Pencilled regular parallelisms

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In memoriam Walter Benz

Abstract

Over any field $\mathbb{K}$, there is a bijection between regular spreads of the projective space $\mathrm{PG}(3,\mathbb{K})$ and 0-secant lines of the Klein quadric in $\mathrm{PG}(5,\mathbb{K})$. Under this bijection, regular parallelisms of $\mathrm{PG}(3,\mathbb{K})$ correspond to hyperflock determining line sets (hfd line sets) with respect to the Klein quadric. An hfd line set is defined to be *pencilled* if it is composed of pencils of lines. We present a construction of pencilled hfd line sets, which is then shown to determine all such sets. Based on these results, we describe the corresponding regular parallelisms. These are also termed as being *pencilled*. Any Clifford parallelism is regular and pencilled. From this, we derive necessary and sufficient algebraic conditions for the existence of pencilled hfd line sets.

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1 Introduction

The topic of our research is *parallelisms* in a three-dimensional projective space $\mathrm{PG}(3,\mathbb{K})$, which we interpret as a point-line geometry $(\mathcal{P}_3,\mathcal{L}_3)$ with point set $\mathcal{P}_3$ and line set $\mathcal{L}_3$; the ground field $\mathbb{K}$ is arbitrary. Recall that a *spread* $\mathcal{C}$ is a partition of $\mathcal{P}_3$ by (disjoint) lines, whereas *parallelism* $\boldsymbol{P}$ is a partition of $\mathcal{L}_3$ by (disjoint) spreads. A spread $\mathcal{C}\in\boldsymbol{P}$ is also called a *parallel class* of $\boldsymbol{P}$. Parallelisms are known as *packings*, when $\mathbb{K}$ is a finite field. For further information about parallelisms we refer to [17], [19], [21], and the exhaustive monograph [20], the last being an indispensable source.

It seems that there is little to say about parallelisms in general. So, in order to obtain "interesting" results about parallelisms, it is common to impose extra constraints, *e.g.* by specifying the ground field or by adding

topological conditions. Recent contributions in this spirit are [2], [3], and [27]; see also the references at the end of Section 2. In the present article we are concerned with *regular parallelisms*, that is, parallelisms that are made from regular spreads. We follow the terminology from [20, Ch. 26], that is, we drop the adverb "totally" appearing in [6] and several other articles. In Section 2 we recall a bijection between regular parallelisms in $\mathrm{PG}(3,\mathbb{K})$ and *hyperflock determining line sets* (hfd line sets for short) in $\mathrm{PG}(5,\mathbb{K})$; the latter projective space is always understood as the ambient space of the Klein quadric representing the lines of $\mathrm{PG}(3,\mathbb{K})$. We make use of this bijection and confine ourselves to regular parallelisms whose corresponding hfd line set is composed of pencils of lines. Regular parallelisms and hfd line sets of this kind are said to be *pencilled*; see Definition 2.1. Examples of pencilled regular parallelisms (with $\mathbb{K}$ being the field $\mathbb{R}$ of real numbers) can be found in [6], even though the term "pencilled" does not appear there. One of our aims is to unify these findings by creating a common basis. Another aim is to develop the theory from its very beginning over an arbitrary ground field rather than over the real numbers only.
The article is organised as follows. We describe the necessary background and definitions in Section 2. Next, in Section 3, we state the main results about pencilled hfd line sets and their corresponding pencilled regular parallelisms. In order to get started, we establish a construction of pencilled hfd line sets in Theorem 3.1. Then we present an explicit description of all hfd line sets in the Main Theorem 3.4. Theorem 3.8 provides necessary and sufficient algebraic conditions in terms of $\mathbb{K}$ for the existence of pencilled regular parallelisms in $\mathrm{PG}(3,\mathbb{K})$. Also some examples are given and a link with the classical Clifford parallelism is established. All proofs and several auxiliary lemmas are postponed to Section 4, which should be read in consecutive order. The final sections 5 and 6 are devoted to the description of pencilled regular parallelisms and to phenomena that arise only in case of characteristic two.

2 Preliminaries

Throughout this paper we stick as close as possible to the notions and the terminology in [8], even though we work over an arbitrary ground $\mathbb{K}$ field rather than over $\mathbb{R}$. By $\lambda\colon \mathcal{L}_3 \to H_5$ we denote Klein's correspondence of line geometry, whose image is the Klein quadric H_5 in $\mathrm{PG}(5,\mathbb{K}) = (\mathcal{P}_5, \mathcal{L}_5)$. There is a widespread literature on this topic. See [4, Sect. 2], [16, Sect. 2] or [22, 2.1] for a short introduction and [9, Sect. 11.4], [17, Sect. 15.4], [25, Ch. 34] and [26, Ch. xv] for detailed expositions.
The polarity of $\mathrm{PG}(5,\mathbb{K})$ associated with H_5 is denoted by π_5. A *subquadric*

of the Klein quadric is the section of H_5 by an r-dimensional subspace of $\mathrm{PG}(5,\mathbb{K})$, $r \in \{-1,0,\dots,5\}$; such a subquadric will usually be denoted by some capital letter with lower index r. We are mainly concerned with three kinds of subquadric. If $x \in H_5$, then $\pi_5(x)$ is a tangent hyperplane, which gives rise to the subquadric $H_5 \cap \pi_5(x)$. This subquadric is a quadratic cone with vertex x and with projective index 2. If $x \in \mathcal{P}_5 \setminus H_5$, then $L_4 := H_5 \cap \pi_5(x)$ is a regular quadric with projective index 1. Over the real numbers L_4 is known to be a model for *Lie circle geometry*, whence it is commonly referred to as the *Lie quadric* [1, p. 155], [12, p. 15]. We maintain this name in the general case, even though there need not be any relationship to circle geometry. Consequently, L_4 will be called a *Lie subquadric* of H_5. On the other hand, the points and lines of L_4 constitute one of the classical generalised quadrangles over any field $\mathbb{K}$ [28, p. 57]. If S is a solid such that $Q_3 := S \cap H_5$ is a regular quadric with projective index 0, then Q_3 is said to be *elliptic*. Planes having empty intersection with H_5 play also an essential role. Such planes are called *zero planes* (*e.g.* in [6]) or *external planes* to the Klein quadric (*e.g.* in [16]). We adopt the second terminology.

The regular spreads in $\mathrm{PG}(3,\mathbb{K})$ correspond under λ precisely to the elliptic subquadrics of H_5. As a consequence, the λ-image of a regular parallelism $\boldsymbol{P}$ is a *hyperflock* of the Klein quadric H_5, that is, a partition of H_5 by (disjoint) elliptic subquadrics [6]. It has proved advantageous to replace such a hyperflock by an equivalent object, namely a certain set of lines in the ambient space of the Klein quadric [6], [17, p. 69]. This approach is based on the following bijection γ from the set $\boldsymbol{C}$ of all regular spreads of $\mathrm{PG}(3,\mathbb{K})$ onto the set $\mathcal{Z}$ of all 0-secants (*i.e.* external lines) of H_5:

$$\gamma\colon \boldsymbol{C} \to \mathcal{Z} \colon \mathcal{C} \mapsto \pi_5\big(\operatorname{span}\lambda(\mathcal{C})\big) =: \gamma(\mathcal{C}). \tag{1}$$

The following results from [6], where $\mathbb{K} = \mathbb{R}$, are easily seen to hold over an arbitrary ground field. By [6, Thm. 1.3], the γ-image of a regular parallelism $\boldsymbol{P}$ of $\mathrm{PG}(3,\mathbb{K})$ is a *hyperflock determining line set* (hfd line set), that is, a set $\mathcal{H} \subset \mathcal{L}_5$ of 0-secants of the Klein quadric H_5 such that each tangent hyperplane of H_5 contains exactly one line of $\mathcal{H}$; cf. [6, Def. 1.2]. Conversely, each hfd line set represents a regular parallelism, and thus the construction of regular parallelisms of $\mathrm{PG}(3,\mathbb{K})$ is equivalent to the construction of hfd line sets in $\mathrm{PG}(5,\mathbb{K})$ [6, Thm. 1.3]; see also [23].

An hfd line set $\mathcal{H}$ allows us to read off and define properties of the corresponding regular parallelism $\gamma^{-1}(\mathcal{H})$, for instance its *dimension* is simply the dimension of the subspace of $\mathrm{PG}(5,\mathbb{K})$ spanned by the union of all lines in $\mathcal{H}$.

Given a point p and an incident plane α in $\mathrm{PG}(n,\mathbb{K})$, $n \in \{3,5\}$, we write

$\mathcal{L}[p, \alpha]$ for the pencil of lines with vertex p and carrier plane α. The crucial notion of the present article is as follows:

Definition 2.1. An hfd line set $\mathcal{H}$ is said to be *pencilled* if $\mathcal{H}$ is composed of line pencils, in other words, if each element of $\mathcal{H}$ belongs to at least one pencil of lines in $\mathcal{H}$. A regular parallelism $\boldsymbol{P}$ of $\mathrm{PG}(3, \mathbb{K})$ is called *pencilled* if the hfd line set $\gamma(\boldsymbol{P})$ is pencilled.

The reader will easily check that the parallelisms constructed in [6] are pencilled; using [6, Rem. 2.9] one shows that also the parallelisms from [4] are pencilled. We observe that over $\mathbb{R}$ pencilled regular parallelisms of dimension 2, 3, 4, and 5 are known. On the other hand, there exist also regular parallelisms that are not pencilled [5, Ex. 16 and 22]. We shall establish in Proposition 3.6 that the *Clifford parallelism* is a pencilled regular parallelism. To this end we need some facts about Clifford parallelism, which we briefly summarise below.
The following is taken from [21, § 14]: Let $\mathbb{K}$ be a field and let $\mathbb{H}$ be a $\mathbb{K}$-algebra such that one of the subsequent conditions, (A) or (B), is satisfied:

$$\left.\begin{array}{ll} \text{(A)} & \mathbb{H} \text{ is a quaternion skew field with centre } \mathbb{K}. \\ \text{(B)} & \mathbb{H} \text{ is an extension field of } \mathbb{K} \text{ with degree } [\mathbb{H} : \mathbb{K}] = 4 \\ & \text{and such that } a^2 \in \mathbb{K} \text{ for all } a \in \mathbb{H}. \end{array}\right\} \qquad (2)$$

We now take $\mathbb{H}$ as the underlying vector space of the projective space $\mathrm{PG}(3, \mathbb{K})$. Every element $c \in \mathbb{H} \setminus \{0\}$ determines the *left translation* $\lambda_c \colon \mathbb{H} \to \mathbb{H} \colon y \mapsto cy$. All left translations $\mathbb{H} \to \mathbb{H}$ constitute a group, which acts on the line set $\mathcal{L}_3$ in a natural way. The orbits of this group action on $\mathcal{L}_3$ are defined to be the classes of *left parallel* lines. In this way a first parallelism is obtained. *Right parallel* lines are defined via *right translations* and give rise to a second parallelism. These two parallelisms turn $\mathrm{PG}(3, \mathbb{K})$ into a projective *double space*; they coincide precisely when (B) applies. Note also that (B) implies that the characteristic of $\mathbb{K}$ is two and that $\mathbb{H}$ is a purely inseparable extension of $\mathbb{K}$.
More generally, a parallelism $\boldsymbol{P}$ of an arbitrary projective space $\mathrm{PG}(3, \mathbb{K})$ is said to be *Clifford* if the underlying vector space of $\mathrm{PG}(3, \mathbb{K})$ can be made into a $\mathbb{K}$-algebra $\mathbb{H}$, subject to (A) or (B), in such a way that $\boldsymbol{P}$ coincides with the left or right parallelism arising from $\mathbb{H}$ [16, Def. 3.4]. We refer to [7], [10], [11], [13], [15], [16], [19, pp. 112–115], [21, § 14] and [24] for surveys, recent results, and a wealth of references on Clifford parallelism.

3 Main results and examples

First, we present a construction of pencilled hfd line sets. We thereby generalise and unify Theorems 5.1, 5.5, and 5.6 in [6]. These theorems are more explicit than our result, but tailored to real projective spaces; see also [8, Rem. 8.1].

Theorem 3.1 (**Construction of pencilled hfd line sets**). *In* PG(5, $\mathbb{K}$), *let D be a line such that*

$$\mathcal{E}_D := \bigl\{\varepsilon \subset \mathcal{P}_5 \mid D \subset \varepsilon \text{ and } \varepsilon \text{ is an external plane to } H_5\bigr\} \tag{3}$$

is non-empty. Then, upon choosing any mapping $f\colon D \to \mathcal{E}_D$, the union

$$\bigcup_{v\in D} \mathcal{L}[v, f(v)] =: \mathcal{H} \tag{4}$$

is a pencilled hfd line set.

In PG(5, $\mathbb{R}$) there is always a line D such that $\mathcal{E}_D \neq \emptyset$; see [6, Sect. 5]. Over an arbitrary field $\mathbb{K}$ this need not be the case. We shall return to this matter after Theorem 3.8. So, for the time being, it remains open whether or not there exists a line D in PG(5, $\mathbb{K}$) such that $\mathcal{E}_D \neq \emptyset$.

Example 3.2. If the mapping f in Theorem 3.1 is constant, then the image of f contains a single plane, say κ_1. Consequently, $\mathcal{H}$ is the plane of lines in κ_1 and D is just one of the lines in κ_1. Therefore the set $\mathcal{H}$ contains also pencils other than those appearing in (4). Indeed, any point of κ_1 is the vertex of a unique pencil in $\mathcal{H}$. The dimension of $\mathcal{H}$ is two.

Example 3.3. Let the image of the mapping f in Theorem 3.1 consist of two distinct planes κ_1, κ_2 only. In a certain way this is the simplest case apart from Example 3.2. The mapping f decomposes the line D into two non-empty subsets D_1 and D_2, namely the pre-images of κ_1 and κ_2, respectively. By (4), the corresponding hfd line set can be written in the form

$$\Bigl(\bigcup_{v\in D_1} \mathcal{L}[v, \kappa_1]\Bigr) \cup \Bigl(\bigcup_{v\in D_2} \mathcal{L}[v, \kappa_2]\Bigr) =: \mathcal{H}_{12}. \tag{5}$$

The dimension of $\mathcal{H}_{12}$ is three. The set D_1 may comprise a single point, or any finite number of distinct points etc. Over the real numbers, f can be chosen in such a way that D_1 is a connected component of D with respect to the standard topology in PG(5, $\mathbb{R}$). Then D_2 is also connected; such a set is illustrated in Figure 1.

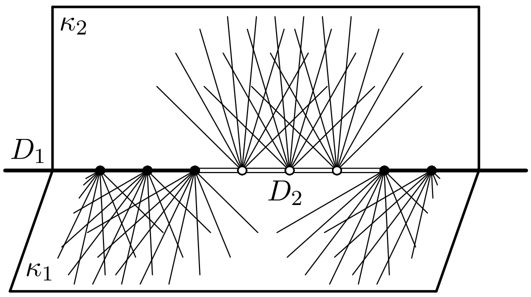

Figure 1: An hfd line set $\mathcal{H}_{12}$

Further extensions and generalisations of the preceding examples are obvious. The main result is a geometric description of *all* pencilled hfd line sets.

Theorem 3.4 (**Main theorem on pencilled hfd line sets**)**.** *In* $\mathrm{PG}(5,\mathbb{K})$, *let* $\mathcal{H}$ *be a pencilled hfd line set. Denote by* $\mathcal{V}$ *the set of all vertices and by* $\mathcal{K}$ *the set of all planes of the pencils in* $\mathcal{H}$. *Then the following hold.*

(i) *All planes of* $\mathcal{K}$ *are external to the Klein quadric* H_5.

(ii) *There exists a surjective mapping* $h\colon \mathcal{V} \to \mathcal{K}$ *that assigns to each* $v \in \mathcal{V}$ *a plane* $h(v) \in \mathcal{K}$ *that is incident with* v *and such that*

$$\mathcal{L}[v, h(v)] = \{X \in \mathcal{H} \mid v \in X\}. \tag{6}$$

(iii) *If* $\mathcal{V}$ *is a set of non-collinear points, then* $\mathcal{V}$ *is a plane,* $\mathcal{K} = \{\mathcal{V}\}$, *and* $\mathcal{H}$ *is the set of lines in the plane* $\mathcal{V}$.

(iv) *If* $\mathcal{V}$ *is a set of collinear points, then* $\mathcal{V}$ *is a line,* $\mathcal{V} \in \mathcal{H}$, *and* $|\mathcal{K}| \geq 2$.

(v) $\mathcal{V} = \bigcap_{\kappa\in\mathcal{K}} \kappa$.

The mapping h allows us to write

$$\mathcal{H} = \bigcup_{v\in\mathcal{V}} \mathcal{L}[v, h(v)]. \tag{7}$$

Remark 3.5. From Theorem 3.4 (ii), the construction in Theorem 3.1 produces *all* pencilled hfd line sets. Indeed, in order to get an appropriate mapping f as in Theorem 3.1 for a given pencilled hfd line set $\mathcal{H}$, it suffices to select some line $D \subset \mathcal{V}$ and to define $f\colon D \to \mathcal{E}_D\colon v \mapsto h(v)$. Clearly, Example 3.2 corresponds to the situation in Theorem 3.4 (iii) and vice versa. On the other hand, Example 3.3, where $|\mathcal{K}| = 2$, is a very particular case of the more general setting in Theorem 3.4 (iv).

So far we have focussed on pencilled hfd line sets in $\mathrm{PG}(5,\mathbb{K})$. We now use the inverse of the bijection γ from (1) in order to obtain results about the corresponding pencilled regular parallelisms in $\mathrm{PG}(3,\mathbb{K})$. (See Section 5 for additional details.) Also, to develop further our theory, we shall make use of results about Clifford parallelism. The following characterisation generalises [6, Lemma 2.7], which is limited to the case $\mathbb{K}=\mathbb{R}$, to an arbitrary ground field.

Proposition 3.6. *A parallelism* $\boldsymbol{P}$ *of* $\mathrm{PG}(3,\mathbb{K})$ *is Clifford if, and only if,* $\boldsymbol{P}$ *is a pencilled regular parallelism and its corresponding hfd line set* $\gamma(\boldsymbol{P})$ *is a plane of lines in* $\mathrm{PG}(5,\mathbb{K})$.

We add in passing that our proof of the proposition above uses [16, Thm. 4.8], which in turn is based upon a series of other results about Clifford parallelism. It would be favourable to have a shorter, more direct proof for the fact that $\gamma(\boldsymbol{P})$ being a plane of lines forces $\boldsymbol{P}$ to be Clifford. The point is, of course, to construct from $\boldsymbol{P}$ a $\mathbb{K}$-algebra $\mathbb{H}$ that makes it possible to verify that $\boldsymbol{P}$ is Clifford.

Remark 3.7. The pencilled hfd line sets from Example 3.2 (based on constant mappings f) are precisely the ones that correspond under γ^{-1} to the Clifford parallelisms of $\mathrm{PG}(3,\mathbb{K})$. This is immediate from Remark 3.5 and Proposition 3.6.
On the other hand, the pencilled regular parallelism $\gamma^{-1}(\mathcal{H}_{12})$ arising from (5) is not Clifford by Proposition 3.6; one might call $\gamma^{-1}(\mathcal{H}_{12})$ a *piecewise Clifford parallelism* (with two pieces).

By the above considerations and in view of the results from [6], Clifford parallelism is just a very particular case within our general theory. Nevertheless, Clifford parallelism is a relevant part of our investigation, because it is used below to establish an algebraic criterion for the existence of arbitrary pencilled regular parallelisms.

Theorem 3.8. *Given any field* $\mathbb{K}$ *the following assertions are equivalent.*

(i) *In* $\mathrm{PG}(3,\mathbb{K})$ *there exists a pencilled regular parallelism that is not Clifford.*

(ii) *In* $\mathrm{PG}(3,\mathbb{K})$ *there exists a Clifford parallelism.*

(iii) *There exists an algebra* $\mathbb{H}$ *over the field* $\mathbb{K}$ *such that one of the conditions,* (A) *or* (B)*, in equation* (2) *is satisfied.*

Remark 3.9. Theorem 3.8 shows, as a by-product, that pencilled regular parallelisms (pencilled hfd line sets) do not exist when $\mathbb{K}$ is quadratically closed or finite, since such a $\mathbb{K}$ does not satisfy Theorem 3.8 (iii). However, this can be seen directly: If $\mathbb{K}$ is quadratically closed, then there are no 0-secants of H_5. If $\mathbb{K}$ is finite, then 0-secants of H_5 do exist, but external planes to the Klein quadric do not; see the proof of Lemma 4.9. Thus in both cases there cannot be pencilled hfd line sets.

We read off from Proposition 3.6 that Theorem 3.8 (i) holds if, and only if, there is a line D in $\mathrm{PG}(5,\mathbb{K})$ such that $|\mathcal{E}_D| \geq 2$. So, again using Theorem 3.8, the construction of a pencilled hfd line set $\mathcal{H}_{12}$ in Example 3.3 can be carried out, precisely when the algebraic condition in Theorem 3.8 (iii) is satisfied by $\mathbb{K}$. We therefore have shown that under this condition there exist, in $\mathrm{PG}(3,\mathbb{K})$, pencilled regular parallelisms with dimension $d = 2$ and with dimension $d = 3$. However, we did not undertake a study of the cases with $d \in \{4, 5\}$. According to [6], pencilled regular parallelisms of the latter dimensions exist over the real numbers; future work should address these cases in the setting of Theorem 3.8 (iii).

4 Proofs

We start with three auxiliary lemmas.

Lemma 4.1. *Let S be a subspace of $\mathrm{PG}(5,\mathbb{K})$. There exists a tangent hyperplane τ of the Klein quadric H_5 with $S \subset \tau$ if, and only if, there exists a subspace M of $\mathrm{PG}(5,\mathbb{K})$ satisfying*

$$M \subset S \cap H_5 \quad \textit{and} \quad \dim M \geq \dim S - 2. \tag{8}$$

Proof. As we noted in Section 2, a tangent hyperplane of the Klein quadric meets H_5 along a quadratic cone with projective index 2. Any other hyperplane of $\mathrm{PG}(5,\mathbb{K})$ intersects H_5 in a Lie subquadric, which has projective index 1. So, a hyperplane θ of $\mathrm{PG}(5,\mathbb{K})$ is tangent to the Klein quadric H_5 precisely when θ contains a plane μ that lies on H_5.
If S is contained in a tangent hyperplane τ, then there is a plane $\mu \subset \tau \cap H_5$. The subspace $M := S \cap \mu$ clearly satisfies the first condition from (8) and also the second one, since $S \vee \mu \subset \tau$ gives

$$\dim M = \dim S + \dim \mu - \dim(S \vee \mu) \geq \dim S + 2 - 4.$$

Conversely, if there is a subspace M subject to (8), then there is a plane of H_5, say μ, that contains M. So, since $M \subset S \cap \mu$, we obtain

$$\dim(S \vee \mu) = \dim S + \dim \mu - \dim(S \cap \mu) \leq \dim S + 2 - (\dim S - 2).$$

This implies that $S \vee \mu$ is contained in a hyperplane of $\mathrm{PG}(5,\mathbb{K})$, which is tangent to H_5 by the above-noted characterisation. $\square$

Corollary 4.2. *In* $\mathrm{PG}(5,\mathbb{K})$, *any subspace* S *with* $\dim S \leq 1$ *is contained in at least one tangent hyperplane of the Klein quadric* H_5.

Lemma 4.3. *In* $\mathrm{PG}(5,\mathbb{K})$, *if a plane* ε *is external to the Klein quadric* H_5, *then so is the polar plane* $\pi_5(\varepsilon)$.

Proof. The plane ε contains no point of H_5. Hence, by Lemma 4.1, there is no tangent hyperplane of H_5 containing ε. Application of π_5 gives that there is no point of H_5 incident with $\pi_5(\varepsilon)$. $\square$

Lemma 4.4. *In* $\mathrm{PG}(5,\mathbb{K})$, *let* $p \notin H_5$ *be a point incident with a line* G. *Then there exists* $x \in H_5$ *with* $p \in \pi_5(x)$ *and* $G \not\subset \pi_5(x)$.

Proof. From $p \in \mathcal{P}_5 \setminus H_5$ and $p \in G$ it follows that $G \not\subset H_5$. Now $\pi_5(p) \cap H_5 =: L_4$ is a Lie subquadric of H_5 and therefore $\mathrm{span}(L_4) = \pi_5(p)$. This shows the existence of a point $x \in L_4$ that is not incident with the solid $\pi_5(G)$. Applying π_5 shows that x has the required properties. $\square$

We proceed with our first proof.

Proof of Theorem 3.1. Since all planes of $\mathcal{E}_D$ are external to H_5, all lines of $\mathcal{H}$ are 0-secants of H_5. There is a point $v_1 \in D$, say. We read off from (3) that $D \subset f(v_1)$, whence (4) shows $D \in \mathcal{L}[v_1, f(v_1)]$. This gives

$$D \in \mathcal{H}. \tag{9}$$

Next, choose any tangent hyperplane of H_5, say τ. From Lemma 4.1, no plane of $\mathcal{E}_D$ is contained in τ, that is,

$$\tau \cap \varepsilon \text{ is a line for all } \varepsilon \in \mathcal{E}_D. \tag{10}$$

If $D \subset \tau$, then by (10), $\tau \cap \varepsilon = D$ for all $\varepsilon \in \mathcal{E}_D$. Using (9), we now see that D is the only line of $\mathcal{H}$ that is incident with τ.
If $D \not\subset \tau$, then $\tau \cap D$ is a point, say p. From (3), for all $v \in D \setminus \{p\}$ there is a unique line of $\mathcal{L}[v, f(v)]$ passing through p, namely the line D, which also is an element of $\mathcal{L}[p, f(p)]$. Therefore, (4) gives

$$\mathcal{L}[p, f(p)] = \{X \in \mathcal{H} \mid p \in X\}. \tag{11}$$

From (10), $\tau \cap f(p)$ is a line incident with τ. More precisely, $\tau \cap f(p)$ is the only line of the pencil $\mathcal{L}[p, f(p)] \subset \mathcal{H}$ lying in τ. According to (11), all lines of $\mathcal{H} \setminus \mathcal{L}[p, f(p)]$ contain some point of D other than p; therefore none

of these lines is contained in τ. Hence $\tau \cap f(p)$ is the only line of $\mathcal{H}$ being incident with τ.
To sum up, we have shown that $\mathcal{H}$ is an hfd line set that, by its definition, is pencilled. □

In the next four lemmas we adopt the assumptions and notations from Theorem 3.4: $\mathcal{H} \subset \mathcal{L}_5$ is a pencilled hfd line set, $\mathcal{V}$ is the set of all vertices, and $\mathcal{K}$ is the set of all planes of the pencils in $\mathcal{H}$.

Lemma 4.5. *The following hold:* (i) $|\mathcal{K}| \geq 1$*;* (ii) $|\mathcal{V}| \geq 2$*.*

Proof. $\mathcal{K} \neq \emptyset$ and $\mathcal{V} \neq \emptyset$ are immediate from the definition of a pencilled hfd line set and the fact that tangent hyperplanes of H_5 do exist. Next, assume to the contrary that $|\mathcal{V}| < 2$. So, from $\mathcal{V} \neq \emptyset$, we obtain $|\mathcal{V}| = 1$. This implies that all lines of $\mathcal{H}$ share a common point $v \in \mathcal{V}$, say. Since $\mathcal{H}$ is an hfd line set, the point v belongs to all tangent hyperplanes of H_5, an absurdity. □

Lemma 4.6. *If $G_1, G_2 \in \mathcal{H}$ are distinct coplanar lines, then the plane $G_1 \vee G_2$ is external to the Klein quadric H_5.*

Proof. From the definition of an hfd line set, we deduce that there exists no tangent hyperplane τ of H_5 with $G_1 \vee G_2 \subset \tau$. Now we apply Lemma 4.1 to $\varepsilon := G_1 \vee G_2$ and obtain that $\emptyset$ is the only subspace of $\mathrm{PG}(5, \mathbb{K})$ being contained in $\varepsilon \cap H_5$. Therefore $\varepsilon \cap H_5 = \emptyset$. □

Lemma 4.7. *Let $\mathcal{L}[v, \kappa] \subset \mathcal{H}$ be a pencil. Then*

$$\mathcal{L}[v, \kappa] = \{X \in \mathcal{H} \mid v \in X\}.$$

Proof. Assume, by way of contradiction, that there exists a line $G \in \mathcal{H}$ satisfying $v \in G$ and $G \not\subset \kappa$. Then $X \vee G$ is an external plane to H_5 for all $X \in \mathcal{L}[v, \kappa]$ according to Lemma 4.6. This implies that $G \vee \kappa$, which has dimension 3, contains no point of H_5. On the other hand, by Corollary 4.2, there is a point $q \in H_5$ such that the tangent hyperplane $\pi_5(q)$ contains the line $\pi_5(G \vee \kappa)$. This means $q \in (G \vee \kappa) \cap H_5$, an absurdity. □

Lemma 4.8. *Let $\mathcal{L}[v_1, \kappa_1]$ and $\mathcal{L}[v_2, \kappa_2]$ be distinct pencils of lines that belong to $\mathcal{H}$. Then the following hold:* (i) $v_1 \neq v_2$*;* (ii) $v_1 \vee v_2 \subset \kappa_1 \cap \kappa_2$*;* (iii) $v_1 \vee v_2 \in \mathcal{H}$*.*

Proof. (i) $v_1 = v_2$ would imply $\kappa_1 \neq \kappa_2$, which would contradict Lemma 4.7.
(ii) *and* (iii). By Corollary 4.2, there is a tangent hyperplane τ of H_5 such that $v_1 \vee v_2 \subset \tau$. Since $\mathcal{H}$ is an hfd line set, this τ cannot contain any of the planes κ_i, $i = 1, 2$. Therefore each of the intersections $\tau \cap \kappa_i$ is a line, which clearly passes through v_i and hence belongs to $\mathcal{H}$. Since τ is incident with a unique line of $\mathcal{H}$, we finally obtain $\tau \cap \kappa_1 = \tau \cap \kappa_2 = v_1 \vee v_2 \in \mathcal{H}$. □

We are now in a position to prove the Main Theorem 3.4.

Proof of Theorem 3.4. (i) Given any plane $\kappa \in \mathcal{K}$ there is a point $v_\kappa \in \mathcal{V}$ with $\mathcal{L}[v_\kappa, \kappa] \subset \mathcal{H}$. As all lines of the pencil $\mathcal{L}[v_\kappa, \kappa]$ are external to the Klein quadric, so is the plane κ.
(ii) Taking into account Lemma 4.8, we define a mapping $h : \mathcal{V} \to \mathcal{K}$ as follows: For each $v \in \mathcal{V}$ there is a unique plane κ with $\mathcal{L}[v, \kappa] \subset \mathcal{H}$, and so we let $h(v) = \kappa$. Lemma 4.7 shows that h satisfies (6). By the definition of $\mathcal{K}$, the mapping h is surjective.
(iii) There exist non-collinear vertices $v_1, v_2, v_3 \in \mathcal{V}$ spanning a plane, say Δ. By (ii), there are well defined planes $h(v_1), h(v_2), h(v_3) \in \mathcal{K}$. For all i, j, k with $\{i, j, k\} = \{1, 2, 3\}$ both lines $v_i \vee v_j$ and $v_i \vee v_k$ are incident with the plane $h(v_i)$ according to (6), hence

$$\Delta = h(v_1) = h(v_2) = h(v_3). \tag{12}$$

Let G be an arbitrary line of $\mathcal{H}$. As $\mathcal{H}$ is pencilled, so there exists a pencil $\mathcal{L}[v_G, h(v_G)] \subset \mathcal{H}$ with $G \in \mathcal{L}[v_G, h(v_G)]$. Without loss of generality, we may assume that v_1, v_2, v_G form a triangle. Using (6), we deduce as above: $h(v_1) = h(v_2) = h(v_G)$. Therefore and by (12), $G \subset h(v_G) = \Delta$. Consequently, $\mathcal{H}$ is contained in the plane of lines in Δ.
Conversely, let F be a line of Δ. By Corollary 4.2, there is a tangent hyperplane τ of H_5 containing F. From (12) and (i), the plane Δ is external to H_5. Now Lemma 4.1 shows that $\Delta \not\subset \tau$. This means that $F = \tau \cap \Delta$. Since all lines of $\mathcal{H}$ are incident with the plane Δ and τ is incident with one of these, we obtain $F \in \mathcal{H}$.
Summing up, $\mathcal{H}$ is the set of lines in the plane Δ, whence $\mathcal{V} = \Delta$ and $\mathcal{K} = \{\mathcal{V}\}$.
(iv) By Lemma 4.5 (ii), there are distinct points $v_1, v_2 \in \mathcal{V}$, whence $D := v_1 \vee v_2$ is the only line containing $\mathcal{V}$. Let $p \in D$ be an arbitrary point. Lemma 4.8 (iii) shows $v_1 \vee v_2 = D \in \mathcal{H} \subset \mathcal{Z}$, and so $p \notin H_5$. Lemma 4.4 implies that there exists a tangent hyperplane τ of H_5 with $p \in \tau$ and $D \not\subset \tau$; hence $D \cap \tau = \{p\}$. By the properties of an hfd line set, there exists a line of $\mathcal{H}$ in τ and, consequently, some vertex $v_\tau \in \mathcal{V}$ lies in τ. Since $\mathcal{V} \subset D$, we obtain $p = v_\tau \in \mathcal{V}$, that is, $\mathcal{V} = D \in \mathcal{H}$.
Now we establish that

$$\mathcal{V} = D = \bigcap_{\kappa \in \mathcal{K}} \kappa. \tag{13}$$

From (ii), the mapping h is surjective. So, given any plane $\kappa \in \mathcal{K}$ there is a point $v_\kappa \in D$ with $h(v_\kappa) = \kappa$. By the foregoing, we have $v_\kappa \in D \in \mathcal{H}$. Thus $D \subset \kappa$ follows from Lemma 4.7. There is a plane $\kappa_1 \in \mathcal{K}$ according to Lemma 4.5 (i). We cannot have $\mathcal{K} = \{\kappa_1\}$, since then, by (ii), we would obtain

$$\mathcal{H} = \bigcup_{v\in\mathcal{V}} \mathcal{L}[v, h(v)] = \bigcup_{v\in\mathcal{V}} \mathcal{L}[v, \kappa_1],$$

that is, $\mathcal{H}$ would comprise all lines in κ_1, which in turn would imply that $\mathcal{V} = D = \kappa_1$, a contradiction to the collinearity of $\mathcal{V}$. So, there are distinct planes $\kappa_1, \kappa_2 \in \mathcal{K}$. Hence $D = \kappa_1 \cap \kappa_2$, which verifies (13) and implies $|\mathcal{K}| \geq 2$.
(v) If $\mathcal{V}$ is collinear, then (13) applies, otherwise the assertion is obvious from (iii). □

Proof of Proposition 3.6. Let $\boldsymbol{P}$ be a pencilled regular parallelism of $\mathrm{PG}(3, \mathbb{K})$ such that $\gamma(\boldsymbol{P})$ is a plane of lines; we denote this plane by κ_1. From Lemma 4.3, applied to κ_1, we obtain that $\pi_5(\kappa_1)$ is also external to H_5. Furthermore, by the action of π_5 on the lattice of subspaces of $\mathrm{PG}(5, \mathbb{K})$, we obtain

$$\bigl\{\mathrm{span}\bigl(\lambda(\mathcal{C})\bigr) \mid \mathcal{C} \in \boldsymbol{P}\bigr\} = \bigl\{S \subset \mathcal{P}_5 \mid S \text{ is a solid and } \pi_5(\kappa_1) \subset S\bigr\}. \qquad (14)$$

This description of $\boldsymbol{P}$ in terms of the Klein correspondence coincides with the definition of a parallelism in [16, Def. 4.2], which relies on the choice of an external plane to H_5; in our context this distinguished external plane is $\pi_5(\kappa_1)$. Finally, by [16, Thm. 4.8], the parallelism $\boldsymbol{P}$ is Clifford.
Conversely, let $\boldsymbol{P}$ be Clifford. From [16, Thm. 5.1] there is an external plane ε_1 to H_5 such that, in our present notation, (14) holds with $\pi_5(\kappa_1)$ to be replaced by ε_1. By the last observation, all parallel classes of $\boldsymbol{P}$ are regular spreads, that is, $\boldsymbol{P}$ is regular. From (1), the polarity π_5 sends the set of solids of $\mathrm{PG}(5, \mathbb{K})$ that contain ε_1 to the hfd line set $\gamma(\boldsymbol{P})$, which therefore is the set of lines in the plane $\pi_5(\varepsilon_1)$. □

The following lemma will be used in order to accomplish the proof of Theorem 3.8.

Lemma 4.9. *In* $\mathrm{PG}(5, \mathbb{K})$*, let* ε_1 *be an external plane to the Klein quadric* H_5*. Then there exists a plane* ε_2 *that is external to* H_5 *and such that* $\varepsilon_1 \cap \varepsilon_2$ *is a line.*

Proof. There is a 1-secant (tangent) T of H_5. This T is not contained in any external plane to H_5. By Lemma 4.3, the plane $\pi_5(\varepsilon_1)$ is also external to H_5. So,

$$\bigl|T \cap \bigl(H_5 \cup \varepsilon_1 \cup \pi_5(\varepsilon_1)\bigr)\bigr| \leq 3. \qquad (15)$$

The existence of an external plane to H_5 is guaranteed by ε_1 and forces $\mathbb{K}$ to be an infinite field; cf. the classification quadrics in $\mathrm{PG}(2, \mathbb{K})$, $\mathbb{K}$ finite, [18, p. 2]. Therefore and by (15), there is a point $q \in T$ that is off the set $H_5 \cup \varepsilon_1 \cup \pi_5(\varepsilon_1)$. This q is the centre of a perspectivity σ of order two that stabilises H_5; the axis of σ is the hyperplane $\pi_5(q)$. We infer from $q \notin \varepsilon_1$ that ε_1 does not contain the centre of σ and from $q \notin \pi_5(\varepsilon_1)$ that ε_1 is not contained in the axis of σ. Hence $\varepsilon_1 \neq \sigma(\varepsilon_1)$ and so $\varepsilon_1 \cap \pi_5(q) = \sigma(\varepsilon_1) \cap \pi_5(q) = \varepsilon_1 \cap \sigma(\varepsilon_1)$ is a line, that is, $\varepsilon_2 := \sigma(\varepsilon_1)$ has the required properties. □

Proof of Theorem 3.8. (i) ⇒ (ii). Let $\boldsymbol{P}$ be a pencilled regular parallelism of $\mathrm{PG}(3,\mathbb{K})$ that is not Clifford. We denote the corresponding pencilled hfd line set $\gamma(\boldsymbol{P})$ by $\mathcal{H}$ and adopt the terminology of the Main Theorem 3.4. So, there is a plane $\kappa_1 \in \mathcal{K}$ and this κ_1 is external to H_5. (There is more than one plane in $\mathcal{K}$, but this fact will not be used.) We now choose some line $D \subset \kappa_1$ and observe $\kappa_1 \in \mathcal{E}_D$. We therefore can carry out the construction of Theorem 3.1 using the constant mapping $f\colon D \to \mathcal{E}_D\colon v \mapsto \kappa_1$; cf. Example 3.2. This gives an hfd line set $\mathcal{H}_1$ that equals the set of lines in κ_1. Proposition 3.6 yields that the parallelism $\gamma^{-1}(\mathcal{H}_1)$ is Clifford.
(ii) ⇒ (i). Let $\boldsymbol{P}$ be a Clifford parallelism of $\mathrm{PG}(3,\mathbb{K})$. By Proposition 3.6, $\gamma(\boldsymbol{P})$ is the set of all lines in an external plane to H_5, say κ_1. Next, we apply Lemma 4.9 and obtain a plane κ_2 that is external to H_5 and such that $\kappa_1 \cap \kappa_2$ is a line. This in turn allows us to proceed as in Example 3.3 in order to obtain a pencilled hfd line set $\mathcal{H}_{12}$ other than a plane of lines. According to Proposition 3.6, $\gamma^{-1}(\mathcal{H}_{12})$ is a pencilled regular parallelism that is not Clifford .
(ii) ⇔ (iii). This follows from [16, Thm. 4.8] and [16, Thm. 5.1]. □

5 Back to $\mathrm{PG}(3,\mathbb{K})$

Our first aim is to state several properties of the bijection $\gamma^{-1}\colon \mathcal{Z} \to \boldsymbol{C}$. From (1), for any 0-secant G of H_5 we obtain the regular spread $\gamma^{-1}(G)$ as follows:

$$G \stackrel{\pi_5}{\longmapsto} \pi_5(G) \longmapsto \pi_5(G) \cap H_5 =: Q_3(G) \xrightarrow{\lambda^{-1}} \gamma^{-1}(G). \tag{16}$$

Here $\pi_5(G)$ is a solid and $Q_3(G)$ denotes an elliptic subquadric of H_5. For any point $p \in \mathcal{P}_5 \setminus H_5$, we may proceed in the same way. This yields

$$p \stackrel{\pi_5}{\longmapsto} \pi_5(p) \longmapsto \pi_5(p) \cap H_5 =: L_4(p) \xrightarrow{\lambda^{-1}} \lambda^{-1}\big(L_4(p)\big) =: \mathcal{G}(p). \tag{17}$$

The hyperplane $\pi_5(p)$ of $\mathrm{PG}(5,\mathbb{K})$ is not tangent to H_5. Thus $L_4(p)$ is a Lie subquadric of H_5 and $\mathcal{G}(p)$ is a general linear complex of lines in $\mathrm{PG}(3,\mathbb{K})$. It is known that (17) defines a bijection of the set $\mathcal{P}_5 \setminus H_5$ onto the set of all general linear complexes of lines in $\mathrm{PG}(3,\mathbb{K})$.

We continue with two definitions. A *flock* of a Lie subquadric $L_4 \subset H_5$ is a partition of L_4 by (disjoint) elliptic subquadrics. Such a flock is said to be *linear* if the members of the flock span solids that constitute a pencil in the ambient space of L_4. For our purposes, it is enough to define a *linear flock of a general linear complex* $\mathcal{G} \subset \mathcal{L}_3$ as the preimage under the Klein correspondence of a linear flock of the Lie subquadric $\lambda(\mathcal{G}) \subset H_5$.
Next, let ε be an external plane to H_5 and let $p \in \varepsilon$. Clearly, $\mathcal{L}[p, \varepsilon]$ contains only 0-secants of H_5 and $p \notin H_5$. By Lemma 4.3, the plane $\pi_5(\varepsilon)$ is also external to H_5. The polarity π_5 takes the pencil $\mathcal{L}[p, \varepsilon]$ to a pencil of solids, namely the set of all solids that contain $\pi_5(\varepsilon)$ and are contained in the hyperplane $\pi_5(p)$. By the previous definition and (16), the set

$$\big\{Q_3(X) \mid X \in \mathcal{L}[p, \varepsilon]\big\}$$

is a linear flock of the Lie subquadric $L_4(p)$. Application of λ^{-1} yields a set of regular spreads:

$$\boldsymbol{F}[p, \varepsilon] := \big\{\lambda^{-1}\big(Q_3(X)\big) \mid X \in \mathcal{L}[p, \varepsilon]\big\}. \tag{18}$$

So, the set $\boldsymbol{F}[p, \varepsilon]$ in (18) is a linear flock of $\mathcal{G}(p)$. It is straightforward to reverse our foregoing arguments. To sum up, we have:

Proposition 5.1. *Under the bijection* $\gamma : \boldsymbol{C} \to \mathcal{Z}$ *from equation* (1)*, the linear flocks of general linear complexes of lines in* $\mathrm{PG}(3, \mathbb{K})$ *are mapped to pencils of* 0*-secants of the Klein quadric* H_5 *in* $\mathrm{PG}(5, \mathbb{K})$*, and vice versa.*

By the above, our definitions and results on hfd line sets in $\mathrm{PG}(5, \mathbb{K})$ are readily translated to the language of line geometry in $\mathrm{PG}(3, \mathbb{K})$.
For example, let us consider a pencilled hfd line set $\mathcal{H}$ other than a plane of lines. From Proposition 3.6, the pencilled regular parallelism $\boldsymbol{P} := \gamma^{-1}(\mathcal{H})$ is not Clifford. Using the Main Theorem 3.4 and the notation from there, we obtain the following description: The hfd line set $\mathcal{H}$ contains the *distinguished line* $D = \mathcal{V}$. From (17), the range of points on D yields

$$\{\mathcal{G}(v) \mid v \in D\};$$

this is a *distinguished pencil of general linear complexes* in $\mathrm{PG}(3, \mathbb{K})$ related with $\boldsymbol{P}$. According to (18), each of the pencils $\mathcal{L}[v, h(v)]$, $v \in D$, yields a linear flock $\boldsymbol{F}[v, h(v)]$ of the general linear complex $\mathcal{G}(v)$. The *distinguished parallel class* $\gamma^{-1}(D)$ of $\boldsymbol{P}$ is the only regular spread that belongs to all these linear flocks. The special role of $\gamma^{-1}(D)$ is also illustrated by

$$\gamma^{-1}(D) = \bigcap_{v \in D} \mathcal{G}(v).$$

Finally, we translate (7) and obtain $\boldsymbol{P} = \bigcup_{v \in D} \boldsymbol{F}[v, h(v)]$.

6 Aspects of characteristic two

In $\mathrm{PG}(5,\mathbb{K})$, let ε be a fixed external plane to H_5 and let G be any line of ε. If $\operatorname{Char}\mathbb{K} \neq 2$, then the polarity π_5 of the Klein quadric is orthogonal, so that every external subspace to H_5 is skew to its π_5-polar subspace. Indeed, any common point of these subspaces would be on H_5. In particular, we always have $\varepsilon \cap \pi_5(\varepsilon) = \emptyset$ and $G \cap \pi_5(G) = \emptyset$.

On the other hand, let us now assume that $\operatorname{Char}\mathbb{K} = 2$. Here π_5 is a null polarity and the situation is less uniform than before. For any subspace S of $\mathrm{PG}(5,\mathbb{K})$ the difference $\dim S - \dim\bigl(S \cap \pi_5(S)\bigr)$ is an even number, since the rank of any alternating bilinear form (on some subspace of $\mathbb{K}^6$) is even. We therefore have to distinguish two cases.

Case 1. $\varepsilon \cap \pi_5(\varepsilon)$ is a point: Letting $\{q\} := \varepsilon \cap \pi_5(\varepsilon)$ it is straightforward to verify that

$$G \subset \pi_5(G) \ \Leftrightarrow\ G \in \mathcal{L}[q,\varepsilon] \quad \text{and} \quad G \cap \pi_5(G) = \emptyset \ \Leftrightarrow\ G \notin \mathcal{L}[q,\varepsilon]. \qquad (19)$$

Therefore G may be contained in its polar solid or be skew to it.

Case 2. $\varepsilon = \pi_5(\varepsilon)$: Here we have $G \subset \varepsilon = \pi_5(\varepsilon) \subset \pi_5(G)$.

Thus, for $\operatorname{Char}\mathbb{K} = 2$, there may be *two kinds of external plane to* H_5 and *two kinds of* 0*-secant of* H_5. As a further consequence, we obtain:

Proposition 6.1. *In case of* $\operatorname{Char}\mathbb{K} = 2$*, every pencil of an hfd line set contains at least one line* N *such that* $N \subset \pi_5(N)$*.*

If N is given as above, then $\pi_5(N) \cap H_5$ is an elliptic subquadric of H_5 and the line N is its *nucleus*; that is, all tangent planes of $\pi_5(N) \cap H_5$ contain the line N.

Next, we sketch, for any characteristic, an algebraic counterpart of the foregoing. So, as before, ε denotes a fixed external plane to H_5 and G is any line of ε. By Proposition 3.6, the set of all lines in ε corresponds under γ^{-1} to a Clifford parallelism $\boldsymbol{P}$ of $\mathrm{PG}(3,\mathbb{K})$. Hence $\boldsymbol{P}$ can be described in terms of a four-dimensional $\mathbb{K}$-algebra $\mathbb{H}$ subject to (2). We assume that the parallel classes of $\boldsymbol{P}$ are the classes of left parallel lines; otherwise the order of factors in the subsequent formula (20) has to be altered.

From now on we consider $\mathbb{H}$ as the underlying vector space of $\mathrm{PG}(3,\mathbb{K})$. The regular spread $\gamma^{-1}(G) \in \boldsymbol{P}$ sends a unique line through that point of $\mathrm{PG}(3,\mathbb{K})$ being spanned by the vector $1 \in \mathbb{H}$. This particular line corresponds to a two-dimensional $\mathbb{K}$-subspace $\mathbb{L}_G$ of $\mathbb{H}$, which actually is a proper intermediate field of $\mathbb{K}$ and $\mathbb{H}$. In terms of the $\mathbb{K}$-vector space $\mathbb{H}$, the regular spread $\gamma^{-1}(G)$ can be represented as

$$\bigl\{c \cdot \mathbb{L}_G \mid c \in \mathbb{H} \setminus \{0\}\bigr\}. \qquad (20)$$

This implies that $\gamma^{-1}(G)$ coincides with the spread that is associated with the quadratic field extension $\mathbb{L}_G/\mathbb{K}$; see, for example, [14]. Hence we obtain for $\operatorname{Char}\mathbb{K} \neq 2$: $\mathbb{H}$ satisfies condition (A) in (2) and $\mathbb{L}_G/\mathbb{K}$ is Galois. Otherwise, one of the following applies:

Case 1. $\operatorname{Char}\mathbb{K} = 2$ and $\mathbb{H}$ satisfies (A): Here $\mathbb{H}$ is a quaternion skew field. Some proper intermediate fields of $\mathbb{K}$ and $\mathbb{H}$ are Galois extensions of $\mathbb{K}$, while others are not. (A characterisation of these intermediate fields among all quadratic extension fields of $\mathbb{K}$ can be found in [11, Thm. 2.2].) Thus, $\mathbb{L}_G/\mathbb{K}$ may be Galois or not.

Case 2. $\operatorname{Char}\mathbb{K} = 2$ and $\mathbb{H}$ satisfies (B): Here all proper intermediate fields of $\mathbb{K}$ and $\mathbb{H}$ are inseparable over $\mathbb{K}$. Therefore $\mathbb{L}_G/\mathbb{K}$ is not Galois.

The announced connection with our previous discussion is as follows: From [14, Lemma 1], $\mathbb{L}_G/\mathbb{K}$ is Galois precisely when the intersection of all tangent planes of the subquadric $Q_3(G) = \pi_5(G) \cap H_5$ is empty; this in turn is equivalent to $G \cap \pi_5(G) = \emptyset$. Therefore, for $\operatorname{Char}\mathbb{K} = 2$ only, $\mathbb{H}$ satisfies (A) if, and only if, $\varepsilon \cap \pi_5(\varepsilon)$ is point, whereas (B) means $\varepsilon = \pi_5(\varepsilon)$.

Finally, it is straightforward to reverse our arguments for any characteristic. As ε varies in the set of all planes of $\mathrm{PG}(3,\mathbb{K})$ that are external to H_5, we obtain (up to $\mathbb{K}$-linear isomorphisms) all $\mathbb{K}$-algebras $\mathbb{H}$ subject to (2). Furthermore, in any such algebra $\mathbb{H}$ the proper intermediate fields of $\mathbb{K}$ and $\mathbb{H}$ are precisely the two-dimensional $\mathbb{K}$-subspaces of $\mathbb{H}$ that contain $1 \in \mathbb{H}$.

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