Projective Representations
II. Generalized chain geometries

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Dedicated to Walter Benz on the occasion of his 70th birthday.

Abstract

In this paper, projective representations of generalized chain geometries are investigated, using the concepts and results of [5]. In particular, we study under which conditions such a projective representation maps the chains of a generalized chain geometry $\Sigma(F, R)$ to reguli; this mainly depends on how the field F is embedded in the ring R. Moreover, we determine all bijective morphisms of a certain class of generalized chain geometries with the help of projective representations.
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1 Introduction

In [5], which is Part I of this publication, we considered the projective line $\mathbb{P}(R)$ over a ring R with 1 and constructed projective representations of $\mathbb{P}(R)$ with the help of unitary (K, R)-bimodules, where K is a not necessarily commutative field. Such a (K, R)-bimodule U gives rise to a projective representation Φ that maps $\mathbb{P}(R)$ into the set $\mathcal{G}$ of those subspaces of the projective space $\mathbb{P}(K, U \times U)$ that are isomorphic to one of their complements.

If R contains a subfield F with $1 \in F$, then the F-sublines turn $\mathbb{P}(R)$ into a *generalized chain geometry* $\Sigma(F, R)$, compare [4]. Ordinary *chain geometries*, where R is an F-algebra, are studied, e.g., in [2] and [10].

We want to investigate the images of the chains of a generalized chain geometry $\Sigma(F, R)$ under projective representations of $\mathbb{P}(R)$. It is well known from [10], Section 4, that chains appear as reguli if R is a finite-dimensional F-algebra, $F = K$, and the representation Φ is injective. In general, this is no longer true. In view of the various examples in the present paper, a unified geometric description of the Φ-images of chains seems very difficult. Hence we focus our attention to those cases where the Φ-images of chains are reguli in the sense of [3], i.e., reguli in not necessarily pappian spaces of arbitrary dimension.

As the Φ-images of any two chains are projectively equivalent, it suffices to discuss the Φ-image of the standard chain $\mathbb{P}(F) \subset \mathbb{P}(R)$. We proceed in two steps:

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In Sections 2 and 3 we discuss arbitrary projective representations of $\mathbb{P}(F)$ and characterize those where $\mathbb{P}(F)$ goes over to a regulus (Theorem 3.1) or a "direct sum" of reguli (Theorem 3.3). Such representations exist for all fields F.

Next, in Section 4, we take into account that $F \subset R$. Then a projective representation Φ of $\mathbb{P}(R)$ determines the projective representation $\Phi|_{\mathbb{P}(F)}$ of $\mathbb{P}(F)$; it depends essentially on how F is embedded in R. There are cases where $\Sigma(F, R)$ does not admit any projective representation mapping its chains to reguli (see Example 4.1).

Finally, in Section 5 we use projective representations in order to determine all bijective morphisms between chain geometries over rings of 2×2-matrices.

Throughout this paper we adopt the notions of [5]. All our rings have a unit element which is preserved by homomorphisms, inherited by subrings, and acts unitally on modules.

2 Transversals

Let R be any ring. We recall from [5], Section 4, that a projective representation of the projective line $\mathbb{P}(R)$ is given by means of a left vector space U over a not necessarily commutative field K and a ring homomorphism $\varphi : R \to \mathrm{End}_K(U)$, or, equivalently, by means of a (K, R)-bimodule U. The projective representation obtained from these data is called Φ and maps $\mathbb{P}(R)$ into the set $\mathcal{G}$ of those subspaces of the projective space $\mathbb{P}(K, U \times U)$ that are isomorphic to one of their complements.

As mentioned before, in the classical case chains go over to reguli. Since reguli have many transversals, we start by defining transversals of an arbitrary subset $\mathcal{S}$ of $\mathcal{G}$.

Definition 2.1 Let T be a line in $\mathbb{P}(K, U \times U)$ and let $\mathcal{S}$ be a subset of $\mathcal{G}$. Then T is a

(a) *weak transversal* of $\mathcal{S}$, if T meets each element of $\mathcal{S}$ in a unique point.

(b) *transversal* of $\mathcal{S}$, if T is a weak transversal and each point of T lies on an element of $\mathcal{S}$.

We know that $\mathbb{P}(R)^\Phi$ contains $U \times \{0\}$, $\{0\} \times U$, and $\{(u, u) \mid u \in U\}$. So one directly sees that a weak transversal of $\mathbb{P}(R)^\Phi$ must have the form

$$T = Ku \times Ku \tag{1}$$

for a suitable $u \in U \setminus \{0\}$. This description shows in particular that any two weak transversals of $\mathbb{P}(R)^\Phi$ are skew. Moreover, if $U = \{0\}$, then of course $\mathbb{P}(R)^\Phi$ does not have any weak transversals; the associated projective space is empty and hence contains no lines.

We can characterize the elements u of U that give rise to (weak) transversals of $\mathbb{P}(R)^\Phi$:

Proposition 2.2 *For $u \in U \setminus \{0\}$ the following statements are equivalent:*

(a) *$T = Ku \times Ku$ is a weak transversal of $\mathbb{P}(R)^\Phi$.*

(b) *u is an eigenvector of all $\rho_a : U \to U : u \mapsto u \cdot a$ $(a \in R)$.*

(c) *Ku is a sub-bimodule of the (K, R)-bimodule U.*

In this case, $\alpha : R \to K$ with $u^{\rho_a} = a^\alpha u$ is a homomorphism of rings, and T is a transversal of $\mathbb{P}(R)^\Phi$ exactly if α is surjective, or, equivalently, if Ku is a cyclic submodule of the right R-module U.

Proof: (a) $\Rightarrow$ (b): The line T meets $\{(u^{\rho_a}, u) \mid u \in U\} \in \mathbb{P}(R)^\Phi$. So $(u^{\rho_a}, u) = (ku, u)$ for some $k \in K$. (b) $\Rightarrow$ (c) is clear. (c) $\Rightarrow$ (a): compare [5], Proposition 4.8. The rest is a straightforward calculation. $\square$

Now we consider the special case that the ring $R =: F$ is a field. If in addition $U \neq \{0\}$, then $\varphi : F \to \mathrm{End}_K(U)$ is injective, i.e., Φ is faithful, and $\mathrm{char}(F) = \mathrm{char}(K)$. Moreover, the homomorphism $\alpha : F \to K$, with $u^{\rho_x} = x^\alpha u$, associated to the eigenvector u of all ρ_x, $x \in F$, is a monomorphism of fields. It is an isomorphism of fields, exactly if $T = Ku \times Ku$ is a transversal. This implies that $\mathbb{P}(F)^\Phi$ cannot have any weak transversals if there is no monomorphism $\alpha : K \to F$, and $\mathbb{P}(F)^\Phi$ cannot have any transversals if $F \ncong K$.
But also in the special case of $F = K$ there are examples where $\mathbb{P}(F)^\Phi = \mathbb{P}(K)^\Phi$ does not have any weak transversals at all. It can also occur that $\mathbb{P}(F)^\Phi$ has no transversals but many weak transversals, that $\mathbb{P}(F)^\Phi$ has exactly one transversal, and that $\mathbb{P}(F)^\Phi$ has at least one transversal:

Examples 2.3 (a) Let L be a commutative field, K a subfield of L and $\alpha \in \mathrm{Aut}(L)$ such that $K^\alpha \not\subset K$. Let $U \neq \{0\}$ be a left vector space over L. Then U becomes a (K, K)-bimodule by setting $k \cdot u := ku$, $u \cdot k := k^\alpha u$. Consider a $k \in K$ with $k^\alpha \notin K$. For each $u \in U \setminus \{0\}$ we have $u^{\rho_k} = k^\alpha u \notin Ku$, so u is not an eigenvector of ρ_k. This means that $\mathbb{P}(K)^\Phi$ has no weak transversals.

(b) Let K be a commutative field and let $\alpha : K \to K$ be a non-surjective monomorphism of fields. Let U be any left vector space over K. Then U becomes a (K, K)-bimodule by setting $u \cdot k := k^\alpha u$. Consider an arbitrary $u \in U \setminus \{0\}$. For each $k \in K$ we see that u is an eigenvector of ρ_k, with associated monomorphism α. So $T = Ku \times Ku$ is a weak transversal of $\mathbb{P}(K)^\Phi$, but no transversal because α is not surjective.

(c) Let K be a commutative field that is an inseparable quadratic extension of a subfield L. Then $K = L + Li$ with $i^2 \in L$, and $\mathrm{char}(K) = 2$. One can easily check that

$$\varphi : K \to \mathrm{M}(2 \times 2, K) : a + bi \mapsto \begin{pmatrix} a + bi & 0 \\ b & a + bi \end{pmatrix}$$

is an injective homomorphism of rings and thus makes the left vector space $U = K^2$ a faithful (K, K)-bimodule. Obviously, $u = (1, 0)$ is an eigenvector of all ρ_k, $k \in K$, with associated automorphism $\alpha = \mathrm{id}$. So $T = Ku \times Ku$ is a transversal of $\mathbb{P}(K)^\Phi$. Moreover, for $k \in K \setminus L$, there is no eigenvector of ρ_k that does not belong to $Ku = K(1, 0)$. Hence $\mathbb{P}(K)^\Phi$ does not have any other transversals.

(d) Let K be a subfield of a ring R. Then $U = R$ is a (K, K)-bimodule with $k \cdot u := ku$, $u \cdot k := uk$. The weak transversals of $\mathbb{P}(K)^\Phi$ are exactly the lines $Ku \times Ku$ with $u \in R \setminus \{0\}$ satisfying $uK \subset Ku$. Such a line is a transversal if, and only if, $uK = Ku$. In particular, $\mathbb{P}(K)^\Phi$ has at least one transversal, namely, $T = K \times K$, where $u = 1$.

Let F be a field, and let U be a (K, F)-bimodule. We want to study the transversals of the projective model $\mathbb{P}(F)^\Phi$ more closely. Because of Proposition 2.2 the existence of a transversal implies that $F \cong K$. So we now restrict ourselves to the case that $F = K$. Since $\mathbb{P}(F)^\Phi = \mathbb{P}(K)^\Phi$ consists of pairwise complementary subspaces, each point of a transversal of $\mathbb{P}(K)^\Phi$ lies on exactly one element of $\mathbb{P}(K)^\Phi$. So for two transversals T_1, T_2 of $\mathbb{P}(K)^\Phi$ it makes sense to define the following mapping

$$\pi_{12} : T_1 \to T_2 : T_1 \cap p^\Phi \mapsto T_2 \cap p^\Phi, \tag{2}$$

where $p \in \mathbb{P}(K)$. We can also describe this mapping algebraically: Let $T_i = Ku_i \times Ku_i$, with $\alpha_i \in \mathrm{Aut}(K)$ associated to the eigenvector u_i $(i = 1, 2)$. Then a point $K(x^{\alpha_1}u_1, y^{\alpha_1}u_1)$ of T_1 (with $x, y \in K$) has the π_{12}-image $K(x^{\alpha_2}u_2, y^{\alpha_2}u_2)$.

We call T_1 and T_2 *projectively linked*, if π_{12} is a projectivity. The relation "projectively linked" is an equivalence relation on the set of all transversals of $\mathbb{P}(K)^\Phi$. Obviously, the transversals T_1 and T_2 are projectively linked if, and only if, $\alpha_1^{-1}\alpha_2$ is an inner automorphism of K.

3 Reguli

We now study the question under which conditions the image of $\mathbb{P}(F)$ under a projective representation Φ is a regulus. As before, we restrict ourselves to the case $F = K$. We refer to [3] for a synthetic definition of a regulus in a not necessarily pappian space of arbitrary dimension. However, this definition is rather involved and for our purposes the description of a regulus in formula (5) below will be sufficient. We always assume that $U \neq \{0\}$. Hence by our global assumptions each representation $K \to \mathrm{End}_K(U)$ is faithful. Note that for the trivial case $U = \{0\}$ the theorems of this section are also true, if one defines that a regulus in the empty projective space $\mathbb{P}(K, U \times U)$ consists exactly of the empty set.

Let $U \neq \{0\}$ be a left vector space over K. The ring $\mathrm{End}_K(U)$ contains many copies of K: Let $(b_i)_{i\in I}$ be a basis of U. Then

$$\lambda : K \to \mathrm{End}_K(U) : k \mapsto k^\lambda, \tag{3}$$

where k^λ is the linear mapping given by

$$b_i \mapsto kb_i, \tag{4}$$

embeds K into $\mathrm{End}_K(U)$. The projective representation $\Lambda : \mathbb{P}(K) \to \mathcal{G}$ associated to λ has the form

$$K(k, l) \mapsto \Big\{ \Big(\sum_{i\in I} x_i k b_i, \sum_{i\in I} x_i l b_i \Big) \mid x_i \in K \Big\}; \tag{5}$$

cf. [5], Theorem 4.2. In [3], Theorem 3.1, it is shown that the image of $\mathbb{P}(K)$ under this projective representation is a regulus. Further, from Proposition 2.2, if $u = \sum_{i\in I} z_i b_i \neq 0$ with z_i in the centre of K, then $T = Ku \times Ku$ is a transversal of the regulus $\mathbb{P}(K)^\Lambda$. Conversely, each weak transversal of $\mathbb{P}(K)^\Lambda$ has this form, whence it is a transversal.

Theorem 3.1 *Let $\varphi : K \to \mathrm{End}_K(U)$ be a representation and let Φ be the associated projective representation of the projective line over K in the projective space $\mathbb{P}(K, U \times U)$. Then the following are equivalent:*

(a) $\mathbb{P}(K)^\Phi$ *is a regulus.*

(b) *Any two transversals of* $\mathbb{P}(K)^\Phi$ *are projectively linked and all transversals of* $\mathbb{P}(K)^\Phi$ *generate* $\mathbb{P}(K, U \times U)$.

(c) *There exists a basis* $(b_i)_{i\in I}$ *of* U *and an* $\alpha \in \mathrm{Aut}(K)$ *such that for all* $k \in K$ *the mapping* $\rho_k = k^\varphi \in \mathrm{End}_K(U)$ *is given by* $b_i \mapsto k^\alpha b_i$ $(i \in I)$.

Proof: (a) $\Rightarrow$ (b): By [3], Lemma 2.8, the regulus $\mathbb{P}(K)^\Phi$ is projectively equivalent to the regulus $\mathbb{P}(K)^\Lambda$, where λ is given by (3) and (4). So it suffices to prove the assertion for $\mathbb{P}(K)^\Lambda$. Let $T = Ku \times Ku$ be a transversal of $\mathbb{P}(K)^\Lambda$, i.e., u is an eigenvector of all mappings (4) with $k \in K$. Put $u = \sum_{i\in I} x_i b_i$, where without loss of generality one coordinate, say x_j, equals 1. We read off from the j-th coordinate that u goes over to ku under each mapping (4). Since T has been chosen arbitrarily, this means that any two transversals are projectively linked. Obviously, the transversals $Kb_i \times Kb_i$ generate the entire space.
(b) $\Rightarrow$ (c): Let $(T_i)_{i\in I}$ be a minimal family of transversals generating $\mathbb{P}(K, U \times U)$. Each T_i can be written as $Ku_i \times Ku_i$, where $(u_i)_{i\in I}$ is a basis of U, and there are automorphisms $\alpha_i \in \mathrm{Aut}(K)$ with $u_i \cdot k = k^{\alpha_i} u_i$. We fix one $j \in I$. As any two transversals are projectively linked, each product $\alpha_j^{-1}\alpha_i$ is an inner automorphism, say $x \mapsto k_i^{-1} x k_i$ with $k_i \in K^*$. Now

$$b_i := k_i u_i, \quad \alpha := \alpha_j \tag{6}$$

have the required properties.
(c) $\Rightarrow$ (a): We observe that the representation λ given by (3) and (4) and the representation φ satisfy $\varphi = \alpha\lambda$. Since $\alpha : K \to K$ is a bijection, we have that $\mathbb{P}(K)^\Phi = \mathbb{P}(K)^\Lambda$. As has been remarked before, this is a regulus. □

Remark 3.2 The existence of a basis $(b_i)_{i\in I}$ of U and of automorphisms $\alpha_i \in \mathrm{Aut}(K)$, such that $u_i \cdot k = k^{\alpha_i} u_i$ holds true for all $i \in I$ and all $k \in K$, is equivalent to the existence of a family of transversals of $\mathbb{P}(K)^\Phi$ which generates the entire space. Let $(T_i)_{i\in I}$ be a minimal generating family of transversals. If we fix one $j \in I$, then

$$\mathbb{P}(K)^\Phi = \left\{ \bigoplus_{i\in I} p^{\pi_{ji}} \mid p \in T_j \right\},$$

where the bijections π_{ji} are defined according to (2).

Note that if the transversals span only a subspace of the entire projective space, then similar statements hold true for the trace of $\mathbb{P}(K)^\Phi$ in this subspace, since this subspace corresponds to a sub-bimodule of the (K, K)-bimodule U (compare [5], Proposition 4.8).

Theorem 3.3 *Assume that* $\mathbb{P}(K, U \times U)$ *is spanned by the transversals of* $\mathbb{P}(K)^\Phi$. *Let* $\mathcal{T}_\vartheta$, $\vartheta \in \Theta$, *be the equivalence classes of projectively linked transversals, and put* $U_\vartheta \times U_\vartheta$ *for the subspace generated by* $\mathcal{T}_\vartheta$. *Then*

$$U \times U = \bigoplus_{\vartheta\in\Theta} (U_\vartheta \times U_\vartheta). \tag{7}$$

For each $\vartheta \in \Theta$, *the trace of* $\mathbb{P}(K)^\Phi$ *in* $\mathbb{P}(K, U_\vartheta \times U_\vartheta)$ *is a regulus.*

Proof: Let $(T_i)_{i\in I}$ be a minimal family of transversals generating the entire space. Then there is a basis $(u_i)_{i\in I}$ of U such that $T_i = Ku_i \times Ku_i$ holds for all $i \in I$. Now suppose that $T = Ku \times Ku$ ($u \in U \setminus \{0\}$) is any transversal. Then $T \in \mathcal{T}_\vartheta$ for some $\vartheta \in \Theta$. There exists a finite subset $I_u \subset I$ such that $u = \sum_{i\in I_u} x_i u_i$ with $x_i \neq 0$ for all $i \in I_u$. For each $k \in K$, the vectors $x_i u_i$ ($i \in I_u$) and u are eigenvectors of $\rho_k \in \mathrm{End}_K(U)$. By calculating u^{ρ_k} in two ways, it follows that these eigenvectors belong to the same eigenvalue, say k^α with $\alpha \in \mathrm{Aut}(K)$. Hence the transversals T_i ($i \in I_u$) and T are projectively linked, so that all T_i ($i \in I_u$) are in class $\mathcal{T}_\vartheta$. This implies that the subspace $U_\vartheta \times U_\vartheta$ is spanned by the set $\{T_i \in \mathcal{T}_\vartheta \mid i \in I\}$. Now (7) is obvious, and the remaining assertion follows from Theorem 3.1.
□

In the finite-dimensional case we have, by Theorem 3.1, that $\mathbb{P}(K)^\Phi$ is a regulus if, and only if, the elements of K are embedded into $\mathrm{End}_K(U)$ as scalar matrices with respect to some basis. The more general case treated in Theorem 3.3 corresponds to an embedding of K into $\mathrm{End}_K(U)$ as diagonal matrices, or, in other words, an embedding where all linear mappings ρ_k, $k \in K$, are simultaneously diagonalizable.

4 Chains

Now we consider projective representations of generalized chain geometries. Let F be a subfield of a ring R. Then the projective line $\mathbb{P}(F)$ is embedded into $\mathbb{P}(R)$ via $F(x,y) \mapsto R(x,y)$. We call the subset $\mathbb{P}(F)$ the *standard chain* of $\mathbb{P}(R)$. Its images under $\mathrm{GL}_2(R)$ are the *chains*, the set of all chains is denoted by $\mathfrak{C}(F,R)$. The incidence structure $\Sigma(F,R) = (\mathbb{P}(R), \mathfrak{C}(F,R))$ is the *generalized chain geometry* over (F,R) as investigated in [4]. Note that some basic properties of such geometries have already been derived in [11], [1]. Recall that two distinct points of $\mathbb{P}(R)$ are *distant*, if there is a $\gamma \in \mathrm{GL}_2(R)$ mapping the first point to $R(1,0)$ and the second point to $R(0,1)$, or, equivalently, if they are joined by a chain.

Let Φ be a projective representation of $\mathbb{P}(R)$, associated to a (K,R)-bimodule U. We want to determine the Φ-images of the chains of $\Sigma(F,R)$. Since $F \subset R$, we have that U is at the same time a (K,F)-bimodule, faithful if $U \neq \{0\}$, which in turn gives rise to the projective representation $\Phi|_{\mathbb{P}(F)}$ of $\mathbb{P}(F)$. By [5], Remark 4.1, the Φ-images of any two chains of $\Sigma(F,R)$ are projectively equivalent. So it suffices to study $\mathbb{P}(F)^\Phi$, and we can make use of the results of the previous sections.

From Theorem 3.1 we know which projective representations of $\mathbb{P}(F)$ map $\mathbb{P}(F)$ to a regulus. Essentially, these representations are given by embeddings of F into some $\mathrm{End}_K(U)$, with $K \cong F$, via a basis, as described in (3), (4). In particular, every left vector space over K can be turned into a suitable (K,F)-bimodule, if $K \cong F$. Now, in addition, F is a subfield of R, and we are looking for (K,R)-bimodules such that the induced (K,F)-bimodule gives rise to reguli. Under certain conditions on $F \subset R$ no projective representation of $\Sigma(F,R)$ maps chains to reguli:

Example 4.1 Let F be commutative, and assume that the multiplicative group F^* is not normal in R^*. Consider a projective representation Φ of $\Sigma(F,R)$ into some non-empty projective space $\mathbb{P}(K, U \times U)$. If $K \ncong F$ then the Φ-images of the chains do not have

any transversals and hence cannot be reguli. So let $K \cong F$. Then any three pairwise complementary subspaces of $\mathbb{P}(K, U \times U)$ lie together in exactly one regulus (compare [7], Proposition). By [4], Theorem 2.4, any three pairwise distant points of $\Sigma(F, R)$ are joined by more than one chain, which again implies that the Φ-images of the chains cannot be reguli.
A class of examples for this is described in [4], Example 2.8: Let $K = \mathrm{GF}(q)$, $F = \mathrm{GF}(q^2)$, $R = \mathrm{M}(2 \times 2, K)$, and $q \neq 2$. Then F^* is not normal in R^*. Note that according to [10], Example 4.5.(2), in this case the images of the chains under the projective representation associated to the (K, R)-bimodule K^2 are regular spreads.
Another class of examples are the geometries $\Sigma(K, L)$, where L is a quaternion skew field and K is one of its maximal commutative subfields, studied in [8].

We proceed with some examples where the chains appear as reguli. This can be checked immediately with the help of Theorem 3.1. We also give an explicit geometric description of all the reguli that are images of chains; the proof of this is left to the reader.

Examples 4.2 Let K be a not necessarily commutative field. In the following cases the Φ-images of the chains of $\Sigma(K, R)$ are reguli in $\mathbb{P}(K, U \times U)$.

(a) Let $R = \mathrm{End}_K(U)$, where K is embedded into R via a basis, according to (3), (4). Then the Φ-images of chains are all reguli in $\mathbb{P}(K, U \times U)$ (compare [3], Theorem 3.4).

(b) Let $R = K^n$, with componentwise addition and multiplication, let K be embedded into R via $k \mapsto (k, \dots, k)$, and let $U = R$. The Φ-images of chains are exactly those reguli in $\mathbb{P}(K, U \times U)$ that have the lines $U_i \times U_i$ among their transversals (compare [5], Example 5.3).

(c) Let $R = K(\varepsilon) = K + K\varepsilon$, the ring of dual numbers over K, and let $U = R$. The Φ-images of chains are exactly those reguli in $\mathbb{P}(K, U \times U)$ that have the line $T = K\varepsilon \times K\varepsilon$ as a transversal and whose unique element through $p \in T$ lies in the plane p^β (compare [5], Example 5.4).

(d) Let R be the ring of upper triangular 2×2-matrices over K, with K embedded as the scalar matrices. Then $U = K^2$ is a (K, R)-bimodule in the natural way. The Φ-images of chains are exactly those reguli that have the line $K(0, 1) \times K(0, 1)$ among their transversals (compare [5], Example 5.5).

In all these examples the Φ-images of the chains of $\Sigma(K, R)$ are exactly the reguli entirely contained in $\mathbb{P}(R)^\Phi$. This is not true in general, as the following example shows. Since in this counterexample R is infinite-dimensional over K, one might conjecture that if R is finite-dimensional over K, and the Φ-images of the chains are reguli, then all reguli in $\mathbb{P}(R)^\Phi$ are obtained in this way.

Example 4.3 (Compare [5], Example 4.7.) Let $R = K[X]$ be the polynomial ring over a commutative field K, and let $U = K(X)$ be its field of fractions. Then U is in a natural way a faithful (K, R)-bimodule and a faithful (K, U)-bimodule. Let $\Phi_1 : \mathbb{P}(R) \to \mathcal{G}$ and $\Phi_2 : \mathbb{P}(U) \to \mathcal{G}$ be the associated faithful representations. Then $\Phi_1 = \bar{\iota}\Phi_2$, where $\bar{\iota} : \mathbb{P}(R) \to \mathbb{P}(U)$ is induced by the natural inclusion $\iota : R \to U$ according to [5], Section 3. One can

easily check that $\bar{\iota}$ is a bijection and that $p = R(1,0)$ and $q = R(1,X)$ are non-distant in $\mathbb{P}(R)$ but $p^{\bar{\iota}}$, $q^{\bar{\iota}}$ are distant in $\mathbb{P}(U)$.
Now we consider the chain geometries $\Sigma(K,R)$ and $\Sigma(K,U)$. The images of the chains under Φ_1 and Φ_2 are reguli contained in $\mathbb{P}(R)^{\Phi_1} = \mathbb{P}(U)^{\Phi_2}$. Recall that "distant" means "joined by a chain". So the images $p^{\Phi_1} = p^{\bar{\iota}\Phi_2}$ and $q^{\Phi_1} = q^{\bar{\iota}\Phi_2}$ lie together on a regulus $\mathcal{R}$ that is entirely contained in $\mathbb{P}(R)^{\Phi_1}$, namely, the Φ_2-image of a chain in $\Sigma(K,U)$ joining $p^{\bar{\iota}}$ and $q^{\bar{\iota}}$. However, since $p, q \in \mathbb{P}(R)$ are non-distant, they are not joined by a chain in $\Sigma(K,R)$, and hence $\mathcal{R}$ does not appear as Φ_1-image of a chain.

In case $K = F$, the ring R itself is a faithful (K,R)-bimodule with respect to the regular representation. The associated projective representation Φ is the identity. We apply Theorem 3.1 to this situation.

Theorem 4.4 *Let $U = R$ be a ring with 1 such that $K \subset R$, $1 \in K$, and let $\varphi : K \to \mathrm{End}_K(U)$ be given by $u \cdot k := uk$. Then $\mathbb{P}(K)^\Phi = \mathbb{P}(K)$ is a regulus if, and only if, there exists a basis $(b_i)_{i\in I}$ of the left vector space $U = R$ over K such that each b_i centralizes K, i.e., $b_i k = k b_i$ holds for all $i \in I$, $k \in K$.*

Proof: Let $\mathbb{P}(K)^\Phi$ be a regulus. By Theorem 3.1(c) there is a basis $(u_i)_{i\in I}$ of $U = R$ and an automorphism $\alpha \in \mathrm{Aut}(K)$ such that $u_i k = k^\alpha u_i$ holds for all $k \in K$, $i \in I$. This means that the lines $T_i = Ku_i \times Ku_i$ are transversals. By 2.3(d), also $T = K \times K$ is a transversal, belonging to the eigenvector $1 \in R$ and the automorphism id_K. Since any two transversals of $\mathbb{P}(K)^\Phi$ are projectively linked by 3.1(c), this means that α is an inner automorphism. By an appropriate change of basis as in (6), the automorphism α goes over to the identity, as desired. The converse follows directly from Theorem 3.1. □

A special case is the one where K belongs to the centre $Z(R)$ of R, i.e., R is a K-algebra. This is the case of ordinary chain geometries.
We consider a wider class of examples:

Example 4.5 Let K be a field and let Z be the centre of K. Let C be any Z-algebra, with basis $(c_i)_{i\in I}$. Then the tensor product $R = K \otimes_Z C$ is a Z-algebra containing $K \cong K \otimes 1$. By [14], Theorem (15.1), the centralizer $Z_R(K)$ of K in R is $C \cong 1 \otimes C$, and $(1 \otimes c_i)_{i\in I}$ is a K-basis of R centralizing K. So in this situation, the chains of $\Sigma(K,R)$ are mapped to reguli by the projective representation associated to the (K,R)-bimodule $U = R$.
Under certain conditions, also the converse holds (cf. [13], Theorem 4.7): Let R be a ring, let K a subfield of R, and let Z be the centre of K. Assume that R possesses a K-basis consisting of elements of the centralizer $C := Z_R(K)$. Then R is a Z-algebra and C is a Z-subalgebra of R. If in addition K is finite-dimensional over Z, then $R \cong K \otimes_Z C$.

This means that we know all pairs (K,R), with K a subfield of R that is finite-dimensional over its centre, where the projective representation associated to $U = R$ maps chains to reguli.

5 An application

In this final section we show how projective models of generalized chain geometries can be used in order to determine all isomorphisms for a certain class of such geometries.

Throughout this section K and K' are fields. The rings of 2×2-matrices over K and K' are denoted by R and R', respectively. The unit matrices are written as E and E'. Each isomorphism $\kappa : K \to K'$ determines an isomorphism $R \to R'$ by putting $(c_{ij}) \mapsto (c_{ij}^{\kappa}) =: (c_{ij})^{\kappa}$. Similarly, each antiautomorphism ω of K gives rise to an antiautomorphism of R with $(c_{ij}) \mapsto (c_{ji}^{\omega}) =: (c_{ij})^{\omega\mathrm{T}}$.

First we show the following preliminary result on mappings preserving the distant-relation $\triangle$ (as defined before Remark 2.6 in [5]):

Theorem 5.1 *Let R and R' be the rings of 2×2-matrices over fields K and K', respectively. Then the following assertions on a mapping $\alpha : \mathbb{P}(R) \to \mathbb{P}(R')$ are equivalent:*

(a) *α is a mapping of the form*

$$\mathbb{P}(R) \to \mathbb{P}(R') : R(A, B) \mapsto R'((A^{\kappa}, B^{\kappa}) \cdot H') \tag{8}$$

with $\kappa : K \to K'$ an isomorphism and $H' \in \mathrm{GL}_2(R')$; or α is the product of a mapping

$$\mathbb{P}(R) \to \mathbb{P}(R) : R(A, B) \mapsto \{(X, Y) \in R^2 \mid -XB^{\omega\mathrm{T}} + YA^{\omega\mathrm{T}} = 0\}, \tag{9}$$

where ω is an antiautomorphism of K, with a mapping (8).

(b) *α is a* distant-preserving bijection, *i.e., $p\triangle q \Rightarrow p^{\alpha}\triangle q^{\alpha}$ for all $p, q \in \mathbb{P}(R)$.*

(c) *α is a bijection which is distant-preserving in both directions.*

Proof: (a) $\Rightarrow$ (c): The mappings (8) are induced by semilinear bijections $R^2 \to R'^2$, whence they are distant-preserving in both directions.

If we are given an antiautomorphism ω of K then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} D^{\omega\mathrm{T}} & -B^{\omega\mathrm{T}} \\ -C^{\omega\mathrm{T}} & A^{\omega\mathrm{T}} \end{pmatrix}^{-1} \tag{10}$$

is an automorphism of $\mathrm{GL}_2(R)$, as it is the product of the contragredient automorphism $(W_{ij}) \mapsto (W_{ji}^{\omega\mathrm{T}})^{-1}$ and an inner automorphism.

Next we show that (9) is a well-defined bijection of $\mathbb{P}(R)$. Let $R(A, B)$ be a point. Then there exists a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(R)$. Put $\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} := \begin{pmatrix} D^{\omega\mathrm{T}} & -B^{\omega\mathrm{T}} \\ -C^{\omega\mathrm{T}} & A^{\omega\mathrm{T}} \end{pmatrix}^{-1}$. With the substitution $(X, Y) = (\widetilde{X}, \widetilde{Y})\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix}$ the linear equation appearing in (9) can be rewritten as

$$(\widetilde{X}, \widetilde{Y}) \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ E \end{pmatrix} = 0.$$

The solutions of this equation form the submodule $R(E,0)$ of R^2, whence the equation appearing in (9) defines a point, namely the point $R(\widetilde{A},\widetilde{B})$.
Similarly, the image of $R(C,D)$ under (9) is $R(\widetilde{C},\widetilde{D})$. From this observation it is immediate that (9) defines a bijection of $\mathbb{P}(R)$ which is distant-preserving in both directions.
(c) $\Rightarrow$ (b): This is obviously true.
(b) $\Rightarrow$ (a): In a first step we translate the first part of this proof into a projective model of $\mathbb{P}(R)$: The ring R acts on the left vector space $U := K^2$ in the natural way and thus turns K^2 into a (K,R)-bimodule. The isomorphism $\varphi := \mathrm{id}_R$ yields a faithful projective representation $\Phi : \mathbb{P}(R) \to \mathcal{G}$, where $\mathcal{G}$ is the set of lines of the projective space $\mathbb{P}(K,K^4)$. Recall that now

$$R(A,B)^\Phi = \text{left rowspace}(A,B).$$

From [5], Remark 4.1, $\mathbb{P}(R)^\Phi = \mathcal{G}$. A bijection $\Phi' : \mathbb{P}(R') \to \mathcal{G}'$ is defined similarly.
It is easily seen that for each α given by (8) the mapping $\Phi^{-1}\alpha\Phi'$ is a bijection $\mathcal{G} \to \mathcal{G}'$ which is induced by the collineation $\mathbb{P}(K,K^4) \to \mathbb{P}(K',K'^4)$ given by the matrix $H' \in \mathrm{GL}_2(R') = \mathrm{GL}_4(K')$ and the isomorphism $\kappa : K \to K'$. Likewise, the Φ-transform of a mapping (9) is a bijection of $\mathcal{G}$ which is induced by the correlation of $\mathbb{P}(K,K^4)$ given by the matrix $\left(\begin{smallmatrix} 0 & -E \\ E & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(R) = \mathrm{GL}_4(K)$ and the antiautomorphism ω of K. The linear equation (9) now simply means the following: If a line is given by two of its points then its correlative image arises as the intersection of two planes.
Obviously, the mappings (8) yield all collineations $\mathbb{P}(K,K^4) \to \mathbb{P}(K',K'^4)$, whence also all correlations $\mathbb{P}(K,K^4) \to \mathbb{P}(K',K'^4)$ are obtained via all products of a mapping (9) with a mapping (8).
Now let α be given according to (b). We read off from [5], Remark 4.1, that the bijection $\Phi^{-1}\alpha\Phi' : \mathcal{G} \to \mathcal{G}'$ maps skew lines to skew lines. By [6], Satz 2, and [9], Theorem 3, the inverse of this bijection of lines is induced either by a collineation or a correlation $\mathbb{P}(K',K'^4) \to \mathbb{P}(K,K^4)$. Hence the assertion follows. $\square$

Note there for the special case $R = R'$ one could also use [12], Theorem 1, in order to prove that the line mapping under consideration is induced by a collineation or correlation. The statement of [12], Theorem 1, is also valid in higher dimensions, but unfortunately it is not applicable to the situation of distant-preserving permutations of the projective line over a ring of $n \times n$-matrices.

Remark 5.2 Since R is a ring of stable rank 2 (cf. [15], 2.6), the points of $\mathbb{P}(R)$ are exactly the submodules of the form $R(A, E+AB)$ with $A,B \in R$ (cf. [1]). Hence the mapping (9) can be written in the explicit form

$$R(A, E+AB) \mapsto R(A^{\omega\mathrm{T}}, E + A^{\omega\mathrm{T}}B^{\omega\mathrm{T}}). \tag{11}$$

Cf. also [10], Theorem 9.1.1, and [1], Theorem 2.4.

Now let $F \subset R$ and $F' \subset R'$ be fields with $E \in F$, $E' \in F'$. Recall that $\mathrm{GL}_2(R')$ operates transitively on the set of chains of $\Sigma(F',R')$. Moreover, the stabilizer in $\mathrm{GL}_2(R')$ of a chain acts 3-transitively on its set of points (see [4], Theorem 2.3). Thus, if we want to find all isomorphisms of the generalized chain geometry $\Sigma(F,R)$ onto $\Sigma(F',R')$, then it is sufficient to determine all fundamental isomorphisms; here a mapping is called *fundamental* if it takes the standard chain of $\Sigma(F,R)$ into the standard chain of $\Sigma(F',R')$ and the points $R(E,0)$, $R(0,E)$, $R(E,E)$ to $R'(E',0')$, $R'(0',E')$, $R'(E',E')$, respectively.

In the next theorem we consider the more general case of bijective morphisms, where a *morphism* maps chains *into* chains. As above, also in this situation it suffices to study the fundamental ones.

Theorem 5.3 *Let $\Sigma(F,R)$ and $\Sigma(F',R')$ be generalized chain geometries, where R and R' are rings of 2×2-matrices over fields K and K', respectively. Then the fundamental bijective morphisms $\Sigma(F,R)\to\Sigma(F',R')$ are exactly the following mappings $\mathbb{P}(R)\to\mathbb{P}(R')$:*

$$R(A,B)\mapsto R'\left((A^\kappa,B^\kappa)\begin{pmatrix} H_1' & 0' \\ 0' & H_1' \end{pmatrix}\right) \tag{12}$$

where $\kappa: K\to K'$ is an isomorphism, $H_1'\in \mathrm{GL}_2(K')$, and ${H_1'}^{-1}F^\kappa H_1'\subset F'$; or the product of a mapping

$$R(A,E+AB)\mapsto R(A^{\omega\mathrm{T}},E+A^{\omega\mathrm{T}}B^{\omega\mathrm{T}}), \tag{13}$$

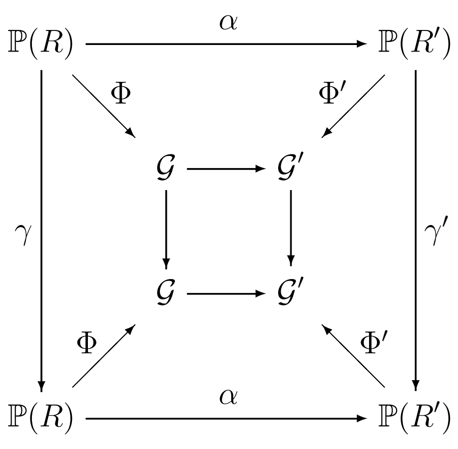
where ω is an antiautomorphism of K, with a mapping (12), where $\kappa: K\to K'$ is an isomorphism and $H_1'\in\mathrm{GL}_2(K')$ such that ${H_1'}^{-1}(F^{\omega\mathrm{T}})^\kappa H_1'\subset F'$.

Proof: Each fundamental bijective morphism α is a distant-preserving bijection. Thus we can apply Theorem 5.1. There are two cases:

Let α be given according to (8). Then H' has necessarily the form $\left(\begin{smallmatrix} H_1' & 0' \\ 0' & H_1'\end{smallmatrix}\right)$ with $H_1'\in \mathrm{GL}_2(K')$, since $R(E,0)$, $R(0,E)$, $R(E,E)$ go over to $R'(E',0')$, $R'(0',E')$, $R'(E',E')$. Any point of the standard chain other than $R(E,0)$ has the form $R(M,E)$, where $M\in F\subset R$. Its image point is $R'({H_1'}^{-1}M^\kappa H_1',E')$, so that ${H_1'}^{-1}F^\kappa H_1'\subset F'$, as required.

Let α be a product of a mapping (9) and a mapping (8). Observe that each mapping (9) fixes $R(E,0)$, $R(0,E)$, and $R(E,E)$. Thus the matrix H' appearing in (8) has the form as in the previous case. Now a similar argument as before together with (11) the yields assertion.

For the proof of the converse we consider the following commutative diagram:



Let $\alpha:\mathbb{P}(R)\to\mathbb{P}(R')$ be as in the assertion, and let γ be a projectivity of $\mathbb{P}(R)$ given by a matrix in $\mathrm{GL}_2(R)$. The same matrix, considered as an element of $\mathrm{GL}_4(K)$, yields a projective collineation π of $\mathbb{P}(K, K^4)$, which gives rise to the permutation $\Phi^{-1}\gamma\Phi$ of the line set $\mathcal{G}$. Note that this does not depend on the choice of the matrix because the kernels of the actions of $\mathrm{GL}_2(R)$ on $\mathbb{P}(R)$ and on $\mathcal{G}$ coincide as the centre of K equals the centre of R. Moreover, the proof of Theorem 5.1 shows that $\Phi^{-1}\alpha\Phi' : \mathcal{G} \to \mathcal{G}'$ is induced by a collineation or correlation $\mu : \mathbb{P}(K, K^4) \to \mathbb{P}(K', K'^4)$. So $\mu^{-1}\pi\mu$ is a projective collineation of $\mathbb{P}(K', K'^4)$, whence it can be described by a matrix in $\mathrm{GL}_4(K') = \mathrm{GL}_2(R')$. As before, this matrix yields the projectivity $\gamma' := \alpha^{-1}\gamma\alpha$ of $\mathbb{P}(R')$.

Now the assertion easily follows, because obviously α maps the standard chain $\mathbb{P}(F)$ of $\Sigma(F, R)$ into the standard chain of $\Sigma(F', R')$, and any other chain of $\Sigma(F, R)$ has the form $\mathbb{P}(F)^\gamma$ for some γ as in the diagram. $\square$

We want to mention here that the second direction of this theorem also follows from the more general statement in [1], Theorem (2.4).

Of course Theorem 5.3 also yields an algebraic description of all fundamental isomorphisms. Then the conditions on the image of F read ${H_1'}^{-1}F^\kappa H_1' = F'$ and ${H_1'}^{-1}(F^{\omega\mathrm{T}})^\kappa H_1' = F'$, respectively.

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