{Gr}_{2n,n}(K)$ , two vertices are joined by an edge if, and only if, they are adjacent. The graph-theoretical distance between two vertices  $W_1, W_2$  of the Grassmann graph on  $\text{Gr}_{2n,n}(K)$  equals their *arithmetical distance*  $\dim(W_1 + W_2) - m$  [16, Proposition 3.32].

However, also the structural approaches can be shown to be equivalent, because adjacency can be expressed in terms of being distant and vice versa: Two points of  $\mathbb{P}(R)$  are distant if, and only if, they are at arithmetical distance  $n$  in the Grassmann graph. The description of  $\sim$  in terms of  $\Delta$  is more subtle, and we refer to [5, Theorem 3.2] for further details.

Even though Bartolone's parametrisation (5) has its origin in ring geometry, the identification from (3) allows its reinterpretation as a surjective parametric representation of  $\text{Gr}_{2n,n}(K)$  in the form

$$R^2 \rightarrow \text{Gr}_{2n,n}(K) : (T_1, T_2) \mapsto \text{left row space of } (T_2 T_1 - I, T_2). \quad (7)$$

We sketch some applications of this result in the geometry of square matrices:

*Remark 1.* The mapping (7) has the disadvantage of being non-injective, but by choosing a fixed matrix  $T_1^{(0)}$  an injective mapping  $R \rightarrow \text{Gr}_{2n,n}(K)$  is obtained from (7). This mapping is easily seen to be an embedding of the matrix space  $R = K^{n \times n}$  in the projective matrix space  $\text{Gr}_{2n,n}(K)$ . For  $T_1^{(0)} = 0$  one gets the “usual” embedding, like in [16].

*Remark 2.* The left row spaces of the matrices  $(I, 0)$  and  $(T_2 T_1 - I, T_2)$  have arithmetical distance  $k$  if, and only if,  $\text{rank } T_2 = k$ . Thus a parametrisation of the “spheres” of  $\text{Gr}_{2n,n}(K)$  with “centre”  $(I, 0)$  and “radius” (arithmetical distance)  $k$  can be obtained from (7) by imposing the extra condition  $\text{rank } T_2 = k$ , while  $T_1 \in R$  is arbitrary. In particular, the case  $k = 1$  can be treated by restricting the choice of  $T_2$  to matrices of the form  $c^T \cdot d$  with  $c, d \in K^n \setminus \{0\}$ .

Similarly, we may also parametrise any maximal set of mutually adjacent elements and any pencil of  $\text{Gr}_{2n,n}(K)$  containing the left row space of  $(I, 0)$  as follows. Firstly, let  $T_2 := c^{(0)\text{T}} \cdot d$  for a fixed vector  $c^{(0)} \in K^{2n} \setminus \{0\}$  and a variable vector  $d \in K^{2n}$ . Then (7) gives the set of all  $n$ -dimensional subspaces which contain the  $(n-1)$ -dimensional subspace given by the linear system  $\sum_{i=1}^n x_i c_i^{(0)} = x_{n+1} = \dots = x_{2n} = 0$ , where the  $c_i^{(0)}$ s are the coordinates of  $c^{(0)}$ . Secondly, let  $T_2 := c^{\text{T}} \cdot d^{(0)}$  for a fixed vector  $d^{(0)} \in K^{2n} \setminus \{0\}$  and a variable vector  $c \in K^{2n}$ . Then (7) gives the set of all  $n$ -dimensional subspaces which are contained in the  $(n+1)$ -dimensional row space of the matrix  $\begin{pmatrix} I & 0 \\ 0 & d^{(0)} \end{pmatrix}$ . Thirdly, let  $T_2 := c^{(0)\text{T}} \cdot t \cdot d^{(0)}$  for fixed vectors  $c^{(0)}, d^{(0)} \in K^{2n} \setminus \{0\}$  and a variable  $t \in K$ . Then (7) gives a pencil of  $n$ -dimensional subspaces or, in the terminology of [16, Definition 3.11], a *line* of our projective matrix space.

*Remark 3.* Let  $\widehat{K^{2n}}$  be the  $2n$ -dimensional right column space over  $K$ . It is the dual of  $K^{2n}$ . For each  $n$ -dimensional subspace  $W$  of  $K^{2n}$  the linear forms (column vectors) which vanish on  $W$  constitute the  $n$ -dimensional annihilating subspace  $W^\circ \subset \widehat{K^{2n}}$ . We may assume that  $W$  is the left row space of  $(T_2 T_1 - I, T_2)$  with  $T_1, T_2 \in R$ . Then

$$W^\circ = \text{right column space of } \begin{pmatrix} -T_2 \\ T_1 T_2 - I \end{pmatrix}. \quad (8)$$

For  $T_1 = 0$  this result is folklore.

*Remark 4.* Let  $\iota : R \rightarrow R$  be any *Jordan isomorphism* (see [11, 9.1] or [16, Definition 3.7], where the term *semi-isomorphism* is used instead). Then the mapping  $\text{Gr}_{2n,n}(K) \rightarrow \text{Gr}_{2n,n}(K)$  given by

$$\text{left row space of } (T_2 T_1 - I, T_2) \mapsto \text{left row space of } (T_2^\iota T_1^\iota - I, T_2^\iota) \quad (9)$$

is well-defined. This follows from [1, Theorem 2.4] or [6, Satz 4.2.11] by removing superfluous conditions about the ground field; see also [11, Theorem 9.1.1]. The well-definedness is also a direct consequence of [4, Theorem 4.4]. In our setting there is an easier proof: The Jordan isomorphism  $\iota$  is either of the form  $X \mapsto Q^{-1} X^\gamma Q$ , with  $\gamma$  an automorphism of  $K$  and  $Q \in \text{GL}_n(K)$ , or of the form  $X \mapsto Q^{-1} (X^\delta)^\text{T} Q$ , with  $\delta$  an antiautomorphism of  $K$  and  $Q \in \text{GL}_n(K)$ . See, e. g., [16, Theorem 3.24]. In the first case (9) coincides with the natural action of the semilinear bijection  $K^{2n} \rightarrow K^{2n} : x \mapsto x^\gamma \cdot \text{diag}(Q, Q)$  on  $\text{Gr}_{2n,n}(K)$ . In the second case we consider the non-degenerate sesquilinear form

$$K^{2n} \times K^{2n} \rightarrow K : (x, y) \mapsto x \cdot \begin{pmatrix} 0 & -Q^{-1} \\ Q^{-1} & 0 \end{pmatrix} \cdot (y^\delta)^\text{T}.$$

It acts on  $\text{Gr}_{2n,n}(K)$  by sending  $W \in \text{Gr}_{2n,n}(K)$  to its perpendicular subspace  $W^\perp$  [16, Proposition 3.42] and has the required properties, since

$$(T_2^t T_1^t - I, T_2^t) \cdot \begin{pmatrix} 0 & -Q^{-1} \\ Q^{-1} & 0 \end{pmatrix} \cdot ((T_2 T_1 - I, T_2)^\delta)^\top = 0.$$

The previous formulas show that (9) together with the natural action of  $\text{GL}_2(R) = \text{GL}_{2n}(K)$  provides a *unified explicit description of adjacency preserving transformations of  $\text{Gr}_{2n,n}(K)$*  which avoids the distinction (like, e. g., in [16, Theorem 3.45]) between semilinear bijections and non-degenerate sesquilinear forms.

### 3 $\sigma$ -Hermitian matrices

Suppose now that the field  $K$  admits an antiautomorphism  $\sigma$  such that  $\sigma^2 = \text{id}_K$ . Such a mapping will be called an *involution*. Observe that we do not adopt any of the extra assumptions on  $\sigma$  from [16, p. 306]. As before, we let  $R = K^{n \times n}$  and the identity matrix of size  $k \times k$  is written as  $I_k$  or simply as  $I$  if  $k$  is understood. The involution  $\sigma$  determines the  $\sigma$ -*transposition*

$$\Sigma : R \rightarrow R : M = (m_{ij}) \mapsto M^\Sigma := (m_{ji}^\sigma).$$

It is an antiautomorphism of  $R$ . The elements of  $H_\sigma := \{X \in R \mid X = X^\Sigma\}$  are the  $\sigma$ -*Hermitian matrices* of  $R$ . If  $M \in R$  is invertible then  $M^{-\Sigma}$  is used as a shorthand for  $(M^{-1})^\Sigma = (M^\Sigma)^{-1}$ . In the special case that  $\sigma = \text{id}_K$  the field  $K$  is commutative, and we obtain the subset of *symmetric matrices* of  $K^{n \times n}$ .

The set  $H_\sigma$  need not be closed under matrix multiplication. In the terminology of [6, 3.1.5]  $H_\sigma$  is a *Jordan system* of  $R$ , where  $R = K^{n \times n}$  is considered as an algebra over the centre  $Z(K)$  of  $K$ . This means that  $H_\sigma$  is a subspace of the  $Z(K)$ -vector space  $R$  which contains  $I$ , and which has the property that

$$A^{-1} \in H_\sigma \quad \text{for all } A \in \text{GL}_n(K) \cap H_\sigma. \quad (10)$$

Moreover,  $H_\sigma$  is *Jordan closed*, i. e., it satisfies the condition

$$ABA \in H_\sigma \quad \text{for all } A, B \in H_\sigma. \quad (11)$$

In [6, Lemma 3.1.11] it is shown that condition (11) follows from (10) under a certain richness assumption on  $H_\sigma$  (called *strongness* there). See also [10].

Generalising the definition in [6, 3.1.14] we define the *projective line* over  $H_\sigma$  (irrespective of whether  $H_\sigma$  is strong or not) by

$$\mathbb{P}(H_\sigma) = \{R(T_2 T_1 - I, T_2) \mid T_1, T_2 \in H_\sigma\}. \quad (12)$$

Note that this definition makes use of multiplication in the ambient matrix ring  $R = K^{n \times n}$  and that  $\mathbb{P}(H_\sigma)$  is a subset of the projective line over the ring  $R$ . Since we do not adopt an assumption on the strongness of  $H_\sigma$ , we cannot apply any results from [6].

We now recall the definition of the *projective space of  $\sigma$ -Hermitian matrices*. Following [9, III § 3] and [16, 6.8] we consider the left vector space  $K^{2n}$  and the non-degenerate  $\sigma$ -anti-Hermitian sesquilinear form  $\beta : K^{2n} \times K^{2n} \rightarrow K$  given (with respect to the standard basis) by the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \text{GL}_{2n}(K). \quad (13)$$

The basic notions and results about sesquilinear forms which will be used below without further reference can be found in [8, 664–666]. See also [9, I §6–11] and [14, §8]. For all  $x = (x_1, x_2, \dots, x_{2n}) \in K^{2n}$  we obtain

$$(x, x)^\beta = w - w^\sigma \quad \text{with} \quad w := \sum_{i=1}^n (x_i x_{i+n}^\sigma) \quad (14)$$

or, in other words,  $\beta$  is *trace-valued*. We read off from the upper left corner of the matrix in (13) that the span of the first  $n$  vectors of the standard basis is totally isotropic (with respect to  $\beta$ ). Therefore all maximal totally isotropic subspaces have dimension  $n$ . The set comprising all maximal totally isotropic subspaces is the point set of the *projective space of  $\sigma$ -Hermitian matrices* or, in another terminology, the point set of the *dual polar space* [7] given by  $\beta$ . Suppose that  $(A, B) \in R^2$  satisfies  $\text{rank}(A, B) = n$ . Then the ( $n$ -dimensional) row space of  $(A, B) \in K^{n \times 2n}$  is totally isotropic if, and only if,

$$AB^\Sigma = BA^\Sigma. \quad (15)$$

This is immediate by multiplying the matrix in (13) by  $(A, B)$  from the left and  $(A, B)^\Sigma$  from the right hand side; see [16, Proposition 6.41].

In terms of our convention from Section 2 our main result is as follows:

**Theorem 1.** *Let  $K$  be any field admitting an involution  $\sigma$ . The point set of the projective space of  $\sigma$ -Hermitian  $n \times n$  matrices over  $K$  coincides with the projective line over the Jordan system  $H_\sigma$  of all  $\sigma$ -Hermitian matrices of  $R = K^{n \times n}$ .*

We postpone the proof until we have established two lemmas. We note that Lemma 2 generalises [6, Satz 10.2.3], where  $\beta$  is assumed to be an alternating bilinear form, i. e.,  $K$  is commutative and  $\sigma = \text{id}_K$ . The proof from there makes use of the fact that all one-dimensional subspaces of  $K^{2n}$  are totally isotropic, but this property does not hold in general. Therefore our proof follows another strategy.

**Lemma 1.** *Let  $U = V \oplus W$  be a maximal totally isotropic subspace which is given as direct sum of subspaces  $V$  and  $W$ . Then there exists a maximal totally isotropic subspace, say  $X$ , such that  $X \cap V^\perp = W$ .*

*Proof.* Let  $\dim V = k$ . Due to  $\dim W = n - k$ ,  $\dim V^\perp = 2n - k$ , and  $U = U^\perp \subset V^\perp$ , there exists a basis  $(b_1, b_2, \dots, b_{2n})$  of  $K^{2n}$  such that

$$\begin{aligned} V &= \text{span}(b_1, b_2, \dots, b_k), \\ W &= \text{span}(b_{k+1}, b_{k+2}, \dots, b_n), \\ V^\perp &= U \oplus \text{span}(b_{n+k+1}, b_{n+k+2}, \dots, b_{2n}). \end{aligned} \tag{16}$$

The remaining basis vectors  $b_{n+1}, b_{n+2}, \dots, b_{n+k}$  can be chosen arbitrarily. The matrix of  $\beta$  with respect to  $(b_i)$  can be written in block form as

$$M = \begin{pmatrix} 0 & 0 & A & 0 \\ 0 & \boxed{\begin{matrix} 0 & B \\ -B^\Sigma & D - D^\Sigma \end{matrix}} & C \\ -A^\Sigma & \boxed{\begin{matrix} -B^\Sigma & D - D^\Sigma \\ -C^\Sigma & -E^\Sigma \end{matrix}} & E \\ 0 & \boxed{\begin{matrix} -C^\Sigma & -E^\Sigma \end{matrix}} & F - F^\Sigma \end{pmatrix} \tag{17}$$

with  $A \in \text{GL}_k(K)$  and  $C \in \text{GL}_{n-k}(K)$ . We remark that appropriate matrices  $D \in K^{k \times k}$  and  $F \in K^{(n-k) \times (n-k)}$  exist because of (14). Our aim is to go over to a new basis as follows: All basis vectors in  $V$ ,  $W$ , and  $V^\perp$  will stay unchanged. The remaining basis vectors  $b_{n+1}, b_{n+2}, \dots, b_{n+k}$  will be replaced in such a way that all entries in the highlighted submatrix turn to zero when performing the associate transformation on  $M$ .

This task can easily be accomplished in terms of several elementary row and column transformations: First one adds appropriate linear combinations of the last  $n - k$  rows of  $M$  to the  $k$  rows of the third horizontal block in order to eliminate  $-B^\Sigma$ . This is possible, since  $-C^\Sigma \in \text{GL}_{n-k}(K)$ . Subsequently, the corresponding column transformations will eliminate  $B$ . Now the (3, 3)-block of the transformed matrix reads  $D - D^\Sigma + (*) - (*)^\Sigma$ . Next, the first  $k$  rows are used to eliminate  $D + (*)$ . This can be carried out, due to  $A \in \text{GL}_k(K)$ . Finally, one applies the corresponding column operations. Altogether the transition from the basis  $(b_i)$  to the new basis  $(b'_j)$  is given by the matrix

$$T := \begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & I_{n-k} & 0 & 0 \\ ((E - B^\Sigma C^{-\Sigma} F) C^{-1} B - D) A^{-1} & 0 & I_k & -B^\Sigma C^{-\Sigma} \\ 0 & 0 & 0 & I_{n-k} \end{pmatrix} \in \text{GL}_{2n}(K).$$



The elimination from above can be summarised as

$$T \cdot M \cdot T^\Sigma = \begin{pmatrix} 0 & 0 & A & 0 \\ 0 & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & C & \\ -A^\Sigma & & E - B^\Sigma C^{-\Sigma}(F - F^\Sigma) & \\ 0 & -C^\Sigma & -E^\Sigma + (F^\Sigma - F)C^{-1}B & F - F^\Sigma \end{pmatrix} \quad (18)$$

and gives the new matrix for  $\beta$ . By our construction and due to the form of the matrix in (18), the basis vectors  $b'_{k+1}, b'_{k+2}, \dots, b'_{n+k}$  generate a subspace  $X$  with the required properties.  $\square$

**Lemma 2.** *Let  $U_1$  and  $U_2$  be two maximal totally isotropic subspaces of  $(K^{2n}, \beta)$ . Then there exists a maximal totally isotropic subspace  $X$  which is a common complement of  $U_1$  and  $U_2$ .*

*Proof.* (a) Let  $V := U_1 \cap U_2$  and put  $k := \dim V$ . Choose subspaces  $W_i$  such that  $U_i = V \oplus W_i$  for  $i \in \{1, 2\}$ . Then  $V^\perp = V \oplus W_1 \oplus W_2$ . The restriction of  $\beta$  to  $V^\perp \times V^\perp$  might be degenerate, with  $V = V^{\perp\perp}$  being the radical of the restricted form. Consequently, the restriction of  $\beta$  to  $(W_1 \oplus W_2) \times (W_1 \oplus W_2)$  is non-degenerate. It will be written as  $\beta_{12}$ . There exist bases  $(b_1, b_2, \dots, b_{n-k})$  and  $(b_{n-k+1}, b_{n-k+2}, \dots, b_{2n-2k})$  of  $W_1$  and  $W_2$ , respectively. The matrix of  $\beta_{12}$  with respect to  $(b_1, b_2, \dots, b_{2n-2k})$  has the form

$$\begin{pmatrix} 0 & A \\ -A^\Sigma & 0 \end{pmatrix} \quad \text{with } A \in \text{GL}_{n-k}(K). \quad (19)$$

Let  $A^{-1}$  be the matrix describing the change from the basis  $(b_1, b_2, \dots, b_{n-k})$  of  $W_1$  to a new basis  $(b'_1, b'_2, \dots, b'_{n-k})$ , say. Thus the matrix

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^\Sigma & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I \end{pmatrix}^\Sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

describes  $\beta_{12}$  with respect to the basis  $(b'_1, \dots, b'_{n-k}, b_{n-k+1}, \dots, b_{2n-2k})$ . Using this new matrix for  $\beta_{12}$  it is straightforward to show that

$$(b'_r + b_{n+r}, b'_s + b_{n+s})^\beta = 0 \quad \text{for all } r, s \in \{1, 2, \dots, n-k\}.$$

Hence

$$W := \text{span}(b'_1 + b_{n+1}, b'_2 + b_{n+2}, \dots, b'_{n-k} + b_{2n-k})$$

is a totally isotropic subspace. Furthermore, we have  $\dim W = n-k$  and  $W_1 \oplus W_2 = W_1 \oplus W = W_2 \oplus W$ .

(b) Let  $U := V \oplus W$ , the sum being direct due to  $V \cap (W_1 \oplus W_2) = 0$ . So  $\dim U = n$ . From  $W \subset V^\perp \cap W^\perp$  follows  $W^\perp \supset V \oplus W = U$ , whereas  $V^\perp \supset$

$V \oplus W = U$  is obvious. Therefore  $U \subset V^\perp \cap W^\perp = (V \oplus W)^\perp = U^\perp$ . Summing up, we have proved that  $U$  is a maximal totally isotropic subspace of  $K^{2n}$ .

By Lemma 1, applied to  $U = V \oplus W$ , there exists a maximal totally isotropic subspace  $X$  with  $X \cap V^\perp = W$ . Consequently,  $X \cap U_i = (X \cap V^\perp) \cap U_i = W \cap U_i = 0$  which in turn shows that  $X$  is a common complement of  $U_1$  and  $U_2$ .  $\square$

We are now in a position to give the promised proof of Theorem 1.

*Proof.* (a) Any point of the projective line over  $H_\sigma$  can be written in the form  $R(T_2T_1 - I, T_2)$  with  $T_1, T_2 \in H_\sigma$  according to (12). Then

$$(T_2T_1 - I)T_2^\Sigma = T_2T_1T_2 - T_2 = T_2(T_2T_1 - I)^\Sigma.$$

Now (15) shows that the left row space of  $(T_2T_1 - I, T_2)$  is a maximal totally isotropic subspace.

(b) Let the left row space of  $(A, B)$  be a maximal totally isotropic subspace. We consider the maximal totally isotropic subspace given as left row space of the matrix  $(I, 0)$ . By Lemma 2 there exists a maximal totally isotropic subspace of  $K^{2n}$  which is a common complement. In matrix form it can be written as  $(C, D)$ . So in terms of  $\mathbb{P}(R)$  we have  $R(I, 0) \triangle R(C, D) \triangle R(A, B)$  or, said differently, each of the matrices

$$\begin{pmatrix} I & 0 \\ C & D \end{pmatrix}, \begin{pmatrix} C & D \\ A & B \end{pmatrix}$$

is invertible. We may thus put  $D = I$  without loss of generality. Clearly,

$$\begin{pmatrix} C & I \\ A - BC & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} \begin{pmatrix} C & I \\ A & B \end{pmatrix} \in \text{GL}_2(R) = \text{GL}_{2n}(K),$$

whence  $A - BC \in \text{GL}_n(K)$ . Defining  $T_1 := C$  and  $T_2 := (BC - A)^{-1}B$  gives

$$\begin{aligned} R(T_2T_1 - I, T_2) &= R((BC - A)^{-1}BC - I, (BC - A)^{-1}B) \\ &= R(BC - (BC - A)I, B) \\ &= R(A, B). \end{aligned}$$

Since the left row space of  $(C, I)$  is totally isotropic, we have  $T_1 = C = C^\Sigma \in H_\sigma$  by (15). Applying (15) to the totally isotropic left row space of  $(T_2T_1 - I, T_2)$  therefore gives

$$T_2T_1T_2^\Sigma - T_2^\Sigma = T_2T_1T_2^\Sigma - T_2, \quad (20)$$

whence  $T_2 \in H_\sigma$ . This completes the proof.  $\square$

*Remark 5.* In Remarks 1–4 we sketched several applications of Bartolone’s parametrisation to the geometry of square matrices. In view of Theorem 1 it is now a straightforward task to carry them over, *mutatis mutandis*, to the geometry of  $\sigma$ -Hermitian matrices. For example, in the projective space of  $\sigma$ -Hermitian matrices we can parametrise any maximal set of mutually adjacent elements containing the left row space of  $(I, 0)$  via matrices of the form  $(T_2T_1 - I, T_2)$  as follows:  $T_1 \in H_\sigma$  is arbitrary, whereas  $T_2 := c^{(0)T} \cdot t \cdot c^{(0)}$  for a fixed vector  $c^{(0)} \in K^{2n} \setminus \{0\}$  and a variable  $t \in K$  satisfying  $t = t^\sigma$ .

In contrast to this analogy the following difference has to be pointed out: It was shown in [12, Section 4] that the characterisation of adjacency in  $\text{Gr}_{2n,n}(K)$  in terms of the distant relation from [5, Theorem 3.2] *cannot be carried over literally* to a projective space of symmetric matrices over a commutative field of characteristic 2. So the following problem arises: Is it possible to express the adjacency relation on any projective space of  $\sigma$ -Hermitian matrices in terms of the distant relation on  $\mathbb{P}(H_\sigma)$ ? An affirmative answer would imply that the distant preserving bijections of  $\mathbb{P}(H_\sigma)$  are precisely the adjacency preserving bijections of the projective matrix space over  $H_\sigma$ .

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