On Sets of Lines Corresponding to Affine Spaces

Hans Havlicek

1. Given a 3-dimensional Pappian projective space, it is well known that the Grassmannian which is representing its set of lines is a quadric (called the Plücker quadric) in a 5-dimensional projective space. The link between the set of lines and the Plücker quadric is the bijective Klein map γ. Under γ pencils of lines become lines of the Plücker quadric. Cf. e.g. [5,287], [15,28], [16,13], [20,176], [21,327].

Now, taking any 3-dimensional projective space, we may ask for an injective map γ from the set of lines into the set of points of (another) projective space, such that every pencil of lines is mapped onto a line. If we have a non-Pappian space, then such a map does not exist [9,172]. For a Pappian space any such map is the product of the Klein map with a suitable collineation [9,174], [22,377].

If ψ denotes a stereographic projection of the Plücker quadric through a point $A = a^{\gamma}$ (for some line a), then the restriction of $\gamma\psi$ to the set of lines which are skew to a is a bijection λ, say, onto a 4-dimensional affine space. The line a has no image under $\gamma\psi$ and all lines which meet a in one point are mapped (in a non-injective way) onto a ruled quadric in the hyperplane at infinity of this affine 4-space. Cf. [10,109], [14] for these results and their generalization to higher dimensions as well as [18].

Any subspace of this affine 4-space may be interpreted as the stereographic image of those points of the Plücker quadric which belong to a subspace passing through A. The corresponding sets of lines are well known (e.g. linear complexes of lines).

However, as has been shown in [13] there is an alternative construction of this bijection λ which will work for Pappian as well as non-Pappian spaces. Thus this λ may serve as a substitute for the Klein map in the non-Pappian case. Throughout this article the inverse of such a bijection λ will be labeled β. The present paper is concerned with those sets of lines which correspond under β to the subspaces of such an affine 4-space. Clearly, only non-Pappian spaces are of interest.

2. If $\mathfrak{B}$ is a right vector space over a field[1] K, then the projective space on $\mathfrak{B}$ has the set of points $\mathcal{P}(\mathfrak{B}) := \{X \mid X=\mathfrak{x}K, \mathfrak{x}\in\mathfrak{B}\setminus\{\mathfrak{o}\}\}$, where $\mathfrak{o}$ denotes the zero vector. If $\mathfrak{U} \subset \mathfrak{B}$ is a subspace, then $\mathcal{P}(\mathfrak{U}) := \{\mathfrak{x}K\in\mathcal{P}(\mathfrak{B}) \mid \mathfrak{x}\in\mathfrak{U}\}$ and $\ell(\mathcal{P}(\mathfrak{U}))$ or $\ell(\mathfrak{U})$ stands for the set of all lines in $\mathcal{P}(\mathfrak{B})$ passing through $\mathcal{P}(\mathfrak{U})$ ($\dim\mathcal{P}(\mathfrak{U})\leq 1$) or contained in $\mathcal{P}(\mathfrak{U})$ ($\dim\mathcal{P}(\mathfrak{U})\geq 3$), respectively. Given two subspaces

[1]The terms *field* and *skew field* will be used for a not necessarily commutative field and a non-commutative field, respectively.

$\mathfrak{U}_1 \subset \mathfrak{U}_2$, then $\ell(\mathcal{P}(\mathfrak{U}_1),\mathcal{P}(\mathfrak{U}_2))$ or $\ell(\mathfrak{U}_1,\mathfrak{U}_2)$ denotes the set of lines through $\mathcal{P}(\mathfrak{U}_1)$ and contained in $\mathcal{P}(\mathfrak{U}_2)$. The centre of K will be written as Z.

In order to show the existence of a bijection β, as has been described in 1, we proceed as follows:

Let $\mathfrak{V}$ and $\mathfrak{W}$ be right vector spaces over the same field K with $\dim\mathcal{P}(\mathfrak{V}) = 4$, $\dim\mathcal{P}(\mathfrak{W}) = 3$. We choose two homomorphisms $p_j \in \mathrm{Hom}_K(\mathfrak{V},\mathfrak{W})$ ($j=0,\infty$), such that their kernels have only $\{\mathfrak{o}\}$ in common, whereas their images are two different 3-dimensional subspaces of $\mathfrak{W}$. Setting

$$p_j := p_0 + j.p_\infty \; (j \in Z) \tag{1}$$

yields a family of homomorphisms $\mathfrak{V} \to \mathfrak{W}$ with $\bar{Z} := Z\cup\{\infty\}$ as set of indices. We shall write $\mathfrak{A}_j := \ker p_j$, $\mathfrak{E}_j := \mathrm{im} p_j$ $(j\in\bar{Z})$, $\mathfrak{J} := \sum(\mathfrak{A}_j \mid j\in\bar{Z})$ and $\mathfrak{A} := \cap(\mathfrak{E}_j \mid j\in\bar{Z})$. In terms of the projective spaces $\mathcal{P}(\mathfrak{V})$ and $\mathcal{P}(\mathfrak{W})$ we get a family $\pi_j:\mathcal{P}(\mathfrak{V})\setminus\mathcal{P}(\mathfrak{A}_j) \to \mathcal{P}(\mathfrak{W})$ of linear maps [8,152] with $(\mathfrak{x}K)^{\pi_j} := (\mathfrak{x}^{p_j})K$.

Then $\mathcal{J} := \mathcal{P}(\mathfrak{J})$ is a hyperplane of $\mathcal{P}(\mathfrak{V})$ which will be regarded as *hyperplane at infinity*. Thus $\mathcal{P}(\mathfrak{V})\setminus\mathcal{J}$ is the set of points of an affine space. The lines $a_j := \mathcal{P}(\mathfrak{A}_j)$ are the directrices of a regulus $\mathcal{R}$ in $\mathcal{J}$ [17], [20,319] which will be called the *absolute regulus*. A line $r \in \ell(\mathfrak{J})$ is an element of $\mathcal{R}$ if, and only if, there exists a point $R \in \mathcal{P}(\mathfrak{W})$ which is the image of r under all mappings π_j. If $\mathcal{P}(\mathfrak{V})$ is Pappian, then $\mathcal{R}$ is one set of generators of a ruled quadric and we have an affine space with orthogonality given by the polar system of the quadric. Cf. e.g. [1,105], [4,122].

In $\mathcal{P}(\mathfrak{W})$ the planes $\mathcal{E}_j := \mathcal{P}(\mathfrak{E}_j)$ belong to a pencil of planes with axis $a := \mathcal{P}(\mathfrak{A})$. We regard this pencil as a line a^* in the dual space of $\mathcal{P}(\mathfrak{W})$. Let a_Z^* denote the family $\{\mathcal{E}_j \mid j\in\bar{Z}\}$. If $K \neq Z$, then a_Z^* is a Z-chain [2,320] (subline) of a^*, else $a_Z^* = a^*$. We shall write $\mathcal{C}_a$ for the set of lines in $\ell(\mathfrak{W})$ which are skew to a.

Summing up yields an injective map

$$\beta:\mathcal{P}(\mathfrak{V})\setminus\mathcal{P}(\mathfrak{J}) \to \mathcal{C}_a,\; X \mapsto X^{\pi_j}\vee X^{\pi_k},\; j,\, k \in \bar{Z},\; j \neq k. \tag{2}$$

By (1), the definition of β does not depend on j, k. Given any two points $Y_0 \in \mathcal{E}_0\setminus a$, $Y_1 \in \mathcal{E}_1\setminus a$, there is a unique point $X \in \mathcal{P}(\mathfrak{V})\setminus\mathcal{J}$ with $X^{\pi_0} = Y_0$, $X^{\pi_1} = Y_1$. Hence β is surjective.

The map β can be used to transfer the affine structure from $\mathcal{P}(\mathfrak{V})\setminus\mathcal{J}$ to $\mathcal{C}_a$ [13]. The stabilizer Φ of the absolute regulus $\mathcal{R}$ in $\mathrm{P\Gamma L}(\mathcal{P}(\mathfrak{V}))$ and the stabilizer Ψ of a_Z^* in $\mathrm{P\Gamma L}(\mathcal{P}(\mathfrak{W}))$ yield isomorphic transformation groups on $\mathcal{P}(\mathfrak{V})\setminus\mathcal{J}$ respectively $\mathcal{C}_a$. An isomorphism is given by $\varphi\ (\in\Phi) \mapsto \beta^{-1}\varphi\beta\ (\in\Psi)$.

3. Let $l \in \ell(\mathfrak{V})\setminus\ell(\mathfrak{J})$ be a line and L its point at infinity. There are three possibilities:

(i) *L is on a directrix* a_j *of* $\mathcal{r}$: Thus $l^{\pi j}$ is a single point and $l^{\pi k}$ $(j\neq k\in\bar{Z})$ is a line. Consequently, $l^{\beta} = (l\setminus\{L\})^{\beta}$ is the intersection of a pencil of lines with c_a. It is easily seen that every pencil of lines whose centre is in a plane of a_Z^* but off a and whose plane does not belong to a^* yields the β-image of a line.

(ii) *L is a point off the absolute regulus* $\mathcal{r}$: Then $l^{\pi o}$, $l^{\pi 1}$ are two skew lines and $(\pi_0|l)^{-1}(\pi_1|l):l^{\pi o} \to l^{\pi 1}$ is a projectivity which generates a regulus $\mathcal{s}$, say. Replacing 0, 1 by any two different elements of $\bar{Z}$ shows that in every plane of a_Z^* there is one directrix of $\mathcal{s}$ and, as follows from [20,321], there are no more directrices. Hence $l^{\beta} = \mathcal{s}\setminus\{a\}$. Conversely, any regulus $\mathcal{s}$ of $\mathcal{P}(\mathfrak{B})$ whose directrices are incident with the planes of a_Z^* contains a and $\mathcal{s}\setminus\{a\}$ is the β-image of a line.

(iii) *L is on a line of* $\mathcal{r}$ *but on no directrix:* There exist $\mathfrak{l}_0 \in \mathfrak{A}_0$, $\mathfrak{l}_1 \in \mathfrak{A}_1$ and $x \in K\setminus Z$ with $\mathfrak{l}_0+\mathfrak{l}_1 \in \mathfrak{A}_\infty$ and $L = \mathfrak{l}K$ where $\mathfrak{l} := \mathfrak{l}_0x+\mathfrak{l}_1$. Since $\mathfrak{A}_\infty = \ker(p_1-p_0)$,

$$\mathfrak{l}^{p_o} = (\mathfrak{l}_0x+\mathfrak{l}_1)^{p_o} = (\mathfrak{l}_0+\mathfrak{l}_1)^{p_o},$$

$$\mathfrak{l}^{p_1} = (\mathfrak{l}_0x+\mathfrak{l}_1)^{p_1} = (\mathfrak{l}_0+\mathfrak{l}_1)^{p_1}x = (\mathfrak{l}_0+\mathfrak{l}_1)^{p_o}x.$$

Thus the spectrum of the automorphism

$$(p_0|\mathfrak{l}K)^{-1}(p_1|\mathfrak{l}K):(\mathfrak{l}^{p_o})K \to (\mathfrak{l}^{p_o})K$$

equals the cross ratio

$$\mathrm{CR}(\mathfrak{l}K,(\mathfrak{l}_0+\mathfrak{l}_1)K,\mathfrak{l}_1K,\mathfrak{l}_0K) = \{c^{-1}xc\,|\,c \in K\setminus\{0\}\}.$$

But $x \in K\setminus Z$ implies that the projectivity $(\pi_0|l)^{-1}(\pi_1|l)$ is no perspectivity. By [11,62], [19], l^{β} is the proper part of a degenerate conic[2] d in the dual plane of $l^{\pi o} \vee l^{\pi 1}$. We adopt the notion *degenerate conic of lines* for a degenerate conic in a ruled plane. Replacing 0, 1 by any two elements j, $k \in \bar{Z}$ $(j\neq k)$ shows that $\{l^{\pi j}|j\in\bar{Z}\}$ is a fundamental chain [11,69] of d. Conversely, suppose that d is any degenerate conic of lines in a plane $\mathcal{F}$ not through a such that $\{\mathcal{F}\cap\mathcal{E}_j|j\in\bar{Z}\}$ is a fundamental chain of d. Then $d \cap c_a$ is the β-image of a line. Cf. [13,49] for a proof of this less obvious result.

4. Assume that $\mathcal{H} \neq \mathcal{I}$ is a hyperplane of $\mathcal{P}(\mathfrak{B})$. There are three possibile cases:

(i) $\mathcal{H} \cap \mathcal{I}$ *contains a directrix* a_j *of* $\mathcal{r}$: Then $f := \mathcal{H}^{\pi j} \subset \mathcal{E}_j$ is a line and $\mathcal{H}^{\pi k} = \mathcal{E}_k$ $(j\neq k\in\bar{Z})$, whence $\mathcal{H}^{\beta}$ is the set of lines in c_a meeting f.

(ii) $\mathcal{H} \cap \mathcal{I}$ *contains no line of* $\mathcal{r}$: It follows from [20,333] that $\{r\cap\mathcal{H}|r\in\mathcal{r}\}$ is a (non-degenerate) conic. Set $L_j := a_j\cap\mathcal{H}$ $(j=0,1)$. Then

[2] In [20,325] this set of points is named *C-configuration*.

$L_1^{\pi_0} \neq L_0^{\pi_1}$ and for every $X \in \mathcal{H}\setminus\mathcal{J}$ we have the plane $\mathcal{S}_X := L_0 \vee L_1 \vee X$ whose images under π_0, π_1 are lines through $L_1^{\pi_0}$, $L_0^{\pi_1}$, respectively. The map

$$\sigma_{\mathcal{H}}: \ell(L_1^{\pi_0}, \mathcal{E}_0) \to \ell(L_0^{\pi_1}, \mathcal{E}_1),\ \mathcal{S}_X^{\pi_0} \mapsto \mathcal{S}_X^{\pi_1}\ (X \in \mathcal{H}\setminus\mathcal{J})$$

is a projectivity of two pencils of lines, as follows from 3 (ii). The line a is fixed under $\sigma_{\mathcal{H}}$. Thus $\mathcal{H}^{\beta}$ is the set of lines in c_a which meet any two lines corresponding under $\sigma_{\mathcal{H}}$ (cf.[3,187]). A detailed discussion of $\mathcal{H}^{\beta}$ is given in [12].

(iii) $\mathcal{H} \cap \mathcal{J}$ *contains a line of* $\mathcal{R}$ *but no directrix*. We deduce from [20,325] that $\{r \cap \mathcal{H} \mid r \in \mathcal{R}\}$ is a degenerate conic and proceed as in (ii) with $L_0^{\pi_1} = L_1^{\pi_0}$ being the only difference. Hence $\mathcal{H}^{\beta}$ equals the set of lines in c_a which belong to any plane $\mathcal{F}$ intersecting $\mathcal{E}_0$ and $\mathcal{E}_1$ in two different lines corresponding under $\sigma_{\mathcal{H}}$. If we regard the star of planes through r^{π_0} as a projective plane within the dual space of $\mathcal{P}(\mathfrak{B})$, then the set of these planes $\mathcal{F}$ is the proper part of a degenerate conic f^*, say. The map $\sigma_{\mathcal{H}}$ is a generating map for f^* and a_Z^* is a fundamental chain of f^*. This follows as in 3 (iii).

5. Let $\mathcal{M} = \mathcal{P}(\mathfrak{M}) \subset \mathcal{P}(\mathfrak{B})$ be a plane and $\mathcal{M} \neq \mathcal{M} \cap \mathcal{J} =: m$. There are nine possibilities.

(i) $m \in \mathcal{R}$: For all $j \in \bar{Z}$ we get a line $\mathcal{M}^{\pi_j} \subset \mathcal{E}_j$ and these lines have $m^{\pi_0} = m^{\pi_j}$ as common point. By (1) all lines of $\mathcal{M}^{\pi_j}$ are lying in one plane $\mathcal{F}$. This implies that $\mathcal{M}^{\beta} = c_a \cap \ell(\mathcal{F})$.

(ii) m *is a directrix* a_j *of* $\mathcal{R}$: Hence there is a point $F := \mathcal{M}^{\pi_j}$ and $\mathcal{M}^{\beta} = \ell(F) \cap c_a$.

In any case other than (i) and (ii) there exist at most two directrices of $\mathcal{R}$ which meet m: If such directrices do exist, then let a_0 be none of them. We see that

$$e_i := (p_0|\mathfrak{M})^{-1}(p_i|\mathfrak{M}): \mathfrak{E}_0 \to \mathfrak{E}_i\ (0 \neq i \in \bar{Z})$$

yields a projective linear map $\varepsilon_i: \mathcal{E}_0 \to \mathcal{E}_i$ with $a^{\varepsilon_i} \subset a$. The image of $\mathcal{M}$ under β is the set of lines in c_a which join a point of $\mathcal{E}_0 \setminus a$ with its image under ε_i.

Choose any $i \in \bar{Z}\setminus\{0\}$. A point $Y = \mathfrak{y}K \in a$ is ε_i-invariant[3] if, and only if, $\mathfrak{y}$ is an eigenvector of e_i. This is equivalent to $Y = X^{\pi_0}$ with $X \in m \cap r$ for some line $r \in \mathcal{R}$. The eigenvalue of $\mathfrak{y}$ is in the centre of K if, and only if, X is on a directrix of $\mathcal{R}$. This follows from 3 (iii). The spectrum of e_i is the union of 0, 1 or 2 conjugacy classes of K [6,153], [7,207].

We infer from $X = m \cap r$ that the hyperplane $\mathcal{H} := \mathcal{M} \vee r$ is of type (i)

[3]We use this as a shorthand for "$Y = Y^{\varepsilon_i}$" or "Y^{ε_i} does not exist". The latter possibility is equivalent to $\mathfrak{y} \in \ker e_i$.

or (iii). The actual type of this $\mathcal{H}$ depends on the existence of a directrix in $\mathcal{H} \cap \mathcal{I}$ rather than the eigenvalue of $\mathfrak{y}$. By 3 (i) or (iii), the hyperplane $\mathcal{H}$ gives rise to a pencil of planes with an axis f, or a degenerate conic of planes. This set of planes is named f^* in both cases. All lines of $\mathcal{M}^\beta$ are incident with at least one plane of $f^* \setminus a^*$. For all $x \in \ell(X,\mathcal{M})\setminus\{m\}$ the plane $x \vee r$ is of type (i), whence x^β is a subset of $\ell(\mathcal{F})$ with $\mathcal{F} \in f^*\setminus a^*$. Since all lines of $\ell(X,\mathcal{M}) \setminus \{m\}$ are parallel, all images x^β are of common type (i) or (iii). Every plane $\mathcal{F} \in f^*\setminus a^*$ arises in this way. If $\mathcal{H}$ is of type (i) or (iii), then we name f a *focal line*[4] of $\mathcal{M}^\beta$ or f^* a *degenerate conic of focal planes* of $\mathcal{M}^\beta$, respectively.

If $\mathfrak{y}$ belongs to a central eigenvalue of any e_i, then let $X = m \cap r = m \cap a_j$ $(0 \neq j \in \bar{Z})$. There is a unique line $r' \in \mathcal{R}$ within the plane $m \vee a_j$. These two lines r, r' coincide if, and only if, Y is the only ε_j-invariant point of a, because $r'^{\pi_0} =: Y'$ is ε_j-invariant. We infer the existence of a hyperplane $\mathcal{H}'$ of type (i) through $\mathcal{M}$ with $m \vee a_j = r' \vee a_j = \mathcal{H}' \cap \mathcal{I}$. The line $f' := \mathcal{H}'^{\pi_j}$ meets all lines of $\mathcal{M}^\beta$. By construction, $Y' \in f' \subset \mathcal{E}_j$. This f' is a focal line of $\mathcal{M}^\beta$. For every $x \in \ell(X,\mathcal{M})\setminus\{m\}$ we deduce that x^β is the intersection of c_a with a pencil of lines $\ell(F,\mathcal{F})$, say. Necessarily $F = x^{\pi_j} \in f'$ and $\mathcal{F} = F \vee x^{\pi_0} \in f^*\setminus a^*$, so that $Y \in \mathcal{F}$.

Now the listing of all possible cases is being continued:

(iii) *m meets no line of $\mathcal{R}$:* If two lines of $\mathcal{M}^\beta$ would span a plane, then it would meet a in an ε_j-invariant point, an absurdity. Hence the lines of $\mathcal{M}^\beta$ are pairwise skew and $\mathcal{M}^\beta$ is a partial spread of $\mathcal{P}(\mathfrak{B})$.

(iv) *m meets a unique line of $\mathcal{R}$ and a directrix a_j:* There is a single ε_j-invariant point Y on a and through this point there is a focal line $f \subset \mathcal{E}_j$. For any point $F \in f$ there is a line in $\ell(\mathcal{E}_0,Y)$ which is being mapped on F under ε_j. The join of this line and f will be labeled F^σ. Writing f^* for the pencil of planes with axis f shows $F^\sigma \in f^*$. Then, by the linearity of e_j, $\sigma: f \to f^*$, $F \mapsto F^\sigma$ is a projectivity. Thus $\mathcal{M}^\beta$ is the intersection of c_a with the union of all pencils $\ell(F,F^\sigma)$ where $F \in f$.

(v) *m meets a unique line of $\mathcal{R}$ but no directrix:* Write Y for the only ε_j-invariant point of a and f^* for the degenerate conic of focal planes whose existence has been shown above. Every plane $\mathcal{F} \in f^*\setminus a^*$ contains a subset of $\mathcal{M}^\beta$ which is the proper part of a degenerate conic of lines, by 3 (iii). Any two of these degenerate conics of lines correspond under the β-transformed map of a

[4] One could also introduce the name *focal pencil of planes* for f^*.

translation of $\mathcal{P}(\mathfrak{B})$.

(vi) *m meets two different directrices a_j, a_k:* There are two focal lines $f_j \subset \mathcal{E}_j$ and $f_k \subset \mathcal{E}_k$. A line of c_a is in $\mathcal{M}^\beta$ if, and only if, it meets f_j and f_k.

(vii) *m meets exactly two lines of $\mathcal{R}$ but only one directrix:* In the terminology introduced above, let $Y = X^{\pi_0}$ and Y' be the only two ε_i-invariant points on a with $X \in m \cap a_j$, say. So we have a focal line $f' \subset \mathcal{E}_j$ with $Y' \in f'$ and a degenerate conic f^* of focal planes all of which pass through Y. The set $\mathcal{M}^\beta$ is the intersection of c_a with the union of all pencils of lines having their centres on f' and their planes in $f^* \setminus a^*$.

(viii) *m meets exactly two lines of $\mathcal{R}$ but no directrix:* Write Y and Y' for the two ε_i-invariant points. There exist two degenerate conics of focal planes which will be named f^* and f'^*, respectively. All planes of f^* pass through Y and all planes of f'^* contain Y'. When fixing one plane $\mathcal{F} \in f^* \setminus a^*$, then the planes of $f'^* \setminus a^*$ will meet $\mathcal{F}$ in a set of lines equal to $\mathcal{M}^\beta \cap \ell(\mathcal{F})$; this is the β-image of a line of type (iii).

(ix) *m meets more than two lines of $\mathcal{R}$:* Let Y and Y' be any two different ε_i-invariant points. Then the arguments of (viii) may be repeated.

6. The stereographic projection of the Plücker quadric (cf. 1) motivates the following definition of a relation ι on $\mathcal{I} \times \ell_a$ with $\ell_a := \ell(\mathfrak{B}) \setminus c_a$:

If $X \in \mathcal{I}$ is off $\mathcal{R}$, then $(X,y) \in \iota :\Leftrightarrow y=a$. If X is on a line $r \in \mathcal{R}$, then $(X,y) \in \iota :\Leftrightarrow y \in \ell(r^{\pi_j},\mathcal{Y})$ ($j \in \bar{Z}$ arbitrary) with $\mathcal{Y} \in a^*$ being subject to the condition

$$\mathrm{CR}(\mathcal{Y},\mathcal{E}_1,\mathcal{E}_0,\mathcal{E}_\infty) = \mathrm{CR}(X, r\cap a_1, r\cap a_0, r\cap a_\infty).$$

The union of $\beta \subset (\mathcal{P}(\mathfrak{B}) \setminus \mathcal{I}) \times c_a$ and $\iota \subset \mathcal{I} \times \ell_a$ is a relation $\rho \subset \mathcal{P}(\mathfrak{B}) \times \ell(\mathfrak{B})$. If $\mathcal{X} \subset \mathcal{P}(\mathfrak{B})$, then we adopt the notation $\mathcal{X}^\rho := \{y \in \ell(\mathfrak{B}) \mid \exists\ X \in \mathcal{X} \text{ with } (X,y) \in \rho\}$.

Every subspace $\mathcal{S} \subset \mathcal{P}(\mathfrak{B})$ gives rise to a set $\mathcal{S}^\rho \subset \ell(\mathfrak{B})$. If $\mathcal{S}$ is not at infinity, then $\mathcal{S}^\rho$ will be called the *closure of $\mathcal{S}^\beta$.* For example, let $l \in \ell(\mathfrak{B}) \setminus \ell(\mathfrak{J})$. Then $l^\beta = \ell(F,\mathcal{F}) \setminus \{a\}$ implies $l^\rho = \ell(F,\mathcal{F})$. If there is a regulus $\mathcal{s}$ with $l^\beta = \mathcal{s} \setminus \{a\}$ then $l^\rho = \mathcal{s}$. Finally, assume that d is a degenerate conic of lines in a plane $\mathcal{F}$ and $l^\beta = d \cap c_a$. Then $l^\rho = d \cup (\cup(\ell(\mathcal{F}\cap a,\mathcal{Y}) \mid \mathcal{Y}=a \vee y, y \in d \cap c_a\})$. It is lengthy but streightforward to discuss all possible cases. If $\mathcal{P}(\mathfrak{B})$ is Pappian, then the image of $\mathcal{S}^\rho$ under the Klein map γ is a (complete) intersection of the Plücker quadric with a subspace through a^γ. This shows that the following definitions are in accordance with the classical ones. Cf. e.g. [3,185], [4,181], [5,322], [15,30], [16,17], [16,31], [20,241], [21,312], [21,319].

A set of lines of $\ell(\mathfrak{B})$ is called a *linear congruence of lines* or *linear complex of lines*, if it corresponds under a collineation of $\mathcal{P}(\mathfrak{B})$ to a set of lines which is in relation ρ with a hyperplane or plane of $\mathcal{P}(\mathfrak{B})$, respectively. By 4, there are three types of linear complexes of lines which will be named *special* (type (i)), *general* (type (ii)), or *degenerate* (type (iii)). By definition of γ, the set $\mathcal{J}^{\gamma}$ is a special linear complex. Besides nine possible types of linear congruences arising from affine planes (cf. 5) we may get three additional types which are related to planes at infinity. In the Pappian case the number of possibilities reduces from twelve to four. (The author is still looking for apropriate names for the various types of linear congruences in the non-Pappian case.)

If $\mathcal{P}(\mathfrak{B})$ is Pappian, then the line a does not play a special role for a set $\mathcal{S}^{\rho}$ whenever $\mathcal{S}$ is not in $\mathcal{J}$, since $\mathcal{S}^{\rho\gamma}$ is not lying in the tangent hyperplane of the Plücker quadric in a^{γ}. However, if K is a skew field, then the line a and the subpencil a_Z^* may be essential. For example, the line a, the chain a_Z^* (and a Z-chain in the line a) are uniquely determined by a general linear complex $\mathcal{H}^{\rho}$ [12].

References

[1] ARTIN, E.: Geometric Algebra, New York, Interscience, 1957.

[2] BENZ, W.: Vorlesungen über Geometrie der Algebren, Grundlehren Bd.197, Berlin-Heidelberg-New York, Springer, 1973.

[3] BRAUNER, H.: Geometrie projektiver Räume I, Mannheim-Wien-Zürich, BI-Wissenschaftsverlag, 1976.

[4] BRAUNER, H.: Geometrie projektiver Räume II, Mannheim-Wien-Zürich, BI-Wissenschaftsverlag, 1976.

[5] BURAU, W.: Mehrdimensionale projektive und höhere Geometrie, Berlin, VEB Dt. Verlag d. Wissenschaften, 1961.

[6] COHN, P.M.: The Similarity Reduction of Matrices over a Skew Field, Math. Z. 132, 151-163 (1973).

[7] COHN, P.M.: Skew Field Constructions, London Math. Soc. Lect. Notes Ser. 27, Cambridge U.P., Cambridge, 1977.

[8] HAVLICEK, H.: Zur Theorie linearer Abbildungen I, J. Geometry 16, 152-167 (1981).

[9] HAVLICEK, H.: Zur Theorie linearer Abbildungen II, J. Geometry 16, 168-180 (1981).

[10] HAVLICEK, H.: Die linearen Geradenabbildungen aus dreidimensionalen projektiven Pappos-Räumen, Sb. österr. Akad. d. Wissensch., math.-naturw. Kl., Abt. II 192, 99-111 (1983).

[11] HAVLICEK, H.: Applications of Results on Generalized Polynomial Identities in Desarguesian Projective Spaces. In: R. KAYA, P. PLAUMANN and K. STRAMBACH (eds.): Rings and Geometry, Dordrecht, D.Reidel, 1985.

[12] HAVLICEK, H.: Lineare Geradenkomplexe über Schiefkörpern, Arch. Math., to appear 1988.

[13] HAVLICEK, H.: Liniengeometrische Modelle affiner 4-Räume, Proc. Conf. on Geometry Thessaloniki, 47-53 (1987).

[14] HERZER, A.: Hom(U,V) und affine Graßmann-Geometrie, Mitt. Math. Sem. Gießen 164 (Coxeter-Festschrift), 199-215 (1984).
[15] HIRSCHFELD, J.W.P.: Finite Projective Spaces of Three Dimensions, Oxford, Clarendon Press, 1985.
[16] HLAVATY, V.: Differential Line Geometry, Groningen, P. Noordhoff, 1953.
[17] KRÜGER, W.: Regelscharen und Regelflächen in dreidimensionalen desarguesschen Räumen, Math. Z. 93, 404-415 (1966).
[18] METZ, R.: Der affine Raum verallgemeinerter Reguli, Geom. Dedicata 10, 337-367 (1981).
[19] RIESINGER, R.: Entartete Steinerkegelschnitte in nichtpapposschen Desarguesebenen, Monatsh. Math. 89, 243-251 (1980).
[20] SEGRE, B.: Lectures on Modern Geometry, Roma, Ed. Cremonese, 1962.
[21] VEBLEN, O. and J.W. YOUNG: Projective Geometry Vol. I, New York-Toronto-London, Blaisdell, 1938.
[22] WELLS, A.L.: Universal Projective Embeddings of the Grassmannian, Half Spinor and Dual Orthogonal Geometries, Quart. J. Math. Oxford 34, 375-386 (1983).

Hans Havlicek
Institut für Geometrie
Technische Universität
Wiedner Hauptstraße 8-10
A-1040 Wien
Austria