

From pentacyclic coordinates to chain geometries, and back*

HANS HAVLICEK

Abstract

Starting with the classical circle geometry of Sophus Lie, we give a survey about some of the developments in the area of chain geometries during the last three decades.

MSC 2000: 51B05, 51B10, 51B15, 51B20, 51B25, 51A45, 51C05, 17C50, 14M15.

Key words: Lie geometry, projective line over a ring, distant graph, Grassmannian, chain geometry, Jordan isomorphism, Jordan system.

1 Pentacyclic Coordinates

1.1 The circle geometry of S. Lie (1842–1899) aims at eliminating the distinction between circles, lines, and points of the Euclidean plane. The idea is that points and lines should be viewed as circles with “zero” and “infinite” radius, respectively. Moreover, lines and circles are endowed with an *orientation*. For our purposes it suffices to think of an *oriented line* (an *oriented circle*) as a line (a circle) with an arrow on it. There are precisely two possibilities of orientation. For circles we can even distinguish between *counterclockwise* and *clockwise* orientation. More precisely, a *Lie cycle* is one of the following:

- An oriented circle. Its *signed radius* $r \neq 0$ is positive (negative) if the orientation is counterclockwise (clockwise).
- A point. Its *radius* is defined to be zero.
- An oriented line.
- The *point at infinity*. It is denoted by the symbol ∞ .

The set of all Lie cycles will be written as \mathcal{N} . It is endowed with a binary *contact relation* \sim , where $x \sim y$ is to be read as “ x touches y ”. This relation is reflexive and symmetric by definition. Touching Lie cycles are depicted in Figure 1. In addition, ∞ is assumed to touch all oriented lines, but no oriented circle and no point.

*This article is an extended version of my lecture “*Jenseits der pentazyklischen Koordinaten*” at the colloquium celebrating the 75th birthday of Prof. Dr. Dr. h. c. Walter Benz, University of Hamburg, June 9th, 2006.

Hans Havlicek

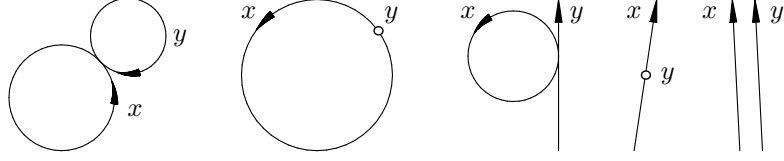


Figure 1: Contact relation

1.2 Let us shortly motivate the “need” for orientation and for the point at infinity: Suppose that we are given three distinct points. There is a *unique* circle or a *unique* line passing through all of them. On the other hand, if we are given the three side lines of a triangle then there are *four* circles touching all of them, the incircle and the three excircles. Thus points and (non-oriented) lines behave totally different. How should they be considered as being “equal”? However, our three given points give rise to precisely *two* distinct Lie cycles touching all of them. So, let us also introduce an orientation on each of the three given lines. Then precisely *one* of the four circles from the above can be oriented in such a way that it is in contact with the oriented lines. This is better than before, but we would like to have *two* such circles. Hence we add an extra *point at infinity* which touches all spears irrespective of their orientation.

1.3 The set \mathcal{N} of Lie cycles can be mapped bijectively onto the point set of a non-degenerate quadric

$$\Lambda : -x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0 \quad (1)$$

in the four-dimensional real projective space $\mathbb{P}_4(\mathbb{R})$ as follows: Choose a Cartesian coordinate system in the plane.

- The image of an oriented circle with midpoint (m_1, m_2) and signed radius $r \neq 0$ is

$$\mathbb{R} \left(\frac{1+N}{2}, \frac{1-N}{2}, m_1, m_2, -r \right),$$

where $N := m_1^2 + m_2^2 - r^2$. The image of a point (m_1, m_2) is given likewise by setting $r = 0$.

- In order to obtain the image point of an oriented line we consider its two equations in Hesse normal form. Precisely one of them, say $a_0 + a_1x_1 + a_2x_2 = 0$, has the property that the unit vector $(-a_2, a_1)$ determines the orientation of the given line. The image point is then defined to be $\mathbb{R}(-a_0, a_0, a_1, a_2, 1)$.
- The image of ∞ is $\mathbb{R}(-1, 1, 0, 0, 0)$.

The *pentacyclic coordinates* of a Lie cycle are, by definition, the homogeneous coordinates of its image point on the *Lie quadric* Λ .

It is easy to check that two Lie cycles are in contact if, and only if, their images on Λ are conjugate with respect to the polarity of Λ or, in other words, if their pentacyclic coordinate vectors are orthogonal with respect to the pseudo-Euclidean dot product of \mathbb{R}^5 given by the matrix $\text{diag}(-1, 1, 1, 1, -1)$.

A *Lie transformation* is a bijection $\mathcal{N} \rightarrow \mathcal{N}$ which preserves \sim in both directions. The *fundamental theorem* of Lie geometry states that all Lie transformations arise from the collineations of $\mathbb{P}_4(\mathbb{R})$ leaving invariant the Lie quadric Λ [6, p. 42].

1.4 In order to illustrate the power of the mapping described in the above, we recall a problem due to Apollonius of Perga (approx. 262–190 B. C.): *Find all circles which touch three given circles.*

In terms of pentacyclic coordinates the solution can be found most easily: First, endow each of the given circles with an orientation. Next, calculate their images on the Lie quadric Λ . Then intersect the tangent hyperplanes of Λ at these points. This gives (up to degenerate cases) a line L , say. Finally, determine $L \cap \Lambda$. This amounts to solving a quadratic equation, whereas all steps before yield linear equations. If the line L and the Lie quadric Λ have points in common (which need not be the case) then their pre-images are the solutions to the Apollonius problem for oriented circles. See Figure 2. Taking into account the various possibilities for orientation of the given circles, the initial problem turns out to have up to eight solutions. By this approach, also all forms of exceptional cases, e. g., when one of the “solutions” is an oriented line or a point, are easily understood.

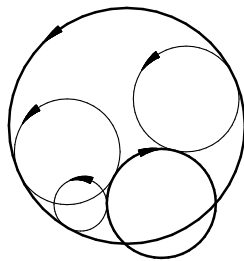


Figure 2: The problem of Apollonius for oriented circles

1.5 We refer to [6] for more details on Lie’s circle geometry and an extensive list of references up to the year 1973. It is worth noting that the point-line geometry of the Lie quadric is one of the classical *generalised quadrangles* [92, pp. 57–58]. For higher-dimensional Lie geometries, differential Lie geometry, and relations to special relativity see [7], [24], [29], [30], [31], [52], [64], [70],

Hans Havlicek

[71], [99], and the references made there. Infinite-dimensional Lie geometry is one of the topics in the recent book of W. Benz [8].

1.6 We now aim at recovering two other classical geometries from Lie's circle geometry.

Firstly, let \mathcal{M} be the set of points in the plane together with ∞ . For each Lie cycle $y \notin \mathcal{M}$ the point set

$$\{x \in \mathcal{M} \mid x \sim y\} \quad (2)$$

is the set of points on a circle or the range of points (including ∞) on a line. This set remains unchanged if the orientation of y is altered. In this way we obtain the Euclidean *Möbius geometry* (August F. Möbius (1790–1868)). In contrast to Lie geometry, which is a set endowed with a binary relation, here we have a set of points together with the family of distinguished subsets (2) carrying the name *Möbius circles* or *chains*. Observe that Möbius circles do not have an orientation.

Secondly, let \mathcal{L} be the set of oriented lines which now will also be called *spears*. Each Lie cycle $y \notin \mathcal{L} \cup \{\infty\}$ gives rise to the set

$$\{x \in \mathcal{L} \mid x \sim y\} \quad (3)$$

which is called a *chain of spears*, shortly a *chain*. This gives the Euclidean *Laguerre geometry*, i. e. the set \mathcal{L} together with the set of all its chains. It is named after Edmond N. Laguerre (1834–1886). The point ∞ is superfluous in Laguerre geometry.

Both geometries allow a unified description. If we consider the usual field of *complex numbers* then the point set of the Euclidean Möbius geometry coincides with the *complex projective line*,

$$\mathbb{P}(\mathbb{C}) := \mathbb{C} \cup \{\infty\} \text{ with } \infty := \frac{1}{0},$$

which is well known from complex analysis. Likewise, the ring of *real dual numbers*,

$$\mathbb{D} := \{x + y\varepsilon \mid (x, y) \in \mathbb{R}^2\}, \text{ where } \varepsilon \notin \mathbb{R} \text{ and } \varepsilon^2 = 0, \quad (4)$$

gives rise to the *dual projective line*. It has the form

$$\mathbb{P}(\mathbb{D}) := \mathbb{D} \cup \left\{ \frac{1}{y\varepsilon} \mid y \in \mathbb{R} \right\}. \quad (5)$$

In contrast to the complex projective line there are *infinitely many* points at infinity; they are given by the second set on the right hand side of (5). This set comprises all formal quotients with enumerator 1 and a zero divisor as denominator. By a classical result, the set of spears of the Euclidean Laguerre geometry can be identified with the dual projective line [6, pp. 26–28].

In either case the chains are precisely the images of the *real projective line* $\mathbb{P}(\mathbb{R}) := \mathbb{R} \cup \{\infty\}$, considered as subset of $\mathbb{P}(\mathbb{C})$ (resp. $\mathbb{P}(\mathbb{D})$), under the action of the complex (resp. dual) linear fractional group

$$z \mapsto \frac{az + b}{cz + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ invertible.}$$

1.7 In the famous book [6] by W. Benz, published in 1973, projective lines over *commutative* rings and chain geometries arising from *commutative* algebras were investigated systematically, thereby generalising the classical results. In this article we focus our attention on the further development of these topics. The book of Benz contains a wealth of further material which we cannot mention here.

2 The projective line over a ring

2.1 We adopt the following conventions: All our rings are associative, with a unit element $1 \neq 0$, which acts unitally on modules, and is inherited by subrings. Multiplication in a *field* need not be commutative. If a ring R contains a field K , as a subring which commutes with all elements of R , then R is called a *K-algebra*. The dimension of R over K may be finite or infinite.

2.2 The crucial task is to find a “good” definition of the *projective line* over a ring R , even when R is not necessarily commutative. In terms of left homogeneous coordinates a *point* of this line should of course have the form $R(a, b)$ with $(a, b) \in R^2$ (considered as a left module over R). But, which pairs (a, b) should be representatives of points? Let us shortly recall some particular cases:

- If R is a field then (a, b) gives rise to a point if, and only if $(a, b) \neq (0, 0)$. (This is mathematical folklore.)
- If R is a *local ring*, i. e., the set of all non-invertible elements of R is an ideal, then (a, b) determines a point if, and only if, a or b is an invertible element. (This was used, e. g., in [60].)
- If R is commutative then a pair (a, b) yields a point if, and only if, it is *unimodular*, i. e., there are elements $x, y \in R$ with $ax + by = 1$. (This definition was adopted in [6].)

All these conditions remain meaningful over any ring. Hence there are *different definitions* for the projective line over a ring. See [58] for a survey and [94, pp. 291–292] for further comments on the problem of obtaining “good” projective geometries from a ring.

2.3 Projective lines over several classes of non-commutative rings, like *skew fields* (see [6]), *matrix rings over commutative fields*, and rings of *ternions* (i. e. upper triangular 2×2 matrices over a commutative field), were exhibited already in the 1960s and before. The Belgians J. Depunt, [26], [27], C. Vanhelleputte [93], X. Hubaut [48], [49], and J. A. Thas [90], [91] were among the first to study projective lines over non-commutative rings other than skew fields. The Italian G. Russo considered the projective line over a ring of upper triangular $m \times m$ matrices over a commutative field. He coined the name “*enioni*” (Italian for “*n-ions*”) for such a ring [72], [73], [74].

We shall stick here to a definition which, to our knowledge, appeared first in [49]. There is even a footnote in this article pointing out the general case of an arbitrary ring R , whereas the paper itself is concerned with finite-dimensional algebras only. The essential ingredient for this definition is the *general linear group* $\mathrm{GL}_2(R)$ in two variables over a ring R . The elements of this group are precisely the invertible 2×2 matrices with entries in R .

Definition 2.4 The *projective line* over a ring R is the set $\mathbb{P}(R)$ of all cyclic submodules $R(a, b)$ of R^2 , where (a, b) is the first row of an invertible 2×2 matrix over R . Such a pair is called *admissible*.

We read off from the 2×2 identity matrix that $R(1, 0)$ is a point. Furthermore, as A ranges in $\mathrm{GL}_2(R)$ all points of $\mathbb{P}(R)$ arise as $R(1, 0) \cdot A$. Thus $\mathbb{P}(R)$ can also be described as the *orbit* of $R(1, 0)$ under the natural action of $\mathrm{GL}_2(R)$. This means that all points of $\mathbb{P}(R)$ are “the same” up to the action of $\mathrm{GL}_2(R)$. It is an easy exercise to show that two admissible pairs represent the same point if, and only if, they are left-proportional by an invertible element in R .

An elegant coordinate-free definition of $\mathbb{P}(R)$ is due to A. Herzer; see [43, p. 785]. We refer also to the recently published book by A. Herzer and A. Blunck [22] for further details.

2.5 The projective line over a ring R has some peculiar properties if there exist elements with a single-sided inverse [14]:

If $s \in R$ has a left-inverse, say l , such that $ls = 1 \neq sl$ then $R(s, 0) = R(ls, 0) = R(1, 0)$, but it is easily seen that there is no matrix in $\mathrm{GL}_2(R)$ with first row $(s, 0)$. Thus a point may have non-admissible representatives. However, from now on *only admissible pairs will be used to represent points*.

If $s \in R$ has a right inverse, say r , such that $sr = 1 \neq rs$ then $R(s, 0) \subsetneq R(1, 0)$, but now $(s, 0)$ is admissible, because

$$\gamma := \begin{pmatrix} s & 0 \\ 1 - rs & r \end{pmatrix} \text{ has the inverse } \gamma^{-1} = \begin{pmatrix} r & 1 - rs \\ 0 & s \end{pmatrix}.$$

This means that there may be *nested points*. So, we are far away from Euclid’s definition: “*A point is that which has no part*” [28, Vol. 1, p. 153].

There is another phenomenon which is only present in certain non-commutative rings. A *unimodular pair need not be admissible* [15, Remark 5.1]. The following example is based on a paper of M. Ojanguren and R. Sridharan [66]: Let K be a skew field and let $R = K[X, Y]$ be the polynomial ring in two independent central indeterminates over K . Given elements $a, b \in K$ with $ab \neq ba$ the pair $(X + a, Y + b)$ turns out to be unimodular, but not admissible.

On the other hand, each admissible pair (a, b) is unimodular, as follows by multiplying an invertible matrix $A = \begin{pmatrix} a & b \\ * & * \end{pmatrix}$ with its inverse. A ring has *stable rank 2* if for each unimodular pair $(a, b) \in R^2$ there is a $c \in R$ such that $a + bc$ is invertible. For such a ring unimodular and admissible pairs are the same [43, p. 785]. F. D. Veldkamp (1931–1999) has repeatedly stressed the importance of rings of stable rank 2 for geometry; see, e. g., [94, p. 293].

3 The distant graph

3.1 On the projective line over a ring R there is an important binary relation: Two points p and q of $\mathbb{P}(R)$ are called *distant*, in symbols $p \triangle q$, if there is a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

with $p = R(a, b)$ and $q = R(c, d)$. This relation is anti-reflexive and symmetric. Distant points are also said to form a *clear*, *spectral* or *regular* pair [45], [49]. The graph of the relation \triangle , i. e. the pair $(\mathbb{P}(R), \triangle)$, is called the *distant graph* of $\mathbb{P}(R)$. It is an undirected graph without loops.

Other authors prefer the negated relation $\not\triangle$ and speak of *neighbouring* or *parallel* points. The term “parallel” stems from parallel spears in Euclidean Laguerre geometry, as depicted on the right hand side of Figure 1. Parallel spears correspond to non-distant points of the dual projective line. The parallelism of spears is an equivalence relation, but $\not\triangle$ is *not* transitive in general.

3.2 We take a closer look at some examples.

- (a) Let \mathbb{Z}_4 be the *ring of integers modulo 4*. We consider also the ring of *dual numbers over the field \mathbb{Z}_2* . These dual numbers are defined as in (4), with \mathbb{R} to be replaced by \mathbb{Z}_2 . The distant graphs of $\mathbb{P}(\mathbb{Z}_4)$ and $\mathbb{P}(\mathbb{Z}_2[\varepsilon])$ are isomorphic to the graph of vertices and edges of an octahedron (Figure 3). But our two rings are not isomorphic, since we have $1 + 1 \neq 0$ in \mathbb{Z}_4 and $1 + 1 = 0$ in $\mathbb{Z}_2[\varepsilon]$. Thus non-isomorphic rings may have isomorphic distant graphs.
- (b) The ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ of *double numbers* over \mathbb{Z}_2 has also four elements. The distant graph of $\mathbb{P}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is depicted in Figure 3, the shaded triangles serve only for better visualisation. There are nine points.

Hans Havlicek

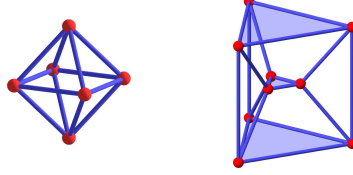


Figure 3: Distant graphs for rings with four elements.

- (c) The projective line over the field \mathbb{C} of complex numbers can be seen as the unit sphere in Euclidean space. Here and in the subsequent examples it will be more intuitive to illustrate the neighbour relation $\not\sim$. Each point $p \in \mathbb{P}(\mathbb{C})$ has a single neighbour, namely p itself (Figure 4).

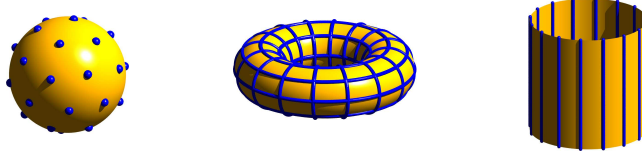


Figure 4: Neighbour relation on $\mathbb{P}(\mathbb{C})$, $\mathbb{P}(\mathbb{R} \times \mathbb{R})$, and $\mathbb{P}(\mathbb{D})$.

- (d) The projective line over the ring $\mathbb{R} \times \mathbb{R}$ of *real double numbers* can be identified with the Cartesian product $\mathbb{P}(\mathbb{R}) \times \mathbb{P}(\mathbb{R})$. By virtue of this identification, two points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ with $p_i, q_j \in \mathbb{P}(\mathbb{R})$ are neighbouring if $p_1 = q_1$ or $p_2 = q_2$. Since the real projective line may be illustrated as a circle, the torus is a point model for the projective line over $\mathbb{R} \times \mathbb{R}$. The longitudinal and latitudinal circles represent the maximal subsets of mutually neighbouring points (Figure 4).
- (e) The *Blaschke cylinder*, named after W. Blaschke (1885–1962), is a point model for the projective line over the real dual numbers. The generators of the cylinder represent the maximal subsets of mutually neighbouring points (Figure 4).
- (f) Let $K^{n \times n}$ be the ring of $n \times n$ matrices over a commutative field K . The projective line over $K^{n \times n}$ can be identified with the Grassmannian \mathcal{G}_n of all n -dimensional subspaces of K^{2n} via the bijection

$$\mathbb{P}(K^{n \times n}) \rightarrow \mathcal{G}_n : R(A, B) \mapsto \text{rowspan of } (A|B), \quad (6)$$

where $(A|B)$ stands for the matrix A augmented by B . Under this bijection distant points correspond to complementary subspaces. See [49, p. 500], where the result is stated in an equivalent form, using the projective space $\mathbb{P}_{2n-1}(K)$. We know today from the work of A. Blunck that

From pentacyclic coordinates to chain geometries, and back

this theorem holds also for matrix rings over skew fields and, *mutatis mutandis*, for *endomorphism rings of infinite-dimensional vector spaces* over arbitrary fields [11, Theorem 2.1].

3.3 Among all examples from above the one in (f) is most important. By the results of E. Artin (1898–1962) and J. H. M. Wedderburn (1882–1948), the Artinian semisimple rings are precisely the direct products of matrix rings over (possibly different) fields [57]. The projective line over such a ring is in one-to-one correspondence with a *product of Grassmannians* [20]. This gives a deep insight into the structure of such a projective line. For example, this allows one to determine the number of points of the projective line over a finite ring, even if the ring is not semisimple [95, pp. 31–36].

3.4 Let us turn back to the general case. The distant graph of a projective line over a ring R is easily seen to be a complete graph if, and only if, R is a field. In this case the diameter of the distant graph equals one. By [43] this diameter is ≤ 2 whenever R is a ring of stable rank 2. For example, finite-dimensional algebras are of stable rank 2. This explains why in most “classical examples” for any two non-distant points there is a point which is distant to both of them. Only few distant graphs with a diameter > 2 seem to be known [15].

The distant graph of $\mathbb{P}(R)$ is connected if, and only if, R is a GE_2 -ring. This means that each matrix in $\text{GL}_2(R)$ is a product of elementary matrices and invertible diagonal matrices. The results of P. M. Cohn [25] provide examples of distant graphs with more than one connected component [15, p. 115].

Only recently, the meaning of the *Jacobson radical* of a ring was expressed in terms of the associated distant graph [18, p. 116]: If $p = R(1, 0)$ and $q \in \mathbb{P}(R)$ then

$$(x \triangle p \Leftrightarrow x \triangle q) \text{ for all } x \in \mathbb{P}(R)$$

holds precisely when $q = R(1, r)$ with r taken from the Jacobson radical of R .

3.5 The group $\text{GL}_2(R)$ acts transitively on the set of mutually distant triples of $\mathbb{P}(R)$ and thereby preserves the distant relation. It is a natural question to ask for all isomorphisms between distant graphs. In [19] an equivalent problem for Grassmannians of vector spaces is exhibited, whereas in [20] all isomorphisms $\mathbb{P}(R) \rightarrow \mathbb{P}(R')$ are determined provided that R and R' are direct products of matrix rings over fields. In both papers there are many interrelations with results about *adjacency preserving transformations* of Grassmannians due to W. Benz, H. Brauner, W.-l. Huang, A. Kreuzer, A. Naumowicz, K. Prażmowski, and others. See [7], [23], [36], [46], [53], and [65]. The distant graph $(\mathbb{P}(R), \triangle)$ turns into a *Plücker space* [7, p. 199] if we add a loop at each point. The *Plücker transformations* are then just the automorphisms of the distant graph.

4 Chain geometries

4.1 Throughout this section R denotes an algebra over a (necessarily commutative) field K . The definition of the *chain geometry* associated with K and R can be taken over literally from [6], since we did already introduce the projective line over R irrespective of commutativity: As R has a unit element, we may assume that $K \subset R$. This allows to identify $\mathbb{P}(K)$ with a subset of $\mathbb{P}(R)$ via $K(a, b) \mapsto R(a, b)$. Every image of $\mathbb{P}(K)$ under a matrix of $\mathrm{GL}_2(R)$ is called a *chain*. Let \mathcal{C} be the set of all chains. Then the incidence structure

$$\Sigma(K, R) := (\mathbb{P}(R), \mathcal{C})$$

is called the *chain geometry* associated with the K -algebra R .

The distant relation on $\mathbb{P}(R)$ can be expressed in terms of $\Sigma(K, R)$ as follows: Two distinct points are distant if, and only if, they are on a common chain. Thus all results about the distant graph are also available in a chain geometry.

4.2 Many basic properties of a chain geometry do not depend on the commutativity of the algebra R . For example, there is a unique chain through any three mutually distant points. A major difference between commutative and non-commutative algebras (or, more generally, rings) concerns the notion of *cross ratio* [43, p. 787]. In the non-commutative case the cross ratio of four points of $\mathbb{P}(R)$ is a class of conjugate elements of R rather than a single element of R . Cross ratios are often useful, since four mutually distinct points are on a common chain precisely when their cross ratio is in K . We refer in particular to the paper of A. Blunck [12] dealing with cross ratios on Grassmannians.

4.3 It is well known that the Möbius geometry $\Sigma(\mathbb{R}, \mathbb{C})$, the Laguerre geometry $\Sigma(\mathbb{R}, \mathbb{D})$, and the *Minkowski geometry* $\Sigma(\mathbb{R}, \mathbb{R} \times \mathbb{R})$ (named after H. Minkowski (1864–1909)) can be represented on an elliptic quadric, a quadratic cone (without its vertex), and a hyperbolic quadric in three-dimensional real projective space $\mathbb{P}_3(\mathbb{R})$. In either case the non-degenerate conics represent the chains. For the Möbius geometry and Laguerre geometry one may also use a Euclidean space and view the elliptic quadric and the cone as a sphere and a cylinder of revolution (Figure 4). However, the hyperbolic quadric modelling the Minkowski geometry cannot be seen as part of the three-dimensional Euclidean space. This is why we used a torus in Figure 4 instead. The grid of longitudinal and latitudinal circles corresponds to the grid of lines on the hyperbolic quadric.

4.4 One of the open problems from [6] was to find *point models* for arbitrary chain geometries. In view of the examples from 4.3, it was quite natural to look for *quadric models*, where chains were represented by non-degenerate conics. H. Hotje showed in [44] and [45] that such models exist for finite-dimensional

From pentacyclic coordinates to chain geometries, and back

quadratic algebras, i. e. K -algebras in which every element has a minimal polynomial over K with degree ≤ 2 . Here the work on these algebras by H. Karzel (who used the term *kinematic algebras* [50], [51]) turned out useful. Several papers on this topic appeared afterwards. The next step was taken in 1980 by W. Benz, H.-J. Samaga, and H. Schaeffer [9]: The chain geometry $\Sigma(K, K^n)$ was shown to be embeddable in a projective space over K as *Segre variety*, with chains going over to *normal rational curves*. The breakthrough was accomplished shortly afterwards by M. Werner [98] using ideas from [49]. We sketch here Werner's approach in a generalised form due to A. Herzer [39]:

- Given a finite-dimensional K -algebra R , determine a *faithful representation* of R in terms of $n \times n$ matrices over K . This amounts to embedding $\Sigma(K, R)$ in $\Sigma(K, K^{n \times n})$ such that chains of $\Sigma(K, R)$ go over to chains of $\Sigma(K, K^{n \times n})$.
- The bijection (6) of $\mathbb{P}(K^{n \times n})$ onto the Grassmannian of n -subspaces of K^{2n} gives a model of $\Sigma(K, R)$ within this Grassmannian. Making use of results by R. Metz [63], the images of chains can be identified as *reguli* of the Grassmannian.
- Finally, this Grassmannian is mapped onto its associated Grassmann variety, lying in a $\binom{2n}{n} - 1$ -dimensional projective space. Here the chains are represented by normal rational curves.

All the classical point models mentioned in the above (excluding the torus) fit into this general concept which may also be described in a coordinate-free way [43, p. 810]. By suitable projections, it is often possible to obtain point models of $\Sigma(K, R)$ in lower-dimensional spaces. These "projected models" are smooth *quasiprojective varieties* and the chains appear there as *rational curves* [40, p. 812].

The case of infinite-dimensional algebras seems to be unsettled.

5 Isomorphisms of chain geometries

5.1 Let $\Sigma(K, R)$ and $\Sigma(K', R')$ be chain geometries. It is one of the basic problems to determine *all* isomorphisms between them. In [6] this problem was solved for various classes of chain geometries over commutative algebras. However, in order to find generalisations one first has to find appropriate mappings of the underlying algebras which can be used to describe all isomorphisms in a second step.

5.2 We start with an obvious example: Let $\alpha : R \rightarrow R'$ be an *isomorphism* of the K -algebra R onto the K' -algebra R' or, said differently, let α be a semilinear

Hans Havlicek

bijection (of vector spaces) such that $(ab)^\alpha = a^\alpha b^\alpha$ for all $a, b \in R$. It is obvious that

$$R(a, b) \mapsto R'(a^\alpha, b^\alpha) \tag{7}$$

defines an *isomorphism* of $\Sigma(K, R)$ onto $\Sigma(K', R')$.

Since we admit non-commutative algebras, we may also consider an *antiisomorphism* of algebras, i. e. a semilinear bijection $\alpha : R \rightarrow R'$ such that $(ab)^\alpha = b^\alpha a^\alpha$ for all $a, b \in R$. It was a longstanding open problem whether or not any antiisomorphism defines “in some natural way” an isomorphism of chain geometries. Observe that the assignment given by (7) is not well-defined in this case, let alone its being an isomorphism: For if u is a unit in R then (a, b) and (ua, ub) represent the same point, whereas (a^α, b^α) and $(a^\alpha u^\alpha, b^\alpha u^\alpha)$ need not be left proportional by a unit in R' . Hence they may represent distinct points.

If we restrict ourselves to local algebras then each point of $\mathbb{P}(R)$ has at least one *normalised* representative $(1, b)$ or $(a, 1)$. Now its image can be defined unambiguously by $R'(1', b^\alpha)$ or $R'(a^\alpha, 1')$. This gives not only a well defined mapping, but also an isomorphism of chain geometries. It was shown in [16] that any antiisomorphism of algebras gives rise to an isomorphism of the associated chain geometries. However, there does not seem to be an explicit formula for this isomorphism in the general case. We refrain from a further discussion, because there are even more general mappings of algebras which deserve our attention!

5.3 Isomorphisms and antiisomorphisms are just particular examples of *Jordan isomorphisms* of algebras (P. Jordan (1902–1980)). A Jordan isomorphism $\alpha : R \rightarrow R'$ is a semilinear bijection taking 1 to 1' such that

$$(aba)^\alpha = a^\alpha b^\alpha a^\alpha \text{ for all } a, b \in R.$$

5.4 For many classes of algebras, e. g. fields or commutative algebras over fields of characteristic $\neq 2$, there are no Jordan isomorphisms other than isomorphisms and antiisomorphisms. On the other hand, proper Jordan isomorphisms (other than isomorphisms and antiisomorphisms) are easy to construct: The mapping which sends each $A \in \mathbb{R}^{2 \times 2}$ to its transpose A^T is an anti-automorphism of $\mathbb{R}^{2 \times 2}$, whence the mapping $(A, B) \mapsto (A, B^T)$ is a Jordan automorphism of the direct product of $\mathbb{R}^{2 \times 2}$ with itself. Further examples of Jordan isomorphisms can be found in [22, pp. 81–82].

5.5 Jordan isomorphisms of finite-dimensional local algebras determine isomorphisms of their chain geometries, as was shown by A. Herzer [41], who could make use of previous results by B. V. Limaye and N. B. Limaye [59], [60], and [61]. Like before, the main problem is the definition of such a mapping. Under the given restrictions this can be done as for antiisomorphisms in 5.2 by using normalised representatives.

From pentacyclic coordinates to chain geometries, and back

In 1989, C. Bartolone made a great step forward by introducing a completely new idea [5]. If R has stable rank 2 (see 2.5) then

$$\mathbb{P}(R) = \{R(xy - 1, x) \mid x, y \in R\}.$$

This means that each point of $\mathbb{P}(R)$ can be written (usually in various ways) with the help of two *parameters* $x, y \in R$. Now, somewhat surprisingly, the assignment

$$R(xy - 1, x) \mapsto R'(x^\alpha y^\alpha - 1', x^\alpha) \text{ with } x, y \in R$$

gives a well defined isomorphism of chain geometries for any Jordan isomorphism $\alpha : R \rightarrow R'$. A generalisation to arbitrary algebras and an interpretation of Bartolone's approach can be found in [17]. However the definition of the point to point mapping arising from a Jordan isomorphism is too involved to be sketched here. We just want to emphasise that this mapping is defined only on the connected component of $R(1, 0)$ in the distant graph. This connected component may, or may not, be the entire projective line $\mathbb{P}(R)$.

5.6 The mappings from the previous paragraphs together with the mappings induced by $\text{GL}_2(R')$ give now all isomorphisms of chain geometries provided that certain assumptions on R and R' are made. The interested reader should consult [43, pp. 832–833]. Also, we would like to add that the results on mappings determined by Jordan isomorphisms can be reformulated in a more general form for *Jordan homomorphisms* of rings.

6 Subspaces of chain geometries

6.1 Another remarkable topic is the investigation of *subspaces* of a chain geometry $\Sigma(K, R)$. A *subspace* has to be closed under chains, but it has also to satisfy a number of extra conditions in order to exclude degenerate cases. For a precise definition one needs a series of notions which are not within the scope of this article. See [22, pp. 59–60]. Hence we have to restrict ourselves to presenting some examples of subspaces together with their interesting algebraic background.

6.2 Given a subalgebra S of a K -Algebra R one would expect the projective line over S to be a subspace of $\Sigma(K, R)$. However, this will only be true if we impose the following extra condition:

$$a \in S \text{ invertible in } R \Rightarrow a^{-1} \in S. \tag{8}$$

If condition (8) is not satisfied for an element $a \in S$ then the distant relation Δ_S on $\mathbb{P}(S)$ does not coincide with the restriction to $\mathbb{P}(S)$ of the distant relation Δ_R coming from $\mathbb{P}(R)$. Indeed, we have $R(a, 1) \Delta_R R(0, 1)$, but $S(a, 1) \not\Delta_S S(0, 1)$. For example, let R be the polynomial algebra $K[X]$, and let S be its field of fractions $K(X)$. Then X is invertible in $K(X)$, but $X^{-1} \notin K[X]$.

6.3 While any subalgebra satisfying the extra condition (8) gives rise to a subspace $\Sigma(K, R)$, there are also more general substructures of R with this property: A *strong Jordan system* of R is defined to be a K -subspace S of R such that (i) S contains the unit element $1 \in R$, (ii) S satisfies condition (8), and (iii) for each $x \in R$ more than half of the elements in the coset $x + K$ are invertible (this is strongness).

Each strong Jordan system S is closed under the *Jordan triple product*, i. e., $aba \in S$ for all $a, b \in S$, but it need not be closed under multiplication. If the characteristic of K is $\neq 2$ then $\frac{1}{2}(ab + ba) \in S$ for all $a, b \in S$. See [22, pp. 61–63].

In order to associate with S a subset of $\mathbb{P}(R)$ the idea from [5] to describe points of $\mathbb{P}(R)$ via parameters $x, y \in R$ is used once more: To each strong Jordan system S corresponds the point set

$$\mathbb{P}(S) := \{R(xy - 1, x) \mid x, y \in S\} \quad (9)$$

which turns out to be a subspace of $\Sigma(K, R)$ [22, p. 67].

For a wide class of algebras all connected subspaces are of the form (9), up to a transformation in $\mathrm{GL}_2(R)$. These results are due to H.-J. Kroll [54], [55], [56], and A. Herzer [42]. Cf. also [22, p. 69–72].

6.4 We consider the following example. Let $\mathbb{R}^{2 \times 2}$ be the algebra of 2×2 matrices over \mathbb{R} . Denote by S the subset of all *symmetric* 2×2 *matrices*. While S is not closed under multiplication, it is a strong Jordan system of $\mathbb{R}^{2 \times 2}$. The chain geometry $\Sigma(\mathbb{R}, \mathbb{R}^{2 \times 2})$ has a quadric model, namely the well known *Klein quadric* in $\mathbb{P}_5(\mathbb{R})$ representing the lines of $\mathbb{P}_3(\mathbb{R})$ (or, equivalently, the 2-subspaces of the vector space \mathbb{R}^4) [43, p. 814]. The subspace $\mathbb{P}(S)$ corresponds to a hyperplane section of the Klein quadric. This hyperplane section can be identified via a collineation with the Lie quadric Λ from (1). This means that Lie’s circle geometry allows an alternative description as the subspace of $\Sigma(\mathbb{R}, \mathbb{R}^{2 \times 2})$ associated to S . The chains correspond to the non-degenerate conics on the Lie quadric. Figure 2 illustrates one such chain within the set \mathcal{N} of Lie cycles: All Lie cycles that touch the two bold cycles form a chain. (Only three of these cycles are actually drawn.) Furthermore, Lie cycles represent distant points if, and only if, they do not touch.

7 Further reading

7.1 There are a lot of topics which would deserve our attention. Among them are *chain geometries over Jordan systems* and the algebraic description of their isomorphisms via *isotopisms* [22], *generalised chain geometries* $\Sigma(K, R)$, where K need not be in the centre of the ring R [13], and *geometries of field extensions* [33], [34], [35], [62], [67]. There is a widespread literature about characterisations of chain geometries and related structures which we did not even touch

upon. The book [22] and the survey article [43] are indispensable sources on these and various other topics. For results and references on *topological circle planes* and other *projective geometries over rings* see [10], [69], [89], and [96]. A series of papers deals with *fractals* in chain geometries [2], [3], [4], [75]. Certain *finite chain geometries* lead to *designs* or *divisible designs* [21], [32], [37], [87], [88]. Connections between chain geometries and the *geometry of matrices* can be found in [38], [47], and [97]. The projective lines over some small rings found attention in quantum physics [68], [76], [77], [78], [79], [80], [81], [82], [83], [84]. For applications in twistor theory of the projective point and the projective line over biquaternions see [1], [85], and [86].

7.2 Acknowledgement. The author is greatly indebted to Andrea Blunck (Hamburg), Herbert Hotje (Hannover), and Armin Herzer (Bodman) for their support in preparing this article.

References

- [1] A. F. Agnew. The twistor structure of the biquaternionic projective point. *Adv. Appl. Clifford Algebras*, 13:231–240, 2003.
- [2] R. Artzy. Dynamics of polynomials in finite and infinite Benz planes. *Matematiche (Catania)*, 47:193–196, 1992. *Combinatorics 92 (Catania, 1992)*.
- [3] R. Artzy. Dynamics of quadratic functions in cycle planes. *J. Geom.*, 44:26–32, 1992.
- [4] R. Artzy. Dynamik von Polynomen in Kreisebenen. *Math. Semesterber.*, 39:195–199, 1992.
- [5] C. Bartolone. Jordan homomorphisms, chain geometries and the fundamental theorem. *Abh. Math. Sem. Univ. Hamburg*, 59:93–99, 1989.
- [6] W. Benz. *Vorlesungen über Geometrie der Algebren*. Springer, Berlin, 1973.
- [7] W. Benz. *Geometrische Transformationen*. BI Wissenschaftsverlag, Mannheim, 1992.
- [8] W. Benz. *Classical geometries in modern contexts*. Birkhäuser, Basel, 2005.
- [9] W. Benz, H.-J. Samaga, and H. Schaeffer. Cross ratios and a unifying treatment of von Staudt’s notion of reeller Zug. In P. Plaumann and K. Strambach, editors, *Geometry – von Staudt’s Point of View*, pages 127–150. Reidel, Dordrecht, 1981.
- [10] W. Bertram. From linear algebra via affine algebra to projective algebra. *Linear Algebra Appl.*, 378:109–134, 2004.
- [11] A. Blunck. Regular spreads and chain geometries. *Bull. Belg. Math. Soc. Simon Stevin*, 6:589–603, 1999.
- [12] A. Blunck. The cross ratio for quadruples of subspaces. *Mitt. Math. Ges. Hamburg*, 22:81–97, 2003.

Hans Havlicek

- [13] A. Blunck and H. Havlicek. Extending the concept of chain geometry. *Geom. Dedicata*, 83:119–130, 2000.
- [14] A. Blunck and H. Havlicek. Projective representations I. Projective lines over rings. *Abh. Math. Sem. Univ. Hamburg*, 70:287–299, 2000.
- [15] A. Blunck and H. Havlicek. The connected components of the projective line over a ring. *Adv. Geom.*, 1:107–117, 2001.
- [16] A. Blunck and H. Havlicek. The dual of a chain geometry. *J. Geom.*, 72:27–36, 2001.
- [17] A. Blunck and H. Havlicek. Jordan homomorphisms and harmonic mappings. *Monatsh. Math.*, 139:111–127, 2003.
- [18] A. Blunck and H. Havlicek. Radical parallelism on projective lines and non-linear models of affine spaces. *Math. Pannonica*, 14:113–127, 2003.
- [19] A. Blunck and H. Havlicek. On bijections that preserve complementarity of subspaces. *Discrete Math.*, 301:46–56, 2005.
- [20] A. Blunck and H. Havlicek. On distant-isomorphisms of projective lines. *Aequationes Math.*, 69:146–163, 2005.
- [21] A. Blunck, H. Havlicek, and C. Zanella. Divisible designs from twisted dual numbers. *Designs Codes Crypt.* submitted.
- [22] A. Blunck and A. Herzer. *Kettengeometrien*. Shaker, Aachen, 2005.
- [23] H. Brauner. Über die von Kollineationen projektiver Räume induzierten Geradenabbildungen. *Sb. österr. Akad. Wiss, Abt. II, math. phys. techn. Wiss.*, 197:327–332, 1988.
- [24] T. E. Cecil. *Lie sphere geometry*. Universitext. Springer, New York, 1992.
- [25] P. M. Cohn. On the structure of the GL_2 of a ring. *Inst. Hautes Etudes Sci. Publ. Math.*, 30:365–413, 1966.
- [26] J. Depunt. Sur la géométrie ternionienne dans le plan. *Bull. Soc. Math. Belg.*, 11:123–133, 1959.
- [27] J. Depunt. Grondslagen van de analytische projectieve ternionenmeetkunde van het platte vlak. *Verh. Konink. Acad. Wetensch. Lett. Schone Kunst. België, Kl. Wetensch.*, 22(63):99 pp., 1960.
- [28] Euclid. *The thirteen books of Euclid's Elements translated from the text of Heiberg. Vol. I: Introduction and Books I, II. Vol. II: Books III–IX. Vol. III: Books X–XIII and Appendix.* Dover Publications Inc., New York, 2nd edition, 1956. Translated with introduction and commentary by Thomas L. Heath.
- [29] C. J. A. Evelyn, G. B. Money-Coutts, and J. A. Tyrrell. *The seven circles theorem and other new theorems*. Stacey International, London, 1974.
- [30] J. P. Fillmore. On Lie's higher sphere geometry. *Enseign. Math. (2)*, 25(1-2):77–114, 1979.

- [31] O. Giering. *Vorlesungen über höhere Geometrie*. Vieweg, Braunschweig, Wiesbaden, 1982.
- [32] S. Giese, H. Havlicek, and R.-H. Schulz. Some constructions of divisible designs from Laguerre geometries. *Discrete Math.*, 301:74–82, 2005.
- [33] H. Havlicek. On the geometry of field extensions. *Aequationes Math.*, 45:232–238, 1993.
- [34] H. Havlicek. Spheres of quadratic field extensions. *Abh. Math. Sem. Univ. Hamburg*, 64:279–292, 1994.
- [35] H. Havlicek. Spreads of right quadratic skew field extensions. *Geom. Dedicata*, 49:239–251, 1994.
- [36] H. Havlicek. On isomorphisms of Grassmann spaces. *Mitt. Math. Ges. Hamburg*, 14:117–120, 1995.
- [37] H. Havlicek. Divisible designs, Laguerre geometry, and beyond. *Quaderni Sem. Mat. Brescia*, 11:iv+63 pp. (electronic), 2006.
www.dmf.bs.unicatt.it/~semmat/preprints/2006.html.
- [38] H. Havlicek and P. Šemrl. From geometry to invertibility preservers. *Studia Math.*, 174:99–109, 2006.
- [39] A. Herzer. On a projective representation of chain geometries. *J. Geom.*, 22:83–99, 1984.
- [40] A. Herzer. Über niedrigdimensionale projektive Darstellungen von Kettengeometrien. *Geom. Dedicata*, 19:287–293, 1985.
- [41] A. Herzer. On isomorphisms of chain geometries. *Note Mat.*, 7:251–270, 1987.
- [42] A. Herzer. On sets of subspaces closed under reguli. *Geom. Dedicata*, 41:89–99, 1992.
- [43] A. Herzer. Chain geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 781–842. Elsevier, Amsterdam, 1995.
- [44] H. Hotje. Einbettung gewisser Kettengeometrien in projektive Räume. *J. Geom.*, 5:85–94, 1974.
- [45] H. Hotje. Zur Einbettung von Kettengeometrien in projektive Räume. *Math. Z.*, 151:5–17, 1976.
- [46] W.-l. Huang. Adjacency preserving transformations of Grassmann spaces. *Abh. Math. Sem. Univ. Hamburg*, 68:65–77, 1998.
- [47] W.-l. Huang. Adjacency preserving mappings of 2×2 Hermitian matrices. *Aequationes Math.*, in print.
- [48] X. Hubaut. Construction d’une droite projective sur une algèbre associative. *Acad. Roy. Belg. Bull. Cl. Sci. (5)*, 50:618–623, 1964.
- [49] X. Hubaut. Algèbres projectives. *Bull. Soc. Math. Belg.*, 17:495–502, 1965.

Hans Havlicek

- [50] H. Karzel. Kinematische Algebren und ihre geometrischen Ableitungen. *Abh. Math. Sem. Univ. Hamburg*, 41:158–171, 1974.
- [51] H. Karzel and G. Kist. Kinematic algebras and their geometries. In R. Kaya, P. Plaumann, and K. Strambach, editors, *Rings and Geometry*, pages 437–509. D. Reidel, Dordrecht, 1985.
- [52] H. Karzel and H.-J. Kroll. *Geschichte der Geometrie seit Hilbert*. Wiss. Buchges., Darmstadt, 1988.
- [53] A. Kreuzer. On isomorphisms of Grassmann spaces. *Aequationes Math.*, 56:243–250, 1998.
- [54] H.-J. Kroll. Unterräume von Kettengeometrien und Kettengeometrien mit Quadrikenmodell. *Resultate Math.*, 19:327–334, 1991.
- [55] H.-J. Kroll. Unterräume von Kettengeometrien. In N. K. Stephanidis, editor, *Proceedings of the 3rd Congress of Geometry (Thessaloniki, 1991)*, pages 245–247, Thessaloniki, 1992. Aristotle Univ. Thessaloniki.
- [56] H.-J. Kroll. Zur Darstellung der Unterräume von Kettengeometrien. *Geom. Dedicata*, 43:59–64, 1992.
- [57] T. Y. Lam. *A First Course in Noncommutative Rings*. Springer, New York, 1991.
- [58] A. Lashkhi. Harmonic maps over rings. *Georgian Math. J.*, 4:41–64, 1997.
- [59] B. V. Limaye and N. B. Limaye. The fundamental theorem for the projective line over commutative rings. *Aequationes Math.*, 16:275–281, 1977.
- [60] B. V. Limaye and N. B. Limaye. Fundamental theorem for the projective line over non-commutative local rings. *Arch. Math. (Basel)*, 28:102–109, 1977.
- [61] B. V. Limaye and N. B. Limaye. Correction to: Fundamental theorem for the projective line over non-commutative local rings. *Arch. Math. (Basel)*, 29:672, 1977.
- [62] H. Mäurer, R. Metz, and W. Nolte. Die Automorphismengruppe der Möbiusgeometrie einer Körpererweiterung. *Aequationes Math.*, 21:110–112, 1980.
- [63] R. Metz. Der affine Raum verallgemeinerter Reguli. *Geom. Dedicata*, 10:337–367, 1981.
- [64] G. Metzger. *Kugelkongruenzen in der Lie-Geometrie*. Thesis, Universität Freiburg, 1987.
- [65] A. Naumowicz and K. Prażmowski. On Segre’s product of partial line spaces and spaces of pencils. *J. Geom.*, 71:128–143, 2001.
- [66] M. Ojanguren and R. Sridharan. Cancellation of Azumaya algebras. *J. Algebra*, 18:501–505, 1971.
- [67] K. Pieconkowski. *Projektive Räume über Schiefkörperpaaren*. W&T Wissenschaft und Technik Verlag, Berlin, 1994. Thesis, TU Berlin.

- [68] M. Planat, M. Saniga, and M. R. Kibler. Quantum entanglement and projective ring geometry. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 2:Paper 066, 14 pp. (electronic), 2006.
- [69] B. Polster and G. Steinke. *Geometry on Surfaces*. Cambridge University Press, Cambridge, 2001.
- [70] J. F. Rigby. The geometry of cycles, and generalized Laguerre inversion. In C. Davis, B. Grünbaum, and F. A. Sherk, editors, *The Geometric Vein*, pages 355–378. Springer, New York, 1981.
- [71] J. F. Rigby. On the Money-Coutts configuration of nine antitangent cycles. *Proc. London Math. Soc.* (3), 43:110–132, 1981.
- [72] G. Russo. Piani “enonari”. *Rend. Mat. e Appl.* (5), 24:377–391, 1965.
- [73] G. Russo. Geometria della “retta” e del piano “enonario”. *Rend. Mat. e Appl.* (5), 26:510–540, 1967.
- [74] G. Russo. Alcune considerazioni sulle trasformazioni lineari di una retta “enonaria”. *Rend. Circ. Mat. Palermo* (2), 21:235–254, 1972.
- [75] H.-J. Samaga. Über fraktale Strukturen in Kreisgeometrien. *Mitt. Math. Ges. Hamburg*, 14:121–131, 1995.
- [76] M. Saniga and M. Planat. Finite geometries in quantum theory: from Galois (fields) to Hjelmslev (rings). *Internat. J. Modern Phys. B*, 20:1885–1892, 2006.
- [77] M. Saniga and M. Planat. Hjelmslev geometry of mutually unbiased bases. *J. Phys. A*, 39:435–440, 2006.
- [78] M. Saniga and M. Planat. On the fine structure of the projective line over $\text{GF}(2) \otimes \text{GF}(2) \otimes \text{GF}(2)$. *Chaos Solitons Fractals*, in print.
- [79] M. Saniga and M. Planat. The projective line over the finite quotient ring $\text{GF}(2)[x]/\langle x^3 - x \rangle$ and quantum entanglement. Theoretical background. *Theor. and Math. Phys.*, in print.
- [80] M. Saniga and M. Planat. Projective planes over “Galois” double numbers and a geometrical principle of complementarity. *Chaos Solitons Fractals*, in print.
- [81] M. Saniga, M. Planat, M. R. Kibler, and P. Pracna. A classification of the projective lines over small rings. *J. Phys. A*, submitted.
- [82] M. Saniga, M. Planat, and M. Minarovjech. The projective line over the finite quotient ring $\text{GF}(2)[x]/\langle x^3 - x \rangle$ and quantum entanglement. The Mermin “magic” square/pentagram. *Theor. and Math. Phys.*, in print.
- [83] M. Saniga, M. Planat, and P. Pracna. A classification of the projective lines over small rings II. Non-commutative case. Manuscript, 2006.
- [84] M. Saniga, M. Planat, and P. Pracna. Projective ring line encompassing two-qubits. Manuscript, 2006.
- [85] V. Souček. The conformal group action on $\mathbb{P}^1(\mathbb{C}\mathbb{H})$. *Twistor Newsletter*, 13:22–25, 1981.

Hans Havlicek

- [86] V. Souček. Complex quaternions, their connection to twistor theory. *Czechoslovak J. Phys. B*, 32:688–691, 1982.
- [87] A. G. Spera. t -Divisible designs from imprimitive permutation groups. *Europ. J. Combin.*, 13:409–417, 1992.
- [88] A. G. Spera. On divisible designs and local algebras. *J. Comb. Designs*, 3:203–212, 1995.
- [89] G. F. Steinke. Topological circle geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry*. Elsevier, Amsterdam, 1995.
- [90] J. A. Thas. Een studie betreffende de projectieve rechte over de totale matrix-algebra $M_3(K)$ der 3×3 matrices met elementen in een algebraïsch afgesloten veld K . *Verh. Konink. Acad. Wetensch. Lett. Schone Kunst. België, Kl. Wetensch.*, 31(112):151 p., 1969.
- [91] J. A. Thas. The m -dimensional projective space $S_m(M_n(GF(q)))$ over the total matrix algebra $M_n(GF(q))$ of the $n \times n$ -matrices with elements in the Galois field $GF(q)$. *Rend. Mat. Roma (6)*, 4:459–532, 1971.
- [92] H. Van Maldeghem. *Generalized Polygons*. Birkhäuser, Basel, 1998.
- [93] C. Vanhelleputte. Een studie betreffende de projectieve meetkunde over de ring der (2×2) -matrices met elementen in een commutatief lichaam. *Verh. Konink. Acad. Wetensch. Lett. Schone Kunst. België, Kl. Wetensch.*, 28(92):93 p., 1966.
- [94] F. D. Veldkamp. Projective ring planes and their homomorphisms. In R. Kaya, P. Plaumann, and K. Strambach, editors, *Rings and Geometry*, pages 289–350. D. Reidel, Dordrecht, 1985.
- [95] F. D. Veldkamp. Projective geometry over finite rings. *Quaderni del Seminario di Geometrie Combinatorie*, 92:1–39, January 1989. Dipartimento di Matematica Istituto G. Castelnuovo, Università degli Studi di Roma.
- [96] F. D. Veldkamp. Geometry over rings. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 1033–1084. Elsevier, Amsterdam, 1995.
- [97] Z.-X. Wan. *Geometry of Matrices*. World Scientific, Singapore, 1996.
- [98] M. Werner. Die klassisch-projektive Natur der Kettengeometrien endlicher Dimension. *Math. Z.*, 181:49–54, 1982.
- [99] I. M. Yaglom. On the circular transformations of Möbius, Laguerre, and Lie. In C. Davis, B. Grünbaum, and F. A. Sherk, editors, *The Geometric Vein*, pages 345–353. Springer, New York, 1981.

Hans Havlicek
Institut für Diskrete Mathematik
und Geometrie
Technische Universität Wien,
Wiedner Hauptstraße 8–10
A-1040 Wien, Austria
havlicek@geometrie.tuwien.ac.at