On invariant notions of Segre varieties in binary projective spaces

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Abstract

Invariant notions of a class of Segre varieties $\mathcal{S}_{(m)}(2)$ of $\mathrm{PG}(2^m - 1, 2)$ that are direct products of m copies of $\mathrm{PG}(1, 2)$, m being any positive integer, are established and studied. We first demonstrate that there exists a hyperbolic quadric that contains $\mathcal{S}_{(m)}(2)$ and is invariant under its projective stabiliser group $G_{\mathcal{S}_{(m)}(2)}$. By embedding $\mathrm{PG}(2^m - 1, 2)$ into $\mathrm{PG}(2^m - 1, 4)$, a basis of the latter space is constructed that is invariant under $G_{\mathcal{S}_{(m)}(2)}$ as well. Such a basis can be split into two subsets whose spans are either real or complex-conjugate subspaces according as m is even or odd. In the latter case, these spans can, in addition, be viewed as indicator sets of a $G_{\mathcal{S}_{(m)}(2)}$-invariant geometric spread of lines of $\mathrm{PG}(2^m - 1, 2)$. This spread is also related with a $G_{\mathcal{S}_{(m)}(2)}$-invariant non-singular Hermitian variety.

The case $m = 3$ is examined in detail to illustrate the theory. Here, the lines of the invariant spread are found to fall into four distinct orbits under $G_{\mathcal{S}_{(3)}(2)}$, while the points of $\mathrm{PG}(7, 2)$ form five orbits.

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1 Introduction

The present note is concerned with *invariant notions* in the ambient space of certain Segre varieties over fields of characteristic two and, in particular, over the smallest Galois field $\mathbb{F}_2$. The attribute *invariant* always refers to the stabiliser of the Segre in the projective group of the ambient space.

Our text is organised as follows: In Section 2 we collect some background results about those Segre varieties $\mathcal{S}_{(m)}(F)$ which are products of m projective lines over a field F. The next section presents an *invariant quadric* of a Segre $\mathcal{S}_{(m)}(F)$ for $m \geq 2$ and a ground field F of characteristic two (Theorem 1). This quadric is regular, of maximal Witt index, contains the given Segre, and its polarity is the fundamental polarity of this Segre. The following sections deal with Segre varieties $\mathcal{S}_{(m)}(2)$ over $\mathbb{F}_2$. By extending the ground field of the ambient space from $\mathbb{F}_2$ to $\mathbb{F}_4$ we find an *invariant basis* (Theorem 2) and an *invariant Hermitian variety* (Section 5). The theory splits according as m is even or odd. In the latter case there is an *invariant geometric line spread* (Corollary 1) which gives also rise to a spread of the invariant quadric. We make use of our previous results in Section 6, where we describe certain line orbits and all point orbits of the stabiliser group of the Segre $S_{(3)}(2)$ in terms of the invariant basis. This complements [14, p. 82], where a completely different description of these point orbits was given without proof.

There is a widespread literature on closely related topics, like notions of *rank of multi-dimensional arrays* [8], *secant varieties of Segre varieties* (mainly over the complex numbers) [1], the very particular properties of certain Segre varieties over $\mathbb{F}_2$ [24], [25], *quantum codes* [14], and *entanglement of quantum bits* in physics [10], [15], [19]. The few sources which are cited here contain a wealth of further references.

2 Notation and background results

Let F be a commutative field and let $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_m$ be $m \geq 1$ two-dimensional vector spaces over F. So $\mathbb{P}(\boldsymbol{V}_k) = \mathrm{PG}(1, F)$ are projective lines over F for $k \in \{1, 2, \ldots, m\}$. We consider the tensor product $\bigotimes_{k=1}^m \boldsymbol{V}_k$ and the projective space $\mathbb{P}(\bigotimes_{k=1}^m \boldsymbol{V}_k) = \mathrm{PG}(2^m - 1, F)$. The non-zero decomposable tensors of $\bigotimes_{k=1}^m \boldsymbol{V}_k$ determine the *Segre variety* (see [7], [17])

$$\mathcal{S}_{\underbrace{1,1,\ldots,1}_{m}}(F) = \{F\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \cdots \otimes \boldsymbol{a}_m \mid \boldsymbol{a}_k \in \boldsymbol{V}_k \setminus \{0\}\}$$

of $\mathbb{P}(\bigotimes_{k=1}^m \boldsymbol{V}_k)$. This Segre will also be denoted by $\mathcal{S}_{(m)}(F)$.

We recall some facts which are well known from the classical case [7, p. 143], where F is the field of complex numbers. They can immediately be carried over to our more general settings. Given a basis $(\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)})$ for each vector space $\boldsymbol{V}_k$, $k \in \{1, 2, \ldots, m\}$, the tensors

$$\boldsymbol{E}_{i_1,i_2,\ldots,i_m} := \boldsymbol{e}_{i_1}^{(1)} \otimes \boldsymbol{e}_{i_2}^{(2)} \otimes \cdots \otimes \boldsymbol{e}_{i_m}^{(m)} \quad \text{with} \quad (i_1, i_2, \ldots, i_m) \in I_m := \{0, 1\}^m \qquad (1)$$

constitute a basis of $\bigotimes_{k=1}^m \boldsymbol{V}_k$. For any multi-index $\boldsymbol{i} = (i_1, i_2, \ldots, i_m) \in I_m$ the *opposite* multi-index, in symbols $\boldsymbol{i}'$, is characterised by $i_k \neq i_k'$ for all $k \in \{1, 2, \ldots, m\}$. In other words, two multi-indices are opposite if, and only if, their Hamming distance is maximal.

Let $f_k \in \mathrm{GL}(\boldsymbol{V}_k)$ for $k \in \{1, 2, \dots, m\}$. Then

$$f_1 \otimes f_2 \otimes \cdots \otimes f_m \in \mathrm{GL}\Big(\bigotimes_{k=1}^{m} \boldsymbol{V}_k\Big) \tag{2}$$

denotes their Kronecker (tensor) product. Each permutation $\sigma \in S_m$ gives rise to linear bijections $\boldsymbol{V}_k \to \boldsymbol{V}_{\sigma(k)}$ sending $(\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)})$ to $(\boldsymbol{e}_0^{(\sigma(k))}, \boldsymbol{e}_1^{(\sigma(k))})$. Also, the symmetric group S_m acts on I_m via $\sigma(\boldsymbol{i}) = \sigma(i_1, i_2, \dots, i_m) = (i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \dots, i_{\sigma^{-1}(m)})$. There is a unique mapping

$$f_\sigma \in \mathrm{GL}\Big(\bigotimes_{k=1}^{m} \boldsymbol{V}_k\Big) \text{ such that } \boldsymbol{E}_{\boldsymbol{i}} \mapsto \boldsymbol{E}_{\sigma(\boldsymbol{i})} \text{ for all } \boldsymbol{i} \in I_m. \tag{3}$$

Clearly, this f_σ depends on the chosen bases. The subgroup of $\mathrm{GL}(\bigotimes_{k=1}^{m} \boldsymbol{V}_k)$ preserving decomposable tensors is generated by all transformations of the form (2) and (3). It induces the *stabiliser* $G_{\mathcal{S}_{(m)}(F)}$ of the Segre $\mathcal{S}_{(m)}(F)$ within the projective group $\mathrm{PGL}(\bigotimes_{k=1}^{m} \boldsymbol{V}_k)$.

Each of the vector spaces $\boldsymbol{V}_k$ admits a symplectic (*i. e.*, non-degenerate and alternating) bilinear form[1] $[\cdot,\cdot] : \boldsymbol{V}_k \times \boldsymbol{V}_k \to F$. Consequently, $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ is equipped with a bilinear form, again denoted as $[\cdot,\cdot]$, which is given by

$$\big[\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \cdots \otimes \boldsymbol{a}_m, \boldsymbol{b}_1 \otimes \boldsymbol{b}_2 \otimes \cdots \otimes \boldsymbol{b}_m\big] := \prod_{k=1}^{m} [\boldsymbol{a}_k, \boldsymbol{b}_k] \text{ for } \boldsymbol{a}_k, \boldsymbol{b}_k \in \boldsymbol{V}_k, \tag{4}$$

and extending bilinearly. Like the forms on $\boldsymbol{V}_k$ this bilinear form on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ is unique up to a non-zero factor in F. In projective terms the form $[\cdot,\cdot]$ on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ (or any proportional one) determines the *fundamental polarity* of $\mathcal{S}_{(m)}(F)$, *i. e.*, a polarity which sends $\mathcal{S}_{(m)}(F)$ to its dual. This polarity is orthogonal when m is even and $\mathrm{Char}\, F \neq 2$, but null otherwise: Indeed, it suffices to consider the tensors of our basis (1). Given $\boldsymbol{i}, \boldsymbol{j} \in I_m$ we have

$$[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}] = \prod_{k=1}^{m} [\boldsymbol{e}_{i_k}^{(k)}, \boldsymbol{e}_{i'_k}^{(k)}] = (-1)^m [\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{E}_{\boldsymbol{i}}] \neq 0, \tag{5}$$

$$[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{j}}] = 0 \quad \text{for all} \quad \boldsymbol{j} \neq \boldsymbol{i}'. \tag{6}$$

Hence $[\cdot,\cdot]$ is symmetric when m is even and $\mathrm{Char}\, F \neq 2$, and it is alternating otherwise.

Let m be even and $\mathrm{Char}\, F \neq 2$. Then $Q : \bigotimes_{k=1}^{m} \boldsymbol{V}_k \to F : \boldsymbol{X} \mapsto [\boldsymbol{X}, \boldsymbol{X}]$ is a quadratic form having Witt index 2^{m-1} and rank 2^m. So, the fundamental polarity of the Segre $\mathcal{S}_{(m)}(F)$ is the polarity of a regular quadric. The Segre coincides with this quadric precisely when $m = 2$.

[1] We use the same symbol for all these forms. Note that $\mathrm{Sp}(\boldsymbol{V}_k, [\cdot,\cdot]) = \mathrm{SL}(\boldsymbol{V}_k)$, since $\dim \boldsymbol{V}_k = 2$ for all $k \in \{1, 2, \dots, m\}$. This coincidence of a symplectic group with a special linear group underpins much of the mathematics used in this article.

3 The invariant quadric

We now focus on the case when F has characteristic two. Here $[\cdot,\cdot]$ is a symplectic bilinear form on $\bigotimes_{k=1}^{m}\boldsymbol{V}_k$ for any $m \geq 1$, whence the fundamental polarity of the Segre $\mathcal{S}_{(m)}(F)$ is always null. Furthermore, (5) simplifies to

$$[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}] = \prod_{k=1}^{m}[\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)}] = [\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{E}_{\boldsymbol{i}}] \neq 0. \tag{7}$$

Proposition 1. *Let $m \geq 2$ and* $\operatorname{Char} F = 2$. *Then there is a unique quadratic form $Q : \bigotimes_{k=1}^{m}\boldsymbol{V}_k \to F$ satisfying the following two properties:*

1. *Q vanishes for all decomposable tensors.*
2. *The symplectic bilinear form $[\cdot,\cdot] : \bigotimes_{k=1}^{m}\boldsymbol{V}_k \times \bigotimes_{k=1}^{m}\boldsymbol{V}_k \to F$ is the polar form of Q.*

Proof. (a) We denote by $I_{m,0}$ the set of all multi-indices $(i_1, i_2, \dots, i_m) \in I_m$ with $i_1 = 0$. In terms of our basis (1) a quadratic form is given by

$$Q : \bigotimes_{k=1}^{m}\boldsymbol{V}_k \to F : \boldsymbol{X} \mapsto \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{X}][\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{X}]}{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}]}. \tag{8}$$

Given an arbitrary decomposable tensor we have

$$\begin{aligned}
Q(\boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_m) &= \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_m][\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_m]}{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}]} \\
&= \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\boldsymbol{e}_0^{(1)}, \boldsymbol{a}_1][\boldsymbol{e}_1^{(1)}, \boldsymbol{a}_1] \cdots [\boldsymbol{e}_0^{(m)}, \boldsymbol{a}_m][\boldsymbol{e}_1^{(m)}, \boldsymbol{a}_m]}{[\boldsymbol{e}_0^{(1)}, \boldsymbol{e}_1^{(1)}] \cdots [\boldsymbol{e}_0^{(m)}, \boldsymbol{e}_1^{(m)}]} \\
&= 2^{m-1} \frac{[\boldsymbol{e}_0^{(1)}, \boldsymbol{a}_1][\boldsymbol{e}_1^{(1)}, \boldsymbol{a}_1] \cdots [\boldsymbol{e}_0^{(m)}, \boldsymbol{a}_m][\boldsymbol{e}_1^{(m)}, \boldsymbol{a}_m]}{[\boldsymbol{e}_0^{(1)}, \boldsymbol{e}_1^{(1)}] \cdots [\boldsymbol{e}_0^{(m)}, \boldsymbol{e}_1^{(m)}]} \\
&= 0,
\end{aligned}$$

where we used (7), $\#I_{m,0} = 2^{m-1}$, $m \geq 2$, and $\operatorname{Char} F = 2$. This verifies property 1.

(b) Let $\boldsymbol{j}, \boldsymbol{k} \in I$ be arbitrary multi-indices. Polarising Q gives

$$\begin{aligned}
Q(\boldsymbol{E}_{\boldsymbol{j}} + \boldsymbol{E}_{\boldsymbol{k}}) + Q(\boldsymbol{E}_{\boldsymbol{j}}) + Q(\boldsymbol{E}_{\boldsymbol{k}}) &= Q(\boldsymbol{E}_{\boldsymbol{j}} + \boldsymbol{E}_{\boldsymbol{k}}) + 0 + 0 \\
&= \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{j}} + \boldsymbol{E}_{\boldsymbol{k}}][\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{E}_{\boldsymbol{j}} + \boldsymbol{E}_{\boldsymbol{k}}]}{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}]}.
\end{aligned}$$

The numerator of a summand of the above sum can only be different from zero if $\boldsymbol{i} \in \{\boldsymbol{j}', \boldsymbol{k}'\}$ and $\boldsymbol{i}' \in \{\boldsymbol{j}', \boldsymbol{k}'\}$. These conditions can only be met for $\boldsymbol{k} = \boldsymbol{j}'$, whence in fact at most one summand, namely for $\boldsymbol{j} \in I_{m,0}$ the one with $\boldsymbol{i} = \boldsymbol{j}$, and for $\boldsymbol{j}' \in I_{m,0}$ the one with $\boldsymbol{i} = \boldsymbol{j}'$, can be non-zero. So

$$Q(\boldsymbol{E_j} + \boldsymbol{E_k}) + Q(\boldsymbol{E_j}) + Q(\boldsymbol{E_k}) = 0 = [\boldsymbol{E_j}, \boldsymbol{E_k}] \quad \text{for} \quad \boldsymbol{k} \neq \boldsymbol{j}'$$

and, irrespective of whether $\boldsymbol{i} = \boldsymbol{j}$ or $\boldsymbol{i} = \boldsymbol{j}'$, we have

$$Q(\boldsymbol{E_j} + \boldsymbol{E_{j'}}) + Q(\boldsymbol{E_j}) + Q(\boldsymbol{E_{j'}}) = \frac{[\boldsymbol{E_j}, \boldsymbol{E_j} + \boldsymbol{E_{j'}}][\boldsymbol{E_{j'}}, \boldsymbol{E_j} + \boldsymbol{E_{j'}}]}{[\boldsymbol{E_j}, \boldsymbol{E_{j'}}]} = [\boldsymbol{E_j}, \boldsymbol{E_{j'}}].$$

But this implies that the quadratic form Q polarises to $[\cdot,\cdot]$, *i. e.*, also the second property is satisfied.

(c) Let $\widetilde{Q}$ be a quadratic form satisfying properties 1 and 2. Hence the polar form of $Q - \widetilde{Q} = Q + \widetilde{Q}$ is zero. We consider F as a vector space over its subfield $F^{\square}$ comprising all squares in F. So $(Q + \widetilde{Q}) : \bigotimes_{k=1}^m \boldsymbol{V}_k \to F$ is a semilinear mapping with respect to the field isomorphism $F \to F^{\square} : x \mapsto x^2$; see, *e. g.*, [9, p. 33]. The kernel of $Q + \widetilde{Q}$ is a subspace of $\bigotimes_{k=1}^m \boldsymbol{V}_k$ which contains all decomposable tensors and, in particular, our basis (1). Hence $Q + \widetilde{Q}$ vanishes on $\bigotimes_{k=1}^m \boldsymbol{V}_k$, and $Q = \widetilde{Q}$ as required. □

From (8) and (7), the quadratic form Q can be written in terms of tensor coordinates $x_{\boldsymbol{j}} \in F$ as

$$Q\Big(\sum_{\boldsymbol{j} \in I_m} x_{\boldsymbol{j}} \boldsymbol{E_j}\Big) = \sum_{\boldsymbol{i} \in I_{m,0}} [\boldsymbol{E_i}, \boldsymbol{E_{i'}}] x_{\boldsymbol{i}} x_{\boldsymbol{i}'} = \prod_{k=1}^m [\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)}] \cdot \sum_{\boldsymbol{i} \in I_{m,0}} x_{\boldsymbol{i}} x_{\boldsymbol{i}'}. \tag{9}$$

The previous results may be slightly simplified by taking symplectic bases, *i. e.*, $[\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)}] = 1$ for all $k \in \{1, 2, \dots, m\}$, whence also $[\boldsymbol{E_i}, \boldsymbol{E_{i'}}] = 1$ for all $\boldsymbol{i} \in I_m$.

Observe also that Proposition 1 fails to hold for $m = 1$. A quadratic form Q vanishing for all decomposable tensors of $\boldsymbol{V}_1$ is necessarily zero, since any element of $\boldsymbol{V}_1$ is decomposable. Hence the polar form of such a Q cannot be non-degenerate.

Theorem 1. *Let* $m \geq 2$ *and* $\operatorname{Char} F = 2$. *There exists in the ambient space of the Segre* $\mathcal{S}_{(m)}(F)$ *a regular quadric* $\mathcal{Q}(F)$ *with the following properties:*

1. *The projective index of* $\mathcal{Q}(F)$ *is* $2^{m-1} - 1$.
2. $\mathcal{Q}(F)$ *is invariant under the group of projective collineations stabilising the Segre* $\mathcal{S}_{(m)}(F)$.

Proof. Any $f_k \in \mathrm{GL}(\boldsymbol{V}_k)$, $k \in \{1, 2, \dots, m\}$, preserves the symplectic form $[\cdot,\cdot]$ on $\boldsymbol{V}_k$ to within a non-zero factor. Any linear bijection f_σ as in (3) is a symplectic transformation of $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$. Hence any transformation from the group $G_{\mathcal{S}_{(m)}(F)}$ preserves the symplectic form (4) up to a non-zero factor. Consequently, also Q is invariant up to a non-zero factor under the action of $G_{\mathcal{S}_{(m)}(F)}$.

From (9) the linear span of the tensors $\boldsymbol{E_j}$ with $\boldsymbol{j}$ ranging in $I_{m,0}$ is a singular subspace with respect to Q. So the Witt index of Q equals 2^{m-1}, because $[\cdot,\cdot]$ being non-degenerate implies that a greater value is impossible.

We conclude that the quadric with equation $Q(\boldsymbol{X}) = 0$ has all the required properties. □

We henceforth call $\mathcal{Q}(F)$ the *invariant quadric* of the Segre $\mathcal{S}_{(m)}(F)$. The case $m = 2$ deserves special mention, as the Segre $\mathcal{S}_{1,1}(F)$ coincides with its invariant quadric $\mathcal{Q}(F)$ given by $Q(\sum_{\boldsymbol{j}\in I_2} x_{\boldsymbol{j}}\boldsymbol{E_j}) = x_{00}x_{11} + x_{01}x_{10} = 0$. This result parallels the situation for $\operatorname{Char} F \neq 2$.

4 The invariant basis

In what follows $\mathbb{F}_q$ will denote the Galois field with q elements. We adopt the notation and terminology from Section 2, but we restrict ourselves to the case $F = \mathbb{F}_2$. Indeed, we shall always identify $\mathbb{F}_2$ with the prime field of $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, where $\omega^2 + \omega + 1 = 0$. For each $k \in \{1, 2, \dots, m\}$ we fix a basis $(\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)})$ so that we obtain the tensor basis (1).

Let $\boldsymbol{V}$ be any vector space over $\mathbb{F}_2$. Then $\boldsymbol{V}$ can be embedded in $\boldsymbol{W} := \mathbb{F}_4 \otimes_{\mathbb{F}_2} \boldsymbol{V}$, which is a vector space over $\mathbb{F}_4$, by $\boldsymbol{a} \mapsto 1 \otimes \boldsymbol{a}$; see, for example, [18, p. 263]. We shall not distinguish between $\boldsymbol{a}$ and $1 \otimes \boldsymbol{a}$. Likewise, if f denotes a linear mapping between vector spaces over $\mathbb{F}_2$, then the unique *linear* extension of f to the corresponding vector spaces over $\mathbb{F}_4$ will also be written as f rather than $1 \otimes f$. Similarly, we use the same symbol for a bilinear form on $\boldsymbol{V}$ and its extension to a bilinear form on $\boldsymbol{W}$. After similar identifications, we have $\mathbb{P}(\boldsymbol{V}) \subset \mathbb{P}(\boldsymbol{W})$, $\mathrm{PGL}(\boldsymbol{V}) \subset \mathrm{PGL}(\boldsymbol{W})$, and so on. We make use of the usual terminology for real and complex spaces also in our setting. We address the vectors of $\boldsymbol{V}$ to be *real*, we speak of *complex-conjugate* vectors, points, and subspaces. In particular, a subspace is said to be *real* if it coincides with its complex-conjugate subspace.

Applying this extension to our vector spaces $\boldsymbol{V}_k$ and their tensor product $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ gives vector spaces $\boldsymbol{W}_k$ and $\mathbb{F}_4 \otimes_{\mathbb{F}_2} (\bigotimes_{k=1}^{m} \boldsymbol{V}_k)$. The last vector space can be identified with $\bigotimes_{k=1}^{m} \boldsymbol{W}_k$ in a natural way, so that the Segre $\mathcal{S}_{(m)}(\mathbb{F}_2) =: \mathcal{S}_{(m)}(2)$ can be viewed as as subset of $\mathcal{S}_{(m)}(\mathbb{F}_4) =: \mathcal{S}_{(m)}(4)$. Likewise, we have $\mathcal{Q}(2) := \mathcal{Q}(\mathbb{F}_2) \subset \mathcal{Q}(\mathbb{F}_4) =: \mathcal{Q}(4)$.

From now on we shall make use of the following observation: When the projective line $\mathbb{P}(\boldsymbol{V}_k)$ is embedded in $\mathbb{P}(\boldsymbol{W}_k)$ advantage can be taken from the fact that there is a *unique* projective basis consisting of the complex-conjugate pair of points, while there is a choice of three different pairs of points for a projective basis of $\mathbb{P}(\boldsymbol{V}_k)$.

Theorem 2. *For each $k \in \{1, 2, \ldots, m\}$ let $\mathbb{F}_4\boldsymbol{u}_0^{(k)}$ and $\mathbb{F}_4\boldsymbol{u}_1^{(k)}$ be the only two points of the projective line $\mathbb{P}(\boldsymbol{W}_k) = \mathrm{PG}(1,4)$ that are not contained in $\mathbb{P}(\boldsymbol{V}_k) = \mathrm{PG}(1,2)$. Then*

$$\mathcal{B}_m := \left\{\mathbb{F}_4\boldsymbol{u}_{i_1}^{(1)} \otimes \boldsymbol{u}_{i_2}^{(2)} \otimes \cdots \otimes \boldsymbol{u}_{i_m}^{(m)} \mid (i_1, i_2, \ldots, i_m) \in I_m\right\}$$

is a basis of $\mathbb{P}(\bigotimes_{k=1}^m \boldsymbol{W}_k) = \mathrm{PG}(2^m - 1, 4)$ which is invariant, as a set, under the stabiliser $G_{\mathcal{S}_{(m)}(\mathbb{F}_2)} =: G_{\mathcal{S}_{(m)}(2)}$ of the Segre $\mathcal{S}_{(m)}(2)$.

Proof. We may assume that

$$\boldsymbol{u}_0^{(k)} = \boldsymbol{e}_0^{(k)} + \omega\boldsymbol{e}_1^{(k)} \quad \text{and} \quad \boldsymbol{u}_1^{(k)} = \boldsymbol{e}_0^{(k)} + \omega^2\boldsymbol{e}_1^{(k)} = (\boldsymbol{e}_0^{(k)} + \boldsymbol{e}_1^{(k)}) + \omega\boldsymbol{e}_1^{(k)}. \tag{10}$$

As $\boldsymbol{u}_0^{(k)}$ and $\boldsymbol{u}_1^{(k)}$ are linearly independent, the 2^m tensors

$$\boldsymbol{U}_{i_1,i_2,\ldots,i_m} := \boldsymbol{u}_{i_1}^{(1)} \otimes \boldsymbol{u}_{i_2}^{(2)} \otimes \cdots \otimes \boldsymbol{u}_{i_m}^{(m)} \quad \text{with} \quad (i_1, i_2, \ldots, i_m) \in I_m \tag{11}$$

constitute a basis of $\bigotimes_{k=1}^m \boldsymbol{W}_k$, whence $\mathcal{B}_m$ is a projective basis. The invariance of $\mathcal{B}_m$ under $G_{\mathcal{S}_{(m)}(2)}$ follows from the fact that the points $\mathbb{F}_4\boldsymbol{u}_0^{(k)}$ and $\mathbb{F}_4\boldsymbol{u}_1^{(k)}$ are determined uniquely up to relabelling. □

We shall refer to $\mathcal{B}_m$ as the *invariant basis* of the Segre $\mathcal{S}_{(m)}(2)$. In order to describe the action of the stabiliser group $G_{\mathcal{S}_{(m)}(2)}$ of the Segre $\mathcal{S}_{(m)}(2)$ on the invariant basis we need a few technical preparations:

First, from now on the set $I_m = \{0, 1\}^m$ of multi-indices will be identified with the vector space $\mathbb{F}_2^m$. Secondly, for any 2-dimensional vector space $\boldsymbol{V}$ over $\mathbb{F}_2$ we can define the $\mathbb{F}_2$-*valued sign function* $\mathrm{sgn}_2 : \mathrm{GL}(V) \to \mathbb{F}_2$ to be 0 if f induces an even permutation of $\boldsymbol{V} \setminus \{0\}$ and 1 otherwise.

Proposition 2. *The stabiliser group $G_{\mathcal{S}_{(m)}(2)}$ of the Segre $\mathcal{S}_{(m)}(2)$ has the following properties:*

1. *Let $f_k \in \mathrm{GL}(\boldsymbol{V}_k)$ for $k \in \{1, 2, \ldots, m\}$ and write*

$$\boldsymbol{s} := (\mathrm{sgn}_2 f_1, \mathrm{sgn}_2 f_2, \ldots, \mathrm{sgn}_2 f_m) \in \mathbb{F}_2^m. \tag{12}$$

   The collineation given by $f_1 \otimes f_2 \otimes \cdots \otimes f_m$ sends any point $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{i}} \in \mathcal{B}_m$ to the point $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{i}+\boldsymbol{s}} \in \mathcal{B}_m$.

2. *$G_{\mathcal{S}_{(m)}(2)}$ acts transitively on the invariant basis $\mathcal{B}_m$.*

3. *Let $\sigma \in S_m$ be a permutation and define f_σ as in* (3)*. Then f_σ sends any point $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{i}} \in \mathcal{B}_m$ to $\mathbb{F}_4\boldsymbol{U}_{\sigma(\boldsymbol{i})} \in \mathcal{B}_m$.*

Proof. (a) Each mapping $f_k \in \mathrm{GL}(\boldsymbol{V}_k) \subset \mathrm{GL}(\boldsymbol{W}_k)$ induces a projectivity of the projective line $\mathbb{P}(\boldsymbol{W}_k) = \mathrm{PG}(1,4)$ which stabilises $\mathbb{P}(\boldsymbol{V}_k) = \mathrm{PG}(1,2)$.

If $\operatorname{sgn}_2 f_k = 0$ then the restriction to $\mathbb{P}(\boldsymbol{V}_k)$ is an even permutation, namely either a permutation without fixed points or the identity on $\mathbb{P}(\boldsymbol{V}_k)$. In the first case the characteristic polynomial of f_k has two distinct zeros over $\mathbb{F}_4$, whence each of the two points $\mathbb{F}_4\boldsymbol{u}_0^{(k)}$ and $\mathbb{F}_4\boldsymbol{u}_1^{(k)}$ remains fixed. In the second case all points of $\mathbb{P}(\boldsymbol{V}_k)$ are fixed.

If $\operatorname{sgn}_2 f_k = 1$ then f_k gives a permutation of $\mathbb{P}(\boldsymbol{V}_k)$ with precisely one fixed point. Such an f_k is an involution, whence the points $\mathbb{F}_4\boldsymbol{u}_0^{(k)}$ and $\mathbb{F}_4\boldsymbol{u}_1^{(k)}$ are interchanged.

We infer from the above results that $f_1 \otimes f_2 \otimes \cdots \otimes f_m$ sends the point of $\mathcal{B}_m$ with multi-index $\boldsymbol{i} \in \mathbb{F}_2^m$ to the point of $\mathcal{B}_m$ with multi-index $(\boldsymbol{i} + \boldsymbol{s}) \in \mathbb{F}_2^m$.

(b) Given $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{F}_2^m$ we let $\boldsymbol{s} := \boldsymbol{i} + \boldsymbol{j}$. In order find a collineation from $G_{\mathcal{S}_{(m)}(2)}$ taking $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{i}}$ to $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{j}}$, it suffices to choose for all $k \in \{1, 2, \ldots, m\}$ some $f_k \in \mathrm{GL}(\boldsymbol{V}_k)$ with $\operatorname{sgn}_2 f_k = s_k$. This can clearly be done, so that $f_1 \otimes f_2 \otimes \cdots \otimes f_m$ yields a collineation with the required properties.

(c) According to the (basis-dependent) definition of f_σ in (3), we have to consider the linear bijections $\boldsymbol{V}_k \to \boldsymbol{V}_{\sigma(k)}$ sending $(\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)})$ to $(\boldsymbol{e}_0^{(\sigma(k))}, \boldsymbol{e}_1^{(\sigma(k))})$. By (10), any such map sends $(\boldsymbol{u}_0^{(k)}, \boldsymbol{u}_1^{(k)})$ to $(\boldsymbol{u}_0^{(\sigma(k))}, \boldsymbol{u}_1^{(\sigma(k))})$. Now $f_\sigma(\boldsymbol{U}_{\boldsymbol{i}}) = \boldsymbol{U}_{\sigma(\boldsymbol{i})}$ follows immediately. ◻

The *parity* of a point $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{i}} \in \mathcal{B}_m$ can be defined as the parity of the multi-index $\boldsymbol{i}$ (*i. e.*, it is even or odd according to the number of 1s among the entries of $\boldsymbol{i}$). We write $\mathcal{B}_m^+$ and $\mathcal{B}_m^-$ for the set of base points with even and odd parity, respectively. Even though we can distinguish points of even and odd parity due to our fixed bases $(\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)})$, a change of bases in the vector spaces $\boldsymbol{V}_k$ may alter the parity of a point. But *having the same parity* is an equivalence relation on $\mathcal{B}_m$ with two equivalence classes, namely $\mathcal{B}_m^+$ and $\mathcal{B}_m^-$, each of cardinality 2^{m-1}.

We define the Hamming distance between $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{i}}$ and $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{j}}$ as the Hamming distance of their multi-indices $\boldsymbol{i}$ and $\boldsymbol{j}$. In particular, we speak of *opposite* points if $\boldsymbol{i}$ and $\boldsymbol{j}$ are opposite. For each point of $\mathcal{B}_m$ there is a unique opposite point. By (10) and (11) opposite points of $\mathcal{B}_m$ are complex-conjugate with respect to the Baer subspace $\mathbb{P}(\bigotimes_{k=1}^m \boldsymbol{V}_k)$ of $\mathbb{P}(\bigotimes_{k=1}^m \boldsymbol{W}_k)$. The opposite point to $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{i}}$ can also be characterised as the only point $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{j}} \in \mathcal{B}_m$ such that $[\boldsymbol{U}_{\boldsymbol{i}}, \boldsymbol{U}_{\boldsymbol{j}}] \neq 0$. We remark that the Hamming distance on $\mathcal{B}_m$ admits another description due to $\mathcal{B}_m \subset \mathcal{S}_{(m)}(4)$. The Hamming distance of $p, q \in \mathcal{B}_m$ equals the number of lines on a shortest polygonal path in $\mathcal{S}_{(m)}(4)$ from p to q.

From the proof of Proposition 2 and the above remarks we immediately obtain:

Theorem 3. *The stabiliser group $G_{\mathcal{S}_{(m)}(2)}$ of the Segre $\mathcal{S}_{(m)}(2)$ acts on the invariant basis $\mathcal{B}_m$ (via the multi-indices of its points) as the group of all affine transformations of $\mathbb{F}_2^m$ having the form $\boldsymbol{i} \mapsto \sigma(\boldsymbol{i}) + \boldsymbol{s}$, where $\sigma \in S_m$ and $\boldsymbol{s} \in \mathbb{F}_2^m$.*

Hence, Hamming distances on $\mathcal{B}_m$ are preserved under $G_{\mathcal{S}_{(m)}(2)}$, and the partition $\mathcal{B}_m = \mathcal{B}_m^+ \,\dot\cup\, \mathcal{B}_m^-$ is a $G_{\mathcal{S}_{(m)}(2)}$-invariant notion.

We now use the invariant basis for describing some other $G_{\mathcal{S}_{(m)}(2)}$-invariant subsets. In the following theorem we also make use of a particular property of Segre varieties over $\mathbb{F}_2$. Recall that (for an arbitrary ground field F) there are precisely m generators through any point p of the Segre $\mathcal{S}_{(m)}(F)$. They span the (m-dimensional) *tangent space* of $\mathcal{S}_{(m)}(F)$ at p. The *tangent* lines at p are the lines through p which lie in its tangent space. For $F = \mathbb{F}_2$ there are $2^m - 1$ tangents at p. Precisely one of them does not lie in any of the $(m-1)$-dimensional subspaces which are spanned by $m-1$ generators through p. This line will be called the *distinguished tangent* at p.

Theorem 4. *The stabiliser group $G_{\mathcal{S}_{(m)}(2)}$ of the Segre $\mathcal{S}_{(m)}(2)$ has the following properties:*

1. *The union of the skew subspaces* $\operatorname{span}\mathcal{B}_m^+$ *and* $\operatorname{span}\mathcal{B}_m^-$ *is a $G_{\mathcal{S}_{(m)}(2)}$-invariant subset of $\mathbb{P}(\bigotimes_{k=1}^m \boldsymbol{W}_k)$.*

2. *The union of the* 2^{m-1} *mutually skew real lines*[2]
$$\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{i}} \vee \mathbb{F}_4\boldsymbol{U}_{\boldsymbol{i}'} \quad \text{with} \quad \boldsymbol{i} \in I_{m,0} \tag{13}$$
is a $G_{\mathcal{S}_{(m)}(2)}$-invariant subset. The $3\cdot 2^{m-1}$ *real points on these lines comprise an orbit of $G_{\mathcal{S}_{(m)}(2)}$.*

3. *If m is even then* $\operatorname{span}\mathcal{B}_m^+$ *and* $\operatorname{span}\mathcal{B}_m^-$ *are real subspaces. Each of the lines from* (13) *is contained in precisely one of them.*

4. *If m is odd then* $\operatorname{span}\mathcal{B}_m^+$ *and* $\operatorname{span}\mathcal{B}_m^-$ *are complex-conjugate subspaces. All lines from* (13) *meet* $\operatorname{span}\mathcal{B}_m^+$ *and* $\operatorname{span}\mathcal{B}_m^-$ *at precisely one point, respectively.*

5. *All distinguished tangents of the Segre* $\mathcal{S}_{(m)}(2)$ *meet* $\operatorname{span}\mathcal{B}_m^+$ *and* $\operatorname{span}\mathcal{B}_m^-$ *at precisely one point, respectively.*

Proof. Ad 1 and 2: The assertions on the invariance of $\operatorname{span}\mathcal{B}_m^+ \,\dot\cup\, \operatorname{span}\mathcal{B}_m^-$ and on the invariance of the union of all lines from (13) are a direct consequence of Theorem 3.

We denote the set of all real points on the lines from (13) by $\mathcal{R}$. Let $\boldsymbol{j} \in I_{m,0}$ and let p be an arbitrary real point on the line $\mathbb{F}_4\boldsymbol{U}_{\boldsymbol{j}} \vee \mathbb{F}_4\boldsymbol{U}_{\boldsymbol{j}'}$. Any collineation from

[2] Any line joining complex-conjugate points is real (cf. the beginning of Section 4). It carries three real points. We use the symbol $\vee$ for the join of points.

$G_{\mathcal{S}_{(m)}(2)}$ takes p to some real point on a line from (13), whence the orbit of p is contained in $\mathcal{R}$.

Conversely, let $q \in \mathcal{R}$. So there is a $\boldsymbol{k} \in I_{m,0}$ with $q \in \mathbb{F}_4\boldsymbol{U_k} \vee \mathbb{F}_4\boldsymbol{U_{k'}}$. By Theorem 3, there exists a collineation in $G_{\mathcal{S}_{(m)}(2)}$ which maps $\mathbb{F}_4\boldsymbol{U_j}$ to $\mathbb{F}_4\boldsymbol{U_k}$ and, consequently, also $\mathbb{F}_4\boldsymbol{U_{j'}}$ to $\mathbb{F}_4\boldsymbol{U_{k'}}$. Furthermore, p is mapped to some real point $\tilde{p}$ on the line $\mathbb{F}_4\boldsymbol{U_k} \vee \mathbb{F}_4\boldsymbol{U_{k'}}$. There exists $f_1 \in \mathrm{GL}(\boldsymbol{V}_1)$ with $\mathrm{sgn}_2\, f_1 = 0$. Then $\boldsymbol{u}_0^{(1)}$ and $\boldsymbol{u}_1^{(1)}$ are eigenvectors of f_1 with eigenvalues λ and λ^2, where $\lambda \in \{\omega, \omega^2\}$. (See the proof of Proposition 2.) The linear bijection $f := f_1 \otimes \mathrm{id}_{\boldsymbol{V}_2} \otimes \cdots \otimes \mathrm{id}_{\boldsymbol{V}_m}$ has $\boldsymbol{U_k}$ and $\boldsymbol{U_{k'}}$ as eigenvectors with eigenvalues λ and λ^2, respectively, due to $k_1 = 0$. Thus the collineation arising from f induces a non-identical even permutation on the three real points of the line $\mathbb{F}_4\boldsymbol{U_k} \vee \mathbb{F}_4\boldsymbol{U_{k'}}$. Such a permutation has only one cycle. So one of f, f^2 or f^3 yields a collineation from $G_{\mathcal{S}_{(m)}(2)}$ which maps $\tilde{p}$ to q.

Ad 3 and 4: Opposite points of $\mathcal{B}_m$ are complex-conjugate and vice versa. Such points share the same parity for m even, but have different parity for m odd.

Ad 5: First, we exhibit the distinguished tangent T of the Segre $\mathcal{S}_{(m)}(2)$ at the point $\mathbb{F}_2\boldsymbol{E}_{1,1,\ldots,1}$. On each of the m generators of the Segre through this point we select one more real point, namely $\mathbb{F}_2\boldsymbol{E}_{0,1,1,\ldots,1}$, $\mathbb{F}_2\boldsymbol{E}_{1,0,1,\ldots,1}$, $\ldots$, $\mathbb{F}_2\boldsymbol{E}_{1,1,\ldots,1,0}$ for facilitating our subsequent reasoning. So, the distinguished tangent T contains the real point

$$\mathbb{F}_2(\boldsymbol{E}_{0,1,1,\ldots,1} + \boldsymbol{E}_{1,0,1,\ldots,1} + \cdots + \boldsymbol{E}_{1,1,\ldots,1,0}). \tag{14}$$

By (10), we have $\boldsymbol{e}_0^{(k)} = \omega^2\boldsymbol{u}_0^{(k)} + \omega\boldsymbol{u}_1^{(k)}$ and $\boldsymbol{e}_1^{(k)} = \boldsymbol{u}_0^{(k)} + \boldsymbol{u}_1^{(k)}$ for all $k \in \{1, 2, \ldots, m\}$. So $\boldsymbol{E}_{1,1,\ldots,1} = \sum_{\boldsymbol{i} \in I_m} \boldsymbol{U_i}$ and

$$\boldsymbol{E}_{0,1,\ldots,1} = \sum_{\boldsymbol{i} \in I_m} x_{\boldsymbol{i}}^{(1)}\boldsymbol{U_i} \quad\text{with}\quad x_{\boldsymbol{i}}^{(1)} = \begin{cases} \omega^2 & \text{for} \quad i_1 = 0, \\ \omega & \text{for} \quad i_1 = 1. \end{cases}$$

Mutatis mutandis, we obtain linear combinations for $\boldsymbol{E}_{1,0,1,\ldots,1}, \ldots, \boldsymbol{E}_{1,1,\ldots,1,0}$ with coefficients $x_{\boldsymbol{i}}^{(2)}, \ldots, x_{\boldsymbol{i}}^{(m)} \in \{\omega^2, \omega\}$. Summing up gives

$$\boldsymbol{E}_{0,1,1,\ldots,1} + \boldsymbol{E}_{1,0,1,\ldots,1} + \cdots + \boldsymbol{E}_{1,1,\ldots,1,0} = \sum_{\boldsymbol{i} \in I_m} y_{\boldsymbol{i}}\boldsymbol{U_i}, \tag{15}$$

where

$$y_{\boldsymbol{i}} = \underbrace{\omega^2 + \omega^2 + \cdots + \omega^2}_{\#\text{ of zeros in } \boldsymbol{i}} + \underbrace{\omega + \omega + \cdots + \omega}_{\#\text{ of ones in } \boldsymbol{i}}.$$

There are two cases:

m even: Due to $\mathrm{Char}\,\mathbb{F}_4 = 2$ we have $y_{\boldsymbol{i}} = 0$ for all $\boldsymbol{i}$ with even parity and $y_{\boldsymbol{i}} = \omega^2 + \omega = 1$ for all $\boldsymbol{i}$ with odd parity. So T meets $\mathrm{span}\,\mathcal{B}_m^-$ at the point (14). The sum of the tensor from (15) and $\boldsymbol{E}_{1,1,\ldots,1}$ determines the point of intersection of T with $\mathrm{span}\,\mathcal{B}_m^+$.

m odd: Due to $\operatorname{Char}\mathbb{F}_4 = 2$ we have $y_{\boldsymbol{i}} = \omega^2$ for all $\boldsymbol{i}$ with even parity and $y_{\boldsymbol{i}} = \omega$ for all $\boldsymbol{i}$ with odd parity. So T meets $\operatorname{span}\mathcal{B}_m^+$ at the point

$$\mathbb{F}_4\Big(\sum_{\boldsymbol{j}} \omega^2 \boldsymbol{U}_{\boldsymbol{j}}\Big) = \mathbb{F}_4\Big(\sum_{\boldsymbol{j}} \boldsymbol{U}_{\boldsymbol{j}}\Big), \tag{16}$$

where $\boldsymbol{j}$ ranges over all elements of I_m with even parity, and the subspace $\operatorname{span}\mathcal{B}_m^-$ at the point $\mathbb{F}_4(\sum_{\boldsymbol{k}} \omega \boldsymbol{U}_{\boldsymbol{k}}) = \mathbb{F}_4(\sum_{\boldsymbol{k}} \boldsymbol{U}_{\boldsymbol{k}})$, where $\boldsymbol{k}$ ranges over all elements of I_m with odd parity.

In either case the two points of intersection are unique, because $\operatorname{span}\mathcal{B}_m^+$ and $\operatorname{span}\mathcal{B}_m^-$ are skew.

Next, we consider an arbitrary distinguished tangent of the Segre. As all points of the Segre comprise a point orbit of $G_{\mathcal{S}_{(m)}(2)}$, also all distinguished tangents are in one line orbit of $G_{\mathcal{S}_{(m)}(2)}$. So, all distinguished tangents share the properties of the tangent T. $\square$

Any pair of skew and complex-conjugate subspaces of $\mathbb{P}(\bigotimes_{k=1}^m \boldsymbol{W}_k)$ determines a *geometric line spread* of $\mathbb{P}(\bigotimes_{k=1}^m \boldsymbol{V}_k)$. This spread comprises all real lines which meet one of the subspaces (and hence both of them in complex-conjugate points). Any of these subspaces is called an *indicator set* of the spread. See [2, p. 74] and [23, p. 29]. So, part of Theorem 4 can be reformulated as follows:

Corollary 1. *If m is odd then the complex-conjugate subspaces* $\operatorname{span}\mathcal{B}_m^+$ *and* $\operatorname{span}\mathcal{B}_m^-$ *are indicator sets of a $G_{\mathcal{S}_{(m)}(2)}$-invariant geometric line spread $\mathcal{L}$ of* $\mathbb{P}(\bigotimes_{k=1}^m \boldsymbol{V}_k) = \mathrm{PG}(2^m - 1, 2)$*. All distinguished tangents of the Segre* $\mathcal{S}_{(m)}(2)$ *and all lines given by* (13) *belong to this spread.*

It is now a straightforward task to establish connections between the fundamental polarity of the Segre $\mathcal{S}_{(m)}(4)$ and the quadric $\mathcal{Q}(4)$ which arises according to Theorem 1. The reader will easily verify the following: The fundamental polarity maps each of the lines from (13) to the span of the remaining ones. For any even m the subspaces $\operatorname{span}\mathcal{B}_m^+$ and $\operatorname{span}\mathcal{B}_m^-$ are interchanged under the fundamental polarity, whereas for m odd each of them is invariant (totally isotropic). The subspaces $\operatorname{span}\mathcal{B}_m^+$ and $\operatorname{span}\mathcal{B}_m^-$ are contained in $\mathcal{Q}(4)$ precisely when $m \geq 2$ is odd.

Example 1. Let $m = 2$. The Segre $\mathcal{S}_{1,1}(2)$ is a hyperbolic quadric of $\mathbb{P}(\boldsymbol{V}_1 \otimes \boldsymbol{V}_2) = \mathrm{PG}(3,2)$. The 15 points of this projective space fall into two orbits under the action of $G_{1,1}(2)$: The first orbit is $\mathcal{S}_{1,1}(2)$ (nine points), the remaining six points form the second orbit. It comprises the real points of the lines $\operatorname{span}\mathcal{B}_m^+$ and $\operatorname{span}\mathcal{B}_m^-$. There are nine distinguished tangents of the quadric. Together they form the hyperbolic linear congruence of lines which arises by joining every real point of $\operatorname{span}\mathcal{B}_m^+$ with every real point of $\operatorname{span}\mathcal{B}_m^-$.

The previous example is clearly just an easy exercise, and it could be mastered without our results about the general case.

5 The invariant Hermitian variety

The symplectic form $[\cdot,\cdot]$ on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ can be extended in exactly two ways to a non-degenerate sesquilinear form[3] on $\bigotimes_{k=1}^{m} \boldsymbol{W}_k$. The bilinear extension is symplectic. In accordance with the notation used elsewhere, it is also denoted by $[\cdot,\cdot]$. The only other extension is sesquilinear with respect to the Frobenius automorphism $z \mapsto z^2$ of $\mathbb{F}_4$. Such Hermitian extension will be written as $[\cdot,\cdot]_H$. We have

$$[\boldsymbol{X},\boldsymbol{Y}]_H = [\overline{\boldsymbol{X}},\boldsymbol{Y}] \tag{17}$$

for all tensors $\boldsymbol{X},\boldsymbol{Y} \in \bigotimes_{k=1}^{m} \boldsymbol{W}_k$, where $\overline{\boldsymbol{X}}$ denote the complex-conjugate tensor of $\boldsymbol{X}$. While $[\cdot,\cdot]$ describes the fundamental polarity of the Segre $\mathcal{S}_{(m)}(4)$, the Hermitian sesquilinear form $[\cdot,\cdot]_H$ yields a unitary polarity of $\mathbb{P}(\bigotimes_{k=1}^{m} \boldsymbol{W}_k)$ and, moreover, the Hermitian variety $\mathcal{H}$ comprising all its absolute points. By its definition, $\mathcal{H}$ is a $G_{\mathcal{S}_{(m)}(2)}$-invariant notion, whence we call it the *invariant Hermitian variety* of the Segre $\mathcal{S}_{(m)}(2)$. Note that $\mathcal{H}$, like the invariant basis and the invariant line spread, is an invariant notion only for $\mathcal{S}_{(m)}(2)$, but not for $\mathcal{S}_{(m)}(4)$.

We remark that the invariant basis $\mathcal{B}_m$ is self-polar with respect to the unitary polarity given by $[\cdot,\cdot]_H$. Indeed, given $\boldsymbol{i},\boldsymbol{j} \in \mathbb{F}_2^m$ we have

$$[\boldsymbol{U_i},\boldsymbol{U_i}]_H = \prod_{k=1}^{m}[\boldsymbol{u}_{i_k}^{(k)},\overline{\boldsymbol{u}}_{i_k}^{(k)}] = (\omega+\omega^2)\prod_{k=1}^{m}[\boldsymbol{e}_0^{(k)},\overline{\boldsymbol{e}}_1^{(k)}] = 1, \tag{18}$$

$$[\boldsymbol{U_i},\boldsymbol{U_j}]_H = \prod_{k=1}^{m}[\boldsymbol{u}_{i_k}^{(k)},\overline{\boldsymbol{u}}_{j_k}^{(k)}] = 0 \quad \text{for all} \quad \boldsymbol{i} \neq \boldsymbol{j}, \tag{19}$$

since, for example, $i_1 \neq j_1$ implies $\boldsymbol{u}_{i_1}^{(1)} = \overline{\boldsymbol{u}}_{j_1}^{(1)}$, whence $[\boldsymbol{u}_{i_1}^{(1)},\overline{\boldsymbol{u}}_{j_1}^{(1)}] = 0$.

The following Proposition establishes a link among the invariant line spread from Corollary 1, the invariant quadric $\mathcal{Q}(2)$, and the invariant Hermitian variety $\mathcal{H}$.

Proposition 3. *Let $m \geq 2$ be odd. A line L of the invariant geometric line spread $\mathcal{L}$ is a generator of the invariant quadric $\mathcal{Q}(4)$ if, and only if, the intersection point $L \cap \operatorname{span}\mathcal{B}_m^+$ belongs to the invariant Hermitian variety $\mathcal{H}$. Otherwise that line L is a bisecant of $\mathcal{Q}(4)$, whence it has no points in common with $\mathcal{Q}(2)$.*

[3]We assume such forms to be linear in the right and semilinear in the left argument.

Proof. Suppose that $L \cap \operatorname{span} \mathcal{B}_m^+ = \mathbb{F}_4\boldsymbol{X}$. By a remark at the end of Section 4, we have $\mathbb{F}_4\boldsymbol{X} \in \operatorname{span} \mathcal{B}_m^+ \subset \mathcal{Q}(4)$ and $\mathbb{F}_4\overline{\boldsymbol{X}} \in \operatorname{span} \mathcal{B}_m^- \subset \mathcal{Q}(4)$. So the line $L = \mathbb{F}_4\boldsymbol{X} \vee \mathbb{F}_4\overline{\boldsymbol{X}}$ is either a generator or a bisecant of $\mathcal{Q}(4)$. The first possibility occurs precisely when $\mathbb{F}_4\overline{\boldsymbol{X}}$ lies in the tangent hyperplane of $\mathcal{Q}(4)$ at $\mathbb{F}_4\boldsymbol{X}$. Employing (17), this in turn is equivalent to $0 = [\overline{\boldsymbol{X}}, \boldsymbol{X}] = [\boldsymbol{X}, \boldsymbol{X}]_H$ which characterises $\mathbb{F}_4\boldsymbol{X}$ as a point of $\mathcal{H}$. □

The above Proposition is a special case of [11, Theorem 1] on quadrics which admit a spread of lines. In our context the invariant spread from Corollary 1 yields a spread of lines on $\mathcal{Q}(2)$, since over $\mathbb{F}_2$ each line of the invariant spread is either external to or contained in that quadric.

6 The Segre variety $\mathcal{S}_{1,1,1}(2)$

In this section we exhibit the ambient space $\mathbb{P}(\boldsymbol{V}_1 \otimes \boldsymbol{V}_2 \otimes \boldsymbol{V}_3) = \mathrm{PG}(7, 2)$ of the Segre $\mathcal{S}_{1,1,1}(2)$. This space has $2^8 - 1 = 255$ points. Furthermore, we have the cardinalities $\#\mathcal{S}_{1,1,1}(2) = 3^3 = 27$, $\#\mathcal{Q}(2) = (2^3 + 1)(2^4 - 1) = 135$ (see [16, Theorem 5.21]), and $\#\mathcal{L} = 255/3 = 85$.

Proposition 4. *Under the action of the stabiliser group $G_{\mathcal{S}_{1,1,1}(2)}$ of the Segre $\mathcal{S}_{1,1,1}(2)$ the lines of the invariant spread $\mathcal{L}$ of $\mathbb{P}(\boldsymbol{V}_1 \otimes \boldsymbol{V}_2 \otimes \boldsymbol{V}_3) = \mathrm{PG}(7, 2)$ fall into four orbits $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$. In terms of the invariant basis $\mathcal{B}_3$ the following characterisation holds: A line from $\mathcal{L}$ is in orbit $\mathcal{L}_r$ if, and only if, its (imaginary) point of intersection with the subspace* $\operatorname{span} \mathcal{B}_3^+$ *lies in $4 - r$ planes of the tetrahedron $\mathcal{B}_3^+$.*

Proof. (a) Throughout this proof the pointwise stabiliser and the stabiliser of $\mathcal{B}_3^+$ in the group $G_{\mathcal{S}_{1,1,1}(2)}$ are abbreviated by G^+_{pw} and G^+, respectively. We observe that G^+_{pw} acts transitively on the points of the Segre $\mathcal{S}_{1,1,1}(2)$: We fix the point $\mathbb{F}_2\boldsymbol{E}_{111} = \mathbb{F}_2(\boldsymbol{e}_1^{(1)} \otimes \boldsymbol{e}_1^{(2)} \otimes \boldsymbol{e}_1^{(3)})$. Given any point of the Segre, say $\mathbb{F}_2\boldsymbol{A}$, where $\boldsymbol{A} = \boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \boldsymbol{a}_3$, there are linear bijections $f_k \in \mathrm{GL}(\boldsymbol{V}_k)$ satisfying $\operatorname{sgn}_2 f_k = 0$ and $\boldsymbol{e}_1^{(k)} \mapsto \boldsymbol{a}_k$ for $k = \{1, 2, 3\}$. So $f_1 \otimes f_2 \otimes f_3$ induces a collineation which sends $\mathbb{F}_2\boldsymbol{E}_{111}$ to $\mathbb{F}_2\boldsymbol{A}$ and belongs to G^+_{pw} by (12).

We write $\mathcal{M}_r$, $r \in \{1, 2, 3, 4\}$, for the subset of $\operatorname{span} \mathcal{B}_3^+ = \mathrm{PG}(3, 4)$ comprising all points which lie in precisely $4 - r$ planes of the tetrahedron $\mathcal{B}_3^+$. So we have $\#\mathcal{M}_1 = 4$ vertices, $\#\mathcal{M}_2 = 3 \cdot 6 = 18$ edge points, $\#\mathcal{M}_3 = 4 \cdot 9 = 36$ face points, and $\#\mathcal{M}_4 = 27$ general points. Clearly, the G^+-orbit of any point from $\operatorname{span} \mathcal{B}_3^+$ is contained in one of the sets $\mathcal{M}_r$.

(b) We show that $\mathcal{M}_4$ is an orbit under G^+_{pw}: By (16), the distinguished tangent of the Segre at $\mathbb{F}_2\boldsymbol{E}_{111}$ meets $\operatorname{span} \mathcal{B}_3^+$ at the point $p := \mathbb{F}_4(\boldsymbol{U}_{000} + \boldsymbol{U}_{011} + \boldsymbol{U}_{101} + \boldsymbol{U}_{110}) \in \mathcal{M}_4$. We infer from the transitive action of G^+_{pw} on the Segre $\mathcal{S}_{1,1,1}(2)$ that all distinguished tangents meet $\operatorname{span}\mathcal{B}_3^+$ in points of $\mathcal{M}_4$. Since $\#\mathcal{S}_{1,1,1}(2) = 27 = \#\mathcal{M}_4$, the group G^+_{pw} acts transitively on $\mathcal{M}_4$.

(c) Any edge of $\mathcal{B}_3^+$ contains precisely three points of $\mathcal{M}_2$. We obtain all of them by projecting $\mathcal{M}_4$ from the opposite edge, whence G^+_{pw} acts transitively on the set of these three points. Likewise, G^+_{pw} acts transitively on the nine points of $\mathcal{M}_3$ in any face of $\mathcal{B}_3^+$.

(d) We know from Proposition 2 that G^+ acts transitively on the set of vertices of $\mathcal{B}_3^+$ via translations $\boldsymbol{i} \mapsto \boldsymbol{i}+\boldsymbol{s}$ on multi-indices. From Theorem 3 the G^+-stabiliser of $\mathbb{F}_4\boldsymbol{U}_{000}$ acts transitively on the remaining vertices of $\mathcal{B}_3^+$ via permutations $\boldsymbol{i} \mapsto \sigma(\boldsymbol{i})$ on multi-indices. Together with our previous results this means that each of the four subsets $\mathcal{M}_r$ is a G^+-orbit. Consequently, each of the sets $\mathcal{L}_r$ is contained in an orbit under the action of $G_{\mathcal{S}_{1,1,1}(2)}$ on the line spread $\mathcal{L}$.

Any collineation from $G_{\mathcal{S}_{1,1,1}(2)} \setminus G^+$ also preserves each of the sets $\mathcal{L}_r$, as it commutes with the Baer involution of $\mathrm{PG}(7,4)$ fixing $\mathbb{P}(\boldsymbol{V}_1 \otimes \boldsymbol{V}_2 \otimes \boldsymbol{V}_3) = \mathrm{PG}(7,2)$ pointwise. This completes the proof. $\square$

From (18) and (19) the equation of the Hermitian variety $\mathcal{H} \cap \operatorname{span}\mathcal{B}_3^+$ with respect to the basis $(\boldsymbol{U}_{000}, \boldsymbol{U}_{011}, \boldsymbol{U}_{101}, \boldsymbol{U}_{110})$ reads

$$x_{000}^3 + x_{011}^3 + x_{101}^3 + x_{110}^3 = 0.$$

Because of $z^3 = 1$ for all $z \in \mathbb{F}_4 \setminus \{0\}$, we get $\mathcal{H} \cap \operatorname{span}\mathcal{B}_3^+ = \mathcal{M}_2 \cup \mathcal{M}_4$. By Proposition 3, the lines from $\mathcal{L}_2 \cup \mathcal{L}_4$ are on the invariant quadric $Q(4)$. More precisely, the lines from $\mathcal{L}_2$ are those generators of $Q(4)$ which do not contain any point of the Segre $\mathcal{S}_{1,1,1}(2)$, whereas the lines from $\mathcal{L}_4$ are the distinguished tangents of $\mathcal{S}_{1,1,1}(2)$. Figure 1, left[4], displays the polar space (point-line incidence

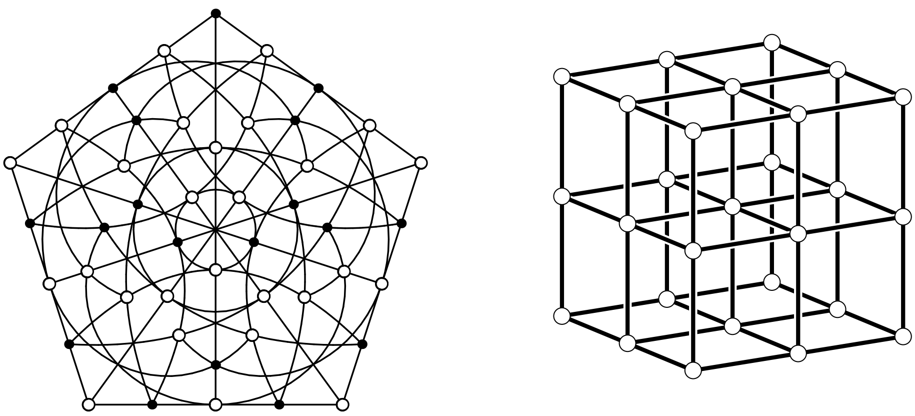

Figure 1: The Hermitian variety $\mathcal{M}_2 \cup \mathcal{M}_4$ (left) and the Segre $\mathcal{S}_{1,1,1}(2)$ (right).

[4]The style of our figure is taken from [21, p. 61].

structure) on $\mathcal{H}$. This Hermitian variety consists of 45 points and carries 27 lines (represented by segments and curves), with five points on any line and three lines on any point. The 27 points represented by small circles are those from $\mathcal{M}_4$, the remaining 18 points are represented by bullets and belong to $\mathcal{M}_2$. The 27 points of $\mathcal{M}_4$ can be viewed as a *skew projection* of the Segre $\mathcal{S}_{1,1,1}(2)$ (Figure 1, right) into $\mathcal{H}$ along the invariant line spread. Under this projection collinearity of points is being preserved.

The lines from $\mathcal{L}_1$ are the four lines from (13). Like the remaining 36 lines from $\mathcal{L}_3$ they are exterior lines (over $\mathbb{F}_2$) of the invariant quadric $Q(2)$.

Proposition 5. *Under the action of the stabiliser group $G_{\mathcal{S}_{1,1,1}(2)}$ of the Segre $\mathcal{S}_{1,1,1}(2)$ the points of $\mathbb{P}(\boldsymbol{V}_1 \otimes \boldsymbol{V}_2 \otimes \boldsymbol{V}_3) = \mathrm{PG}(7,2)$ fall into five orbits $O_1, O_2, \dots, O_5$. For $r \in \{1,2,3\}$ the points of O_r are precisely the real points on the lines of $\mathcal{L}_r$. The orbit O_4 comprises those real points on the lines from $\mathcal{L}_4$ which are off the Segre $\mathcal{S}_{1,1,1}(2)$, whereas O_5 equals the Segre $\mathcal{S}_{1,1,1}(2)$.*

Proof. It is clear that O_5 is an orbit under $G_{\mathcal{S}_{1,1,1}(2)}$. The points of O_1 form an orbit according to Theorem 4. In order to show that O_2 and O_3 are orbits, we shall select one line of $\mathcal{L}_2$ and $\mathcal{L}_3$, respectively. By Proposition 4, it suffices then to show that all real points of this line are in one orbit. This task will be accomplished with mappings $f_k \in \mathrm{GL}(V_k)$, $k \in \{1,2,3\}$, given by $\boldsymbol{e}_0^{(k)} \mapsto \boldsymbol{e}_1^{(k)}$, $\boldsymbol{e}_1^{(k)} \mapsto \boldsymbol{e}_0^{(k)} + \boldsymbol{e}_1^{(k)}$. From (10), we have $f_k(\boldsymbol{u}_0^{(k)}) = \omega \boldsymbol{u}_0^{(k)}$ and $f_k(\boldsymbol{u}_1^{(k)}) = \omega^2 \boldsymbol{u}_1^{(k)}$.

Let $L_2 \in \mathcal{L}_2$ be the line joining $\mathbb{F}_4(\boldsymbol{U}_{000} + \boldsymbol{U}_{011}) \in \mathcal{M}_2$ with its complex-conjugate point $\mathbb{F}_4(\boldsymbol{U}_{111} + \boldsymbol{U}_{100})$. The mapping $f_1 \otimes \mathrm{id}_{\boldsymbol{V}_2} \otimes \mathrm{id}_{\boldsymbol{V}_3}$ has $\boldsymbol{U}_{000} + \boldsymbol{U}_{011}$ as eigentensor with eigenvalue ω. Its complex-conjugate tensor is therefore an eigentensor with eigenvalue ω^2. From the proof of Proposition 2, this implies that $f_1 \otimes \mathrm{id}_{\boldsymbol{V}_2} \otimes \mathrm{id}_{\boldsymbol{V}_3}$ induces a non-trivial even permutation on the set of real points of L_2. So, under the powers of this permutation the three real points of L_2 are permuted in one cycle.

Let $L_3 \in \mathcal{L}_3$ be the line joining $\mathbb{F}_4(\boldsymbol{U}_{011} + \boldsymbol{U}_{101} + \boldsymbol{U}_{110}) \in \mathcal{M}_3$ with its complex-conjugate point. Here $f_1 \otimes f_2 \otimes f_3$ possesses $\boldsymbol{U}_{011} + \boldsymbol{U}_{101} + \boldsymbol{U}_{110}$ as eigentensor with eigenvalue $\omega^5 = \omega^2$. Its complex-conjugate tensor is therefore an eigentensor with eigenvalue ω. Now the assertion follows as above.

The distinguished tangent of the Segre $\mathcal{S}_{1,1,1}(2)$ at the point $\mathbb{F}_2 \boldsymbol{E}_{111}$ contains two precisely two points of O_4. From (14), these points are $\mathbb{F}_2(\boldsymbol{E}_{011} + \boldsymbol{E}_{101} + \boldsymbol{E}_{110})$ and $\mathbb{F}_2(\boldsymbol{E}_{111} + \boldsymbol{E}_{011} + \boldsymbol{E}_{101} + \boldsymbol{E}_{110})$. Let $g_1 \in \mathrm{GL}(\boldsymbol{V}_1)$ be defined by $\boldsymbol{e}_0^{(1)} \mapsto \boldsymbol{e}_0^{(1)} + \boldsymbol{e}_1^{(1)}$, $\boldsymbol{e}_1^{(1)} \mapsto \boldsymbol{e}_1^{(1)}$. Then $g_1 \otimes \mathrm{id}_{\boldsymbol{V}_2} \otimes \mathrm{id}_{\boldsymbol{V}_3}$ will interchange these two points, whence we may argue as before. □

Let us close this section with a few remarks: The orbits of the stabiliser group $G_{\mathcal{S}_{1,1,1}(2)}$ are described (without proof) in a completely different way in [14, p. 82]. The a, b, c, d, e-orbits from there are in our terminology the sets O_5 (27 points), O_2

(54 points), $\mathcal{O}_3$ (108 points), $\mathcal{O}_4$ (54 points), and $\mathcal{O}_1$ (12 points), respectively. The union $\mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5$ is the invariant quadric $\mathcal{Q}(2)$. With respect to the tensor basis (1) the equation of $\mathcal{Q}(2)$ reads

$$x_{000}x_{111} + x_{001}x_{110} + x_{010}x_{101} + x_{011}x_{100} = 0. \tag{20}$$

The square of the left hand side of (20) is *Cayley's hyperdeterminant* of the $3\times3\times3$ array $(x_{\boldsymbol{i}})_{\boldsymbol{i}\in I_3}$; see [14, Theorem 5.45] and compare with [12] and [13].

By virtue of the fundamental polarity of $\mathcal{S}_{1,1,1}(2)$, Proposition 5 provides a classification of the hyperplanes of $\mathbb{P}(\boldsymbol{V}_1 \otimes \boldsymbol{V}_2 \otimes \boldsymbol{V}_3)$ under the action of the group $G_{\mathcal{S}_{1,1,1}(2)}$. Moreover, it gives a classification of the *geometric hyperplanes* (or *primes*) of $\mathcal{S}_{1,1,1}(2)$, since any geometric hyperplane of this Segre arises as intersection with a unique hyperplane of the ambient space [22]. This is a rather particular property of Segre varieties $\mathcal{S}_{(m)}(2)$ which is not shared by Segre varieties $\mathcal{S}_{(m)}(F)$ in general [4].

The Segre $\mathcal{S}_{1,1,1}(2)$ (as a point-line geometry) appears in the literature in various guises, namely as the $(27_3, 27_3)$ *Gray configuration* [20] or as the *smallest slim dense near hexagon* [6]. It is also a point model of the *chain geometry* based on the $\mathbb{F}_2$-algebra $\mathbb{F}_2\times\mathbb{F}_2\times\mathbb{F}_2$, the chains being the twisted cubics on $\mathcal{S}_{1,1,1}(2)$ (*i. e.* triads of points with mutual Hamming distance 3); see [3, (5.4)] or [5, p. 272]. We add in passing that the tangent lines of these twisted cubics are just our distinguished tangents of $\mathcal{S}_{1,1,1}(2)$.

7 Conclusion

We established several invariant notions for Segre varieties $\mathcal{S}_{(m)}(2)$ over the field $\mathbb{F}_2$. For $m \leq 3$ these invariants provide sufficient information for the classification of the points and hyperplanes of the ambient space of $\mathcal{S}_{(m)}(2)$. For larger values of m the situation seems to be much more intricate. For example, when m is odd then the lines of the invariant spread will fall into at least 2^{m-1} orbits, as follows from a straightforward generalisation of Proposition 4. However, this gives only a lower bound for the number of orbits. Indeed, for $m \geq 3$ there are 3^m distinguished tangents of $\mathcal{S}_{(m)}(2)$, but $3^{2^{m-1}-1}$ points of span $\mathcal{B}_m^+$ which belong to no face of the simplex $\mathcal{B}_m^+$. These two cardinalities coincide only when $m = 3$, whence for all odd $m > 3$ we no longer have a one-one correspondence between the set of distinguished tangents and the set of all points of span $\mathcal{B}_m^+$ which belong to no face of $\mathcal{B}_m^+$.

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