Quadratic forms and their duals

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Dedicated to Friedrich Manhart on the occasion of his 70th birthday

Abstract

There are many specific results, spread over the literature, regarding the dualisation of quadrics in projective spaces and quadratic forms on vector spaces. In the present work we aim at generalising and unifying some of these. We start with a quadratic form Q that is defined on a subspace $\boldsymbol{S}$ of a finite-dimensional vector space $\boldsymbol{V}$ over a field $\mathbb{F}$. Whenever Q satisfies a certain condition, which comes into effect only when $\mathbb{F}$ is of characteristic two, Q gives rise to a *dual quadratic form* $\hat{Q}$. The domain of the latter is a particular subspace $\hat{\boldsymbol{S}}$ of the dual vector space of $\boldsymbol{V}$. The connection between Q and $\hat{Q}$ is given by a binary relation between vectors of $\boldsymbol{S}$ and linear forms belonging to $\hat{\boldsymbol{S}}$.

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1 Introduction

Each finite-dimensional vector space $\boldsymbol{V}$ over a field $\mathbb{F}$ comes along with its dual space $\boldsymbol{V}^*$ and, up to a canonical identification, the dual of $\boldsymbol{V}^*$ is again $\boldsymbol{V}$. Beyond that, there are many instances where some "feature" defined on $\boldsymbol{V}$ determines in a natural way an analogous "dual feature" on $\boldsymbol{V}^*$ and, moreover, the initial feature coincides with its second dual. For example, each basis of $\boldsymbol{V}$ gives rise to a unique basis of $\boldsymbol{V}^*$, which is known as its dual basis. In this note we address the problem whether or not an analogous situation occurs when dealing with quadratic forms.

There are several results that contribute to the above problem. Most of them are given in terms of quadrics of a finite-dimensional projective space $\mathbb{P}$ over $\mathbb{F}$ and its dual space $\mathbb{P}^*$. Thereby homogenous coordinates are used to describe not only the points of $\mathbb{P}$ but also the hyperplanes of $\mathbb{P}$, which are identified with the points of

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$\mathbb{P}^*$. A quadric locus (resp. quadric envelope) is specified as the set of those points of $\mathbb{P}$ (resp. hyperplanes of $\mathbb{P}$) the homogeneous coordinates of which belong to the zero set of a chosen quadratic form. A very general construction is given by Hodge and Pedoe [14, pp. 213–214] under the assumption that $\mathbb{F}$ is an algebraically closed field of characteristic zero: It is shown that each quadric locus of a subspace of $\mathbb{P}$ gives rise to a quadric envelope of a certain subspace of $\mathbb{P}^*$. Loosely speaking, the latter quadric envelope comprises those hyperplanes of $\mathbb{P}$ that are "tangent" to the quadric locus. The initial quadric locus can be reproduced, *mutatis mutandis*, from its corresponding quadric envelope. Constructions of the same kind (under varying assumptions on the initial quadric locus and the ground field) can be found in many books, for example, [4, § 46.3], [8, p. 98 Satz 2], [8, pp. 129–134], [16, Thm. 4.2.3], [16, Thm. 4.2.10] and [21, pp. 269–270]. We refer also to [23, p. 155], where a lattice-theoretical approach is used to describe the dual of a projective-metric space. Furthermore, the so-called homogeneous model of an affine-metric space follows these lines. See [1], [11], [13], [20] and the many references given there.

In view of the above, we present a self-contained approach that provides a one-to-one correspondence between certain quadratic forms defined on subspaces of $\boldsymbol{V}$ and certain quadratic forms defined on subspaces of $\boldsymbol{V}^*$. There will be no restriction whatsoever on the ground field $\mathbb{F}$. This correspondence incorporates *all* quadratic forms on subspaces of $\boldsymbol{V}$ resp. $\boldsymbol{V}^*$ provided that $\mathbb{F}$ is not of characteristic two.

The paper is organised as a kind of guided tour. In Section 2, we collect basic results. Next, in Section 3, we exhibit specific features (in $\boldsymbol{V}$ and $\boldsymbol{V}^*$) arising from a quadratic form Q that is defined on a subspace $\boldsymbol{S}$ of $\boldsymbol{V}$. Section 4 starts by requiring a condition to be satisfied by Q. This condition holds in any case when the characteristic of $\mathbb{F}$ is unequal two. Then the *dual quadratic form* of Q, which is written as $\hat{Q}$, is defined on a particular subspace $\hat{\boldsymbol{S}}$ of $\boldsymbol{V}^*$ in a coordinate-free way. As one might expect, $\hat{Q}$ admits a dual quadratic form, too, which coincides with Q. It is worth noting that the subspace $\hat{\boldsymbol{S}}$ of $\boldsymbol{V}^*$ depends only on the radical $\boldsymbol{R}$ of the polar form of Q and not on the subspace $\boldsymbol{S}$. The latter determines the radical $\hat{\boldsymbol{R}}$ of the polar form of $\hat{Q}$. In particular, $\hat{\boldsymbol{R}}$ is the zero subspace of $\boldsymbol{V}^*$ precisely when $\boldsymbol{S} = \boldsymbol{V}$. This is why we consider from the very beginning quadratic forms on subspaces of $\boldsymbol{V}$ rather than quadratic forms defined on all of $\boldsymbol{V}$. Still, coordinates must not be avoided. Therefore, Section 5 contains the transition from Q to its dual quadratic form $\hat{Q}$ in terms of coordinates. Also, some references to closely related work are given there. The ties between Q and $\hat{Q}$ are fostered in Section 6 by discussing a relationship between the similarities of $(\boldsymbol{S}, Q)$ and $(\hat{\boldsymbol{S}}, \hat{Q})$.

2 Preliminaries

In this section, we establish notation and collect basic results. For proofs and further details we refer, among others, to [2, Ch. II § 2.3–2.7], [12, Kap. 4], [18, Ch. 3], [22, 3.7], [24, p. 18] and [24, pp. 50–51].

Hereafter, $\mathbb{F}$ denotes a (commutative) field. The characteristic of $\mathbb{F}$, which is abbreviated as $\operatorname{Char}\mathbb{F}$, is arbitrary unless explicitly stated otherwise. A vector space always means a finite-dimensional vector space over $\mathbb{F}$.

Given a vector space $\boldsymbol{V}$, say, we write $\boldsymbol{V}^*$ for the *dual space* of $\boldsymbol{V}$, that is the vector space formed by all linear mappings $\boldsymbol{V} \to \mathbb{F}$. The elements of $\boldsymbol{V}^*$ will be addressed as *linear forms*. The *canonical pairing* $\langle\,\cdot\,,\cdot\,\rangle_{\boldsymbol{V}}\colon \boldsymbol{V}^* \times \boldsymbol{V} \to \mathbb{F}$ sends any pair $(\boldsymbol{a}^*, \boldsymbol{x}) \in \boldsymbol{V}^* \times \boldsymbol{V}$ to the scalar $\langle \boldsymbol{a}^*, \boldsymbol{x}\rangle_{\boldsymbol{V}} := \boldsymbol{a}^*(\boldsymbol{x})$. Since the dimension of $\boldsymbol{V}$ is finite, we may consider $\boldsymbol{V}$ as the dual space of $\boldsymbol{V}^*$ by identifying each vector $\boldsymbol{u} \in \boldsymbol{V}$ with the mapping $\langle\,\cdot\,,\boldsymbol{u}\rangle_{\boldsymbol{V}} \in (\boldsymbol{V}^*)^*$ sending $\boldsymbol{a}^* \mapsto \langle \boldsymbol{a}^*, \boldsymbol{u}\rangle_{\boldsymbol{V}}$ for all $\boldsymbol{a}^* \in \boldsymbol{V}^*$.

If $\boldsymbol{M}$ is a subset of $\boldsymbol{V}$, then the *annihilator* of $\boldsymbol{M}$ with respect to $\boldsymbol{V}$ is given as

$$\operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{M}) := \{\boldsymbol{a}^* \in \boldsymbol{V}^* \mid \langle \boldsymbol{a}^*, \boldsymbol{m}\rangle_{\boldsymbol{V}} = 0 \text{ for all } \boldsymbol{m} \in \boldsymbol{M}\}. \tag{2.1}$$

We note that $\operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{M})$ is a subspace of $\boldsymbol{V}^*$. Likewise, the annihilator with respect to $\boldsymbol{V}^*$ of any subset of $\boldsymbol{V}^*$ is a subspace of $\boldsymbol{V}$. If $\boldsymbol{T}$ is a subspace of $\boldsymbol{V}$, in symbols $\boldsymbol{T} \leq \boldsymbol{V}$, then the dimensions of $\boldsymbol{T}$ and $\operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{T})$ satisfy $\dim \boldsymbol{T} + \dim(\operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{T})) = \dim \boldsymbol{V}$. By going over to the annihilator of $\operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{T})$ with respect to $\boldsymbol{V}^*$, we obtain $\operatorname{ann}_{\boldsymbol{V}^*}(\operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{T})) = \boldsymbol{T}$.

Let $\dim \boldsymbol{V} =: n$ and let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$ be a basis of $\boldsymbol{V}$. The corresponding *dual basis* of $\boldsymbol{V}^*$ is written as $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \dots, \boldsymbol{e}_n^*\}$. Thus, in terms of the Kronecker symbol δ_{ij}, we have $\langle \boldsymbol{e}_i^*, \boldsymbol{e}_j\rangle_{\boldsymbol{V}} = \delta_{ij}$ for all $i, j \in \{1, 2, \dots, n\}$. If J is a subset of $\{1, 2, \dots, n\}$, then

$$\operatorname{ann}_{\boldsymbol{V}}(\operatorname{span}\{\boldsymbol{e}_j \mid j \in J\}) = \operatorname{span}\{\boldsymbol{e}_i^* \mid i \in \{1, 2, \dots, n\} \setminus J\}. \tag{2.2}$$

Suppose that $\lambda\colon \boldsymbol{V} \to \boldsymbol{W}$ denotes a linear mapping into some vector space $\boldsymbol{W}$. The kernel and the image of λ are written as $\ker\lambda$ and $\operatorname{im}\lambda$, respectively. The *transpose* of λ is defined to be $\lambda^{\mathrm{T}}\colon \boldsymbol{W}^* \to \boldsymbol{V}^*\colon \boldsymbol{c}^* \mapsto \boldsymbol{c}^* \circ \lambda$. Thus

$$\langle \lambda^{\mathrm{T}}(\boldsymbol{c}^*), \boldsymbol{x}\rangle_{\boldsymbol{V}} = \langle \boldsymbol{c}^*, \lambda(\boldsymbol{x})\rangle_{\boldsymbol{W}} \ \text{ for all } \ \boldsymbol{c}^* \in \boldsymbol{W}^*,\ \boldsymbol{x} \in \boldsymbol{V}.$$

Consequently, λ^{T} is linear and $(\lambda^{\mathrm{T}})^{\mathrm{T}} = \lambda$. Furthermore,

$$\ker \lambda^{\mathrm{T}} = \operatorname{ann}_{\boldsymbol{W}}(\operatorname{im}\lambda) \ \text{ and } \ \operatorname{im}\lambda^{\mathrm{T}} = \operatorname{ann}_{\boldsymbol{V}}(\ker\lambda). \tag{2.3}$$

The *general linear group* of $\boldsymbol{V}$, denoted by $\mathrm{GL}(\boldsymbol{V})$, comprises all linear bijections from $\boldsymbol{V}$ to itself. Let any $\psi \in \mathrm{GL}(\boldsymbol{V})$ be given. We read off from (2.3), applied to ψ, that $\psi^{\mathrm{T}} \in \mathrm{GL}(\boldsymbol{V}^*)$. If $\boldsymbol{T} \leq \boldsymbol{V}$, then

$$\psi^{\mathrm{T}}\big(\operatorname{ann}_{\boldsymbol{V}}(\psi(\boldsymbol{T}))\big) = \operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{T}). \tag{2.4}$$

For the remainder of this note, $\boldsymbol{V}$ denotes a vector space that contains a *distinguished subspace* $\boldsymbol{S} \leq \boldsymbol{V}$. We also assume that $\boldsymbol{S}$ is equipped with a *quadratic form* $Q\colon \boldsymbol{S} \to \mathbb{F}$. So Q satisfies two properties: (i) $Q(c\boldsymbol{x}) = c^2 Q(\boldsymbol{x})$ for all $c \in \mathbb{F}$ and all $\boldsymbol{x} \in \boldsymbol{S}$. (ii) The mapping

$$B\colon \boldsymbol{S} \times \boldsymbol{S} \to \mathbb{F}\colon (\boldsymbol{x}, \boldsymbol{y}) \mapsto Q(\boldsymbol{x} + \boldsymbol{y}) - Q(\boldsymbol{x}) - Q(\boldsymbol{y}) \tag{2.5}$$

is bilinear. We follow the convention to call $(\boldsymbol{S}, Q)$ a *metric vector space*. Only at the beginning of Section 4, one additional assumption concerning Q will be made.

When dealing with $\boldsymbol{S}$ or Q, we have to draw a clear distinction between intrinsic and extrinsic notions. An *intrinsic* notion rests upon $\boldsymbol{S}$ being a vector space in its own right and disregards $\boldsymbol{V}$. An *extrinsic* notion involves the vector space $\boldsymbol{V}$. For example, the vector space $\boldsymbol{S}^*$ dual to $\boldsymbol{S}$ and the corresponding canonical pairing $\langle\,\cdot\,,\cdot\,\rangle_{\boldsymbol{S}}\colon \boldsymbol{S}^* \times \boldsymbol{S} \to \mathbb{F}$ are intrinsic notions. Each subset $\boldsymbol{M} \subseteq \boldsymbol{S}$ has the (extrinsic) annihilator $\operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{M})$ as in (2.1). By replacing $\boldsymbol{V}$ with $\boldsymbol{S}$ and $\langle\,\cdot\,,\cdot\,\rangle_{\boldsymbol{V}}$ with $\langle\,\cdot\,,\cdot\,\rangle_{\boldsymbol{S}}$ in (2.1), we obtain the (intrinsic) annihilator of $\boldsymbol{M}$ with respect to $\boldsymbol{S}$, which is written as $\operatorname{ann}_{\boldsymbol{S}}(\boldsymbol{M})$.

3 Intrinsic and extrinsic notions coming from Q

We proceed by recalling well-established intrinsic notions that arise under the assumptions made at the end of the previous section; see, for example, [3, pp. 54–56], [7, pp. 39–40], [10, Ch. 4], [10, Ch. 12], [19, § 7.B–C] and [24, pp. 54–57].

The bilinear form B, as defined in (2.5), is addressed as the *polar form* of Q. We note that B is symmetric and satisfies

$$B(\boldsymbol{x}, \boldsymbol{x}) = 2Q(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \boldsymbol{S}. \tag{3.1}$$

The *radical* of B is the subspace

$$\boldsymbol{R} := \{\boldsymbol{x} \in \boldsymbol{S} \mid B(\boldsymbol{x}, \boldsymbol{y}) = 0 \text{ for all } \boldsymbol{y} \in \boldsymbol{S}\} \leq \boldsymbol{S}. \tag{3.2}$$

We say that B is *non-degenerate* if, and only if, $\boldsymbol{R} = \{\boldsymbol{o}\}$. The bilinearity of B implies that the mapping

$$D\colon \boldsymbol{S} \to \boldsymbol{S}^*\colon \boldsymbol{x} \mapsto D(\boldsymbol{x}) := B(\boldsymbol{x}, \,\cdot\,) \tag{3.3}$$

is well-defined and linear. We therefore obtain

$$B(\boldsymbol{x}, \boldsymbol{y}) = \langle D(\boldsymbol{x}), \boldsymbol{y}\rangle_{\boldsymbol{S}} \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S} \tag{3.4}$$

and

$$\ker D = \{\boldsymbol{x} \in \boldsymbol{S} \mid \langle D(\boldsymbol{x}), \boldsymbol{y}\rangle_{\boldsymbol{S}} = 0 \text{ for all } \boldsymbol{y} \in \boldsymbol{S}\} = \boldsymbol{R}. \tag{3.5}$$

From B being symmetric, we have $\langle D(\boldsymbol{x}), \boldsymbol{y}\rangle_{\boldsymbol{S}} = B(\boldsymbol{x}, \boldsymbol{y}) = B(\boldsymbol{y}, \boldsymbol{x}) = \langle D(\boldsymbol{y}), \boldsymbol{x}\rangle_{\boldsymbol{S}}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}$. Therefore and due to the identification of $\boldsymbol{S}^{**}$ with $\boldsymbol{S}$, the mapping D coincides with its transpose $D^{\mathrm{T}}\colon \boldsymbol{S} \to \boldsymbol{S}^*$. From the second equation in (2.3), applied to $D = D^{\mathrm{T}}$, and from (3.5), we get

$$\operatorname{im} D = \operatorname{ann}_{\boldsymbol{S}}(\ker D) = \operatorname{ann}_{\boldsymbol{S}}(\boldsymbol{R}) \leq \boldsymbol{S}^*. \tag{3.6}$$

Remark 3.1. Let $\operatorname{Char}\mathbb{F} = 2$. Then (3.1) implies $B(\boldsymbol{x}, \boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \boldsymbol{S}$, that is, B is an alternating bilinear form. Hence $\dim \boldsymbol{S} - \dim \boldsymbol{R}$ is even.

Remark 3.2. Let $\operatorname{Char}\mathbb{F} \neq 2$. Now Q can be recovered from B, since (3.1) shows $Q(\boldsymbol{x}) = \frac{1}{2}B(\boldsymbol{x}, \boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{S}$. Therefore, under the premise $\operatorname{Char}\mathbb{F} \neq 2$, it is customary to address $\frac{1}{2}B$ rather than B as the polar form of Q. However, in order to obtain a unified approach, we do not follow this convention.

Next, we introduce several extrinsic notions coming from $(\boldsymbol{S}, Q)$ and $\boldsymbol{V}$. In view of results following behind, we thereby adopt the "reverse" notation

$$\hat{\boldsymbol{S}} := \operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{R}) \leq \boldsymbol{V}^* \;\text{ and }\; \hat{\boldsymbol{R}} := \operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{S}) \leq \hat{\boldsymbol{S}}. \tag{3.7}$$

A crucial concept is the following binary relation on $\hat{\boldsymbol{S}} \times \boldsymbol{S}$, which is based upon the polar form B of the given quadratic form Q.

Definition 3.3. A linear form $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$ is said to be *B-linked* to a vector $\boldsymbol{x} \in \boldsymbol{S}$, in symbols $\boldsymbol{a}^* \between \boldsymbol{x}$, precisely when

$$\langle \boldsymbol{a}^*, \boldsymbol{y}\rangle_{\boldsymbol{V}} = B(\boldsymbol{x}, \boldsymbol{y}) \;\text{ for all }\; \boldsymbol{y} \in \boldsymbol{S}. \tag{3.8}$$

It will be advantageous to describe the relation $\between$ in a different way. To this end, we establish the mapping $N\colon \hat{\boldsymbol{S}} \to \boldsymbol{S}^*\colon \boldsymbol{a}^* \mapsto N(\boldsymbol{a}^*)$ by setting

$$\langle N(\boldsymbol{a}^*), \boldsymbol{y}\rangle_{\boldsymbol{S}} := \langle \boldsymbol{a}^*, \boldsymbol{y}\rangle_{\boldsymbol{V}} \;\text{ for all }\; \boldsymbol{y} \in \boldsymbol{S}. \tag{3.9}$$

Thus $N(\boldsymbol{a}^*)$ is just the restriction of the linear form $\boldsymbol{a}^*$ to $\boldsymbol{S}$ or, informally speaking, $N(\boldsymbol{a}^*)$ is a "narrowed" version of $\boldsymbol{a}^*$. The mapping N is linear and

$$\ker N = \operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{S}) = \hat{\boldsymbol{R}}. \tag{3.10}$$

By the definition of N, the image of N is contained in $\operatorname{ann}_{\boldsymbol{S}}(\boldsymbol{R})$. On the other hand, each linear form belonging to $\operatorname{ann}_{\boldsymbol{S}}(\boldsymbol{R})$ has a pre-image under N, since it can be extended to at least one linear form $\boldsymbol{V} \to \mathbb{F}$. Thus $\operatorname{im} N = \operatorname{ann}_{\boldsymbol{S}}(\boldsymbol{R})$. Now, from (3.6), we get

$$\operatorname{im} N = \operatorname{ann}_{\boldsymbol{S}}(\boldsymbol{R}) = \operatorname{im} D. \tag{3.11}$$

Next, we express four properties of the relation $\between$ using the mappings D and N, as described in (3.3) and (3.9), respectively. The last two properties can be thought of as a variant form of the *Riesz representation theorem*; see, for example, [18, Thm. 11.5].

Lemma 3.4. *The relation $\Diamond$ from Definition 3.3 satisfies the following properties:*

(a) *Let $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$ and $\boldsymbol{x} \in \boldsymbol{S}$. Then $\boldsymbol{a}^* \Diamond \boldsymbol{x}$ is equivalent to $N(\boldsymbol{a}^*) = D(\boldsymbol{x})$.*

(b) *Let $\boldsymbol{f}_i^* \Diamond \boldsymbol{s}_i$ and $c_i \in \mathbb{F}$ for $i \in \{1, 2, \dots, k\}$. Then $(\sum_{i=1}^k c_i \boldsymbol{f}_i^*) \Diamond (\sum_{i=1}^k c_i \boldsymbol{s}_i)$.*

(c) *Let $\boldsymbol{f}^* \in \hat{\boldsymbol{S}}$. Then there is $\boldsymbol{s} \in \boldsymbol{S}$ such that $\{\boldsymbol{x} \in \boldsymbol{S} \mid \boldsymbol{f}^* \Diamond \boldsymbol{x}\}$ equals the coset $\boldsymbol{s} + \boldsymbol{R}$ of the subspace $\boldsymbol{R} = \ker D \leq \boldsymbol{S}$.*

(d) *Let $\boldsymbol{s} \in \boldsymbol{S}$. Then there is $\boldsymbol{f}^* \in \hat{\boldsymbol{S}}$ such that $\{\boldsymbol{a}^* \in \hat{\boldsymbol{S}} \mid \boldsymbol{a}^* \Diamond \boldsymbol{s}\}$ equals the coset $\boldsymbol{f}^* + \hat{\boldsymbol{R}}$ of the subspace $\hat{\boldsymbol{R}} = \ker N \leq \hat{\boldsymbol{S}}$.*

Proof. (a) We have $\boldsymbol{a}^* \Diamond \boldsymbol{x}$ precisely when (3.8) is satisfied. Using (3.4), the latter condition can be rewritten as $\langle N(\boldsymbol{a}^*), \boldsymbol{y}\rangle_{\boldsymbol{S}} = \langle D(\boldsymbol{x}), \boldsymbol{y}\rangle_{\boldsymbol{S}}$ for all $\boldsymbol{y} \in \boldsymbol{S}$ or, said differently, $N(\boldsymbol{a}^*) = D(\boldsymbol{x})$.

(b) Taking into account the linearity of the mappings N and D, the claim is an immediate consequence of (a).

(c) From (3.11), there exists $\boldsymbol{s} \in \boldsymbol{S}$ with $N(\boldsymbol{f}^*) = D(\boldsymbol{s})$. According to (a), the given linear form $\boldsymbol{f}^* \in \hat{\boldsymbol{S}}$ is B-linked to a vector $\boldsymbol{x} \in \boldsymbol{S}$ precisely when $D(\boldsymbol{x}) = D(\boldsymbol{s})$, which is equivalent to $\boldsymbol{x} \in \boldsymbol{s} + \ker D$. Now the assertion follows, since (3.5) shows $\ker D = \boldsymbol{R}$.

(d) We follow the idea of proof from (c) with N and D changing their roles. The set of all $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$ that are B-linked to $\boldsymbol{s}$ equals the (non-empty) pre-image of $D(\boldsymbol{s})$ under N. From (3.10), this pre-image can be written as $\boldsymbol{f}^* + \ker N = \boldsymbol{f}^* + \hat{\boldsymbol{R}}$ for some $\boldsymbol{f}^* \in \hat{\boldsymbol{S}}$. □

Example 3.5. Let $\dim \boldsymbol{V} = 3$ and $\dim \boldsymbol{S} = 2$. We may choose a basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of $\boldsymbol{V}$ such that $\boldsymbol{S} = \operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. Also, we assume that $Q \colon \boldsymbol{S} \to \mathbb{F}$ satisfies

$$Q(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2) = x_2^2 \ \text{ for all } \ x_1, x_2 \in \mathbb{F}.$$

In order to describe the relation $\Diamond$ by following Lemma 3.4 (a), we have to specify the mappings N and D. From (3.10) and (2.2), $\ker N = \hat{\boldsymbol{R}} = \operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{S}) = \operatorname{span}\{\boldsymbol{e}_3^*\}$. Next, we determine $\ker D = \boldsymbol{R}$, that is, the radical of B. From (2.5), we obtain

$$B\,(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2, y_1\boldsymbol{e}_1 + y_2\boldsymbol{e}_2) = 2x_2y_2 \ \text{ for all } \ x_1, x_2, y_1, y_2 \in \mathbb{F}.$$

The factor $2 \in \mathbb{F}$ in the preceding formula makes us distinguish two cases.

Case 1. Let $\operatorname{Char} \mathbb{F} = 2$. Here B is a zero form. Consequently, $\boldsymbol{R} = \boldsymbol{S}$, D is a zero mapping and $\hat{\boldsymbol{S}} = \hat{\boldsymbol{R}}$. The last equation implies that N is a zero mapping, too. Thus $D(\boldsymbol{x}) = N(\boldsymbol{a})$ or, equivalently, $\boldsymbol{a}^* \Diamond \boldsymbol{x}$ holds for all pairs $(\boldsymbol{a}^*, \boldsymbol{x}) \in \hat{\boldsymbol{S}} \times \boldsymbol{S}$.

Case 2. Let $\operatorname{Char} \mathbb{F} \neq 2$. Here B has the radical $\boldsymbol{R} = \operatorname{span}\{\boldsymbol{e}_1\} = \ker D$, whence (2.2) yields $\hat{\boldsymbol{S}} = \operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{R}) = \operatorname{span}\{\boldsymbol{e}_2^*, \boldsymbol{e}_3^*\}$. As we noted above, $\ker N = \operatorname{span}\{\boldsymbol{e}_3^*\}$. Therefore, $\{N(\boldsymbol{e}_2^*)\}$ is a basis of $\operatorname{im} N$ and, by virtue of (3.11), also a basis of $\operatorname{im} D$. Indeed, $D(\boldsymbol{e}_2) = 2N(\boldsymbol{e}_2^*)$. To sum up, the following holds for all $a_2, a_3, x_1, x_2 \in \mathbb{F}$:

The linear form $\boldsymbol{a}^* := a_2\boldsymbol{e}_2^* + a_3\boldsymbol{e}_3^* \in \hat{\boldsymbol{S}}$ is B-linked to the vector $\boldsymbol{x} := x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 \in \boldsymbol{S}$ if, and only if, $a_2 N(\boldsymbol{e}_2^*) = N(\boldsymbol{a}^*) = D(\boldsymbol{x}) = 2x_2 N(\boldsymbol{e}_2^*)$. In the present setting, this is equivalent to $a_2 = 2x_2$.

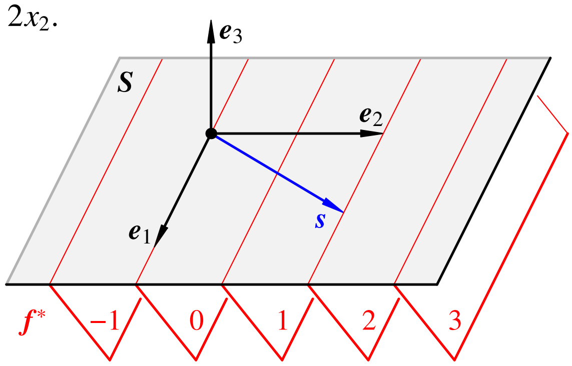

Figure 1: B-linked elements $\boldsymbol{f}^* \between \boldsymbol{s}$

Figure 1 illustrates the second case over the real numbers $\mathbb{R}$. A linear form $\boldsymbol{f}^* := 2\boldsymbol{e}_2^* + f_3\boldsymbol{e}_3^*$ with $f_3 \in \mathbb{R} \setminus \{0\}$ is visualised by some of its level surfaces (parallel planes). Each of these planes is labelled with a number equal to the constant value assumed there by $\boldsymbol{f}^*$. The linear form $\boldsymbol{f}^*$ is B-linked to all vectors of the coset $\boldsymbol{e}_2 + \operatorname{span}\{\boldsymbol{e}_1\}$, in particular to the highlighted vector $\boldsymbol{s} := s_1\boldsymbol{e}_1 + \boldsymbol{e}_2$ with $s_1 \in \mathbb{R} \setminus \{0\}$.

Remark 3.6. The findings in Lemma 3.4 allow for another interpretation: The linear mapping N gives rise to a linear bijection N_0 of the factor space $\hat{\boldsymbol{S}}/\hat{\boldsymbol{R}} = \hat{\boldsymbol{S}}/\ker N$ onto the subspace $\operatorname{im} N \leq \boldsymbol{S}^*$. Explicitly, $N_0(\boldsymbol{a}^* + \ker N) = N(\boldsymbol{a}^*)$ for all $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$. This result is known under the name *first isomorphism theorem*; see, for example, [18, Thm. 3.5]. Likewise, D yields a linear bijection D_0 of $\boldsymbol{S}/\boldsymbol{R} = \boldsymbol{S}/\ker D$ onto $\operatorname{im} D \leq \boldsymbol{S}^*$. From (3.11), the product $D_0^{-1} \circ N_0 \colon \hat{\boldsymbol{S}}/\hat{\boldsymbol{R}} \to \boldsymbol{S}/\boldsymbol{R}$ is a linear bijection, too. The latter mapping takes $\boldsymbol{a}^* + \hat{\boldsymbol{R}} \in \hat{\boldsymbol{S}}/\hat{\boldsymbol{R}}$ to $\boldsymbol{x} + \boldsymbol{R} \in \boldsymbol{S}/\boldsymbol{R}$ precisely when $\boldsymbol{a}^* \between \boldsymbol{x}$. Thus, informally speaking, the relation $\between$ describes the linear bijection $D_0^{-1} \circ N_0$ by drawing on representatives of corresponding cosets.

Also, it seems worth noting that $\hat{\boldsymbol{S}}/\hat{\boldsymbol{R}}$ is canonically isomorphic to the dual space of $\boldsymbol{S}/\boldsymbol{R}$. The canonical pairing then satisfies

$$\langle \boldsymbol{a}^* + \hat{\boldsymbol{R}}, \boldsymbol{x} + \boldsymbol{R} \rangle_{\boldsymbol{S}/\boldsymbol{R}} = \langle \boldsymbol{a}^*, \boldsymbol{x} \rangle_V \quad \text{for all} \quad (\boldsymbol{a}^*, \boldsymbol{x}) \in \hat{\boldsymbol{S}} \times \boldsymbol{S}.$$

This follows from a more general result which can be found in [9, p. 67] at the very end of Section 2.23.

4 The dual quadratic form of Q

While we still adhere to the global settings adopted at the end of Section 2, from now on there will be an additional assumption regarding the non-zero vectors in the radical $\boldsymbol{R}$, namely

$$Q(\boldsymbol{x}) = 0 \ \text{ for all } \ \boldsymbol{x} \in \boldsymbol{R} \setminus \{\boldsymbol{o}\}. \tag{4.1}$$

The reason behind this will be clarified in Remark 4.3. If $\operatorname{Char}\mathbb{F} = 2$, then there are quadratic forms that fail to satisfy (4.1). One instance can be read off from Example 3.5, Case 1: The vector $\boldsymbol{e}_2 \in \boldsymbol{R}$ appearing there satisfies $Q(\boldsymbol{e}_2) = 1$. If $\operatorname{Char}\mathbb{F} \neq 2$, then the circumstances are quite different: Indeed, (3.2) and (3.1) show $Q(\boldsymbol{x}) = \frac{1}{2}B(\boldsymbol{x}, \boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \boldsymbol{R}$, so that (4.1) holds in any case.

We are now in a good position to present our first main result.

Theorem 4.1. *Let $\boldsymbol{S} \leq \boldsymbol{V}$ and let $Q\colon \boldsymbol{S} \to \mathbb{F}$ be a quadratic form subject to* (4.1)*. Then there exists a unique mapping $\hat{Q}\colon \hat{\boldsymbol{S}} \to \mathbb{F}$ such that*

$$\boldsymbol{a}^* \between \boldsymbol{x} \ \textit{ implies } \ \hat{Q}(\boldsymbol{a}^*) = Q(\boldsymbol{x}). \tag{4.2}$$

This mapping $\hat{Q}$ is a quadratic form.

Proof. Upon choosing $\boldsymbol{f}^* \in \hat{\boldsymbol{S}}$ arbitrarily, there is a vector $\boldsymbol{s} \in \boldsymbol{S}$ with $\boldsymbol{f}^* \between \boldsymbol{s}$ according to Lemma 3.4 (c). So there is at most one function $\hat{Q}$ subject to (4.2). Furthermore, $\boldsymbol{f}^* \between \boldsymbol{x}$ forces $\boldsymbol{x} - \boldsymbol{s} \in \boldsymbol{R}$. Now, from (4.1) and (3.2), we obtain

$$Q(\boldsymbol{x}) = Q((\boldsymbol{x} - \boldsymbol{s}) + \boldsymbol{s}) = \underbrace{Q(\boldsymbol{x} - \boldsymbol{s})}_{=0} + Q(\boldsymbol{s}) + \underbrace{B(\boldsymbol{x} - \boldsymbol{s}, \boldsymbol{s})}_{=0} = Q(\boldsymbol{s}). \tag{4.3}$$

Consequently, (4.2) provides a consistent definition of $\hat{Q}$.

It remains to establish that $\hat{Q}$ is a quadratic form. First, with $\boldsymbol{f}^* \between \boldsymbol{s}$ as above, let any $c \in \mathbb{F}$ be given. In order to verify that $\hat{Q}(c\boldsymbol{f}^*) = c^2\hat{Q}(\boldsymbol{f}^*)$, we make use of Lemma 3.4 (b) to obtain $(c\boldsymbol{f}^*) \between (c\boldsymbol{s})$. Thus

$$\hat{Q}(c\boldsymbol{f}^*) = Q(c\boldsymbol{s}) = c^2 Q(\boldsymbol{s}) = c^2\hat{Q}(\boldsymbol{f}^*),$$

as required. Next, we consider the mapping

$$\hat{B}\colon \hat{\boldsymbol{S}} \times \hat{\boldsymbol{S}} \to \mathbb{F}\colon (\boldsymbol{a}^*, \boldsymbol{b}^*) \mapsto \hat{Q}(\boldsymbol{a}^* + \boldsymbol{b}^*) - \hat{Q}(\boldsymbol{a}^*) - \hat{Q}(\boldsymbol{b}^*).$$

We claim that

$$(\boldsymbol{f}_1^* \between \boldsymbol{s}_1 \ \text{ and } \ \boldsymbol{f}_2^* \between \boldsymbol{s}_2) \ \text{ implies } \ \hat{B}(\boldsymbol{f}_1^*, \boldsymbol{f}_2^*) = B(\boldsymbol{s}_1, \boldsymbol{s}_2). \tag{4.4}$$

The assumptions in (4.4) and Lemma 3.4 (b) yield $(\boldsymbol{f}_1^* + \boldsymbol{f}_2^*) \between (\boldsymbol{s}_1 + \boldsymbol{s}_2)$. Hence, from (4.2), we obtain

$$\begin{aligned} \hat{B}(\boldsymbol{f}_1^*, \boldsymbol{f}_2^*) &= \hat{Q}(\boldsymbol{f}_1^* + \boldsymbol{f}_2^*) - \hat{Q}(\boldsymbol{f}_1^*) - \hat{Q}(\boldsymbol{f}_2^*) \\ &= Q(\boldsymbol{s}_1 + \boldsymbol{s}_2) - Q(\boldsymbol{s}_1) - Q(\boldsymbol{s}_2) = B(\boldsymbol{s}_1, \boldsymbol{s}_2). \end{aligned}$$

By virtue of (4.4) and Lemma 3.4 (b), it is now straightforward to derive the bilinearity of $\hat{B}$ from the bilinearity of B. □

We are thus led to the following definition.

Definition 4.2. Let a subspace $\boldsymbol{S} \leq \boldsymbol{V}$ and a quadratic form $Q\colon \boldsymbol{S} \to \mathbb{F}$ be given as in Theorem 4.1. Then the quadratic form $\hat{Q}\colon \hat{\boldsymbol{S}} \to \mathbb{F}$ appearing there, which is characterised by (4.2), is called the *dual quadratic form* of Q.

The dual space of $\boldsymbol{V}$ is an essential part of the definition from above. For example, it has been used in (3.7) to define $\hat{\boldsymbol{S}} \leq \boldsymbol{V}^*$.

Remark 4.3. The constraint (4.1) has been used in the proof of Theorem 4.1 in order to establish equation (4.3), which ensures that (4.2) provides a well-defined mapping $\hat{Q}$. An analogous constraint appears in the closely related context of induced quadratic forms on factor spaces; see [3, pp. 56–57] or [24, pp. 56–57]. If a quadratic form $Q\colon \boldsymbol{S} \to \mathbb{F}$ does not meet (4.1), then our definition of the mapping $\hat{Q}$ fails. Indeed, suppose that $Q(\boldsymbol{s}) \neq 0$ for some $\boldsymbol{s} \in \boldsymbol{R} \setminus \{\boldsymbol{o}\}$. According to Lemma 3.4 (c), the zero form $\boldsymbol{o}^* \in \hat{\boldsymbol{S}}$ satisfies not only $\boldsymbol{o}^* \between \boldsymbol{o}$ but also $\boldsymbol{o}^* \between \boldsymbol{s}$, whereas $Q(\boldsymbol{o}) = 0 \neq Q(\boldsymbol{s})$.

We proceed by taking a closer look at the dual quadratic form of Q.

Proposition 4.4. *Let Q be given as in Theorem 4.1. Then $\hat{Q}$, the dual quadratic form of Q, and the polar form $\hat{B}$ of $\hat{Q}$ satisfy the following properties:*

(a) *If $\boldsymbol{f}^* \between \boldsymbol{s}$, then $\hat{B}(\boldsymbol{a}^*, \boldsymbol{f}^*) = \langle \boldsymbol{a}^*, \boldsymbol{s}\rangle_V$ for all $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$.*

(b) *The radical of $\hat{B}$ equals $\hat{\boldsymbol{R}} = \operatorname{ann}_V(\boldsymbol{S})$ and $\hat{Q}(\boldsymbol{a}^*) = 0$ for all $\boldsymbol{a}^* \in \hat{\boldsymbol{R}} \setminus \{\boldsymbol{o}^*\}$.*

Proof. (a) From Lemma 3.4 (c), for each $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$ there is an auxiliary vector $\boldsymbol{u} \in \boldsymbol{S}$ (depending on $\boldsymbol{a}^*$) with $\boldsymbol{a}^* \between \boldsymbol{u}$. Now (4.4) gives $\hat{B}(\boldsymbol{a}^*, \boldsymbol{f}^*) = B(\boldsymbol{u}, \boldsymbol{s})$, whereas (3.8) implies $B(\boldsymbol{u}, \boldsymbol{s}) = \langle \boldsymbol{a}^*, \boldsymbol{s}\rangle_V$.

(b) First, we determine the radical of $\hat{B}$. Let $\boldsymbol{f}^* \in \hat{\boldsymbol{S}}$. By virtue of Lemma 3.4 (c), we may pick $\boldsymbol{s} \in \boldsymbol{S}$ with $\boldsymbol{f}^* \between \boldsymbol{s}$. The given linear form $\boldsymbol{f}^*$ is in the radical of $\hat{B}$ if, and only if, $\hat{B}(\boldsymbol{a}^*, \boldsymbol{f}^*) = 0$ for all $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$. According to (a), this condition is equivalent to $\langle \boldsymbol{a}^*, \boldsymbol{s}\rangle_V = 0$ for all $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$, which in turn characterises $\boldsymbol{s}$ as being in $\boldsymbol{R}$. From (3.5) and Lemma 3.4 (a), $\boldsymbol{s} \in \boldsymbol{R} = \ker D$ holds if, and only if, $D(\boldsymbol{s}) = N(\boldsymbol{f}^*)$ is a zero linear form. Using (3.10), the latter turns out to be equivalent to $\boldsymbol{f}^* \in \ker N = \hat{\boldsymbol{R}}$.

Next, let $\boldsymbol{a}^* \in \hat{\boldsymbol{R}} \setminus \{\boldsymbol{o}^*\}$. Then $\langle \boldsymbol{a}^*, \boldsymbol{y}\rangle_V = 0$ for all $\boldsymbol{y} \in \boldsymbol{S}$ implies that $\boldsymbol{a}^*$ is B-linked to the zero vector $\boldsymbol{o} \in \boldsymbol{S}$. Therefore, $\hat{Q}(\boldsymbol{a}^*) = Q(\boldsymbol{o}) = 0$. □

In view of Proposition 4.4 (b), we may now apply, *mutatis mutandis*, our previous definitions and results to the quadratic form $\hat{Q}$. Thereby $\boldsymbol{V}$ and $\boldsymbol{V}^*$ swap their roles. The same applies to $\boldsymbol{S}$ and $\hat{\boldsymbol{S}}$ as well as $\boldsymbol{R}$ and $\hat{\boldsymbol{R}}$, as follows from

$\mathrm{ann}_{\boldsymbol{V}^*}(\hat{\boldsymbol{R}}) = \boldsymbol{S}$ and $\mathrm{ann}_{\boldsymbol{V}^*}(\hat{\boldsymbol{S}}) = \boldsymbol{R}$, respectively. In addition, we obtain linear mappings $\hat{D}\colon \hat{\boldsymbol{S}} \to (\hat{\boldsymbol{S}})^*$ and $\hat{N}\colon \boldsymbol{S} \to (\hat{\boldsymbol{S}})^*$ together with a relation $\hat{\Diamond}$ on $\boldsymbol{S} \times \hat{\boldsymbol{S}}$. We note that, in analogy to (3.8), $\boldsymbol{y} \mathbin{\hat{\Diamond}} \boldsymbol{b}^*$ is equivalent to

$$\langle \boldsymbol{a}^*, \boldsymbol{y}\rangle_V = \hat{B}(\boldsymbol{a}^*, \boldsymbol{b}^*) \ \text{ for all } \ \boldsymbol{a}^* \in \hat{\boldsymbol{S}}. \tag{4.5}$$

We now show our second main result.

Theorem 4.5. *Let $Q\colon \boldsymbol{S} \to \mathbb{F}$ be a quadratic form as in Theorem 4.1 and let $\hat{Q}\colon \hat{\boldsymbol{S}} \to \mathbb{F}$ be its dual quadratic form. Then the following hold:*

(a) *The relation $\Diamond$ on $\hat{\boldsymbol{S}} \times \boldsymbol{S}$, which arises from the polar form of Q, has the relation $\hat{\Diamond}$ on $\boldsymbol{S} \times \hat{\boldsymbol{S}}$, which arises from the polar form of $\hat{Q}$, as its converse relation.*

(b) *The dual quadratic form of $\hat{Q}$ coincides with Q.*

Proof. (a) Let any $\boldsymbol{f}^* \in \hat{\boldsymbol{S}}$ be given. Applying Lemma 3.4 (c) and (d) to the relations $\Diamond$ and $\hat{\Diamond}$, respectively, yields that there are $\boldsymbol{s}, \boldsymbol{s}' \in \boldsymbol{S}$ with

$$\{\boldsymbol{x} \in \boldsymbol{S} \mid \boldsymbol{f}^* \Diamond \boldsymbol{x}\} = \boldsymbol{s} + \boldsymbol{R} \ \text{ and } \ \{\boldsymbol{x} \in \boldsymbol{S} \mid \boldsymbol{x} \mathbin{\hat{\Diamond}} \boldsymbol{f}^*\} = \boldsymbol{s}' + \boldsymbol{R}. \tag{4.6}$$

It remains to establish that the above sets coincide. To this end, we apply Proposition 4.4 (a) to $\boldsymbol{f}^* \Diamond \boldsymbol{s}$. This gives $\hat{B}(\boldsymbol{a}^*, \boldsymbol{f}^*) = \langle \boldsymbol{a}^*, \boldsymbol{s}\rangle_V$ for all $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$ or, in view of (4.5), $\boldsymbol{s} \mathbin{\hat{\Diamond}} \boldsymbol{f}^*$. Now the second equation in (4.6) shows $\boldsymbol{s} \in \boldsymbol{s}' + \boldsymbol{R}$, which in turn yields $\boldsymbol{s} + \boldsymbol{R} = \boldsymbol{s}' + \boldsymbol{R}$.

(b) According to part (a), the transition from $\hat{Q}$ to its dual quadratic form will reproduce Q. □

Remark 4.6. The bilinear form B (resp. $\hat{B}$) is non-degenerate precisely when $\hat{\boldsymbol{S}} = \boldsymbol{V}^*$ (resp. $\boldsymbol{S} = \boldsymbol{V}$). If both B and $\hat{B}$ are non-degenerate, then (in the terminology of [3, p. 23 Déf. 8]) each of the bilinear forms B and $\hat{B}$ is the *inverse* of the other one. Furthermore, $D^{-1} = \hat{D}$ and, in accordance with Remark 3.6, the relation $\Diamond$ (resp. $\hat{\Diamond}$) is nothing more than the graph of D^{-1} (resp. $\hat{D}^{-1}$).

Remark 4.7. Let $\mathrm{Char}\,\mathbb{F} \neq 2$. As we recalled in Remark 3.2, it is common here to work with $\frac{1}{2}B$ rather than B. This suggests to introduce a relation of being $(\frac{1}{2}B)$-*linked* by replacing B with $\frac{1}{2}B$ in (3.8) and to modify Theorem 4.1 accordingly. Since $\boldsymbol{a}^* \Diamond \boldsymbol{s}$ holds precisely when $\boldsymbol{a}^*$ is $(\frac{1}{2}B)$-linked to $2\boldsymbol{s}$, and due to $Q(2\boldsymbol{s}) = 4Q(\boldsymbol{s})$, in this way $4\hat{Q}$ (rather than $\hat{Q}$) will arise from the initial quadratic form Q. Applying this modified construction to $4\hat{Q}$ reproduces Q. We shall come across $4\hat{Q}$ again in Remark 5.3.

5 Q and $\hat{Q}$ in terms of coordinates

In this section the bridge between $(\boldsymbol{S}, Q)$ and $(\hat{\boldsymbol{S}}, \hat{Q})$, as considered in Theorem 4.1, will be elucidated using coordinates. We thereby put $\dim \boldsymbol{V} =: n$, $\dim \boldsymbol{S} =: m$, $\dim \boldsymbol{R} =: d$ and we introduce the (possibly empty) index sets

$$\begin{aligned} I &:= \{1, 2, \dots, n\}, & I_1 &:= \{1, 2, \dots, d\}, \\ I_2 &:= \{d+1, d+2, \dots, m\}, & I_3 &:= \{m+1, m+2, \dots, n\}. \end{aligned} \tag{5.1}$$

There is at least one basis $\{\boldsymbol{e}_i \mid i \in I\}$ of $\boldsymbol{V}$ satisfying

$$\boldsymbol{R} = \operatorname{span}\{\boldsymbol{e}_i \mid i \in I_1\} \text{ and } \boldsymbol{S} = \operatorname{span}\{\boldsymbol{e}_i \mid i \in I_1 \cup I_2\}. \tag{5.2}$$

Let $g_i := Q(\boldsymbol{e}_i)$ and $g_{ij} := B(\boldsymbol{e}_i, \boldsymbol{e}_j)$ for all $i, j \in I_1 \cup I_2$. Then, for all $i, j \in I_1 \cup I_2$, formula (2.5) gives $g_{ij} = B(\boldsymbol{e}_i, \boldsymbol{e}_j) = B(\boldsymbol{e}_j, \boldsymbol{e}_i) = g_{ji}$ and, furthermore, (3.1) gives $2g_i = 2Q(\boldsymbol{e}_i) = B(\boldsymbol{e}_i, \boldsymbol{e}_i) = g_{ii}$. (If $\operatorname{Char} \mathbb{F} = 2$, then the last equation reduces to $0 = g_{ii}$; otherwise, it means $g_i = \frac{1}{2} g_{ii}$. See Remarks 3.1 and 3.2.) Due to (4.1) and from $\boldsymbol{R}$ being the radical of B, we have $g_i = g_{ij} = g_{ji} = 0$ whenever $i \in I_1$. Consequently, for all $x_1, x_2, \dots, x_m \in \mathbb{F}$, it follows that

$$Q\left(\sum_{h=1}^{m} x_h \boldsymbol{e}_h\right) = \sum_{i=d+1}^{m} g_i x_i^2 + \sum_{i=d+1}^{m-1} \sum_{j=i+1}^{m} g_{ij} x_i x_j. \tag{5.3}$$

We now change over to the dual space $\boldsymbol{V}^*$. The initially chosen basis of $\boldsymbol{V}$ determines the dual basis $\{\boldsymbol{e}_i^* \mid i \in I\}$ of $\boldsymbol{V}^*$. We infer from (5.2) and (2.2) that

$$\hat{\boldsymbol{R}} = \operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{S}) = \operatorname{span}\{\boldsymbol{e}_i^* \mid i \in I_3\} \text{ and } \hat{\boldsymbol{S}} = \operatorname{ann}_{\boldsymbol{V}}(\boldsymbol{R}) = \operatorname{span}\{\boldsymbol{e}_i^* \mid i \in I_2 \cup I_3\}.$$

Let $\hat{g}_i := \hat{Q}(\boldsymbol{e}_i^*)$ and $\hat{g}_{ij} := \hat{B}(\boldsymbol{e}_i^*, \boldsymbol{e}_j^*)$ for all $i, j \in I_2 \cup I_3$. From Proposition 4.4 (b), we get $\hat{g}_i = \hat{g}_{ij} = \hat{g}_{ji} = 0$ whenever $i \in I_3$. Consequently, for all $a_{d+1}, a_{d+2}, \dots, a_n \in \mathbb{F}$, it follows that

$$\hat{Q}\left(\sum_{h=d+1}^{n} a_h \boldsymbol{e}_h^*\right) = \sum_{i=d+1}^{m} \hat{g}_i a_i^2 + \sum_{i=d+1}^{m-1} \sum_{j=i+1}^{m} \hat{g}_{ij} a_i a_j. \tag{5.4}$$

Our goal is to express the coefficients $\hat{g}_i$ and $\hat{g}_{ij}$ appearing in (5.4) using their analogues from (5.3). To this end, we introduce the auxiliary subspaces

$$\boldsymbol{T} := \operatorname{span}\{\boldsymbol{e}_k \mid k \in I_2\} \le \boldsymbol{S} \text{ and } \hat{\boldsymbol{T}} := \operatorname{span}\{\boldsymbol{e}_k^* \mid k \in I_2\} \le \hat{\boldsymbol{S}}$$

together with the symmetric matrices

$$G_{22} := (g_{ij})_{i,j \in I_2} \text{ and } \hat{G}_{22} := (\hat{g}_{ij})_{i,j \in I_2}.$$

From $\boldsymbol{S} = \boldsymbol{R} \oplus \boldsymbol{T}$ (resp. $\hat{\boldsymbol{S}} = \hat{\boldsymbol{R}} \oplus \hat{\boldsymbol{T}}$), each element of $\boldsymbol{T}$ (resp. $\hat{\boldsymbol{T}}$) is contained in a single coset of $\boldsymbol{R}$ (resp. $\hat{\boldsymbol{R}}$). Thus Lemma 3.4 implies that the restriction of the relation $\lozenge$ to $\hat{\boldsymbol{T}} \times \boldsymbol{T}$ is the graph of a linear bijection $\tau\colon \hat{\boldsymbol{T}} \to \boldsymbol{T}$. (See also Remark 3.6.) According to (3.3), for all $k \in I_2$, we have $D(\boldsymbol{e}_k) = \sum_{l=d+1}^{m} g_{kl} N(\boldsymbol{e}_l^*)$. By virtue of Lemma 3.4 (a),

$$\left(\sum_{l=d+1}^{m} g_{kl}\boldsymbol{e}_l^*\right) \lozenge \boldsymbol{e}_k = \tau\left(\sum_{l=d+1}^{m} g_{kl}\boldsymbol{e}_l^*\right) \text{ for all } k \in I_2.$$

So, in terms of the given bases, G_{22} is the matrix of τ^{-1}. Likewise, the restriction of $\hat{\lozenge}$ to $\boldsymbol{T}\times\hat{\boldsymbol{T}}$ is the graph of a linear bijection $\hat{\tau}\colon \boldsymbol{T} \to \hat{\boldsymbol{T}}$, whose inverse is described by $\hat{G}_{22}$. From Theorem 4.5 (a), $\hat{\tau}^{-1} = \tau$, whence

$$\hat{G}_{22} = G_{22}^{-1}. \tag{5.5}$$

Furthermore, (5.5) implies $\boldsymbol{e}_i^* \lozenge \tau(\boldsymbol{e}_i^*) = \sum_{k=d+1}^{m} \hat{g}_{ik}\boldsymbol{e}_k$ for all $i \in I_2$. Thus, from (4.2) and (5.3), we finally arrive at

$$\hat{g}_i = Q\left(\sum_{k=d+1}^{m} \hat{g}_{ik}\boldsymbol{e}_k\right) = \sum_{k=d+1}^{m} g_k\hat{g}_{ik}^2 + \sum_{k=d+1}^{m-1}\sum_{l=k+1}^{m} \hat{g}_{ik}\hat{g}_{il}g_{kl} \text{ for all } i \in I_2. \tag{5.6}$$

This rather cumbersome formula incorporates not only the coefficients appearing in (5.3) but also the entries of the matrix G_{22}^{-1}. In the following part, the preceding results will be simplified under certain additional prerequisites. However, we have to distinguish two cases.

Case 1. Let $\operatorname{Char}\mathbb{F} = 2$. Now the restriction of B to $\boldsymbol{T} \times \boldsymbol{T}$ is a non-degenerate alternating bilinear form, whence $\dim \boldsymbol{T} = \dim \boldsymbol{S} - \dim \boldsymbol{R} = m - d$ is even; see Remark 3.1. There is a choice of the basis vectors $\boldsymbol{e}_k$ with $k \in I_2$ such that the entries of G_{22} along its minor diagonal are 1, whereas all remaining entries equal 0; see [3, p. 81 Cor. 3], [10, Thm. 2.10], [12, Satz 9.8.5], [18, Thm. 11.14] or [24, p. 69] (sometimes up to a reordering of the basis vectors). Thus, with i, j ranging in I_2, we have

$$g_{ij} = \begin{cases} 1 & \text{if } j = d + m + 1 - i, \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Formula (5.3) turns into

$$Q\left(\sum_{h=1}^{m} x_h\boldsymbol{e}_h\right) = \sum_{i=d+1}^{m} g_i x_i^2 + \sum_{i=d+1}^{(d+m)/2} x_i x_{d+m+1-i}.$$

By virtue of $G_{22} = G_{22}^{-1}$, formula (5.5) now reads $\hat{G}_{22} = G_{22}$. Consequently, by taking into account (5.7), formula (5.6) reduces to

$$\hat{g}_i = \sum_{k=d+1}^{m} g_k g_{ik}^2 + \sum_{k=d+1}^{m-1}\sum_{l=k+1}^{m} \underbrace{g_{ik}g_{il}}_{=0}\, g_{kl} = g_{d+m+1-i} \text{ for all } i \in I_2,$$

whence (5.4) can be rewritten as

$$\hat{Q}\left(\sum_{h=d+1}^{n} a_h \boldsymbol{e}_h^*\right) = \sum_{i=d+1}^{m} g_{d+m+1-i} a_i^2 + \sum_{i=d+1}^{(d+m)/2} a_i a_{d+m+1-i}.$$

Case 2. Let $\operatorname{Char} \mathbb{F} \neq 2$. Now, according to Remark 3.2, we have $Q(\boldsymbol{x}) = \frac{1}{2}B(\boldsymbol{x}, \boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{S}$ and $\hat{Q}(\boldsymbol{a}^*) = \frac{1}{2}\hat{B}(\boldsymbol{a}^*, \boldsymbol{a}^*)$ for all $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$. In particular, $Q(\boldsymbol{e}_i) = g_i = \frac{1}{2}g_{ii}$ and $\hat{Q}(\boldsymbol{e}_i) = \hat{g}_i = \frac{1}{2}\hat{g}_{ii}$ for all $i \in I_2$. Moreover, we are able to avoid the usage of (5.6) by replacing (5.3) with

$$Q\left(\sum_{h=1}^{m} x_h \boldsymbol{e}_h\right) = \frac{1}{2}\sum_{i=d+1}^{m}\sum_{j=d+1}^{m} g_{ij} x_i x_j \tag{5.8}$$

and (5.4) with

$$\hat{Q}\left(\sum_{h=d+1}^{n} a_h \boldsymbol{e}_h^*\right) = \frac{1}{2}\sum_{i=d+1}^{m}\sum_{j=d+1}^{m} \hat{g}_{ij} a_i a_j. \tag{5.9}$$

The relationship among the coefficients g_{ij} and $\hat{g}_{ij}$ appearing in the above formulas is governed solely by the matrix equation (5.5).

Next, we choose the basis vectors $\boldsymbol{e}_k$ with $k \in I_2$ in such a way that G_{22} is a diagonal matrix; see [3, pp. 91–92 Cor. 2], [10, Thm. 4.2], [12, Satz 9.8.10], [18, Thm. 11.21] or [22, Thm. 6.10]. By doing so, (5.8) simplifies to

$$Q\left(\sum_{h=1}^{m} x_h \boldsymbol{e}_h\right) = \frac{1}{2}\sum_{i=d+1}^{m} g_{ii} x_i^2.$$

According to (5.5), the matrix $\hat{G}_{22} = G_{22}^{-1}$ now is also diagonal, whence $\hat{g}_{ii} = g_{ii}^{-1}$ for all $i \in I_2$. Consequently, (5.9) can be rewritten as

$$\hat{Q}\left(\sum_{h=d+1}^{n} a_h \boldsymbol{e}_h^*\right) = \frac{1}{2}\sum_{i=d+1}^{m} g_{ii}^{-1} a_i^2.$$

Example 5.1. Let $\boldsymbol{V}$ be a five-dimensional vector space over the field $\mathbb{R}$ of real numbers. Also, let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_5\}$ be a basis of $\boldsymbol{V}$ and let $\boldsymbol{S} := \operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. In view of (5.8), we define a quadratic form $Q\colon \boldsymbol{S} \to \mathbb{R}$ by

$$Q\left(\sum_{h=1}^{3} x_h \boldsymbol{e}_h\right) := \frac{1}{2}\left(x_2^2 + 2x_2x_3 + 2x_3x_2 + 3x_3^2\right) \quad \text{for all } \ x_1, x_2, x_3 \in \mathbb{R}.$$

The first row of the 3×3 symmetric matrix

$$\left(B(\boldsymbol{e}_i, \boldsymbol{e}_j)\right)_{i,j\in\{1,2,3\}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

shows that $\boldsymbol{e}_1$ belongs to the radical $\boldsymbol{R}$ of B. The above matrix has rank two, whence $\boldsymbol{R}$ has dimension $3 - 2 = 1$. Consequently, $\boldsymbol{R} = \operatorname{span}\{\boldsymbol{e}_1\}$ so that (5.2) is satisfied. Therefore, using the terminology and results of the current section, we now have: $n = 5$, $m = 3$, $d = 1$, $I_1 = \{1\}$, $I_2 = \{2, 3\}$, $I_3 = \{4, 5\}$, $\hat{\boldsymbol{R}} = \operatorname{span}\{\boldsymbol{e}_4^*, \boldsymbol{e}_5^*\}$ and $\hat{\boldsymbol{S}} = \operatorname{span}\{\boldsymbol{e}_2^*, \boldsymbol{e}_3^*, \boldsymbol{e}_4^*, \boldsymbol{e}_5^*\}$. Furthermore,

$$G_{22} = (g_{ij})_{i,j\in I_2} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \text{ and } \hat{G}_{22} = (\hat{g}_{ij})_{i,j\in I_2} = G_{22}^{-1} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

according to (5.5). Thus, from (5.9), the dual quadratic form $\hat{Q} \colon \hat{\boldsymbol{S}} \to \mathbb{R}$ satisfies

$$\hat{Q}\left(\sum_{h=2}^{5} a_h \boldsymbol{e}_h^*\right) = \frac{1}{2}\left(-3a_2^2 + 2a_2a_3 + 2a_3a_2 - a_3^2\right) \text{ for all } a_2, a_3, a_4, a_5 \in \mathbb{R}.$$

Remark 5.2. The results of the preceding paragraphs provide (up to scaling factors in $\mathbb{F} \setminus \{0\}$) explicit connections to the literature about quadric loci and their corresponding quadric envelopes, as quoted in Section 1. The same is true for the work on the homogeneous model of an affine metric space cited directly beneath. Any such model fits into our approach by choosing $\boldsymbol{S}$ as a hyperplane of $\boldsymbol{V}$ and Q as a quadratic form with a non-degenerate polar form B. Then Q satisfies (4.1) in a trivial way. The domain of $\hat{Q}$ turns out to be the entire vector space $\boldsymbol{V}^*$ rather than a proper subspace of $\boldsymbol{V}^*$. The polar form $\hat{B}$ has a one-dimensional radical.

Remark 5.3. Let $\operatorname{Char} F \neq 2$. Following Remark 4.7, let us exhibit the ties between Q and $4\hat{Q}$ in a different way. Formulas (5.8) and (5.9) are actually based upon the matrices $\frac{1}{2}G_{22}$ and $\frac{1}{2}\hat{G}_{22}$, respectively. By analogy to (5.9), the quadratic form $4\hat{Q}$ can be expressed using the matrix $\frac{1}{2}(4\hat{G}_{22})$, which in turn coincides with the inverse of the matrix $\frac{1}{2}G_{22}$ describing Q. So, in terms of the specific coordinates underlying our calculations, Q and $4\hat{Q}$ come, respectively, from an *invertible symmetric matrix of size* $m \times m$ and from its *inverse matrix*.

When $d = 0$ and $m = n$, then the previous observation is well known from the literature about the equations of quadrics; see the references in Section 1. In addition, it leads to an alternative construction which relies two facts. First, the inverse of any invertible $n \times n$ matrix M can be written as $(\det M)^{-1} \operatorname{adj}(M)$, where $\operatorname{adj}(M)$ denotes the *adjoint* of M (the transpose of the cofactor matrix of M). Second, a quadric of a projective space remains unchanged if its defining quadratic form is multiplied by a non-zero scalar. Therefore, if a quadric locus is defined (by analogy to the above) in terms of some symmetric (but not necessarily invertible) $n \times n$ matrix M, then the matrix $\operatorname{adj}(M)$ can be used to define a quadric envelope; see [15, pp. 58–74], [17, Sect. 9.3–9.4], [21, pp. 102–118] or [21, pp. 262–278]. We refer also to [5, p. 32] for generalisations in the realm of algebraic geometry.

6 Extensions of similarities

We start by recalling a few more intrinsic notions coming from the metric vector space $(\boldsymbol{S}, Q)$. See, among others, [6], [7, p. 40], [10, Ch. 4], [10, Ch. 12], [19, § 7.G–H] and [24, p. 55]. A mapping $\varphi \in \mathrm{GL}(\boldsymbol{S})$ is a *similarity* of $(\boldsymbol{S}, Q)$ whenever there exists a constant $c \in \mathbb{F} \setminus \{0\}$, known as *ratio* of φ, such that $Q(\varphi(\boldsymbol{x})) = cQ(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{S}$. If Q is not the zero form, then such a ratio c is determined uniquely by φ. Otherwise, in order to avoid ambiguity, only $c = 1$ will be considered as ratio of φ. All similarities of $(\boldsymbol{S}, Q)$ constitute the *general orthogonal group* $\mathrm{GO}(\boldsymbol{S}, Q)$. An *isometry* of $(\boldsymbol{S}, Q)$ is, by definition, a similarity of ratio 1. All isometries of $(\boldsymbol{S}, Q)$ form the *orthogonal group* $\mathrm{O}(\boldsymbol{S}, Q)$. The *weak orthogonal group* $\mathrm{O}'(\boldsymbol{S}, Q)$ consists of those isometries of $(\boldsymbol{S}, Q)$ which fix the radical $\boldsymbol{R}$ elementwise.

Next, we adopt the extrinsic point of view. For each $\varphi \in \mathrm{GL}(\boldsymbol{S})$ there exists at least one $\psi \in \mathrm{GL}(\boldsymbol{V})$ with $\psi(\boldsymbol{x}) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{S}$. Any such ψ will be called an *extension* of φ.

The above definitions and results carry over to $(\hat{\boldsymbol{S}}, \hat{Q})$ and $\mathrm{GL}(\boldsymbol{V}^*)$ in an obvious way. The following theorem establishes a relationship between the groups $\mathrm{GO}(\boldsymbol{S}, Q)$ and $\mathrm{GO}(\hat{\boldsymbol{S}}, \hat{Q})$ by demonstrating that $\psi \in \mathrm{GL}(\boldsymbol{V})$ is an extension of a similarity with ratio c of $(\boldsymbol{S}, Q)$ if, and only if, its transpose $\varphi^{\mathrm{T}} \in \mathrm{GL}(\boldsymbol{V}^*)$ is an extension of a similarity with ratio c of $(\hat{\boldsymbol{S}}, \hat{Q})$.

Theorem 6.1. *Let $Q\colon \boldsymbol{S} \to \mathbb{F}$ be a quadratic form as in Theorem 4.1 and let $\hat{Q}\colon \hat{\boldsymbol{S}} \to \mathbb{F}$ be its dual quadratic form. Furthermore, let $\psi \in \mathrm{GL}(\boldsymbol{V})$ and $c \in \mathbb{F}\setminus\{0\}$. Then the following statements are equivalent:*

(a) $\psi(\boldsymbol{S}) = \boldsymbol{S}$ *and* $Q(\psi(\boldsymbol{x})) = cQ(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \boldsymbol{S}$.

(b) $\psi^{\mathrm{T}}(\hat{\boldsymbol{S}}) = \hat{\boldsymbol{S}}$ *and* $\hat{Q}(\psi^{\mathrm{T}}(\boldsymbol{a}^*)) = c\hat{Q}(\boldsymbol{a}^*)$ *for all* $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$.

Proof. (a) $\Rightarrow$ (b) By expressing B in terms of Q as in (2.5), it follows

$$B(\psi(\boldsymbol{x}), \psi(\boldsymbol{y})) = cB(\boldsymbol{x}, \boldsymbol{y}) \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}. \tag{6.1}$$

Consequently, $\boldsymbol{x} \in \boldsymbol{R}$ is equivalent to $\psi(\boldsymbol{x}) \in \boldsymbol{R}$, that is, $\psi(\boldsymbol{R}) = \boldsymbol{R}$. From (2.4), applied to $\boldsymbol{R}$, and $\hat{\boldsymbol{S}} = \mathrm{ann}_{\boldsymbol{V}}(\boldsymbol{R})$, we obtain $\psi^{\mathrm{T}}(\hat{\boldsymbol{S}}) = \hat{\boldsymbol{S}}$.

From Lemma 3.4 (c) and $\psi(\boldsymbol{S}) = \boldsymbol{S}$, for each $\boldsymbol{a}^* \in \hat{\boldsymbol{S}}$ there exists at least one $\boldsymbol{u} \in \boldsymbol{S}$ (depending on $\boldsymbol{a}^*$) such that $\boldsymbol{a}^* \between \psi(\boldsymbol{u})$. Then, for all $\boldsymbol{y} \in \boldsymbol{S}$, it follows that

$$B(c\boldsymbol{u}, \boldsymbol{y}) = cB(\boldsymbol{u}, \boldsymbol{y}) = B(\psi(\boldsymbol{u}), \psi(\boldsymbol{y})) = \langle \boldsymbol{a}^*, \psi(\boldsymbol{y})\rangle_{\boldsymbol{V}} = \langle \psi^{\mathrm{T}}(\boldsymbol{a}^*), \boldsymbol{y}\rangle_{\boldsymbol{V}},$$

where we rest on B being linear in its first argument and on (6.1). So, according to Definition 3.3, $\psi^{\mathrm{T}}(\boldsymbol{a}^*) \between (c\boldsymbol{u})$. Thus we get

$$\hat{Q}(\psi^{\mathrm{T}}(\boldsymbol{a}^*)) = Q(c\boldsymbol{u}) = c^2 Q(\boldsymbol{u}) = c^2 \cdot c^{-1} Q(\psi(\boldsymbol{u})) = c\hat{Q}(\boldsymbol{a}^*).$$

(b) $\Rightarrow$ (a) In view of Theorem 4.5 (b) and $(\psi^{\mathrm{T}})^{\mathrm{T}} = \psi$, the proof of the converse runs in the same manner by interchanging the roles of Q and $\hat{Q}$. $\square$

Next, we present an example which illustrates not only Theorem 6.1 but also yet another property of certain B-linked elements.

Example 6.2. Suppose that $\boldsymbol{f}^* \in \hat{\boldsymbol{S}}$ and $\boldsymbol{s} \in \boldsymbol{S}$ satisfy $\boldsymbol{f}^* \between \boldsymbol{s}$ and $Q(\boldsymbol{s}) \neq 0$. Then $\hat{Q}(\boldsymbol{f}^*) = Q(\boldsymbol{s}) \neq 0$, so that $\ker \boldsymbol{f}^*$ is a hyperplane of $\boldsymbol{V}$. The linear mapping

$$\psi_{s,f^*} \colon \boldsymbol{V} \to \boldsymbol{V} \colon \boldsymbol{x} \mapsto \boldsymbol{x} - Q(\boldsymbol{s})^{-1} \langle \boldsymbol{f}^*, \boldsymbol{x} \rangle_V \boldsymbol{s}$$

fixes all vectors in $\ker \boldsymbol{f}^*$ and sends $\boldsymbol{s}$ to $-\boldsymbol{s}$, since $\boldsymbol{f}^* \between \boldsymbol{s}$ entails $\langle \boldsymbol{f}^*, \boldsymbol{s} \rangle_V = B(\boldsymbol{s}, \boldsymbol{s}) = 2Q(\boldsymbol{s})$. Using the last equation, one readily verifies that ψ_{s,f^*}^2 equals the identity mapping on $\boldsymbol{V}$. Consequently, $\psi_{s,f^*} \in \mathrm{GL}(\boldsymbol{V})$ is a *dilatation* or a *transvection*; see [10, p. 7] and [24, p. 20]. The transpose of ψ_{s,f^*} can be written as

$$\psi_{s,f^*}^{\mathrm{T}} \colon \boldsymbol{V}^* \to \boldsymbol{V}^* \colon \boldsymbol{a}^* \mapsto \boldsymbol{a}^* - \hat{Q}(\boldsymbol{f}^*)^{-1} \langle \boldsymbol{a}^*, \boldsymbol{s} \rangle_V \boldsymbol{f}^*.$$

The *reflection* of $(\boldsymbol{S}, Q)$ along $\boldsymbol{s}$ is that isometry $\varphi_s \in \mathrm{O}'(\boldsymbol{S}, Q)$ which is given as

$$\varphi_s \colon \boldsymbol{S} \to \boldsymbol{S} \colon \boldsymbol{x} \mapsto \boldsymbol{x} - Q(\boldsymbol{s})^{-1} B(\boldsymbol{s}, \boldsymbol{x}) \boldsymbol{s};$$

see [7, p. 40], [10, p. 40], [24, p. 145] or [19, p. 36]. Likewise, the reflection of $(\hat{\boldsymbol{S}}, \hat{Q})$ along $\boldsymbol{f}^*$ reads

$$\hat{\varphi}_{f^*} \colon \hat{\boldsymbol{S}} \to \hat{\boldsymbol{S}} \colon \boldsymbol{a}^* \mapsto \boldsymbol{a}^* - \hat{Q}(\boldsymbol{f}^*)^{-1} \hat{B}(\boldsymbol{a}^*, \boldsymbol{f}^*) \boldsymbol{f}^*$$

and belongs to the weak orthogonal group $\mathrm{O}'(\hat{\boldsymbol{S}}, \hat{Q})$. From $\boldsymbol{f}^* \between \boldsymbol{s}$, we have $\langle \boldsymbol{f}^*, \boldsymbol{x} \rangle_V = B(\boldsymbol{s}, \boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{S}$. Hence ψ_{s,f^*} is an extension of φ_s. Theorem 6.1 ensures that $\psi_{s,f^*}^{\mathrm{T}}$ extends an isometry of $(\hat{\boldsymbol{S}}, \hat{Q})$. Looking at Proposition 4.4 (a) reveals that this isometry equals $\hat{\varphi}_{f^*}$.

We wish to express the outcome of Theorem 6.1 in matrix form. In doing so, $\psi \in \mathrm{GL}(\boldsymbol{V})$ is to satisfy condition (a) for some $c \in \mathbb{F} \setminus \{0\}$. Hence condition (b) holds for ψ^{T}. We use again the bases of $\boldsymbol{V}$ and $\boldsymbol{V}^*$, as introduced at the beginning of Section 5, as well as the index sets I, I_1, I_2 and I_3 from (5.1). Let

$$p_{ij} := \langle \boldsymbol{e}_i^*, \psi(\boldsymbol{e}_j) \rangle_V = \langle \psi^{\mathrm{T}}(\boldsymbol{e}_i^*), \boldsymbol{e}_j \rangle_V \ \text{ for all } \ i, j \in I.$$

The matrix $P := (p_{ij})_{i,j \in I}$ allows for two interpretations: Reading P column-wise, it describes ψ with respect to basis $\{\boldsymbol{e}_j \mid j \in I\}$ of $\boldsymbol{V}$. Reading P row-wise (or the transpose of P column-wise), it describes ψ^{T} with respect to the basis $\{\boldsymbol{e}_i^* \mid i \in I\}$ of $\boldsymbol{V}^*$. For all $r, s \in \{1, 2, 3\}$, we denote as P_{rs} that submatrix of P which is formed by all p_{ij} with i and j ranging in I_r and I_s, respectively. Thereby empty matrices may occur. Since $\psi(\boldsymbol{R}) = \boldsymbol{R}$ and $\psi(\boldsymbol{S}) = \boldsymbol{S}$, all entries of the submatrices P_{21}, P_{31} and P_{32} are zero. So the matrix P can be written in block form

$$P = \left(\begin{array}{cc|c} P_{11} & P_{12} & P_{13} \\ \cline{3-3} 0 & \multicolumn{1}{|c|}{P_{22}} & P_{23} \\ \cline{1-2} 0 & \multicolumn{1}{|c}{0} & P_{33} \end{array} \right),$$

where 0 serves as an abbreviation for a zero-matrix of an appropriate size. Moreover, two (overlapping) square submatrices are highlighted. The one in the upper left corner is based upon the index set $I_1 \cup I_2$ and describes (column-wise) that similarity of $(\boldsymbol{S}, Q)$ which is extended by ψ. The one in the lower right corner is based upon the index set $I_2 \cup I_3$ and describes (row-wise) that similarity of $(\hat{\boldsymbol{S}}, \hat{Q})$ which is extended by ψ^{T}. The radical $\boldsymbol{R}$ of B (resp. $\hat{\boldsymbol{R}}$ of $\hat{B}$) is fixed elementwise under ψ (resp. ψ^{T}) precisely when P_{11} (resp. P_{33}) is an identity matrix. Thus, in general, Theorem 6.1 provides no relationship between the weak orthogonal groups $\mathrm{O}'(\boldsymbol{S}, Q)$ and $\mathrm{O}'(\hat{\boldsymbol{S}}, \hat{Q})$.

Remark 6.3. A particular case of Theorem 6.1 is omnipresent in literature about the homogeneous model of an affine-metric space, as quoted in Section 1. Under the premises sketched in Remark 5.2, the theorem leads to an isomorphism of the motion group of the affine-metric space arising from $(\boldsymbol{S}, Q)$ and the weak orthogonal group $\mathrm{O}'(\boldsymbol{V}^*, \hat{Q})$.

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